

A Study on Semi* - Open Sets in Topological Spaces

MEERA M

(17PMA011)

Thesis Submitted to

**Avinashilingam Institute for Home Science and Higher Education
for Women**

Coimbatore - 641043

In Partial Fulfilment of the Requirements for the Degree of

Master of Science in Mathematics

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P. Jeyalaxmi

Signature of the Head of the department

K. Sivakumari

Signature of the supervisor

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INTRODUCTION

In the literature of General Topology, weak forms of open(closed) sets, called preopen (preclosed) sets, semiopen (semiclosed)sets, α -open(α -closed) sets and semi preopen (semi-preclosed)sets were available.Again , there is a very vast developed field of sets and their neighborhoods in general topology, is the field of generalized open sets and generalized closed sets of the above said sets respectively.

N.Levine [14] introduced semi-open sets in topological spaces in 1963. N.Biswas [2] defined semi-closed sets. S.G.Crossley and S.K.Hildebrand [3] defined semi-closure of a set. N.Levine [15] defined and studied generalized closed sets in 1970. P.Das [4] defined semi-limit point of a subset. The semi-derived set of a subset of a topological space was also defined and studied by him. W.Dunham [8] introduced the concept of generalized closure and defined a new topology τ^* and investigated some of their properties. G.DiMaio and T.Noiri [5] defined and studied semi-regular subsets. Quite recently the A.Robert and S.PiousMissier [12] introduced and studied semi*-open sets using the generalized closure operator.

In 1974, Das [4] defined the concept of semi-connectedness in topology and investigated its properties. Compactness is one of the most important, useful and fundamental concepts in topology. In 1981, Dorsett [6] introduced and studied the concept of semi-compact spaces. Since then, Hanna and Dorsett, Ganster [11] and Mohammad S. Sarsak[16] investigated the properties of semi-compact spaces. In 1990, Ganster [9] defined and investigated semi-Lindelöf spaces. In the article, “On semi*-Connected spaces and Semi*-compact spaces”,by A.Robert and S.PiousMissier [31] they introduced the concepts of semi*-connected spaces, semi*-compact spaces and semi*Lindelöf spaces. They investigate their basic properties and relationship with already existing concepts.

In 1963, Norman Levine [14] introduced the concept of semi-open set in topological spaces which is a weaker form of open sets. Velicko introduced the notion of Regular open sets in topological spaces. Levine [15] generalized the concept of closed sets to generalized closed sets in 1970. Using generalized closed sets, Dunham [8] introduced the concept of closure operator Cl^* and the generalized closure of a set A is denoted by $Cl^*(A)$. Recently S.PiousMissier and A.Robert [27] introduced a nearly open sets, namely $Semi^*\alpha$ -open sets in a topological space. They introduced a new class of nearly open sets, $Semi^*$ regular open sets in topological spaces and studied their properties.

In 1973, Das [4] defined the concept of semi-connectedness in topological spaces and investigated its properties. Compactness is one of the most important, useful and fundamental concepts in topology. In 1981, Dorsett [6] introduced and studied the concept of semi-compact spaces. Since then, Hanna and Dorsett [11], Ganster [9] investigated the properties of semi-compact spaces. PasunkiliPandian.S [22] introduced $semi^*$ -pre-compact spaces and investigated their properties. Robert, A. and Pious Missier, S.[31] recently introduced and studied $semi^*$ -connectedness and $semi^*$ -compactness in topological spaces. A.Robert and R. krishnaveni [23] have defined $semi^*$ regular open sets and $semi^*$ regular closed sets and investigated their properties. They introduced the concept of $semi^*$ regular connected spaces and also discuss their relationship with already existing concepts namely connectedness, semi-connectedness, semi-pre connectedness and $semi^*\alpha$ -connectedness. Further we define $semi^*$ regular compact spaces and investigate their properties. They also show the relationship of $semi^*$ r-compactness with each of the concepts of compactness, semi-compactness $semi^*$ -compactness and $semi^*$ -pre compactness.

The main aim of this thesis is to study the newly introduced $semi^*$ - open sets and its property from the following articles.

- (i) “A New class of Nearly Open sets” by A. Robert and S. Pious Missier (2012).
- (ii) “On Semi*- Closed sets” by A. Robert and S. Pious Missier (2012).
- (iii) “On Semi*-Connected Spaces and Semi*- Compact Spaces” by A.Robert and S. Pious Missier (2012).
- (iv) “A Stronger form of Semi*-Open sets” by S. Pious Missier and R. Krishnaveni(2016).
- (v) “Connectedness and Compactness via Semi*-Regular open sets” by S. Pious Missier , R. Krishnaveni and G.Mahadevan (2017).

Chapter – I deals with preliminaries definitions and results in topological spaces that are required for this study.

Chapter - II consists of 3 sections namely

2.1 Semi* - open sets

2.2 Semi* - closed sets

2.3 Semi*-Connected and Compact spaces

In **section 2.1**, the concept of semi*-open sets introduced by A.Robert and S. Pious Missieris reviewed. The characterization of semi* - open sets and already existing open sets in topological spaces were studied and semi*-interior point were defined. They investigated fundamental properties of Semi*-open sets.

In **section 2.2**, the concept of Semi* - closed sets introduced by A.Robert and S. Pious Missier is reviewed. The relationship of Semi* - closed sets and already existing open sets in topological spaces were studied. Some interesting properties were analysed in parallel to open sets and closed sets in general topology.

In **section 2.3**,A.Robert and S. Pious Missierintroduced the concepts of semi*-connected spaces, semi*-compact spaces and semi*-Lindelof spaces. They investigated their basic properties and discussed their relationship with already existing concepts.

Chapter – III consists of 3 sections namely

3.1 Semi* - regular open sets

3.2 Connected via Semi*regular - closed sets

3.3 Compactness via Semi*regular - closed sets

In **section 3.1**, a new class of sets, namely Semi star regular open sets introduced by S.PiousMissier and R.Krishnaveni, using regular open sets and the generalized closure operator due to W.Dunham is reviewed and were analyzed. The characterizations of Semi star regular open sets and some fundamental properties of Semi star regular open sets and their interrelationships with their known star generalized closed sets were discussed.

In the **section 3.2** and **section 3.3**, S.PiousMissier, R.Krishnaveni and G.Mahadevan introduced the new concepts namely, semi*r-connectedness and semi*r-compactness using semi*regular open sets which are reviewed here. Their relationships with already existing concepts of connectedness and compactness are discussed.

REVIEW OF LITERATURE

Topology is considered to be one of the three linchpins of abstract mathematics along with analysis and algebra. In the history of topology, results were motivated by investigation of real world problems. In the later years the emphasis was given to its abstract development. General Topology is a branch of topology that studies the relationship between the basic topological properties that spaces may possess. It has shown its fruitfulness in both pure and applied directions. It leads the path of research into fuzzy topology, bitopology and digital topology.

Topological ideas are present in almost all areas of today's Mathematics. The term "Topological spaces" defined by Felix Hausdorff stands for what is called Hausdorff spaces. In current usage a topological space is a generalisation of Hausdorff spaces given by Kazimierz Kuratowski. Topological spaces are mathematical structures that allow the formal definition of concepts such as convergence, connectedness and continuity. The starting point of great theories is often an exact and exhaustive investigation of a simple concept, as is the case of the notion of closed sets in topology.

In 1970, Levine introduced generalized closed sets (briefly g -closed sets) in a topological space in order to extend many of the important properties of closed sets to a larger family. In the recent past, there has been considerable interest in the study of various forms of generalized closed sets.

Primarily, the topological spaces were characterized by open sets. Later Stone (1937) introduced regular openness which is stronger than openness. In 1963, Norman Levine introduced the notion of semi-openness which is weaker than the notion of openness in topological spaces. Since then several interesting generalized open sets have been introduced by many topologists.

Njasted (1965) established α -closed sets. Norman Levine (1970) introduced the notion of generalized closed (briefly g-closed) sets in topological spaces and showed that compactness, countably compactness, para compactness and normality were all g-closed hereditary. Generalized closed sets are a strong tool in the characterization of topological spaces satisfying weak separation axioms.

since the number of research papers published on various closed sets in topological spaces is numerous, a brief review of literature on some of the important article published on this topic.

Title : A new class of nearly Open sets (2012) .

Author : A. Robert and S. Pious Missier

Inference observed : In this paper they introduce a new class of sets, namely semi*-open sets, using the generalized closure operator due to Dunham. They gave a characterization of semi*-open sets. They also define semi*-interior point and the semi*-interior of a subset. Further, they investigate fundamental properties of semi*-open sets.

Title : On semi*-closed sets (2012).

Author : A. Robert and S. Pious Missier

Inference observed : In this paper they introduce a new class of sets, namely semi*-closed sets. They gave a characterization of semi*-open sets and define semi*-closure, semi*-limit points and the semi*-derived set of a subset are defined. Further, they investigate fundamental properties of semi*-closed sets.

Title : On Semi*-connected and Semi*-compact spaces(2012).

Author : A. Robert and S. Pious Missier

Inference observed : The purpose of this paper is to introduce the concepts of semi*-connected spaces, semi*-compact spaces and semi*-Lindelof. They investigate their basic properties. They also discuss their relationship with already existing concepts.

Title : Semi-star-alpha-open sets and Associated functions

(2014).

Author : A. Robert and S. Pious Missier

Inference observed: In aim of this paper is to introduce variations functions associated with semi* α -open sets. Here semi* α -continuous, semi* α -irresolute, contra-semi* α -continuous and contra-semi* α -irresolute functions are defined. Characterization for these functions are given. Further their fundamental properties are investigated. Many other functions associated with semi* α -open sets and their contra versions are introduced and their properties are studied. In addition strongly semi* α -irresolute functions, contra-strongly semi* α -irresolute functions, semi* α -totally continuous , totally semi* α -continuous functions and semi* α -homeomorphisms are introduced and their properties are investigated.

