SINGLE SERVER RETRIAL QUEUEING MODELS UNDER MULTIPLE WORKING VACATION POLICY

By

S.Selvi Ajitha

REG.NO. 20PMA014

Supervisor

Dr. N.Sangeetha

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore-641043

In Partial Fulfillment of the Requirement for the Degree of

Master of Science in Mathematics

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Signature of the Head of the Department

N. S 19/05/2022

Signature of the Supervisor

DECLARATION

I declare that the dissertation, entitled on "**SINGLE SERVER RETRIAL QUEUEING MODELS UNDER MULTIPLE WORKING VACATION POLICY**" submitted by me for the Degree of Master of Science is a record of work carried out by me during the period from December 2021 to May 2022 under the guidance of Dr. N. SANGEETHA Teaching and Research Fellow, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, and it has not formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or other similar title in this University or any other University or Institution of Higher Learning.

S. Selvi Tirtha

Signature of the Candidate

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ACKNOWLEDGEMENT

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CHAPTER 1

1. INTRODUCTION

Queuing theory is the mathematical study of waiting lines or queues. In queuing theory, a model is constructed so that queue lengths and waiting times can be predicted. A flow of customers from a finite or infinite population towards the service facility forms a queue on account of the lack of capability to serve them all at a time. The arrival unit that requires some service to be performed is called 'customer'. The customer may be persons, machines, vehicles and so on. Queue stands for the number of customers waiting to be served. This does not include the customers being serviced. The process or system that performs services to the customers is termed by service channel or service facility. The objective of the queuing system is that the average waiting time of the customer is minimized and the percentage utilization of the server is maintained above the desired level.

APPLICATION AREAS OF QUEUEING SYSTEM:

- Bank counters
- Tollgate
- Ration shop
- Mainframe computer centre
- Library
- Traffic signal
- Final inspection station of television assembly line
- Airport runways
- Telephone booth
- Maintenance shop

1.1 Classical Queueing System

A classical queueing system can be described as customers arriving for service, waiting for service if it is not immediate, and if having waited for service, leaving the system after being served. The general structure of a queueing system is presented in the Fig. 1.1.

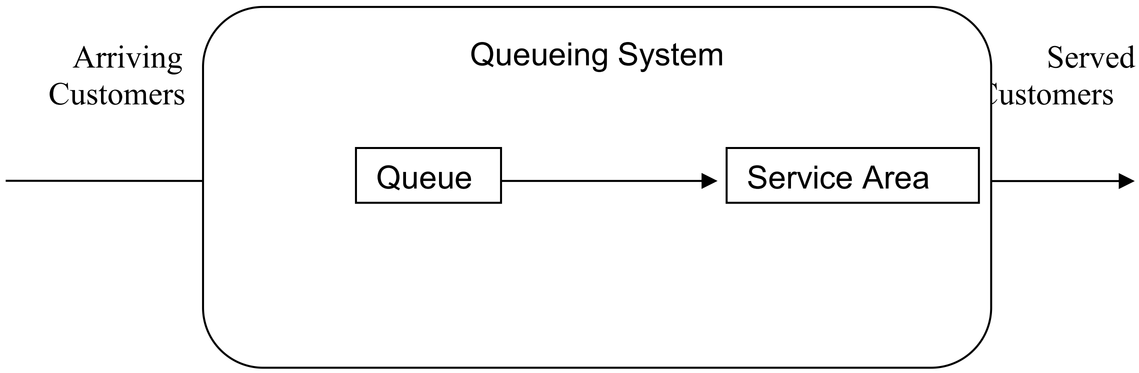

Fig. 1.1 General Structure of a Queueing System

1.2 Characteristics of a Queueing System

The basic characteristics of a queueing system which provide an adequate description are arrival pattern, service pattern, queue discipline, system capacity and service channels.

Arrival Pattern

Arrival pattern describes the way how units arrive and join the system. The arrival may be either single or batches of variable or fixed size. The source of units may be finite or infinite. The arrival patterns are often measured in terms of an average number of arrivals per unit time. The inter-arrival time is the interval between two consecutive arrivals. In case, the arrival times are known with certainty, the queueing problems are characterized as deterministic models. However, in usual queueing situations, the process of arrivals is stochastic and it is necessary to know the probability distribution associated with the successive arrivals.

The most common stochastic queueing models assume that inter-arrival times follow an Exponential distribution. The arrival pattern also describes the behaviour of the customers as some customers may wait patiently in the queue and some may be impatient if it takes a long time to receive the desired service. If an arriving customer decides not to join the queue, the customer is said to have balked. If a customer leaves the queue after joining due to impatience, it is known as reneging. In case there are two or more parallel waiting lines and a customer moves from one queue to another, the customer is said to have jockeyed. An arrival process could be stationary or non-stationary according to the probability distribution describing the arrival pattern being time-independent or dependent of time.

Service Pattern

Service pattern describes the manner in which the service is rendered. The customers may be served in single or in batches of variable or fixed size and the time required for serving one customer or customers in batches is called the service time. The service pattern of customers may be stationary or non-stationary with respect to time and state-dependent or independent with respect to the number of customers waiting for service. The service time may be deterministic or probabilistic.

Queue Discipline

Queue discipline refers to the manner in which customers are selected for service from the queue. The most common disciplines are first come, first served according to which the customers are served in order of their arrival. The following are the various queue disciplines:

- FIFO: first in first out (or first come first served FCFS)
- LIFO: last in, first out, usually seen in a warehouse where the items that come last are
  taken out first
- SIRO: service in random order

- Priority disciplines: Under this discipline, the service is offered to customers depending on the priority with the other customers. There are two situations in priority discipline called pre-emptive and non-pre-emptive priority. In the pre-emptive case, the customer with higher priority is allowed to enter service immediately suspending the service in progress to a customer with lower priority. In non-pre-emptive case the higher priority goes to the head of the queue but cannot get into the service until the customer presently in service is completed.

System Capacity

In some queueing processes there is a physical limitation to the amount of waiting room so that when the line reaches a certain length, no further customers are allowed to enter until space becomes available, as a result of service completion. These are referred to as finite queueing situations. A queue with a limited waiting room can be viewed as one with forced balking.

Service Channels

The number of servers in a queueing model may be finite or infinite. The number of servers may be arranged in series, parallel or a combination of both, depending upon the nature of the services required. In parallel channels, all the channels provide identical services so that several customers may be served simultaneously. In series channels, a customer must pass through successively in several ordered channels before service is completed.

1.3 Kendall's Notation

A queueing process can be represented by the notation introduced by Kendall (1951) as A / B / C / X / Y, where A represents the inter-arrival time distribution of customers, B denotes the service time distribution, C the number of parallel servers, X represents the capacity of the system and Y denotes the queue discipline. If the queueing system has infinite capacity and the queue discipline is FIFO, then the system is denoted as A/B/C without mentioning X and Y.

The notation $M^X/G/1$ indicates a batch arrival queueing system with exponential inter-arrival times, general service times, single server, infinite system capacity and first come first served queue discipline.

1.4 Retrial Queueing System

In conventional queueing theory it is usually assumed that an arriving customer who cannot get service immediately either joins the waiting line or leaves the system forever. Sometimes impatient customers leave the queue but it is also assumed that they are leaving the system forever. In real life situation, such customers after some random period of time return to the system and try to get service. The standard queueing models do not consider the phenomenon of retrials and therefore cannot be applied in solving a number of practical problems. Retrial queues have been introduced to meet this inadequacy. Retrial queueing systems are characterized by the feature that arriving customers who cannot receive service immediately may join a virtual queue called orbit to try their request after some random time. General structure of a retrial queueing system is presented in Fig. 1.2.

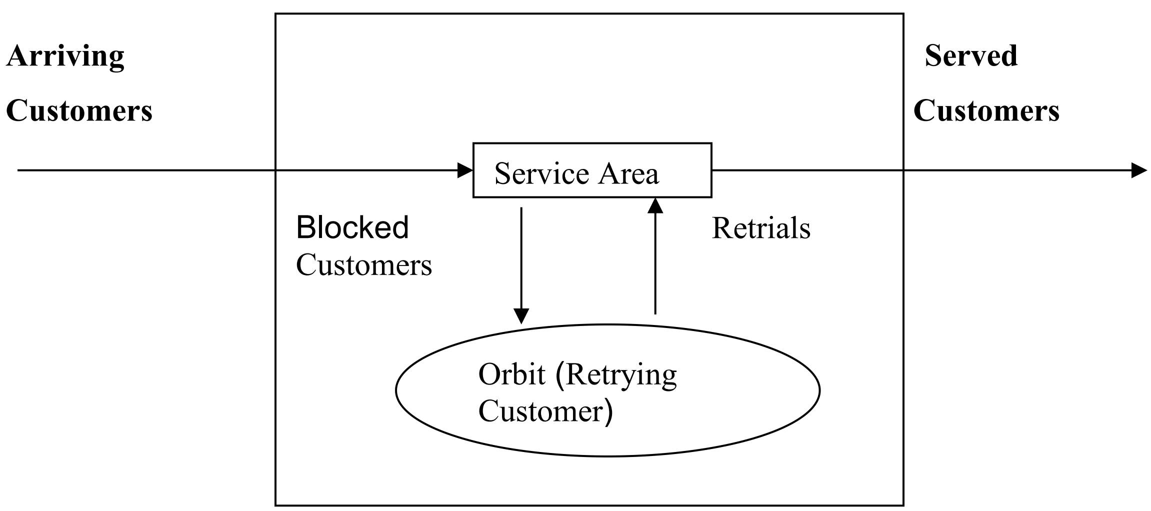

Fig.1.2 General Structure of a Retrial Queueing System

1.5 Literature survey:

Queueing theory has advancement in many disciplines since its inception. The theory of Probabilities and Telephone Conversations of Erlang (1909) has historic importance in Queueing Theory due to its exact mathematical treatment.