Title : Connectedness and Compactness via semi-star-alpha-open sets

(2014).

Author : A. Robert and S. Pious Missier

Inference observed: In this paper, they introduced new concepts, namely semi* α -connectedness, semi* α -compactness using semi* α -open sets. They investigate their basic properties. They also discuss their relationship with already existing concepts of connectedness and compactness.

Title : Between α -closed sets and semi α -closed sets (2014).

Author : A. Robert and S. Pious Missier.

Inference observed: In general topology many strong and weak form of open and closed sets have been defined and studied. Govindappa Navalagi introduced the concepts of semi α -open sets which is a weaker form of α -open sets. Semi* α -open sets is defined analogously by replacing the closure operator by the generalized closure operator due to Dunham in the definition of semi α -open sets. In this paper they introduced a new class of sets namely, semi* α -closed sets, as the complement of semi* α -open sets. They found the characterization of semi* α -closed sets and also defined the semi* α -closure of a subset. Further , they investigated the fundamental properties of the semi* α -closure. They define the semi* α - derived set of a subset and studied its properties.

Title : A stronger form of semi*-open sets (2016).

Author : S. Pious Missier and R.Krishnaveni.

Inference observed: In this paper, they introduced a new class of sets, namely Semi*-regular open sets, using regular open sets and the generalized closure operator due to W.Dunham and analyzed the characterizations of Semi star regular open sets. Further, they studied some fundamental properties of Semi star regular open sets and their interrelationships with their known star generalized closed sets.They also define Semi*- regular interior.

Title : Connectedness and Compactness Via semi*-regular open sets (2017).

Author : S. Pious Missier , R.Krishnaveni and G.Mahadevan.

Inference observed: In this paper, they introduced new concepts namely, semi*r-connectedness and semi*r-compactness using semi*regular open sets and investigated their basic properties. They also discuss their relationships with already existing concepts of connectedness and compactness.

CHAPTER - I

PRELIMINARIES

Definition 1.1.1 [36]

A subset A of a topological space (X, τ) is **regular open** if $A = \text{Int}(\text{Cl}(A))$.

Definition 1.1.2 [14]

A subset A of a topological space (X, τ) is **semi-open** if there is an open set U in X such that $U \subseteq A \subseteq \text{Cl}(U)$ or equivalently if $A \subseteq \text{Cl}(\text{Int}(A))$. The class of all semi-open sets in (X, τ) is denoted by $\text{SO}(X, \tau)$.

Definition 1.1.3 [17]

A subset A of a topological space (X, τ) is **pre-open** if $A \subseteq \text{Int}(\text{Cl}(A))$.

Definition 1.1.4 [18]

A subset A of a topological space (X, τ) is **α -open** if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$.

Definition 1.1.5 [14]

A subset A of a topological space (X, τ) is **semi-closed** if $\text{Int}(\text{Cl}(A)) \subseteq A$. The class of all semi-closed sets is denoted by $\text{SC}(X, \tau)$.

Definition 1.1.6 [14]

If A is a subset of a space X , the **semi-interior** of A is defined as the union of all semi-open sets of X contained in A . It is denoted by $s\text{Int}(A)$.

Definition 1.1.7 [17]

A subset A of a topological space (X, τ) is **pre-closed** if $\text{Cl}(\text{Int}(\text{Cl}(A))) \subseteq A$.

Definition 1.1.8 [18]

A subset of topological space (X, τ) is **α -closed** if $\text{Cl}(\text{Int}(\text{Cl}(A))) \subseteq A$.

Definition 1.1.9 [15]

A subset A of a topological space (X, τ) is **generalized closed** (briefly **g-closed**) if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

Definition 1.1.10 [15]

If A is a subset of X , the **generalized interior** of A is defined as the union of all **g-open** sets in X that are contained in A and it is denoted by $\text{Int}^*(A)$.

Definition 1.1.11 [15]

If A is a subset of a space X , the **generalized closure** of A is defined as the intersection of all **g-closed** sets in X that are containing in A and it is denoted by $\text{Cl}^*(A)$.

Theorem 1.1.12 [15]

The union of two **g-closed** sets is **g-closed**.

Lemma 1.1.13 [8]

If $E \subseteq X$, then $E \subseteq \text{Cl}^*(E) \subseteq \text{Cl}(E)$.

Proof :

Since, A closed set is **g-closed**. The closures follow the inequality $\text{Cl}^*(E) \subseteq \text{Cl}(E) \forall E \subseteq (X, \tau)$.

Hence we get $E \subseteq \text{Cl}^*(E) \subseteq \text{Cl}(E)$.

Remark 1.1.14 [8]

Both containment relation in the previous lemma may be proper. Consider $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then $\text{Cl}^*(\{a\}) = \{a, c\}$ since the only **g-closed** supersets of $\{a\}$ are $\{a, c\}$ and X , while $\text{Cl}(\{a\}) = X$. That is, $\{a\} \subsetneq \text{Cl}^*(\{a\}) \subsetneq \text{Cl}(\{a\})$.

Theorem 1.1.15 [8]

Cl^* is a kuratowski closure operator on X .

Proof :

- (i) $Cl^*(\phi) = \phi$ and $E \subseteq Cl^*(E)$ from the lemma 1.1.13
- (ii) If $E_1 \cup E_2 \subseteq A \in \mathcal{D}$ where \mathcal{D} is set of all g-closed sets in (X, τ) , then $E_i \subseteq A$ and so $Cl^*(E_i) \subseteq A$ for $i=1,2$.
Thus $Cl^*(E_1) \cup Cl^*(E_2) \subseteq \bigcap \{A : E_1 \cup E_2 \subseteq A \in \mathcal{D}\} = Cl^*(E_1 \cup E_2)$.
Conversely, to prove that $Cl^*(E_1 \cup E_2) \subseteq Cl^*(E_1) \cup Cl^*(E_2)$. For , if there is an $x \in Cl^*(E_1 \cup E_2)$ with $x \notin Cl^*(E_1) \cup Cl^*(E_2)$, then there are g-closed sets A_1 and A_2 with $E_1 \subseteq A_1, E_2 \subseteq A_2$ and $x \notin A_1 \cup A_2$. But then $E_1 \cup E_2 \subseteq A_1 \cup A_2$, a g-closed by theorem 1.1.12, contradicting $x \in Cl^*(E_1 \cup E_2)$. This concludes that $Cl^*(E_1 \cup E_2) = Cl^*(E_1) \cup Cl^*(E_2)$.
- (iii) Finally, if $E \subseteq A \in \mathcal{D}$, then $Cl^*(E) \subseteq A$ and $Cl^*(Cl^*(E)) \subseteq A$ by definition of Cl^* .

Hence $Cl^*(Cl^*(E)) \cap \{A : E \subseteq A \in \mathcal{D}\} = Cl^*(E)$.

By (i)-(iii), Cl^* is a closure operator on X .

Definition 1.1.16 [8]

Let τ^* be the topology on X generated by Cl^* in the usual manner. That is,

$$\tau^* = \{O^* : Cl^*(O^*)^c = (O^*)^c\}.$$

Definition 1.1.17 [15]

A topological space X is $T_{1/2}$ if every g-closed set in X is closed.

Theorem 1.1.18 [8]

$\tau \subseteq \tau^*$ with equality iff (X, τ) is $T_{1/2}$.

Proof :

If E is τ -closed, $E \subseteq Cl^*(E) \subseteq Cl(E)$ implies that E is τ^* -closed. Thus $\tau \subseteq \tau^*$. Further, suppose $\tau = \tau^*$ and let $A \subseteq X$ be g-closed in (X, τ) . Then $A = Cl^*(A)$ and so A is closed in $\tau^* = \tau$. Thus (X, τ) is $T_{1/2}$.

Conversely, if (X, τ) is $T_{1/2}$, its closed sets and g-closed sets coincide and so $Cl = Cl^*$.

Hence $\tau = \tau^*$.

Theorem 1.1.19 [15]

For each $x \in X$, either $\{x\}$ is closed or $\{x\}^c$ is g-closed

Proof

If $\{x\}$ is not closed, then the only open superset of $\{x\}^c$, is X itself.

Thus the closure of $\{x\}^c$ is contained in each of its neighborhoods and $\{x\}^c$ is g-closed.

Theorem 1.1.20 [8]

(X, τ) is $T_{1/2}$ iff every singleton in X either open or closed iff every subset of X is the intersection of all open sets and all closed sets containing it.

Theorem 1.1.21 [8]

For any space (X, τ) , $x \neq y$ implies $Cl^*(x) \neq Cl^*(y)$.

Proof :

If $\{x\}$ is closed, $y \notin \{x\} = Cl\{x\} = Cl^*(x)$. Otherwise, $y \in \{x\}^c$, a g-closed set by theorem 1.1.19,

Thus $y \in Cl^*(y) \subseteq \{x\}^c$, and so $x \notin Cl^*(y)$.

Remark 1.1.22 [8]

The previous result shows that (X, τ^*) is always T_0 . In fact, a stronger result is established as given below:

Theorem 1.1.23 [8]

For any space (X, τ) , (X, τ^*) is $T_{1/2}$.

Proof :

If $\{x\}$ is τ -closed, $\{x\}$ is τ^* -closed as well. Otherwise, $\{x\}^c$ is g -closed and so $Cl^*(\{x\}^c) = \{x\}^c$, which implies that $\{x\}$ is τ^* -open. By theorem 1.1.20,

(X, τ^*) is $T_{1/2}$.