Molina (1927) published his paper, Application of the theory of probability to telephone trunking problems. Kendall (1953) on the embedded chain, Takacs (1955) on waiting time, Cox (1955) on supplementary variables, Champernowne (1956) on the uses of random walks, Saaty (1960) on absorbing barriers and time-dependent solutions, Little (1961) on deriving the queueing formula $L = \lambda W$, Conway and Maxwell (1962) on state-dependent service and Neuts (1973) on the discrete-time queueing model are some remarkable work in queueing literature. A detailed account of the investigations made by these authors may be found in the books by Takagi (1991), etc.The applications of retrial queues in science and engineering are given by Kulkarni and Liang (1997).

Retrial Queues:

Queueing systems in which arriving customers who find all servers and waiting positions occupied may retry for service after a period of time are called retrial queues. Retrial queues have been widely used to model many problems in telephone switching systems, telecommunication networks, computer networks and computer systems. One of the earliest papers in retrial queues was On the Influence of Repeated Calls in the Theory of Probabilities of Blocking by Kosten (1947). Keilson et al. (1968) published the first result on M/G/1 retrial queue using the method of a supplementary variable. The first investigation with general retrial time was done by Kapyrin (1977), in which each customer in the orbit generates a stream of repeated attempts that are independent of the customers in the orbit and the state of the server. A review of a retrial queueing literature can be found in the survey papers of Yang and Templeton (1987), Falin (1990), Kim and Kim (2016), Shekar et al. (2016) and Tuan Phung-Duc (2019), the bibliographies of Artalejo (1999a, 1999b and 2010) and the books by Falin and Templeton (1997) and Artalejo and Gomez Corral (2008). The applications of retrial queues in science and engineering are given by Kulkarni and Liang (1997).

Queues that feature multiple entities arriving simultaneously are among the oldest models in queueing theory and are often referred to as batch arrival queueing systems. The batch arrival retrial queueing model was introduced by Falin (1976). Kulkarni (1986) and Falin (1988) analyzed multiple classes of customers with batch arrivals. Artalejo and Atentia (2004) analyzed a single server retrial queue with batch arrivals. A detailed study on batch arrival retrial queues under both classical and constant retrial policies was done by Jain et al. (2008). Many authors including Kalyanaraman (2012), Yamamuro (2012), Singh et al. (2015) and Zirem et al. (2019) discussed retrial queueing situations with batch arrivals.

Retrial queueing systems with **batch arrival** are common in many practical situations. In digital communication systems, messages which are transmitted could consist of a random number of packets. Falin (1976) introduced the single server batch arrival retrial queueing model. He used the embedded Markov chain technique to derive the joint distribution of the channel state and the queue length. Kulkarni (1986) and Falin (1988) examined multiple classes of customers with batch arrivals. Artalejo and Atencia (2004) presented a single server retrial queue with batch arrivals and carried out an extensive analysis of the system, including existence of the stationary regime, embedded Markov chain and stochastic decomposition. A detailed study on batch arrival queue under both classical and constant retrial policies was considered by Jain et al. (2008). Yamamuro (2012) analyzed an M/G/1 retrial queue 7 with batch arrivals and obtained the expected queue length. Many authors including Florea and Nanau (2015), Kim (2017), Niranjan et al. (2019), Krishnamoorthy et al. (2021) and Sadhna Singh and Srivastava (2021) discussed retrial queueing situations with batch arrival.

Impatient Customers

Due to numerous applications in computer and communication systems, the retrial queues with impatient customers have nowadays become the point of attraction to the queue theorists. Sometimes, the customers may get discouraged on the unavailability of the server, therefore either the customers would not join the orbit, i.e., they would like to balk or leave the system after waiting for some time, i.e., renege. Both reneging and balking can be visualized in day-to-day congestion situations where some impatient customers quit the system without getting the service. Such a scenario is prevalent in computer networks, telecommunication systems and call centers. The Retrial Queueing model with impatient customers was first studied by Cohen (1957). Yang et al. (1990) investigated the M/G/1 retrial queue with impatient customers and obtained system performance measures in terms of server utilization. A multi-server retrial queue with balking was considered by Falin and Artalejo (1995). Further Artalejo and Lopez-Herrero (2000) presented an M/G/1 retrial queue with balking to obtain the limiting distribution of the number of customers in the system. Retrial queues with a constant retrial rate, server downs and impatient customers analyzed by Li and Zhao (2005). Artalejo and Pla (2009) modelled a Markovian multi-server retrial queue with customer balking and illustrated the effect of impatience in the problem arising from call center management. Kasturi Ramanath and Lakshmi (2011) provided a two phase retrial queue with Bernoulli feedback and non persistent customers and derived the steady state probability generating functions of the orbit size and system size using supplementary variable technique. Arrar et al. (2012) investigated the asymptotic behavior of batch arrival retrial queue with the impatient phenomenon. Explicit expressions for the average queue length of orbit and the system of an unreliable bulk arrival retrial queue with impatient customers multi-optional service were analysed by Bhagat and Jain (2013). Suganthi and Pavai Madheswari (2015) discussed retrial queueing system with customer impatience and derived performance measures such as mean orbit size and system size. Sumitha and Udaya Chandrika (2016) considered two phase retrial queues with impatient customers, Bernoulli vacation and orbital search. Using the supplementary variable technique, the authors derived steady-state probabilities and important system measures. Gao et al. (2017) studied a Markovian single server queue with impatient customers under constant retrial time and provided some important performance indices. A finite capacity single server interdependent retrial queueing model with controllable arrival rates, balking and reneging was proposed by Nisha et al. (2019). Analysis of a retrial queue with group service of impatient customers was examined by D'Arienzo et al. (2020). Sztrik et al. (2021) suggested an unreliable retrial queueing system of M/G/1 type with impatient customers and collisions and performed a stochastic simulation to check changes in system characteristics when other distributions of service times are used.

Server Breakdown and Repair

Breakdown is a remarkable and unavoidable phenomenon in the service facility of a queueing system. Retrial queues with servers subject to breakdown and repair are often encountered in many practical applications. The study of a retrial queueing models with server breakdown and repair can be found in the articles of Aissani (1988), Kulkarni and Choi (1990) and Aissani and Artalejo (1998). Djellab (2002) studied a single server retrial queueing system subject to breakdowns and used its stochastic decomposition property to approximate the model performance in the case of general retrial times. Sherman and Kharoufeh (2006) presented an M/M/1 retrial queue with an unreliable server and demonstrated the stochastic decomposability of the orbit and system size distributions and provided expressions for the limiting distribution of the server status and standard queueing performance measures. Choudhury et al. (2010) obtained explicit expressions of availability, failure frequency and mean busy period of two-phase batch arrival unreliable retrial queueing system. Senthil Kumar and Arumuganathan (2013) described an unreliable batch arrival retrial queue with two types of repair with second optional service and derived the orbit size and system size distributions with reliability measures. An MX/G/1 non-Markovian retrial queueing model with a second optional service, multiple vacations, breakdown and repair was proposed by Suganya and Ayyappan (2014). Singh et al. (2016) determined the queue size distribution of an unreliable batch arrival retrial queue with additional service under the Bernoulli vacation policy by incorporating the supplementary variables corresponding to service time, repair time and retrial time. Singh and Kaur (2017) extended the same model by incorporating multi phase repair. Batch arrival retrial queue with delay time and additional multi optional repair was addressed by Sumitha and Udaya Chandrika (2017). Kuki et al. (2020) analysed numerically a finite source Markov retrial system with non-reliable server, collision and impatient customers. Revathi et al. (2020) suggested a single server batch arrival retrial queue with optional re-service under modified Bernoulli vacation which consists of a breakdown and delay time and derived the probability generating functions of the number of customers in the orbit for different states using supplementary variable technique. Jain and Kaur (2021) considered an unreliable bulk arrival retrial queue with multiphase optional service by incorporating the features of balking, Bernoulli vacation and Bernoulli feedback. The authors evaluated analytical results for the queue size distribution and system metrics and performed a comparative study of the exact waiting time derived by using the supplementary variable technique and the approximate waiting time derived by using the maximum entropy principle.

Working vacation

In the working vacation period, the server gives service to a customer at a lower service rate, but the server stops the service completely during the normal vacation period. This queueing system has major applications in providing network service, web service, file transfer and mail service, etc. Servi and Finn (2002) have introduced an M/M/1 queueing system with working vacations.Wu and Takagi (2006) have extended the M/M/1/WV queue to an M/G/1/WV queue.Liu and Song(2013) have discussed a discrete time retrial queue with non-persistent customers and working vacations.Zhang and Liu(2015) have analyzed a single server retrial queue with working vacation and interruption. Yang and Wu(2019) analyzed a steady state analysis of an M/M/1 retrial queue with working vacations, in which the server is subject to starting failures.Jain et al. (2021) have discussed a Markovian queue with working vacation, retrial and impatient customers including the feature of imperfect service during working vacation.

Feedback Queueing Models:

Feedback in retrial queueing literature represents customer dissatisfaction because of inappropriate quality of service. In case of feedback, after getting partial or incomplete service, the customer retries for service. In computer communication, the transmission of a protocol data unit is sometimes repeated due to the occurrence of an error. Rework in industrial operations is also an example of a queue with feedback. Takacs (1963) was the first to study feedback queueing models. The author studied queues with feedback to determine the distribution of the queue size and the first two moments of the Distribution function of the total time spent in the system by a customer. The authors Choi and Kulkarni (1992), Choi et al. (1998), Lee (2005) and Chen et al. (2009) discussed the concept of feedback in retrial queueing systems. Using supplementary variable technique, the joint distribution of the server state and orbit length of a non-Markovian single server feedback retrial queue with collisions and general retrial times was investigated by Krishna Kumar et al. (2010). Boualem et al. (2012) used stochastic ordering techniques to establish various monotonicity results with respect to arrival rates, service time distributions and retrial parameters for an M/G/1 retrial queue with a Bernoulli feedback. Tao et al. (2014) considered an M/M/1 retrial queue with working vacations, vacation interruption, Bernoulli feedback under N-policy and derived the stationary probability distribution with performance measures by matrix analytic method. A retrial queueing system with two classes of jobs with constant retrial, abandonment and Feedback customers were studied by Bouchentouf et al. (2015). Prakash Rani et al. (2017) obtained mean queue length and mean system length of a batch arrival retrial queueing system with additional optional service and customer feedback.Pankaj Sharma (2018) investigated an unreliable single server retrial queueing model with discouragement, Bernoulli feedback, and repair and modified vacation policy. Chang et al. (2018) studied an unreliable single server retrial queue with geometric loss and feedback under threshold-based policy to derive the formulae for computing rate matrix and stationary probabilities. Single server bulk arrival retrial queue with M stages of heterogeneous service and feedback was presented by Sangeetha et al. (2020). Jain and Kaur (2021) analysed a batch arrival unreliable retrial queue with multiphase optional service by incorporating the features of Bernoulli feedback, vacation and balking.

PROFILE OF WORK

Introduction and Review of literature are presented in chapter 1.

In chapter 2, the performance analysis of a single server batch arrival retrial queue with working vacation and vacation interruption is studied.

In chapter 3, single server feedback retrial queue with impatient customers, server breakdown and repair under multiple working vacation policy is analyzed.

CHAPTER 2

2. BATCH ARRIVAL RETRIAL QUEUE WITH WORKING VACATION AND VACATION INTERRUPTION

A batch arrival retrial queue with working vacation and vacation interruption is considered. Customers arrive in batches according to the Poisson process. If the server is idle, one of the customers from the batch starts the service immediately and the remaining customers go to the orbit. Otherwise all the arriving customers join the orbit. After each service completion, the customer can join the orbit as a feedback customer or depart the system. As soon as the completion of regular service, if the orbit becomes empty, then the server goes for a working vacation. The server works at a lower service rate during working vacation period. The retrial time, service time and working vacation time are generally distributed. Using supplementary variable technique various performance measures are derived.

2.1 Model Description

Consider a single server batch arrival retrial queueing system in which customers arrive in groups or batches according to the Poisson process with parameter λ.The batch size Y of positive customers is a random variable with distribution function P(Y=k)= C_k , k = 1, 2,..., ∞, and probability generating function $C(z)=\sum_{k=1}^{\infty} C_k z^k$ and first two moments m_1 and m_2 . Upon arrival, if the server is busy , all the customers join the orbit. Otherwise, one of the customers in the batch starts the service and the rest join the orbit and try their service after some random amount of time. The inter-retrial times form an arbitrary distribution $A(x)$ with the Laplace-Stieltjes transform $A^*(s)$ and the hazard rate function $\eta(x)=\dfrac{dA(x)}{1-A(x)}$. The service time is generally distributed with the distribution function $B(x)$ and Laplace-Stieltjes transform $B^*(s)$ and the hazard rate function $\mu(x) = \dfrac{dB(x)}{1-B(x)}$.

As soon as the completion of regular service, if the orbit becomes empty, then the server goes for a working vacation. The server works at a lower service rate during working vacation period. The working vacation period is the period when the server takes a vacation in order to attend secondary jobs at different service rates in the empty system and the vacation follows an exponential distribution with parameter ν. If any customers in the orbit are at service completion, the server will stop the vacation and come back to the normal busy period which means vacation interruptions happens. During the working vacation period, the service time is generally distributed with distribution function $S(x)$ and Laplace-Stieltjes transform $S^*(x)$ and the condition completion rate $\gamma(x) = \dfrac{dS(x)}{1-S(x)}$.

2.2 Mathematical Analysis of the System

In this section, the steady state difference differential equations for the retrial system under consideration are formulated by considering the elapsed retrial time, the elapsed service time and the elapsed working vacation time as supplementary variables. Further, the probability generating function for the server state and the number of customers in the orbit and in the system are derived.

2.2.1 Server State Probabilities and Notations

The server state $X(t)$ at time t is defined as

$$X(t) = \begin{cases} 0, \text{ server is idle} \\ 1, \text{ server is busy in a regular service period} \\ 2, \text{ server is busy in working vacation period} \end{cases}$$

Let $\xi_r(t), r = 0,1 or 2$ respectively denote the elapsed retrial time, regular service time or working vacation time.

The supplementary variables are introduced in order to obtain a bivariate Markov process $\{Nt), X(t), t \geq 0\}$,where $N(t)$ denotes the number of customers in the orbit at time t and $X(t)$ denotes the server state.

The joint distributions of the server state and queue size are defined as

$I_0(t) = P\{X(t) = 0, N(t) = 0\}$

$I_n(x,t) = P\{X(t) = 0, N(t) = n, x \leq \xi_0(t) < x + dx\}, \quad n \geq 1$

$P_n(x,t) = P\{X(t) = 1, N(t) = n, x \leq \xi_1(t) < x + dx\}, \quad n \geq 0$

$W_n(x,t) = P\{X(t) = 2, N(t) = n, x \leq \xi_2(t) < x + dx,\} \quad n \geq 0$

2.2.2 Governing Equations

The system of equations that governs the model under consideration is given below

$$\frac{d}{dt} I_0(t) = -\lambda I_0 + \int_0^\infty P_0(x,t)\mu(x)dx + \int_0^\infty W_0(x,t)\gamma(x)dx \tag{2.1}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) I_n(x,t) = -(\lambda + \eta(x)) I_n(x,t), \quad n \geq 1 \tag{2.2}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) P_n(x,t) = -(\lambda + \mu(x)) P_n(x,t) + \lambda(1 - \delta_{0n}) \sum_{k=1}^{n} C_k P_{n-1}(x,t), \quad n \geq 0 \tag{2.3}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) W_n(x,t) = -(\lambda + \nu + \gamma(x)) W_n(x,t) + \lambda(1 - \delta_{0n}) \sum_{k=1}^{n} C_k W_{n-1}(x,t), \quad n \geq 0 \tag{2.4}$$

with boundary conditions

$$I_n(0,t) = \int_0^\infty P_n(x,t)\mu(x)dx + \int_0^\infty W_n(x,t)\gamma(x)dx, \qquad n \geq 1 \tag{2.5}$$

$$P_0(0,t) = \int_0^\infty I_1(x,t)\eta(x)dx + \nu \int_0^\infty W_0(x,t)dx \tag{2.6}$$

$$P_n(0,t) = \int_0^\infty I_{n+1}(x,t)\eta(x)dx + \nu\int_0^\infty W_n(x,t)dx + \lambda C_{n+1}I_0 + \lambda\sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x,t)dx \qquad (2.7)$$

$$W_n(0,t) = \begin{cases} \lambda I_0, & n = 0 \\ 0, & n \geq 1 \end{cases} \qquad (2.8)$$