Definition 1.1.24 [36]

A space X is **locally indiscrete** if every open set in X is closed.

Definition 1.1.25

A space X is **extremally disconnected** if the closure of every open set in X is open.

Definition 1.1.26 [36]

A topological space X is said to be **connected** if X cannot be expressed as the union of two disjoint nonempty open sets in X .

Definition 1.1.27 [4]

A topological space X is said to be **semi-connected** if X cannot be expressed as the union of two disjoint nonempty semi-open sets in X .

Definition 1.1.28

A subset A of a topological space (X, τ) is called **clopen** if it is both open and closed in X .

Theorem 1.1.29

A topological space X is connected if and only if the only clopen subsets of X are ϕ and X .

Definition 1.1.30

A collection B of open sets in X is called an **open cover** of $A \subseteq X$ if $A \subseteq \bigcup \{U_\alpha : U_\alpha \in B\}$ holds.

Definition 1.1.31

A collection B of semi-open sets in X is called an **semi-open cover** of $A \subseteq X$ if $A \subseteq \bigcup \{U_\alpha : U_\alpha \in B\}$ holds.

Definition 1.1.32

A space X is said to be **compact** if every open cover of X has a finite subcover.

Definition 1.1.33 [10]

A space X is said to be **semi-compact** if every semi-open cover of X has a finite subcover.

Definition 1.1.34

A space X is said to be **Lindelöf** if every cover of X by open sets contains a countable sub cover.

Definition 1.1.35 [10]

A space X is said to be **semi-Lindelöf** if every cover of X by semi-open sets contains a countable sub cover.

Definition 1.1.36 [13]

The topology on the set of integers generated by the set S of all triplets of the form $\{2n-1, 2n, 2n+1\}$ as sub base is called the **Khalimsky topology or digital topology** and it is denoted by κ .

Definition 1.1.37 [13]

The collection $SU\{\{2n+1\}: n \in \mathbb{Z}\}$ is a base for the topology κ . The digital line equipped with the Khalimsky topology is called the **Khalimsky line or digital line**.

Definition 1.1.38 [13]

The topological product of two Khalimsky lines (\mathbb{Z}, κ) is called the **Khalimsky plane or digital plane** and is denoted by (\mathbb{Z}^2, κ^2) .

CHAPTER – II

SEMI*-OPEN SETS

SECTION 2.1: SEMI*-OPEN SETS

Definition 2.1.1

A subset of a topological space (X, τ) is called a **semi*-open set** if there is an open sets U in X such that $U \subseteq A \subseteq Cl^*(U)$.

Notations:

The set of all semi*-open sets in (X, τ) is denoted by $S^*O(X, \tau)$ or simply $S^*O(X)$.

Definition 2.1.2

The **semi*-interior** of A is defined as the union of all semi*-open sets of X contained in A . It is denoted by $s^*Int(A)$.

Definition 2.1.3

Let A be a subset of X . A point x in X is called a **semi*-Interior point** of A if A contains a semi*-open set containing x .

Definition 2.1.4

A subset A of a space X is said to be **nowhere dense** if $Int(cl(A)) = \phi$.

Definition 2.1.5

A space X is **locally indiscrete** if every open set in X is closed.

Definition 2.1.6

The **Sierpinski space** is a finite topological space with two points, only one of which is closed. It is the smallest example of a topological space which is neither trivial nor discrete.

Theorem 2.1.7

A subset A of X is semi*-open if and only if $A \subseteq cl^*(Int(A))$.

Proof:

Given that A is semi*-open

To prove: $A \subseteq \text{cl}^*\text{Int}(A)$

By definition of semi*-open set, there is an open set U in X such that $U \subseteq A \subseteq \text{cl}^*(U)$.

$$\Rightarrow U \subseteq A$$

$$\Rightarrow U = \text{Int}(U) \subseteq \text{Int}(A)$$

$$\therefore U = \text{Int}(A) \quad (\text{since } A \subseteq \text{cl}^*(U))$$

$$\therefore A \subseteq \text{cl}^*\text{Int}(A).$$

Conversely $A \subseteq \text{cl}^*\text{Int}(A)$.

To prove : A is semi*- open

Take $U = \text{Int}(A)$

We know that $\text{Int}(A) \subseteq A$.

Given : $A \subseteq \text{cl}^*(U)$

Then $\text{Int}(A) \subseteq A \subseteq \text{cl}^*(U)$

$$\therefore U \subseteq A \subseteq \text{cl}^*(U).$$

Hence A is semi*-open .

Remark 2.1.8

- i) In any space (X, τ) , ϕ and X are semi*-open sets . Every nonempty semi*- open set must contain at least one nonempty open set and hence cannot be nowhere dense .
- ii) In any topological space, a singleton set is a semi*-open set if and only if it is open and hence a subset A of X is pointwise dense if and only if $A = \cup \{Cl\{x\} : x \in A \text{ and } \{x\} \text{ is semi}^* - \text{open}\}$

Theorem 2.1.9

If $\{A_\alpha\}$ is a collection of semi*-open sets in X , then $\cup A_\alpha$ is also semi*-open in X .

Proof :

Since A_α is semi*-open in X for each α , there is an open set U_α in X such that

$$U_\alpha \subseteq A_\alpha \subseteq Cl^*(U_\alpha).$$

It can be written as, $\cup U_\alpha \subseteq \cup A_\alpha \subseteq \cup Cl^*(U_\alpha)$

$\therefore \cup U_\alpha \subseteq \cup A_\alpha \subseteq Cl^*(\cup U_\alpha)$, Since $\cup U_\alpha$ is open

$\therefore \cup A_\alpha$ is semi*-open in X .

Theorem 2.1.10

If A is semi*-open in X and B is open in X , $A \cap B$ is semi*-open in X .

Proof :

Given that A is semi*-open in X , there is an open set U in X such that

$$U \subseteq A \subseteq Cl^*(U).$$

Since B is open in X , $U \cap B$ is also open in X .

$$\Rightarrow U \cap B \subseteq A \cap B \subseteq Cl^*(U) \cap B.$$

$$\therefore U \cap B \subseteq A \cap B \subseteq Cl^*(U \cap B)$$

Hence $A \cap B$ is semi*-open in X .

Remark 2.1.11

The intersection of two semi*-open sets need not be semi*-open as seen from the following examples. But the intersection of a semi*-open set and an open set is semi*open as show in theorem 2.1.10.

Example 2.1.12

Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. In the space (X, τ) , the subsets $A = \{a, d\}$ and $B = \{b, d\}$ are semi*-open but $A \cap B = \{d\}$ is not semi*-open.

Example 2.1.13

Consider the subspace (X, τ) of the digital plane where $X = \{1,2\} \times \{1,2,3\}$. In (X, τ) , the subsets $A = \{(1,1), (2,2)\}$ and $B = \{(1,3), (2,2)\}$ are semi*-open but $A \cap B = \{(2,2)\}$ is not semi*-open.

Theorem : 2.1.14:

A subset A of X is semi*-open if and only if A contains a semi*-open set about each of its points.

Proof :

Necessity follow from the definition 2.1.3.

Conversely, to prove: A is semi* -open set in X .

Let $x \in A$. We assumed that A is semi*open set in X then there is a semi*-open set in U_x containing x such that $U_x \subseteq A$ then we have $\cup \{U_x : x \in A\}$.

By theorem 2.1.8, If $\{U_x\}$ is a collection of semi*- open sets in X , then $\cup \{U_x\}$ is semi*open in X .

$\therefore A$ is a semi*-open set.

Theorem 2.1.15:

$S^*O(X, \tau)$ forms a topology on X if and only if it is closed under finite intersection.

Proof :

Follows from the remark :2.1.8(i) and theorem 2.1.9.

Theorem 2.1.16:

If A is any subset of X , $s^*Int(A)$ is semi*-open. In fact $s^*Int(A)$ is the largest semi*-open set contained in A .

Proof:

Follows from Definition 2.1.2 and Theorem 2.1.9.

Theorem 2.1.17 :

A subset A of X is semi*-open if and only if $s^*Int(A)=A$.

Proof:

A is semi*-open in X. By definition, Union of semi *open sets contained in A and A itself is a member in the union.

Hence $s^*Int(A)=A$.

Conversely let $s^*Int(A)=A$.

By theorem 2.1.15, $s^*Int(A)$ is semi*-open and hence A is semi*-open.

Theorem 2.1.18:

If A is a subset of X, then $s^*Int(A)$ is the set of all semi*-Interior points of A.

Proof:

By definition 2.1.3, $x \in s^*Int(A)$ if and only if x belongs to some semi*-open subset U of A. That is, if and only if x is a semi*Interior point of A.

Corollary 2.1.19:

A subset A of X is semi*-open if and only if every point of A is a semi*- Interior point of A.

Proof:

Follows from Theorem 2.1.15 and Theorem 2.1.16.

Theorem 2.1.20: Every open set is semi*-open.

Proof:

Let U be open set in X. Then $Int(U)=U$. Therefore $U \subseteq Cl^*(U)=Cl^*(Int(U))$. Hence by Theorem 2.1.7, U is semi*open.

Corollary 2.1.21:

If a subset A is semi*-open and U is open, then $A \cup U$ is semi*-open.

Proof:

Follows from Theorem 2.1.20 and Theorem 2.1.9.

Remark 2.1.22:

The converse of Theorem 2.1.20. is not true as shown in the following examples.

Example 2.1.23:

Consider the topological space (X, τ) in Example 2.1.11. The subsets $\{a, d\}$,

$\{b, d\}$ and $\{a, b, d\}$ are semi*-open in X but not open.