Assuming the existence of steady state probabilities, the steady state equation corresponding to the equation (2.1) to (2.8)

$$\lambda I_0 = \int_0^\infty P_0(x)\mu(x)dx + \int_0^\infty W_0(x)\gamma(x)dx \qquad (2.9)$$

$$\frac{dI_n(x)}{dx} = -(\lambda + \eta(x))I_n(x), \; n \geq 1 \qquad (2.10)$$

$$\frac{dP_n(x)}{dx} = -(\lambda + \mu(x))P_n(x) + \lambda(1-\delta_{0n})\sum_{k=1}^{n} C_k P_{n-1}(x), \quad n \geq 0 \qquad (2.11)$$

$$\frac{dW_n(x)}{dx} = -(\lambda + \nu + \gamma(x))W_n(x) + \lambda(1-\delta_{0n})\sum_{k=1}^{n} C_k W_{n-1}(x), \quad n \geq 0 \qquad (2.12)$$

where δ_{0n} is the Kronecker delta

with boundary conditions

$$I_n(0) = \int_0^\infty P_n(x)\mu(x)dx + \int_0^\infty W_n(x)\gamma(x)dx, \; n \geq 1 \qquad (2.13)$$

$$P_0(0) = \int_0^\infty I_1(x)\eta(x)dx + \nu\int_0^\infty W_0(x)dx \qquad (2.14)$$

$$P_n(0) = \int_0^\infty I_{n+1}(x)\eta(x)dx + \nu\int_0^\infty W_n(x)dx + \lambda C_{n+1}I_0 + \lambda\sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x)dx \qquad (2.15)$$

$$W_n(0) = \begin{cases} \lambda I_0, & n = 0 \\ 0, & n \geq 1 \end{cases} \qquad (2.16)$$

2.2.3 Orbit Size Distribution at Random Epoch

To analyze the model, the following probability generating function are defined.

$$I(x,z)=\sum_{n=1}^{\infty} I_n(x)z^n \quad ; \quad P(x,z)=\sum_{n=0}^{\infty} P_n(x)z^n \quad ; \quad W(x,z)=\sum_{n=0}^{\infty} W_n(x)z^n \tag{2.17}$$

Multiplying the equations (2.9) to (2.16) by z^n and summing over all possible values of n, we get

$$\left(\frac{\partial}{\partial x}+\lambda+\eta(x)\right)I(x,z)=0 \tag{2.18}$$

$$\left(\frac{\partial}{\partial x}+\lambda(1-C(z))+\mu(x)\right)P(x,z)=0 \tag{2.19}$$

$$\left(\frac{\partial}{\partial x}+\lambda(1-C(z))+\nu+\gamma(x)\right)W(x,z)=0 \tag{2.20}$$

$$I(0,z)=\int_0^{\infty} P(x,z)\mu(x)dx+\int_0^{\infty} W(x,z)\gamma(x)dx-\lambda I_0 \tag{2.21}$$

$$P(0,z)=\frac{1}{z}\int_0^{\infty} I(x,z)\eta(x)dx+\frac{\lambda}{z}C(z)I_0+\frac{\lambda}{z}C(z)\int_0^{\infty} I(x,z)dx+\nu\int_0^{\infty} W(x,z)dx \tag{2.22}$$

$$W(0,z)=\lambda I_0 \tag{2.23}$$

Solving the partial differential equations (2.18) to (2.20), we get

$$I(x,z)=I(0,z)e^{-\lambda x}(1-A(x)) \tag{2.24}$$

$$P(x,z)=P(0,z)e^{-\lambda(1-C(z))x}(1-B(x)) \tag{2.25}$$

$$W(x,z)=W(0,z)e^{-(\lambda(1-C(z))+\nu)x}(1-S(x)) \tag{2.26}$$

Using the expressions in equations (2.24), (2.25) and (2.26) in the equations (2.21) and (2.22) yield

$$I(0,z)=\int_0^\infty P(0,z)e^{-\lambda(1-C(z))x}(1-B(x))\frac{dB(x)}{1-B(x)}dx$$
$$+\int_0^\infty W(0,z)e^{-(\lambda(1-C(z))+\nu)x}(1-S(x))\frac{dS(x)}{1-S(x)}dx-\lambda I_0$$
$$=P(0,z)B^*(\lambda(1-C(z)))+\lambda I_0[S^*(\nu+\lambda(1-C(z)))-1] \qquad (2.27)$$

$$P(0,z)=\int_0^\infty I(0,z)e^{-\lambda x}(1-A(x))\frac{dA(x)}{1-A(x)}dx+\frac{\lambda}{z}C(z)I_0$$
$$+\frac{\lambda}{z}C(z)\int_0^\infty I(0,z)e^{-\lambda x}(1-A(x))dx+\nu\int_0^\infty W(0,z)e^{-(\lambda(1-C(z)+\nu))x}(1-S(x))dx$$

$$=\frac{1}{z}[A^*(\lambda)+C(z)(1-A^*(\lambda))]I(0,z)+\frac{\lambda}{z}C(z)I_0$$
$$+\nu\lambda I_0\frac{[1-S^*(\nu+\lambda(1-C(z)))]}{\lambda(1-C(z))+\nu} \qquad (2.28)$$

Inserting the equation (2.27) in the equation (2.28) and on simplifying, we get

$$P(0,z)=\frac{\begin{array}{l}\lambda I_0\{C(z)+[S^*(\nu+\lambda(1-C(z)))-1](A^*(\lambda)+C(z)(1-A^*(\lambda)))\\ \quad+\nu z[1-S^*(\nu+\lambda(1-C(z)))]/(\lambda(1-C(z))+\nu)\}\end{array}}{z-(A^*(\lambda)+C(z)(1-A^*(\lambda)))B^*(\lambda(1-C(z)))} \qquad (2.29)$$

Substituting the equation (2.29) in the equation (2.27), we obtain

$$I(0,z)=\frac{\begin{array}{l}\lambda I_0[C(z)+\nu z(1-S^*(\nu+\lambda(1-C(z))))B^*(\lambda(1-z))/(\lambda(1-C(z))+\nu)\\ \quad+z(1-S^*(\nu+\lambda(1-C(z))))]\end{array}}{z-[A^*(\lambda)+C(z)(1-A^*(\lambda))]B^*(\lambda(1-C(z)))} \qquad (2.30)$$

Using the solutions the probability generating functions of the orbit size at different states of the server are derived.

- The probability generating function of the orbit size when the sever is idle in the non-empty system is given by

$$I(z)=\int_0^\infty I(x,z)dx$$

$$=I(0,z)\int_0^\infty e^{-\lambda x}(1-A(x))dx$$

$$=\frac{I_0(1-A^*(\lambda))[C(z)+(vz(1-S^*(v+\lambda(1-C(z))))\;\; B^*(\lambda(1-C(z))))/(\lambda(1-C(z))+v))+z(1-S^*(v+\lambda(1-C(z))))]}{z-(A^*(\lambda)+C(z)(1-A^*(\lambda)))B^*(\lambda(1-C(z)))}$$

(2.31)

- The probability generating function of the orbit size when the server is busy is given by

$$P(z)=\int_0^\infty P(x,z)dx$$

$$=P(0,z)\int_0^\infty e^{-\lambda(1-C(z))}(1-B(x))dx$$

$$=\frac{I_0(1-B^*(\lambda(1-C(z))))[C(z)+(S^*(v+\lambda(1-C(z)))-1)\;\;(A^*(\lambda)+C(z)(1-A^*(\lambda))\;+vz(1-S^*(v+\lambda(1-C(z))))/(\lambda(1-C(z))+v)]}{(1-C(z))(z-(A^*(\lambda)+C(z)(1-A^*(\lambda)))B^*(\lambda(1-C(z))))}$$

(2.32)

- The probability generation function of the orbit size when the server is on working vacation is given by

$$W(0,z)=\int_0^\infty W(x,z)dx$$

$$=W(0,z)\int_0^\infty e^{-(\lambda(1-C(z))+\nu)x}(1-S(x))dx$$

$$=\frac{\lambda I_0(1-S^*(\nu+\lambda(1-C(z))))}{\lambda(1-C(z))+\nu} \tag{2.33}$$

Using the normalizing condition, $I_0+\lim_{z\to 1} I(z)+\lim_{z\to 1} P(z)+\lim_{z\to 1} W(z)=1$, the analytical expression for I_0 is obtained as

$$I_0=\frac{1-m_1(1-A^*(\lambda))-\lambda m_1\mu^{(1)}}{\begin{aligned}&(1-\lambda m_1\mu^{(1)}-(1-A^*(\lambda))m_1)(1+(\lambda/\nu)(1-S^*(\nu+\lambda(1-C(z)))))\\&+(1-A^*(\lambda))(m_1+(1-S^*(\nu+\lambda(1-C(z)))/\nu)(1-\lambda m_1+\nu\lambda m_1\mu^{(1)})\\&+((1/\nu)+1)f^{(1)}+S^*(\nu+\lambda(1-C(z)))-1)+\lambda\mu^{(1)}(m_1+((\lambda m_1/\nu)+1)\\&-m_1(1-A^*(\lambda))(1-S^*(\nu+\lambda(1-C(z)))))\end{aligned}} \tag{2.34}$$

The probability generating function of the orbit size $P_q(z)$ and the system size $P_s(z)$ are respectively given by

$$P_q(z)=I_0+I(z)+P(z)+W(z)$$

$$\begin{aligned}=I_0\{&(1-C(z))[(z-(A^*(\lambda)+C(z)(1-A^*(\lambda))))(1+(\lambda(1-S^*(\nu+\lambda(1-z)))\\&/(\lambda(1-C(z))+\nu))+(1-A^*(\lambda))(C(z)+(\nu z(1-S^*(\nu+\lambda(1-z)))B^*(\lambda(1-C(z))\\&+(S^*(\nu+\lambda(1-C(z)))-1)(A^*(\lambda)+C(z)(1-A^*(\lambda))+\nu z(1-S^*(\nu+\lambda(1-C(z))))\\&/(\lambda(1-C(z))+\nu)]\}\end{aligned} \tag{2.35}$$

$$P_s(z)=I_0+I(z)+z(P(z)+W(z))$$

$$=I_0\{(1-C(z))[(z-(A^*(\lambda)+C(z)(1-A^*(\lambda))))(1+z(\lambda(1-S^*(\nu+\lambda(1-z)))$$

$$/(\lambda(1-C(z))+\nu))+(1-A^*(\lambda))(C(z)+(\nu z(1-S^*(\nu+\lambda(1-z)))B^*(\lambda(1-C(z))$$

$$+(S^*(\nu+\lambda(1-C(z)))-1)(A^*(\lambda)+C(z)(1-A^*(\lambda))+\nu z(1-S^*(\nu+\lambda(1-C(z))))$$

$$/(\lambda(1-C(z))+\nu)]\}$$

(2.36)

2.3 Stability condition

The necessary and sufficient condition for the system to be stable is

$$1-m_1(1-A^*(\lambda))-\lambda m_1\mu^{(1)}<1$$

2.4 Performance Measures

- The probability that the server is idle in the non-empty system is given by

$$I=\lim_{z\to 1} I(z)$$

$$=\frac{I_0(1-A^*(\lambda))(m_1+(1-S^*(\nu+\lambda(1-z))/\nu)(1-\lambda m_1+\nu\lambda m_1\mu^{(1)}) + ((1/\nu)+1)f^{(1)}+S^*(\nu+\lambda(1-z))-1)}{1-\lambda m_1\mu^{(1)}-(1-A^*(\lambda))m_1} \tag{2.37}$$

- The probability that the server is regular busy is given by

$$P=\lim_{z\to 1} P(z)$$

$$=\frac{\lambda\mu^{(1)}(m_1+((\lambda m_1/\nu)+1)-m_1(1-A^*(\lambda))(1-S^*(\nu+\lambda(1-z))))}{1-\lambda m_1\mu^{(1)}-(1-A^*(\lambda))m_1} \tag{2.38}$$

- The probability that the server is on working vacation is given by

$$W=\lim_{z\to 1} W(z)$$

$$=\lambda I_0(1-S^*(\nu))/\nu \tag{2.39}$$

- Expected orbit size is given by

$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z)$$

$$= \frac{N_q'''D'' - D'''N_q''}{3D''^2} \tag{2.40}$$

where

$$N_q'' = I_0\{-2[(1+((\lambda/\nu)(1-S^*(\nu+\lambda(1-C(z)))))(1-\lambda m_1\mu^{(1)} - m_1(1-A^*(\lambda)))$$

$$+(1-A^*(\lambda))(m_1 + f^{(1)} + (S^*(\nu+\lambda(1-C(z)))-1)) - ((1/\nu)$$

$$(1-S^*(\nu+\lambda(1-C(z)))))\,(\lambda m_1 + \nu\lambda m_1\mu^{(1)} + \nu) - f^{(1)}) - \lambda m_1\mu^{(1)}]\}$$

$$\begin{aligned}N_q''' = I_0\{&3[(1+((\lambda/\nu)(1-S^*(\nu+\lambda(1-C(z))))))(\lambda m_1\mu^{(2)} - 2\lambda m_1\mu^{(1)}m_1(1-A^*(\lambda))\\
&-m_2(1-A^*(\lambda))) - 2((\lambda\nu f^{(1)} - \lambda^2 m_1(1-S^*(\nu+\lambda(1-C(z))))))\\
&(1-\lambda m_1\mu^{(1)} + m_1(1-A^*(\lambda)))] - (1-A^*(\lambda))[m_2 + f^{(2)} + 2f^{(1)}\\
&-(1/\nu)(\lambda m_2(1-S^*(\nu+\lambda(1-C(z))))) - 2\lambda m_1 f^{(1)} + \nu(1-S^*(\nu+\lambda(1-C(z))))\\
&(\lambda m_1\mu^{(1)} + \lambda m_1\mu^{(2)} + m_1\mu^{(1)})) - 2\nu f^{(1)} - f^{(2)} - 2\nu f^{(1)}\lambda m_1\mu^{(1)}] - \lambda m_1\mu^{(1)}[m_2\\
&+(S^*(\nu+\lambda(1-C(z)))-1)m_2(1-A^*(\lambda)) - 2(1-A^*(\lambda))m_1 f^{(1)} + f^{(2)}\\
&+((1/\nu)(\lambda m_1(1-S^*(\nu+\lambda(1-C(z))))) + \lambda m_2(1-S^*(\nu+\lambda(1-C(z)))\\
&-\lambda m_1 f^{(1)} + 2\nu f^{(1)} + \nu f^{(2)} + \nu(\lambda m_1(1-S^*(\nu+\lambda(1-C(z))))) - f^{(1)}]]\}\end{aligned}$$

$$D'' = -2m_1(1-\lambda m_1\mu^{(1)} - m_1(1-A^*(\lambda)))$$

$$D''' = 3m_1(m_2(1-A^*(\lambda)) - \lambda m_1\mu^{(2)} + 2\lambda m_1\mu^{(1)}(1-A^*(\lambda)))$$

- Expected system size is given by

$$L_s = \lim_{z \to 1} \frac{d}{dz} P_s(z)$$

$$= L_q + P + W \tag{2.41}$$

CHAPTER 3

3. M/G/1 FEEDBACK RETRIAL QUEUE WITH IMPATIENT CUSTOMERS, SERVER BREAKDOWN AND REPAIR UNDER MULTIPLE WORKING VACATION POLICY

Single server feedback retrial queue with impatient customers is considered. If the arriving customer finds the server available, the customer receives the service immediately. Otherwise, the arrivals either balk the system with certain probability or join the orbit. As soon as the completion of regular service, if the orbit becomes empty, then the server goes for a working vacation. The server works at a lower service rate during working vacation period. The normal busy server may breakdown at any instant. The repair of the failed server starts immediately and the interrupted customers remains in the system until the service gets completed. A mathematical model is constructed and the steady state distributions of the server state including the average number of customers in the system are derived. Reliability measures are also obtained.

3.1 Mathematical Description of the Model

A single server unreliable retrial queueing system with balking and feedback under multiple working vacation policy is considered.

Arrival process

Single server retrial queueing system is considered where the customers arrive at the system according to a Poisson process with arrival rate λ .

Retrial process

If an arriving customer finds that the server is free, the customer begins his service immediately. If the server is not available, then the customer join the orbit with probability b or balks the system with the probability $\overline{b}$ $(=1-b)$. The customers joining the orbit attempt to take the service after some random amount of time. The inter-retrial time is generally distributed with distribution function A(x) and Laplace Stieltjes Transform $A^*(S)$.

Regular Service Process and Feedback Rule

Upon the arrival of the customers, if the server is free, the customer receives the regular service immediately. After the service completion, the customer either joins the orbit as a feedback customer or departs the system. The service time follows a general distribution with Distribution function $B(x)$ and Laplace Stieltjes Transform $B^*(S)$ with first two moments $\mu^{(1)}$ and $\mu^{(2)}$.

Working vacation process:

The working vacation period is the period when the server takes a vacation in order to attend secondary jobs at different service rates in the empty system and the vacation follows an exponential distribution with parameter ν. If any customers in the orbit are at service completion, the server will stop the vacation and come back to the normal busy period which means vacation interruptions happens. During the working vacation period, the service time follows general distribution with distribution function $S(x)$ and Laplace Stieltjes Transform $S^*(S)$ with first two moments $\gamma^{(1)}$ and $\gamma^{(2)}$.