Example 2.1.24:

Consider the subspace (X, τ) of the digital plane given in Example 2.1.13. In (X, τ) , the subsets $\{(1,1),(1,3),(2,2)\}$, $\{(1,1),(1,3),(2,1),(2,2)\}$ and $\{(1,1),(1,2),(1,3),(2,2),(2,3)\}$ are semi*-open but not open.

Definition 2.1.25:

For a topological space (X, τ) , let $\tau_{s^*} = \{U \in S^*O(X, \tau) : U \cap A \in S^*O(X, \tau) \text{ for all } A \in S^*O(X, \tau)\}$.

Theorem 2.1.26:

If (X, τ) is a topological space, then τ_{s^*} is a topology on X finer than τ .

Proof:

Clearly $\varphi, X \in \tau_{s^*}$. Let $U_\alpha \in \tau_{s^*}$ and $U = \cup U_\alpha$. Since $U_\alpha \in S^*O(X, \tau)$

By using Theorem 2.1.9, $U \in S^*O(X, \tau)$.

Let $A \in S^*O(X, \tau)$. Then $U_\alpha \cap A \in S^*O(X, \tau)$, for each α (since $U_\alpha \in \tau_{s^*}$)

hence by Theorem 2.1.9,

$$U \cap A = (\cup U_\alpha) \cap A = \cup (U_\alpha \cap A) \in S^*O(X, \tau).$$

Therefore $U \in \tau_{s^*}$.

Now let $U_1, U_2, \dots, U_n \in \tau_{s^*}$. Then $U_1, U_2, \dots, U_n \in S^*O(X, \tau)$ and by definition

of τ_{s^*} , $\bigcap_{i=1}^n U_i \in S^*O(X, \tau)$. If $A \in S^*O(X, \tau)$, then by repeated application of the condition, we

have $\bigcap_{i=1}^n U_i \cap A \in S^*O(X, \tau)$.

Hence $\bigcap_{i=1}^n U_i \in \tau_{S^*}$.

This shows that τ_{S^*} is a topology on X .

Let $V \in \tau$. By using Theorem 2.1.20, $V \in S^*O(X, \tau)$.

Also by Theorem 2.1.13, $V \cap A \in S^*O(X, \tau)$ for all $A \in S^*O(X, \tau)$. Hence $V \in \tau_{S^*}$.

Thus τ_{S^*} is finer than τ .

Theorem 2.1.27:

Every semi*-open set is semi-open.

Proof:

Let A be a semi*-open set. Then there is an open set U in X such that $U \subseteq A \subseteq Cl^*(U)$.

Note that $Cl^*(U) \subseteq Cl(U)$. Therefore $U \subseteq A \subseteq Cl(U)$. Hence A is semi-open.

Remark 2.1.28:

The converse of Theorem 2.1.27 is not true as shown in the following examples.

Example 2.1.29:

Consider the topological space (X, τ) given in Example 2.1.11. The subsets $\{a, c, d\}$ and $\{b, c, d\}$ are semi-open in X but not semi*-open.

Example 2.1.30:

Consider the subspace (X, τ) of the digital plane where $X = \{0, 1\} \times \{1, 2, 3\}$. In (X, τ) , the subsets $\{(1,1), (1,2)\}$, $\{(0,2), (1,1), (1,2)\}$ and $\{(0,3), (1,2), (1,3)\}$ are semi-open but not semi*-open.

Theorem 2.1.31:

In any topological space (X, τ) , $\tau \subseteq S^*O(X, \tau) \subseteq SO(X, \tau)$. That is, the class of semi*-open sets is placed between the class of open sets and the class of semi-open sets.

Proof:

Follows from Theorem 2.1.20 and Theorem 2.1.27.

Remark 2.1.32:

- (i) If (X, τ) is a locally indiscrete space, then $\tau = S^*O(X, \tau) = SO(X, \tau)$.
- (ii) In the Sierpinski space (X, τ) , where $X = \{0, 1\}$ and $\tau = \{\emptyset, \{1\}, X\}$, $\tau = S^*O(X, \tau) = SO(X, \tau)$.
- (iii) The inclusions in Theorem 2.1.31 may be strict and equality may also hold. This can be seen from the following examples.

Example 2.1.33:

In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b, c, d\}, X\}$,

$$SO(X, \tau) = S^*O(X, \tau) = \{\emptyset, \{a\}, \{b, c, d\}, X\}.$$

$$\text{Here } \tau = S^*O(X, \tau) = SO(X, \tau).$$

Example 2.1.34:

In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$,

$$SO(X, \tau) = S^*O(X, \tau) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}.$$

$$\text{Here } \tau \subsetneq S^*O(X, \tau) = SO(X, \tau).$$

Example 2.1.35:

In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$,

$$SO(X, \tau) = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}; S^*O(X, \tau) = \{\emptyset, \{a, b\}, X\}.$$

$$\text{Here } \tau = S^*O(X, \tau) \subsetneq SO(X, \tau).$$

Example 2.1.36:

Consider the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{a, b, c\}, X\}$.

$$SO(X, \tau) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

$$S^*O(X, \tau) = \{ \phi, \{a\}, \{a, d\}, \{a, b, c\}, X \}.$$

Here $\tau \subsetneq S^*O(X, \tau) \subsetneq SO(X, \tau)$

Example 2.1.37:

Consider the subspace (X, τ) of the digital plane where $X = \{1, 2, 3\} \times \{0, 1\}$.

If a, b, c, d, e, f denote the points $(1,0), (1,1), (2, 0), (2,1), (3,0), (3,1)$ respectively, then

$$\tau = \{ \phi, \{b\}, \{f\}, \{a,b\}, \{b,f\}, \{e,f\}, \{a,b,f\}, \{b,d,f\}, \{b,e,f\}, \{a,b,d,f\}, \{a,b,e,f\}, \{b,d,e,f\}, \{a,b,d,e,f\}, X \}.$$

$$SO(X) = \{ \phi, \{b\}, \{f\}, \{a,b\}, \{b,c\}, \{b,d\}, \{b,f\}, \{c,f\}, \{d,f\}, \{e,f\}, \{a,b,c\}, \{a,b,d\}, \{a,b,f\}, \{b,c,d\}, \{b,c,f\}, \{b,d,f\}, \{b,e,f\}, \{c,d,f\}, \{c,e,f\}, \{d,e,f\}, \{a,b,c,d\}, \{a,b,c,f\}, \{a,b,d,f\}, \{a,b,e,f\}, \{b,c,d,f\}, \{b,c,e,f\}, \{b,d,e,f\}, \{c,d,e,f\}, \{a,b,c,d,f\}, \{a,b,c,e,f\}, \{a,b,d,e,f\}, \{b,c,d,e,f\}, X \}.$$

$$S^*O(X) = \{ \phi, \{b\}, \{f\}, \{a,b\}, \{b,c\}, \{b,f\}, \{c,f\}, \{e,f\}, \{a,b,c\}, \{a,b,f\}, \{b,c,f\}, \{b,d,f\}, \{b,e,f\}, \{c,e,f\}, \{a,b,c,f\}, \{a,b,d,f\}, \{a,b,e,f\}, \{b,c,d,f\}, \{b,c,e,f\}, \{b,d,e,f\}, \{a,b,c,d,f\}, \{a,b,c,e,f\}, \{a,b,d,e,f\}, \{b,c,d,e,f\}, X \}.$$

Here $\tau \subsetneq S^*O(X, \tau) \subsetneq SO(X, \tau)$.

Example 2.1.38:

Consider the subspace (X, τ) of the digital plane where $X = \{0, 1, 2\} \times \{1, 2\}$.

If a, b, c, d, e, f denote the points $(0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2)$ respectively, then

$$\tau = \{ \phi, \{c\}, \{a,c\}, \{c,d\}, \{c,e\}, \{a,c,d\}, \{a,c,e\}, \{c,d,e\}, \{a,b,c,d\}, \{a,c,d,e\}, \{c,d,e,f\}, \{a,b,c,d,e\}, \{a, c, d, e, f\}, X \}.$$

$$SO(X) =$$

$$S^*O(X) = \{ \phi, \{c\}, \{a,c\}, \{b,c\}, \{c,d\}, \{c,e\}, \{c,f\}, \{a,b,c\}, \{a,c,d\}, \{a,c,e\}, \{a,c,f\}, \{b,c,d\}, \{b, c, e\}, \{b, c, f\}, \{c,d,e\}, \{c,d,f\}, \{c,e,f\}, \{a,b,c,d\}, \{a,b,c,e\}, \{a,c,d,e\}, \{a,b,c,f\}, \{a,c,d,e\}, \{a,c,d,f\}, \{a, c, e, f\}, \{b, c, d, e\}, \{b, c, d, f\}, X \}.$$

$\{b,c,e,f\}, \{c,d,e,f\}, \{a,b,c,d,e\}, \{a,b,c,d,f\}, \{a,c,d,e,f\}, \{a,b,c,e,f\}, \{b, c, d, e, f\}, X$.

Here $\tau \subseteq S^*O(X, \tau) = SO(X, \tau)$.

Remark 2.1.39:

If X is a $T_{1/2}$ space, the g -closed sets and the closed sets coincide and hence $Cl^*(U) = Cl(U)$. Therefore the class of semi*-open sets and the class of semi-open sets coincide. In particular, in the Khalimsky line and in the real line with usual topology, the semi*-open sets and the semi-open sets coincide. But the converse is not true. That is, a space, in which the class of semi*-open sets and the class of semi-open sets coincide, need not be $T_{1/2}$ and this can be seen from the Example 2.1.34 and Example 2.1.38. In these spaces the class of semi*-open sets and the class of semi-open sets coincide but they are not $T_{1/2}$.