Breakdown and Repair

The server is subject to active breakdown when the server is busy. The lifetime of the server is assumed to be exponentially distributed with rate τ .The repair of the failed server starts immediately and the interrupted customer waits in the service area until the remaining service gets completed. The repair time is generally distributed with distribution function $R(x)$ and Laplace Stieltjes Transform $R^*(S)$ with first two moments $\beta^{(1)}$ and $\beta^{(2)}$.

The states of the system at time t can be described by the Markov process $\{X(t), N(t); t \geq 0\}$ where $X(t)$ denotes the server state 0,1,2 or 3 depending on the server is free or busy in regular service period, working vacation period and repair. Define the supplementary variables $\xi_1(t), \xi_2(t)$ and $\xi_3(t)$ as follows

$\xi_1(t)$ = elapsed retrial time, if $X(t)=0$

$\xi_2(t)$ = elapsed service time, if $X(t)=1$ and 2

$\xi_3(t)$ =elapsed repair time, if $X(t)=3$

The functions $\eta(x), \mu(x), \gamma(x)$ and $\beta(y)$ are the conditional completion rates for repeated attempts, regular service and service during vacation period and repair respectively.

Then $\eta(x)=\dfrac{dA(x)}{1-A(x)}$, $\mu(x)=\dfrac{dB(x)}{1-B(x)}$, $\gamma(x)=\dfrac{dS(x)}{1-S(x)}$ and $\beta(y)=\dfrac{dR(y)}{1-R(y)}$

3.2 Orbit Size Distribution at Random Epoch

For the process $\{N(t), t \geq 0\}$ the probability densities are defined as follows $I_0(t)=P\{X(t)=0, N(t)=0\}$

$I_n(x,t)dx = P\{X(t)=1, N(t)=n, x \leq \xi_0(t) < x+dx\}$, $n \geq 1$

$P_n(x,t)dx = P\{X(t)=2, N(t)=n, x \leq \xi_1(t) < x+dx\}$, $\quad n \geq 0$

$W_n(x,t)dx = P\{X(t)=3, N(t)=n, x \leq \xi_1(t) < x+dx\}$, $\quad n \geq 0$

$R_n(x,y,t)dx = P\{X(t)=4, N(t)=n, x \leq \xi_2(t) < x+dx\}$, $\quad n \geq 0$

The steady state governing equations of the model under consideration are given below

$$\frac{d}{dt} I_0(t) = \lambda I_0 = q\left[\int_0^\infty P_0(x,t)\mu(x)dx + \int_0^\infty W_0(x,t)\gamma(x)dx\right] \tag{3.1}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right) I_n(x,t) = -\,[\lambda+\eta(x)]I_n(x,t), \quad n \geq 1 \tag{3.2}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right) P_0(x,t) = -[\lambda+\tau+\mu(x)]P_0(x,t) + \lambda(1-b)P_0(x,t), \qquad n=0 \tag{3.3}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)P_n(x,t)=-[\lambda+\tau+\mu(x)]P_n(x,t)+\lambda(1-b)P_n(x,t)$$

$$+\lambda bP_{n-1}(x,t)+\int_0^\infty R_n(x,y,t)\beta(y)dy,\quad n\geq 1 \tag{3.4}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)W_0(x,t)=-[\lambda+\tau+\mu(x)]W_0(x,t)+\lambda(1-b)W_0(x,t),\quad n=0 \tag{3.5}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)W_n(x,t)=-[\lambda+\tau+\mu(x)]W_n(x,t)+\lambda(1-b)W_n(x,t)+\lambda bW_{n-1}(x,t),\quad n\geq 1 \tag{3.6}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)R_0(x,y,t)=-[\lambda+\beta(y)]R_0(x,y,t)+\lambda(1-b)R_0(x,y,t),\qquad n=0 \tag{3.7}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)R_n(x,y,t)=-[\lambda+\beta(y)]R_n(x,y,t)+\lambda(1-b)R_n(x,y,t)+\lambda bR_{n-1}(x,y,t),\quad n\geq 1 \tag{3.8}$$

with boundary conditions

$$I_n(0,t)=\delta\left[\int_0^\infty P_{n-1}(x,t)\mu(x)dx+\int_0^\infty W_{n-1}(x,t)\gamma(x)dx\right]+q[\int_0^\infty P_n(x,t)\mu(x)dx+\int_0^\infty W_n(x,t)\gamma(x)dx] \tag{3.9}$$

$$P_0(0,t)=\int_0^\infty I_1(x,t)\eta(x)dx+\nu\int_0^\infty W_0(x,t)dx,\qquad n=0 \tag{3.10}$$

$$P_n(0,t)=\int_0^\infty I_{n+1}(x,t)\eta(x)dx+\lambda\int_0^\infty I_n(x,t)dx+\nu\int_0^\infty W_n(x,t)dx,\qquad n\geq 1 \tag{3.11}$$

$$W_n(0,t)=\begin{cases}\lambda I_0, & n=0\\ 0, & n\geq 1\end{cases} \tag{3.12}$$

$$R_n(x,0,t) = \tau P(x,t), \qquad n \ge 1 \tag{3.13}$$

Assuming the existence of steady state probabilities, the steady state equation corresponding to the equations (3.1) to (3.13) are

$$\lambda I_0 = q\int_0^\infty P_0(x)\mu(x)dx + q\int_0^\infty W_0(x)\gamma(x)dx \tag{3.14}$$

$$\frac{dI_n(x)}{dx} = -[\lambda + \eta(x)]I_n(x), \qquad n \ge 1 \tag{3.15}$$

$$\frac{dP_0(x)}{dx} = -[\lambda + \tau + \mu(x)]P_0(x) + \lambda(1-b)P_0(x), \qquad n = 0 \tag{3.16}$$

$$\frac{dP_n(x)}{dx} = -[\lambda + \tau + \mu(x)]P_n(x) + \lambda(1-b)P_n(x) + \lambda b P_{n-1}(x) + \int_0^\infty R_n(x,y)\beta(y)dy, n \ge 1 \tag{3.17}$$

$$\frac{dW_0(x)}{dx} = -[\lambda + \tau + \mu(x)]W_0(x) + \lambda(1-b)W_0(x), \qquad n = 0 \tag{3.18}$$

$$\frac{dW_n(x)}{dx} = -[\lambda + \tau + \mu(x)]W_n(x) + \lambda(1-b)W_n(x) + \lambda b W_{n-1}(x), \qquad n \ge 1 \tag{3.19}$$

$$\frac{dR_0(x,y)}{dx} = -[\lambda + \beta(y)]R_0(x,y) + \lambda(1-b)R_0(x,y), \qquad n = 0 \tag{3.20}$$

$$\frac{dR_n(x,y)}{dx} = -[\lambda + \beta(y)]R_n(x,y) + \lambda(1-b)R_n(x,y) + \lambda b R_{n-1}(x,y), \qquad n \ge 1 \tag{3.21}$$

The steady state boundary conditions at x=0 and y=0 are

$$I_n(0) = \delta\int_0^\infty P_{n-1}(x)\mu(x)dx + q\int_0^\infty P_n(x)\mu(x)dx + \delta\int_0^\infty W_{n-1}(x)\gamma(x)dx + q\int_0^\infty W_n(x)\gamma(x)dx \tag{3.22}$$

$$P_0(0) = \int_0^\infty I_1(x)\eta(x)dx + v\int_0^\infty W_0(x)dx, \qquad n = 0 \tag{3.23}$$

$$P_n(0) = \int_0^\infty I_{n+1}(x)\eta(x)dx + \lambda\int_0^\infty I_n(x)dx + v\int_0^\infty W_n(x)dx, \qquad n \ge 1 \tag{3.24}$$

$$W_n(0) = \begin{cases} \lambda I_0, & n = 0 \\ 0, & n \geq 1 \end{cases} \tag{3.25}$$

$$R_n(x,0) = \tau P(x), \qquad n \geq 1 \tag{3.26}$$

Define the probability generating functions $I(x,z), P(x,z), W(x,z)\, and\, R(x,y,z)$ as

$$\left.\begin{aligned} I(x,z) &= \sum_{n=1}^{\infty} I_n(x) z^n \quad ; \quad P(x,z) = \sum_{n=01}^{\infty} P_n(x) z^n; \\ W(x,z) &= \sum_{n=0}^{\infty} W_n(x) z^n \quad \text{and} \quad R(x,y,z) = \sum_{n=0}^{\infty} R_n(x,y) z^n \end{aligned}\right\} \tag{3.27}$$

Multiplying the equations (3.14) to (3.26) by z^n and summing over all possible values of n,we get

$$\left(\frac{\partial}{\partial x} + \lambda + \eta(x)\right) I(x,z) = 0 \tag{3.28}$$

$$\left(\frac{\partial}{\partial x} + \lambda b(1-z) + \tau + \mu(x)\right) P(x,z) = \int_0^{\infty} R(x,y,z)\beta(y)dy \tag{3.29}$$

$$\left(\frac{\partial}{\partial x} + \lambda b(1-z) + \nu + \mu(x)\right) W(x,z) = 0 \tag{3.30}$$

$$\left(\frac{\partial}{\partial x} + \lambda b(1-z) + \beta(y)\right) R(x,y,z) = 0 \tag{3.31}$$

$$I(0,z) = (\delta z + q)\int_0^{\infty} P(x,z)\mu(x)dx + (\delta z + q)\int_0^{\infty} W(x,z)\gamma(x)dx - \lambda I_0 \tag{3.32}$$

$$P(0,z) = \frac{1}{z}\int_0^{\infty} I(x,z)\eta(x)dx + \lambda\int_0^{\infty} I(x,z)dx + \nu\int_0^{\infty} W(x,z)dx \tag{3.33}$$

$$W(0,z) = \lambda I_0 \tag{3.34}$$

$$R(x,0,z) = \tau\, P(x,z) \tag{3.35}$$

Solving the partial differential equations (3.15)-(3.18), we obtain

$$I(x,z) = I(0,z)e^{-\lambda x}[1 - A(x)] \tag{3.36}$$

$$P(x,z) = P(0,z)e^{-g(z)x}[1 - B(x)] \tag{3.37}$$

$$W(x,z) = W(0,z)e^{-(\lambda b(1-z)+\gamma)x}[1 - S(x)] \tag{3.38}$$

$$R(x,y,z) = R(x,0,z)e^{-\lambda b(1-z)y}[1 - R(y)] \tag{3.39}$$

where

$$g(z) = \lambda b(1-z) + \tau(1 - R^*(\lambda b(1-z)))$$

Using the expressions in equations (3.37) and (3.38), the equation (3.32) yields

$$\begin{aligned} I(0,z) &= (\delta z + q)\int_0^\infty P(0,z)(1 - B(x))e^{-g(z)x}\beta(x)dx \\ &\quad + (\delta z + q)\int_0^\infty W(0,z)(1 - S(x)e^{-(\nu + \lambda b(1-z))x}\mu(x)dx - \lambda I_0 \\ &= (\delta z + q)P(0,z)B^*(g(z)) + (\delta z + q)W(0,z)S^*(\nu + \lambda b(1-z)) - \lambda I_0 \end{aligned} \tag{3.40}$$

Substituting the equations (3.36) to (3.38) in (3.32) and (3.33),we get

$$\begin{aligned} P(0,z) &= \frac{1}{z}\int_0^\infty I(0,z)(1 - R(x))e^{-\lambda x}\eta(x)dx + \lambda\int_0^\infty I(0,z)(1 - R(x))e^{-\lambda x}dx \\ &\quad + \nu\int_0^\infty W(0,z)(1 - S(x))e^{-(\nu+\lambda b(1-z))x}dx \\ &= \frac{I(0,z)}{z}[A^*(\lambda) + z(1 - A^*(\lambda)] + \lambda I_0(\nu/(1 - S^*(\nu + \lambda b(1-z))) \end{aligned} \tag{3.41}$$

$$\begin{aligned} R(x,0,z) &= \tau\, P(x,z) \\ &= \tau\, P(0,z)e^{-g(z)x}(1 - B(x)) \end{aligned} \tag{3.42}$$

Using the equation (3.34) in the equation (3.40), we obtain

$$I(0,z) = (\delta z + q)[P(0,z)B^*(g(z)) + \lambda I_0 S^*(\nu + \lambda b(1-z))] - \lambda I_0 \tag{3.43}$$

Using the expression $P(0,z)$ in equation (3.41), the equation (3.43) becomes

$$I(0,z)=\frac{z\lambda I_0[(\delta z+q)(S^*(\nu+\lambda b(1-z)))+(\nu/(\nu+\lambda b(1-z))(1-S^*(\nu+\lambda b(1-z)))B^*(g(z)))-1]}{z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z)))} \tag{3.44}$$

Inserting the equation (3.44) in the equation (3.41) and simplifying, we have

$$P(0,z)=\frac{\lambda I_0[((\delta z+q)S^*(\nu+\lambda b(1-z)))-1)(A^*(\lambda)+z(1-A^*(\lambda))) + z((\nu/(\nu+\lambda b(1-z))(1-S^*(\nu+\lambda b(1-z))))]}{z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z))} \tag{3.45}$$

Substituting the equation (3.45) in (3.42), we have

$$R(x,0,z)=\frac{\lambda\tau I_0[((\delta z+q)S^*(\nu+\lambda b(1-z))-1)(A^*(\lambda)+z(1-A^*(\lambda))) + z(\nu/(\nu+\lambda b(1-z)))(1-S^*(\nu+\lambda b(1-z)))[1-B(x)]e^{-g(z)x}]}{z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z))} \tag{3.46}$$

Using (3.34), (3.44) to (3.46) in the equations (3.36)-(3.39), we get

$$I(x,z)=\frac{z\lambda I_0[(\delta z+q)(S^*(\nu+\lambda b(1-z))+(\nu/(\nu+\lambda b(1-z))(1-S^*(\nu+\lambda b(1-z))))B^*(g(z)))-1](1-A(x))e^{\lambda x}]}{z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z))} \tag{3.47}$$

$$P(x,z)=\frac{\lambda I_0[(\delta z+q)S^*(\nu+\lambda b(1-z))-1)(A^*(\lambda)+z(1-A^*(\lambda))) + z((\nu/(\nu+\lambda b(1-z))(1-S^*(\nu+\lambda b(1-z))))][1-B(x)]e^{-g(z)x}}{z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z))} \tag{3.48}$$

$$W(x,z)=\lambda I_0[1-S(x)]e^{-(\nu+\lambda b(1-z))x} \tag{3.49}$$

$$R(x,y,z)=\frac{\begin{array}{c}\lambda\tau I_0(1-R^*(\lambda b(1-z)))[((\delta z+q)S^*(\nu+\lambda b(1-z))-1)(A^*(\lambda)+z(1-A^*(\lambda)))\\ +z(\nu/(\nu+\lambda b(1-z))(1-S^*(\nu+\lambda b(1-z))))][1-B(x)]e^{-g(z)x}\end{array}}{\lambda b(1-z)(z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z)))}$$

(3.50)

- The probability generating function of the orbit size when the server is idle in the non-empty system is given by

$$I(z)=\int_0^\infty I(x,z)dx$$

$$=\frac{\begin{array}{c}z(1-A^*(\lambda))I_0[(\delta z+q)(S^*(\nu+\lambda b(1-z))+((\nu/(\nu+\lambda b(1-z)))\\ (1-S^*(\nu+\lambda b(1-z))))B^*(g(z))-1]\end{array}}{z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z))}$$

(3.51)

- The probability generating function of the orbit size when the server is busy in providing regular service is given by

$$P(z)=\int_0^\infty P(x,z)dx$$

$$=\frac{\begin{array}{c}\lambda I_0(1-B^*(g(z)))[((\delta z+q)S^*(\nu+\lambda b(1-z))-1)(A^*(\lambda)+z(1-A^*(\lambda)))\\ +z((\nu/(\nu+\lambda b(1-z)))(1-S^*(\nu+\lambda b(1-z))))]\end{array}}{g(z)(z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z)))}$$

(3.52)

- The probability generating function of the orbit size when the server is on working vacation is given by

$$W(z)=\int_0^\infty W(x,z)dx$$

$$=\{\lambda I_0(\nu/(\nu+\lambda b(1-z)))(1-S^*(\nu+\lambda b(1-z)))\}/\tau$$

(3.53)

- The probability generating function of the orbit size when the server is under repair is given by

$$R(z) = \int_0^\infty \int_0^\infty R(x,y,z)dxdy$$

$$= \frac{\lambda \tau I_0(1-B^*(g(z)))(1-R^*(g(z)))[((\delta z+q)S^*(\nu+\lambda b(1-z))-1)(A^*(\lambda) + z(1-A^*(\lambda))) + z((\nu/(\nu+\lambda b(1-z)))(1-S^*(\nu+\lambda b(1-z))))]}{(\lambda b(1-z)\times g(z))(z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z)))}$$

(3.54)

Using the normalizing condition $I_0 + \lim_{z\to 1} I(z) + \lim_{z\to 1} P(z) + \lim_{z\to 1} W(z) + \lim_{z\to 1} R(z) = 1$,

we obtain

$$I_0 = \frac{A^*(\lambda) - \delta - \lambda b\mu^{(1)}(1+\tau\beta^{(1)})}{qA^*(\lambda) + (\lambda/\nu)(1-S^*(\nu))[(b-\delta)+(1-b)A^*(\lambda)] + \lambda\mu^{(1)}(1+\tau\beta^{(1)})((\delta-b)S^*(\nu)+(1-b)(1-S^*(\nu))A^*(\lambda))} \quad (3.55)$$