Theorem 2.1.40:

If (X, τ) is any topological space, then $S^*O(X, \tau^*) = SO(X, \tau^*)$.

Proof:

Follows from the fact that the space (X, τ^*) is $T_{1/2}$ and theorem states that “if (X, τ) ” and Remark 2.1.39.

Lemma 2.1.41:

If A be semi*-open, then $Cl^*(A) = Cl^*(Int(A))$.

Proof:

Since A is semi*-open, $A \subseteq Cl^*(Int(A))$. Hence $Cl^*(A) \subseteq Cl^*(Int(A))$ which proves the lemma.

Theorem 2.1.42:

Let A be semi*-open and $B \subseteq X$ such that $A \subseteq B \subseteq Cl^*(A)$. Then B is semi*-open.

Proof:

Since A is semi*-open, $A \subseteq Cl^*(Int(A))$. Since $Int(A) \subseteq Int(B)$, $Cl^*(Int(A)) \subseteq Cl^*(Int(B))$. Therefore by the above lemma, $B \subseteq Cl^*(Int(B))$. Hence by Theorem 2.1.7, B is semi*-open.

Theorem 2.1.43:

Let β be a collection of subsets in (X, τ) satisfying

- (i) $\tau \subseteq \beta$
- (ii) If $B \in \beta$ and $D \subseteq X$ such that $B \subseteq D \subseteq Cl^*(B)$ implies $D \in \beta$.

Then $S^*O(X, \tau) \subseteq \beta$. Thus $S^*O(X, \tau)$ is the smallest collection satisfying the conditions (i) and (ii).

Proof:

Let $A \in S^*O(X, \tau)$.

Then there is an open set U in X such that $U \subseteq A \subseteq Cl^*(U)$.

By (i), $U \in \beta$. By (ii), $A \in \beta$. Thus $S^*O(X, \tau) \subseteq \beta$.

Also by Theorem 2.1.20 and Theorem 2.1.11, $S^*O(X, \tau)$ satisfies (i) and (ii).

Thus $S^*O(X, \tau)$ is the smallest collection satisfying (i) and (ii).

Theorem 2.1.44:

If (X, τ) is a topological space, then $S^*O(X, \tau) \subseteq SO(X, \tau^*)$. That is, every semi*-open set in (X, τ) is semi-open in (X, τ^*) .

Proof:

If A is a semi*-open set in (X, τ) , then there is an open set U in (X, τ) such that $U \subseteq A \subseteq Cl^*(U)$. Since U is open in (X, τ) , U is open in (X, τ^*) . Thus A is semi-open in (X, τ^*) .

Remark 2.1.45:

The inclusion in Theorem 2.1.44 can be strict and equality also holds as seen from the following examples:

Example 2.1.46:

Consider the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{ \phi, \{a, b\}, \{a, b, c\}, X \}$

$$S^*O(X, \tau) = \{ \phi, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X \}.$$

$GC(X, \tau) = \{ \phi, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X \}.$

$\tau^* = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X \}.$

$SO(X, \tau^*) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X \} = \wp(X) \setminus \{\{d\}\}.$

Here $S^*O(X, \tau) \subsetneq SO(X, \tau^*)$.

Example 2.1.47:

Consider the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X \}.$

$S^*O(X, \tau) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X \};$ $GC(X, \tau) = \{ \phi, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X \}.$

$\tau^* = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X \}.$

Here $SO(X, \tau^*) = S^*O(X, \tau) = \wp(X) \setminus \{\{d\}\}.$

Remark 2.1.48:

The concepts of semi*-open sets and α -open sets are independent as seen from the following examples:

Example 2.1.49:

In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{ \phi, \{a\}, \{a, b, c\}, X \}$, the subsets $\{a, b\}$, $\{a, c\}$, $\{a, b, d\}$ and $\{a, c, d\}$ are α -open but not semi*-open.

Example 2.1.50:

In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{ \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X \}$, the subsets $\{a, d\}$ and $\{b, d\}$ are semi*-open but not α -open.

Remark 2.1.51:

The concepts of semi*-open sets and pre-open sets are independent as seen from the following examples:

Example 2.1.52:

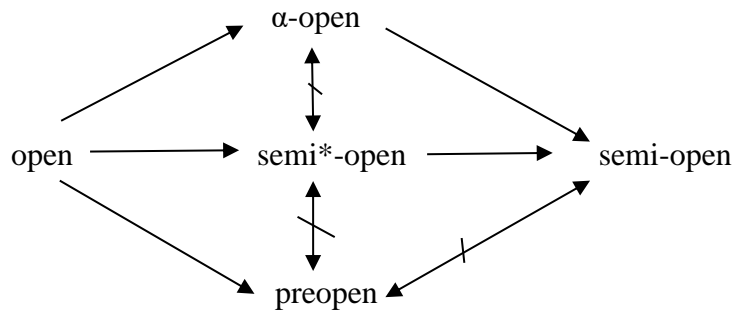
In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, X\}$, the subsets $\{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}$ and $\{a, c, d\}$ are pre-open but not semi*-open.

Example 2.1.53:

In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$, the subsets $\{a, d\}, \{b, d\}$ and $\{b, c, d\}$ are semi*-open but not pre-open.

From the above discussions we have the following diagram:

Diagram 2.1.54:



Theorem 2.1.55:

In any topological space (X, τ) the following hold:

- i. $s^*Int(\phi) = \phi$.
- ii. $s^*Int(X) = X$.

If A and B are subsets of X ,

- iii. $s^*Int(A) \subseteq A$.
- iv. $A \subseteq B \Rightarrow s^*Int(A) \subseteq s^*Int(B)$.

- v. $s^*Int(s^*Int(A))=s^*Int(A)$. That is, the operator s^*Int is idempotent.
- vi. $Int(A) \subseteq s^*Int(A) \subseteq sInt(A) \subseteq A$.
- vii. $s^*Int(A \cup B) \supseteq s^*Int(A) \cup s^*Int(B)$.
- viii. $s^*Int(A \cap B) \subseteq s^*Int(A) \cap s^*Int(B)$.
- ix. $Int(s^*Int(A)) = Int(A)$.
- x. $s^*Int(Int(A))=Int(A)$.

Proof:

(i), (ii), (iii) and (iv) follow from Definition 2.1.2.

(v) follows from Theorem 2.1.19 and Theorem 2.1.20.

(vi) follows from Theorem 2.1.23 and Theorem 2.1.30.

(vii) and (viii) follow from (iv) above.

Since $s^*Int(A) \subseteq A$, $Int(s^*Int(A)) \subseteq Int(A)$.

Also from (vi), $Int(A) \subseteq s^*Int(A)$ and so $Int(A) \subseteq Int(s^*Int(A))$.

Therefore $Int(A)(s^*Int(A))=Int(A)$. This proves (ix).

(x) follows from the fact that $Int(A)$ is open and hence semi*-open and by invoking Theorem 2.1.20, $s^*Int(Int(A))=Int(A)$.

Remark 2.1.56:

In (vi) of Theorem 2.1.53, each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 2.1.57:

In the space (X, τ) where $X=\{a, b, c, d, e, f, g\}$ and $\tau=\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, e\}, \{a, c, d\}, \{b, f, g\}, \{a, b, c, d\}, \{a, b, f, g\}, \{a, b, c, d, e\}, \{a, b, e, f, g\}, \{a, b, c, d, f, g\}, X\}$. Let $A= \{a, b, c, d\}$.

Then $Int(A)=s^*Int(A)=sInt(A)=\{a, b, c, d\}=A$.

Let $B=\{a, e\}$. Then $Int(B)=\{a\}$; $s^*Int(B)=sInt(B)=\{a, e\}$.

Here $Int(B) \subsetneq s^*Int(B)=sInt(B)=B$.

Let $C= \{a, b, c, d, e, f\}$. Then $Int(C)=s^*Int(C)=\{a, b, c, d, e\}$; $sInt(C)=\{a, b, c, d, e, f\}$.

Here $Int(C) =s^*Int(C) \subsetneq sInt(C) =C$.

Let $D = \{b, d, f, g\}$. Then $\text{Int}(D) = s^*\text{Int}(D) = s\text{Int}(D) = \{b, f, g\}$. Here $\text{Int}(D) = s^*\text{Int}(D) = s\text{Int}(D) \subsetneq D$.

Let $E = \{a, c, e\}$. Then $\text{Int}(E) = \{a\}$; $s^*\text{Int}(E) = \{a, e\}$; $s\text{Int}(E) = \{a, c, e\}$.

Here $\text{Int}(E) \subsetneq s^*\text{Int}(E) \subsetneq s\text{Int}(E) = E$.

Let $F = \{b, c, d, e\}$. Then $\text{Int}(F) = \{b\}$; $s^*\text{Int}(F) = s\text{Int}(F) = \{b, e\}$.

Here $\text{Int}(F) \subsetneq s^*\text{Int}(F) = s\text{Int}(F) \subsetneq F$.

Let $G = \{a, d, f\}$. Then $\text{Int}(G) = s^*\text{Int}(G) = \{a\}$; $s\text{Int}(G) = \{a, d\}$.

Here $\text{Int}(G) = s^*\text{Int}(G) \subsetneq s\text{Int}(G) \subsetneq G$. Let $H = \{b, c, d, e, f\}$.

Then $\text{Int}(H) = \{b\}$; $s^*\text{Int}(H) = \{b, e\}$; $s\text{Int}(H) = \{b, e, f\}$.

Here $\text{Int}(H) \subsetneq s^*\text{Int}(H) \subsetneq s\text{Int}(H) \subsetneq H$.

Example 2.1.58:

Consider the subspace (X, τ) of the digital plane where $X = \{1, 2, 3\} \times \{1, 2\}$. If a, b, c, d, e, f denote the points $(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)$ respectively, then

$\tau = \{ \phi, \{a\}, \{e\}, \{a,b\}, \{a,e\}, \{e,f\}, \{a,b,e\}, \{a,c,e\}, \{a,e,f\}, \{a,b,c,e\}, \{a,b,e,f\}, \{a,c,e,f\}, \{a,b,c,e,f\}, X \}$.

$\text{SO}(X) = \{ \phi, \{a\}, \{e\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{c,e\}, \{d,e\}, \{e,f\}, \{a,b,c\}, \{a,b,d\}, \{a,b,e\}, \{a,c,d\}, \{a,c,e\}, \{a,d,e\}, \{a,e,f\}, \{c,d,e\}, \{c,e,f\}, \{d,e,f\}, \{a,b,c,d\}, \{a,b,c,e\}, \{a,b,d,e\}, \{a,b,e,f\}, \{a,c,d,e\}, \{a,c,e,f\}, \{a,d,e,f\}, \{c,d,e,f\}, \{a,b,c,d,e\}, \{a,b,c,e,f\}, \{a,b,d,e,f\}, \{a,c,d,e,f\}, X \}$.

$\text{S}^*\text{O}(X) = \{ \phi, \{a\}, \{e\}, \{a,b\}, \{a,d\}, \{a,e\}, \{d,e\}, \{e,f\}, \{a,b,d\}, \{a,b,e\}, \{a,c,e\}, \{a,d,e\}, \{a,e,f\}, \{d,e,f\}, \{a,b,c,e\}, \{a,b,d,e\}, \{a,b,e,f\}, \{a,c,d,e\}, \{a,c,e,f\}, \{a,d,e,f\}, \{a,b,c,d,e\}, \{a,b,c,e,f\}, \{a,b,d,e,f\}, \{a,c,d,e,f\}, X \}$.

Let $A = \{a, b, c, d, f\}$. Then $\text{Int}(A) = \{a, b\}$; $s^*\text{Int}(A) = \{a, b, d\}$; $s\text{Int}(A) = \{a, b, c, d\}$.

Here $\text{Int}(A) \subsetneq s^*\text{Int}(A) \subsetneq s\text{Int}(A) \subsetneq A$.

Let $B = \{b, c, e, f\}$. Then $\text{Int}(B) = s^*\text{Int}(B) = \{e, f\}$; $s\text{Int}(B) = \{c, e, f\}$.

Here $\text{Int}(B) = s^*\text{Int}(B) \subsetneq s\text{Int}(B) \subsetneq B$.

Let $C = \{a, b, d, f\}$. Then $\text{Int}(C) = \{a, b\}$; $s^*\text{Int}(C) = s\text{Int}(C) = \{a, b, d\}$.

Here $\text{Int}(C) \subsetneq s^*\text{Int}(C) = s\text{Int}(C) \subsetneq C$.

Let $D = \{a, c, d\}$. Then $\text{Int}(D) = \{a\}$; $s^*\text{Int}(D) = \{a, d\}$; $s\text{Int}(D) = \{a, c, d\}$.

Here $\text{Int}(D) \subsetneq s^*\text{Int}(D) \subsetneq s\text{Int}(D) = D$.

Let $E = \{a, b, c, d, e\}$. Then $\text{Int}(E) = \{a, b, c, e\}$; $s^*\text{Int}(E) = s\text{Int}(E) = \{a, b, c, d, e\}$.

Here $\text{Int}(E) \subsetneq s^*\text{Int}(E) = s\text{Int}(E) = E$.

Let $F = \{a, c\}$. Then $\text{Int}(F) = s^*\text{Int}(F) = \{a\}$; $s\text{Int}(F) = \{a, c\}$.

Here $\text{Int}(F) = s^*\text{Int}(F) \subsetneq s\text{Int}(F) = F$.

Let $G = \{b, e, f\}$. Then $\text{Int}(G) = s^*\text{Int}(G) = s\text{Int}(G) = \{e, f\}$.

Here $\text{Int}(G) = s^*\text{Int}(G) = s\text{Int}(G) \subsetneq G$.

Let $H = \{a, b, e, f\}$. Then $\text{Int}(H) = s^*\text{Int}(H) = s\text{Int}(H) = H$.

Remark 2.1.59:

The inclusions in (vii) and (viii) of Theorem 2.1.53 may be strict and equality may also hold. This can be seen from the following examples.

Example 2.1.60:

Consider the space (X, τ) in Example 2.1.55

Let $A = \{b, c, e, f, g\}$ and $B = \{a, b, c, f, g\}$ then $A \cup B = \{a, b, c, e, f, g\}$ and $A \cap B = \{b, c, f, g\}$ $s^*\text{Int}(A) = \{b, e, f, g\}$; $s^*\text{Int}(B) = \{a, b, f, g\}$;
 $s^*\text{Int}(A \cup B) = \{a, b, e, f, g\}$; $s^*\text{Int}(A \cap B) = \{b, f, g\}$

Here $s^*\text{Int}(A \cup B) = s^*\text{Int}(A) \cup s^*\text{Int}(B)$ and $s^*\text{Int}(A \cap B) = s^*\text{Int}(A) \cap s^*\text{Int}(B)$

Let $C = \{a, c, d, e, g\}$ and $D = \{b, d, e, f, g\}$ then $C \cap D = \{d, e, g\}$

$s^*\text{Int}(C) = \{a, c, d, e\}$; $s^*\text{Int}(D) = \{b, e, f, g\}$; $s^*\text{Int}(C \cap D) = \emptyset$; $s^*\text{Int}(C) \cap s^*\text{Int}(D) = \{e\}$

Here $s^*\text{Int}(C \cap D) \subsetneq s^*\text{Int}(C) \cap s^*\text{Int}(D)$

Let $E = \{b, c, d, f, g\}$ and $F = \{a, b, d, g\}$ then $E \cup F = \{a, b, c, d, f, g\}$; $s^*Int(E) = \{b, f, g\}$;

$s^*Int(F) = \{a, b\}$; $s^*Int(E \cup F) = \{a, b, c, d, f, g\}$; $s^*Int(E) \cup s^*Int(F) = \{a, b, f, g\}$;

Here $s^*Int(E) \cup s^*Int(F) \subsetneq s^*Int(E \cup F)$

SECTION 2.2 : ON SEMI*-CLOSED SETS

Definition 2.2.1:

A subset A of a space (X, τ) is called a **semi*-closed set** if $X \setminus A$ is semi*-open.

Notation:

The class of all semi*-closed sets in (X, τ) is denoted by $S^*C(X, \tau)$ or simply $S^*C(X)$.

Definition 2.2.2:

The **semi*-closure** of A is defined as the Intersection of all semi*-closed sets in X containing A . It is denoted by $s^*Cl(A)$.

Definition 2.2.3:

Let $A \subseteq X$. An element $x \in X$ is called a **semi*adherent point** of A if every semi*-open set in X containing x intersects A .

Definition 2.2.4:

Let $A \subseteq X$. An element $x \in X$ is called a **semi*limit point** of A if every semi*-open set in X containing x intersects A in a point different from x .

Definition 2.2.5:

The set of all semi*-limit points of A is called the **semi*-Derived set** of A . It is denoted by $D_{s^*}[A]$.

Definition 2.2.6:

A subset A of X is called **semi*-regular** if it is both semi*-open and semi*-closed.

Theorem 2.2.7:

A subset A of a space (X, τ) is semi*-closed if and only if there is a closed set F in (X, τ) such that $\text{Int}^*(F) \subseteq A \subseteq F$.

Proof:

Suppose A is semi*-closed. Then $X \setminus A$ is semi*-open.

Then there exists an open set U in X such that $U \subseteq X \setminus A \subseteq \text{Cl}^*(U)$

$$\Rightarrow X \setminus U \supseteq A \supseteq X \setminus \text{Cl}^*(U).$$

Note that in any space, $X \setminus \text{Cl}^*(U) = \text{Int}^*(X \setminus U)$.

Therefore $X \setminus U \supseteq A \supseteq \text{Int}^*(X \setminus U)$ where $X \setminus U$ is closed in X .

conversely, Suppose there is a closed set F in (X, τ) such that $\text{Int}^*(F) \subseteq A \subseteq F$

$$\Rightarrow X \setminus \text{Int}^*(F) \supseteq X \setminus A \supseteq X \setminus F.$$

Since $X \setminus \text{Int}^*(F) = \text{Cl}^*(X \setminus F)$, we have $\text{Cl}^*(X \setminus F) \supseteq X \setminus A \supseteq X \setminus F$ where $X \setminus F$ is an open set.

Hence by Definition 2.1.1, $X \setminus A$ is semi*-open. Therefore A is semi*-closed.

Remark 2.2.8:

In any space (X, τ) , ϕ and X are semi*-closed sets.

Theorem 2.2.9:

$A \subseteq X$ is semi*-closed if and only if $\text{Int}^*(\text{Cl}(A)) \subseteq A$.

Proof:

If $A \subseteq B$ then $X \setminus A \supseteq X \setminus B$ and

$$\text{Cl}^*(X \setminus A) = X \setminus \text{Int}^*(A) \text{ and}$$

$$X \setminus \text{Cl}^*(A) = \text{Int}^*(X \setminus A).$$

A is semi*-closed $\Leftrightarrow X \setminus A$ is semi*-open.

$$\Leftrightarrow X \setminus A \subseteq \text{Cl}^*(\text{Int}(X \setminus A)) \text{ (By definition 2.1.1).}$$

$$\Leftrightarrow \text{Int}^*(\text{Cl}(A)) \subseteq A.$$

Theorem 2.2.10:

If A is semi*-closed in X and $B \subseteq X$ is such that $\text{Int}^*(A) \subseteq B \subseteq A$. Then B is semi*-closed in X .