The probability generating function of the orbit size is given by

$$P_q(z) = \frac{N_q(z)}{D(z)} = I_0 + I(z) + P(z) + W(z) + R(z) \quad (3.56)$$

where $N_q(z) = I_0\{b(1-z)[z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z)))$

$$((\lambda(\nu/(\nu+\lambda b(1-z)))(1-S^*(\nu+\lambda b(1-z))))/\nu))+1)+z(1-A^*(\lambda))$$

$$[(\delta z+q)(S^*(\nu+\lambda b(1-z))+(\nu/(\nu+\lambda b(1-z)))\ (1-S^*(\nu+\lambda b(1-z))$$

$$B^*(g(z))-1]+(1-B^*(g(z)))[((\delta z+q)\ S^*(\nu+\lambda b(1-z))-1)(A^*(\lambda)$$

$$z(1-A^*(\lambda)))+z((\nu/(\nu+\lambda b(1-z)))\quad (1-S^*(\nu+\lambda b(1-z)))]\}$$

$$D(z) = b(1-z)(z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z)))$$

- The probability generating function of the system size is given by

$$P_s(z) = \frac{N_s(z)}{D(z)} = I_0 + I(z) + z[P(z) + W(z) + R(z)] \quad (3.57)$$

where

$$N_s(z) = I_0\{b(1-z)[z-(\delta z+q)(A^*(\lambda)+z(1-A^*(\lambda)))B^*(g(z)))(\lambda z(\nu/(\nu+\lambda b(1-z)))$$

$$(1-S^*(\nu+\lambda b(1-z)))/\nu)+1)+z(1-A^*(\lambda))[(\delta z+q)S^*(\lambda b(1-z)+\nu)$$
$$+(\nu/(\nu+\lambda b(1-z))(1-S^*(\nu+\lambda b(1-z)))(B^*(g(z)))-1] \;\; +z(1-B^*(g(z)))$$
$$[((\delta z+q)S^*((\nu+\lambda b(1-z))))-1) \quad (A^*(\lambda)+z(1-A^*(\lambda)))$$
$$+z((\nu/(\nu+\lambda b(1-z)))(1-S^*(\nu+\lambda b(1-z)))]\}$$

3.3 Stability Condition

The necessary and sufficient condition for the system to be stable is

$$A^*(\lambda)-\delta-\lambda b\mu^{(1)}(1+\tau g^{(1)})<1.$$

3.4 Performance Measures

- The probability of the server is idle during the retrial time is given by

$$I = \lim_{z\to 1} I(z)$$

$$= \frac{I_0(1-A^*(\lambda))[\delta+f^{(1)}+(1-S^*(\nu))((\lambda b/\nu)+\lambda b\mu^{(1)}(1+\tau\beta^{(1)}))]}{A^*(\lambda)-\delta-\lambda b\mu^{(1)}(1+\tau\beta^{(1)})} \quad (3.58)$$

- The probability of the server is busy is given by

$$P = \lim_{z\to 1} P(z)$$

$$= \frac{\lambda I_0 \mu^{(1)} (\delta S^*(\nu) + f^{(1)} + (1 - S^*(\nu))((\lambda b/\nu) + A^*(\lambda)))}{A^*(\lambda) - \delta - \lambda b \mu^{(1)} (1 + \tau \beta^{(1)})} \tag{3.59}$$

- The probability of the server is on working vacation is given by

$$W = \lim_{z \to 1} W(z)$$

$$= \lambda I_0 (1 - S^*(\nu))/\nu \tag{3.60}$$

- The probability of the server is under repair is given by

$$R = \lim_{z \to 1} R(z)$$

$$= \frac{\tau \lambda I_0 \beta^{(1)} [\delta S^*(\nu) + f^{(1)} + (1 - S^*(\nu))((\lambda b/\nu) + A^*(\lambda))] \mu^{(1)}}{A^*(\lambda) - \delta - \lambda b \mu^{(1)} (1 + \tau \beta^{(1)})} \tag{3.61}$$

3.5 Mean system size and orbit size

- Expected orbit size is given by

$$L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z)$$

$$= \frac{N_q''' D'' - D''' N''}{3 D''^2} \tag{3.62}$$

where

$$N_q'' = I_0 \{-2b[(1 + (\lambda(1 - S^*(\nu)/\nu))(A^*(\lambda) - \delta - \lambda b \beta^{(1)} (1 + \tau g^{(1)})$$

$$+ (1 - A^*(\lambda))(\delta S^*(\nu) + f^{(1)} + (\lambda b/\nu)(1 - S^*(\nu)) + (1 - S^*(\nu))(\lambda b \beta^{(1)} (1 + \tau g^{(1)}))]\}$$

$$Nr_q''' = I_0\{-3(\lambda b)^2[\mu^{(2)}(1+\tau\beta^{(1)})^2 + \tau\mu^{(1)}\beta^{(2)}][(\delta-b)S^*(\nu)+(1-b)A^*(\lambda)(1-S^*(\nu))]$$

$$-3\lambda b\mu^{(1)}(1+\tau\beta^{(1)})(2(\lambda b/\nu)(q+\lambda b)(1-(1-\nu)S^*(\nu))+S^*(\nu)(1-A^*(\lambda))$$
$$(\delta+(\lambda-2)b)+2b(1-A^*(\lambda))((1-\nu)-(\lambda+bS^*(\nu)))) \; -6(\lambda b/\nu)^2(1-S^*(\tau)$$
$$+\nu S^{*'}(\nu)))(b-\delta)+(1-b)A^*(\lambda) + b\mu^{(1)}(1+\tau\beta^{(1)})(1-A^*(\lambda)-\lambda))-6b$$
$$((1-A^*(\lambda))/\tau)(\lambda(1-S^*(\tau))(\delta(b-\nu)+b)-\delta)\}$$

$$D'' = -2b(A^*(\lambda)-\delta-\lambda b\mu^{(1)}(1+\tau\beta^{(1)}))$$

$$D''' = 3b[\lambda b\mu^{(1)}(\delta+(1-A^*(\lambda))(1+\tau\beta^{(1)}) + 2\delta(1-A^*(\lambda))+(\lambda b)^2(\mu^{(2)}(1+\tau\beta^{(1)})^2+\tau\mu^{(1)}\beta^{(2)})]$$

- Expected system size is given by

$$L_s = \lim_{z\to 1}\frac{d}{dz}P_s(z) \tag{3.63}$$
$$= L_q + P + W$$

3.4.1 Reliability Measures

In this section, the reliability measures like availability of the server and failure frequency are derived in the following theorem.

Theorem: 3.1

The steady state availability ($\mathcal{A}$) and failure frequency ($\mathcal{F}$) of the server are

$$\mathcal{A} = \frac{\tau\lambda I_0\beta^{(1)}\mu^{(1)}[\delta S^*(\nu)+f^{(1)}+(1-S^*(\nu))((\lambda b/\nu)+A^*(\lambda))]}{A^*(\lambda)-\delta-\lambda b\mu^{(1)}(1+\tau\beta^{(1)})} \tag{3.64}$$

$$\mathcal{F} = \frac{\tau\lambda I_0\mu^{(1)}[\delta S^*(\nu)+f^{(1)}+(1-S^*(\nu))((\lambda b/\nu)+A^*(\lambda))]}{A^*(\lambda)-\delta-\lambda b\mu^{(1)}(1+\tau\beta^{(1)})} \tag{3.65}$$

Proof.

The availability of the server in the system is given by

$$\mathcal{A} = I_0 + \lim_{z \to 1}\left[\int_0^\infty I(x,z)dx + \int_0^\infty P(x,z)dx + \int_0^\infty W(x,z)dx\right]$$

$$= I_0 + \lim_{z \to 1}[I(z) + P(z) + W(z)]$$

$$= I_0 + I + P + W$$

Result in equation (3.64) can be obtained by using equations (3.54) and (3.61).

The failure frequency of the server in the system is given by

$$\mathcal{F} = \lim_{z \to 1}\int_0^\infty P(x,z)dx$$

$$= \tau \lim_{z \to 1} P(z)$$

$$= \tau P$$

Result in the equation (3.65) can be obtained by using equations (3.59) and (3.52).

SUMMARY AND CONCLUSION

In this dissertation, a single server retrial queueing system with multiple working vacation is considered.

Batch arrival retrial queue with working vacation and vacation interruption is discussed in Chapter 2. In Chapter 3, single server feedback retrial queue with impatient customers, server breakdown and repair under multiple working vacation policy is analyzed. The models considered are investigated using supplementary variable technique to obtain performance measures of the system and reliability indices of interest.

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