Proof:

Since A is semi*-closed, $X \setminus A$ is semi*-open.

$$\text{Now } \text{Int}^*(A) \subseteq B \subseteq A \Rightarrow X \setminus \text{Int}^*(A) \supseteq X \setminus B \supseteq X \setminus A.$$

That is, $\text{Cl}^*(X \setminus A) \supseteq X \setminus B \supseteq X \setminus A$.

Therefore by Theorem 2.1.41, $X \setminus B$ is semi*-open. Hence B is semi*-closed.

Theorem 2.2.11:

If $\{A_\alpha\}$ is a collection of semi*-closed sets, then $\bigcap A_\alpha$ is also a semi*-closed set.

Proof:

Since each A_α is semi*-closed, $X \setminus A_\alpha$ is semi*-open.

By Theorem 2.1.14, $\bigcup (X \setminus A_\alpha)$ is semi*-open.

That is, $X \setminus (\bigcap A_\alpha)$ is semi*-open.

Hence $\bigcap A_\alpha$ is semi*-closed.

Corollary 2.2.12:

If A is semi*-closed and U is semi*-open in X , then $A \setminus U$ is semi*-closed in X .

Proof:

$$\text{Follows from } A \setminus U = A \cap (X \setminus U).$$

Remark 2.2.13:

Union of two semi*-closed sets need not be semi*-closed as seen from the following example.

Example 2.2.14:

Let $X=\{a, b, c, d\}$ with $\tau=\{ \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. In the space (X, τ) , the subsets $A=\{a, c\}$ and $B=\{b, c\}$ are semi*-closed but $A \cup B=\{a, b, c\}$ is not semi*-closed.

Theorem 2.2.15:

If A is semi*-closed in X and B is closed in X , then $A \cup B$ is semi*-closed in X .

Proof:

Since A is semi*-closed, $X \setminus A$ is semi*-open in X .

Also $X \setminus B$ is open.

By Theorem 2.1.10, $(X \setminus A) \cap (X \setminus B) = X \setminus (A \cup B)$ is semi*-open in X .

Hence $A \cup B$ is semi*-closed in X .

Theorem 2.2.16:

If A is any subset of a space X , then $s^*Cl(A)$ is semi*-closed. In fact $s^*Cl(A)$ is the smallest semi*-closed set in X containing A .

Proof:

Follows from Definition 2.2.1 and Theorem 2.2.11.

Theorem 2.2.17:

A is semi*-closed if and only if $s^*Cl(A)=A$.

Proof:

A is semi*-closed implies $s^*Cl(A)=A$ is obvious. On the other hand, let $s^*Cl(A)=A$.

By Theorem 2.2.16, $s^*Cl(A)$ is semi*-closed and hence A is semi*-closed.

Theorem 2.2.18:

Every closed set is semi*-closed.

Proof:

Let A be closed. Then $X \setminus A$ is open.

By Theorem 2.1.20 which states that “every open set is semi*-open”, $X \setminus A$ is semi*-open.

Hence A is semi*-closed.

Remark 2.2.19:

The converse of Theorem 2.2.18 is not true as shown by the following example.

Example 2.2.20:

In the space (X, τ) of Example 2.2.14, the subsets $A = \{a, c\}$ and $B = \{b, c\}$ are semi*-closed but not closed.

Corollary 2.2.21:

If A is semi*-closed and F is closed in X , then $A \cap F$ is semi*-closed in X .

Proof:

Since F is closed, by Theorem 2.2.18, F is semi*closed.

Then by Theorem 2.2.11, $A \cap F$ is semi*-closed.

Corollary 2.2.22:

If A is semi*-closed and U is open in X , then $A \setminus U$ is semi*-closed in X .

Proof:

Follows from $A \setminus U = A \cap (X \setminus U)$ and from Corollary 2.2.21.

Theorem 2.2.23:

Every semi*-closed set is semi-closed.

Proof:

If A is semi*-closed, $X \setminus A$ is semi*-open.

By Theorem 2.1.27 which states that “every semi*-open set is semi-open”, $X \setminus A$ is semi-open.

Hence A is semi-closed.

Remark 2.2.24:

The converse of Theorem 2.2.23 is not true as shown by the following example.

Example 2.2.25:

Consider the space in Example 2.2.14. In the space (X, τ) , the subsets $\{a, d\}$ and $\{b, d\}$ are semi-closed but not semi*-closed.

Theorem 2.2.26:

In any topological space (X, τ) , $\mathcal{F} \subseteq S^*C(X, \tau) \subseteq SC(X, \tau)$.

Proof:

Follows from Theorem 2.2.18 and Theorem 2.2.23.

Remark 2.2.27:

(i) If (X, τ) is a locally indiscrete space, then $\mathcal{F} = S^*C(X, \tau) = SC(X, \tau)$.

(ii) In the Sierpinski space (X, τ) , where $X = \{0, 1\}$ and $\tau = \{\emptyset, \{1\}, X\}$, $\mathcal{F} = S^*C(X, \tau) = SC(X, \tau)$.

(iii) The inclusions in Theorem 2.2.26 may be strict and equality may also hold.

This can be seen from the following examples.

Example 2.2.28:

Consider the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b, c, d\}, X\}$. Here $\mathcal{F} = S^*C(X, \tau) = SC(X, \tau)$.

Example 2.2.29:

Consider the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$.

Here $\mathcal{F} = \{\emptyset, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

$SC(X, \tau) = S^*C(X, \tau) = \{\emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Here $\mathcal{F} \subsetneq S^*C(X, \tau) = SC(X, \tau)$.

Example 2.2.30:

Consider the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a, b\}, X\}$.

$$\mathcal{F}=\{\phi, \{c, d\}, X\};$$

$$SC(X, \tau)=\{\phi, \{c\}, \{d\}, \{c, d\}, X\};$$

$$S^*C(X, \tau)=\{\phi, \{c, d\}, X\}.$$

Here $\mathcal{F}=S^*C(X, \tau)\subsetneq SC(X, \tau)$.

Example 2.2.31:

Consider the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a\}, \{a, b, c\}, X\}$

$$\mathcal{F}=\{\phi, \{d\}, \{b, c, d\}, X\};$$

$$SC(X, \tau)=\{\phi, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}, X\};$$

$$S^*C(X, \tau)=\{\phi, \{d\}, \{b, c\}, \{b, c, d\}, X\}.$$

Here $\mathcal{F}\subsetneq S^*C(X, \tau)\subsetneq SC(X, \tau)$.

Remark 2.2.32:

In a $T_{1/2}$ space, the class of semi*-closed sets and the class of semi-closed sets coincide. In particular, in the Khalimsky line and in the real line with usual topology the semi*-closed sets and the semi-closed sets coincide. In any topological space, $X\setminus\{x\}$ is semi*-closed if and only if it is closed.

Remark 2.2.33:

The concepts of semi*-closed sets and α -closed sets are independent as seen from the following examples:

Example 2.2.34:

In the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a\}, \{a, b, c\}, X\}$, the subsets $\{b\}$, $\{c\}$, $\{b, d\}$ and $\{c, d\}$ are α -closed but not semi*-closed.

Example 2.2.35:

In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, the subsets $\{a, c\}$ and $\{b, c\}$ are semi*-closed but not α -closed.

Remark 2.2.36:

The concepts of semi*-closed sets and pre-closed sets are independent as seen from the following examples:

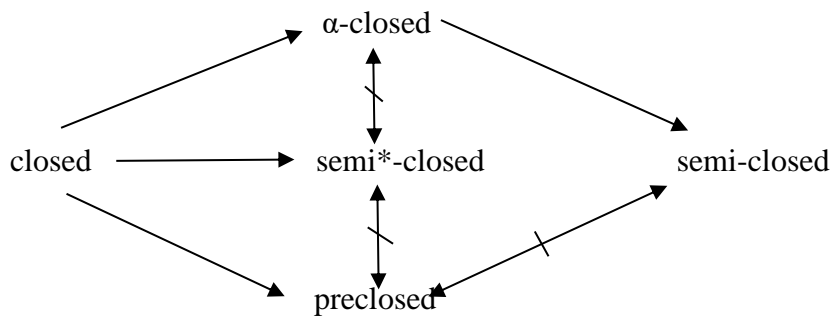
Example 2.2.37:

In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, X\}$, the subsets $\{b, d\}$ and $\{c, d\}$ are pre-closed but not semi*-closed.

Example 2.2.38:

In the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, X\}$, the subsets $\{a, c\}$ and $\{b, c\}$ are semi*-closed but not pre-closed. From the above discussions we have the following diagram:

Diagram 2.2.39:



Theorem 2.2.40:

In any topological space (X, τ) , the following results hold:

- (i) $s^*Cl(\phi) = \phi$.
- (ii) $s^*Cl(X) = X$. If A and B are subsets of X ,
- (iii) $A \subseteq s^*Cl(A)$.
- (iv) $A \subseteq B \implies s^*Cl(A) \subseteq s^*Cl(B)$.
- (v) $s^*Cl(s^*Cl(A)) = s^*Cl(A)$. That is, the operator s^*Cl is idempotent.
- (vi) $A \subseteq sCl(A) \subseteq s^*Cl(A) \subseteq Cl(A)$.
- (vii) $s^*Cl(A \cup B) \supseteq s^*Cl(A) \cup s^*Cl(B)$.
- (viii) $s^*Cl(A \cap B) \subseteq s^*Cl(A) \cap s^*Cl(B)$.
- (ix) $Cl(s^*Cl(A)) = Cl(A)$.
- (x) $s^*Cl(Cl(A)) = Cl(A)$.

Proof:

(i), (ii), (iii) and (iv) follow from Definition 2.2.2.

(v) follows from Theorem 2.2.16 and Theorem 2.2.17.

(vi) follows from Theorem 2.2.18 and Theorem 2.2.23.

(vii) and (viii) follow from (iv) above.

From (iii) above, we have $A \subseteq s^*Cl(A)$ and hence $Cl(A) \subseteq Cl(s^*Cl(A))$.

Also from (vi) above, we have $Cl(A) \supseteq s^*Cl(A)$ and hence $Cl(A) \supseteq Cl(s^*Cl(A))$.

Therefore $Cl(s^*Cl(A)) = Cl(A)$.

This proves (ix). Clearly, (x) follows from the fact that $Cl(A)$ is closed and hence semi*-closed and by invoking Theorem 2.2.17.

Remark 2.2.41:

In (vi) of Theorem 2.2.40, the inclusions may be strict and equality may also hold. This can be seen from the following example.

Example 2.2.42:

Let $X = \{a, b, c, d, e, f, g\}$ be endowed with $\tau = \{ \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, e\}, \{a, c, d\}, \{b, f, g\}, \{a, b, c, d\}, \{a, b, f, g\}, \{a, b, c, d, e\}, \{a, b, e, f, g\}, \{a, b, c, d, f, g\}, X \}$.

Here $\mathcal{F} = \{ \phi, \{e\}, \{c, d\}, \{f, g\}, \{c, d, e\}, \{e, f, g\}, \{a, c, d, e\}, \{b, e, f, g\}, \{c, d, f, g\}, \{c, d, e, f, g\}, \{a, c, d, e, f, g\}, \{b, c, d, e, f, g\}, X \}$;

$SC(X, \tau) = \{ \phi, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{c, d\}, \{c, e\}, \{c, f\}, \{c, g\}, \{d, e\}, \{d, f\}, \{d, g\}, \{e, f\}, \{e, g\}, \{f, g\}, \{a, c, d\}, \{b, f, g\}, \{c, d, e\}, \{c, d, f\}, \{c, d, g\}, \{c, e, f\}, \{c, e, g\}, \{c, f, g\}, \{d, e, f\}, \{d, e, g\}, \{d, f, g\}, \{e, f, g\}, \{a, c, d, e\}, \{a, c, d, f\}, \{a, c, d, g\}, \{b, c, f, g\}, \{b, d, f, g\}, \{b, e, f, g\}, \{c, d, e, f\}, \{c, d, e, g\}, \{c, d, f, g\}, \{c, e, f, g\}, \{d, e, f, g\}, \{a, c, d, e, f\}, \{a, c, d, e, g\}, \{a, c, d, f, g\}, \{b, c, d, f, g\}, \{b, c, e, f, g\}, \{b, d, e, f, g\}, \{c, d, e, f, g\}, \{a, c, d, e, f, g\}, \{b, c, d, e, f, g\}, X \}$;

$S^*C(X, \tau) = \{ \phi, \{e\}, \{c, d\}, \{f, g\}, \{a, c, d\}, \{b, f, g\}, \{c, d, e\}, \{e, f, g\}, \{a, c, d, e\}, \{b, e, f, g\}, \{c, d, f, g\}, \{a, c, d, f, g\}, \{b, c, d, f, g\}, \{c, d, e, f, g\}, \{a, c, d, e, f, g\}, \{b, c, d, e, f, g\}, X \}$.

Let $A = \{e, f, g\}$. Then $Cl(A) = s^*Cl(A) = sCl(A) = A$.

Let $B = \{b, c, d, f, g\}$. Then $Cl(B) = \{b, c, d, e, f, g\}$; $s^*Cl(B) = sCl(B) = \{b, c, d, f, g\}$.

Here $B = sCl(B) = s^*Cl(B) \subsetneq Cl(B)$.

Let $C = \{f\}$. Then $sCl(C) = \{f\}$; $s^*Cl(C) = Cl(C) = \{f, g\}$.

Here $C = sCl(C) \subsetneq s^*Cl(C) = Cl(C)$.

Let $D = \{a, c, e\}$. Then $sCl(D) = s^*Cl(D) = Cl(D) = \{a, c, d, e\}$. Here $D \subsetneq sCl(D) = s^*Cl(D) = Cl(D)$

Let $E = \{b, d, f, g\}$. Then $sCl(E) = \{b, d, f, g\}$; $s^*Cl(E) = \{b, c, d, f, g\}$; $Cl(E) = \{b, c, d, e, f, g\}$.

Here $E = sCl(E) \subsetneq s^*Cl(E) \subsetneq Cl(E)$.

Let $F = \{a, f, g\}$. Then $sCl(F) = s^*Cl(F) = \{a, c, d, f, g\}$; $Cl(F) = \{a, c, d, e, f, g\}$.

Here $F \subsetneq sCl(F) = s^*Cl(F) \subsetneq Cl(F)$.

Let $G = \{b, c, e, g\}$. Then $sCl(G) = \{b, c, e, f, g\}$; $s^*Cl(G) = Cl(G) = \{b, c, d, e, f, g\}$.

Here $G \subsetneq sCl(G) \subsetneq s^*Cl(G) = Cl(G)$.

Let $H = \{a, g\}$. Then $sCl(H) = \{a, c, d, g\}$; $s^*Cl(H) = \{a, c, d, f, g\}$; $Cl(H) = \{a, c, d, e, f, g\}$.

Here $H \subsetneq sCl(H) \subsetneq s^*Cl(H) \subsetneq Cl(H)$.

Remark 2.2.43:

The inclusions in (vii) and (viii) of Theorem 2.2.40, may be strict and equality may also hold. This can be seen from the following examples.

Example 2.2.44:

Consider the space (X, τ) in Example 2.2.42.

Let $A = \{a, d\}$ and $B = \{d, e\}$ then $A \cup B = \{a, d, e\}$ and $A \cap B = \{d\}$.
 $s^*Cl(A) = \{a, c, d\}$; $s^*Cl(B) = \{c, d, e\}$; $s^*Cl(A \cup B) = \{a, c, d, e\}$;
 $s^*Cl(A \cap B) = \{c, d\}$.

Here $s^*Cl(A \cup B) = s^*Cl(A) \cup s^*Cl(B)$ and $s^*Cl(A \cap B) = s^*Cl(A) \cap s^*Cl(B)$.

Let $C = \{b, f\}$ and $D = \{a, c\}$ then $C \cup D = \{a, b, c, f\}$; $s^*Cl(C) = \{b, f, g\}$;
 $s^*Cl(D) = \{a, c, d\}$; $s^*Cl(C \cup D) = X$; $s^*Cl(C) \cup s^*Cl(D) = \{a, b, c, d, f, g\}$.

Here $s^*Cl(C) \cup s^*Cl(D) \subsetneq s^*Cl(C \cup D)$.

Let $E = \{a, e\}$ and $F = \{c, e, f\}$ then $E \cap F = \{e\}$; $s^*Cl(E) = \{a, c, d, e\}$;
 $s^*Cl(F) = \{c, d, e, f, g\}$; $s^*Cl(E \cap F) = \{e\}$; $s^*Cl(E) \cap s^*Cl(F) = \{c, d, e\}$.

Here $s^*Cl(E \cap F) \subsetneq s^*Cl(E) \cap s^*Cl(F)$.

Theorem 2.2.45:

Let β be a collection of subsets in (X, τ) satisfying

- (i) $\mathcal{F} \subseteq \beta$ where \mathcal{F} is the collection of closed sets in (X, τ)
- (ii) If $B \in \beta$ and $D \subseteq X$ such that $Int^*(B) \subseteq D \subseteq B$ implies $D \in \beta$. Then $S^*C(X, \tau) \subseteq \beta$. Thus $S^*C(X, \tau)$ is the smallest collection satisfying the (i) and (ii).

Proof:

Let $A \in S^*C(X, \tau)$. Then by Theorem 2.2.7, there is a closed set F in (X, τ) such that $Int^*(F) \subseteq A \subseteq F$.

By (i), $F \in \beta$. By (ii), $A \in \beta$.

Hence $S^*C(X, \tau) \subseteq \beta$. By using Theorem 2.2.18 and Theorem 2.2.10, $S^*C(X, \tau)$ satisfies (i) and (ii).

Thus $S^*C(X, \tau)$ is the smallest collection satisfying (i) and (ii).

Theorem 2.2.46:

Let $A \subseteq X$ and let $x \in X$. Then $x \in s^*Cl(A)$ if and only if every semi*-open set in X containing x intersects A .

Proof:

Suppose $x \notin s^*Cl(A)$. Then $X \setminus s^*Cl(A)$ is a semi*-open set containing x that does not intersect A .

Conversely, Suppose U is a semi*-open set containing x that does not intersect A . Then $X \setminus U$ is a semi*-closed set containing A .

Therefore $s^*Cl(A) \subseteq X \setminus U$.

Hence $x \notin s^*Cl(A)$.

Thus $x \notin s^*Cl(A)$ if and only if there is a semi*-open set containing x that does not intersect A . This proves the theorem.

Corollary 2.2.47:

Let A be a subset of X . $s^*Cl(A)$ is the set of all semi*-adherent points of A .

Proof:

Follows from Definition 2.2.3.

Theorem 2.2.48:

If A is a subset of X , then

- (i) $s^*Cl(X \setminus A) = X \setminus s^*Int(A)$.
- (ii) $s^*Int(X \setminus A) = X \setminus s^*Cl(A)$.

Proof:

- (i) Let $x \in X \setminus s^*Int(A)$. Then $x \notin s^*Int(A)$.

This implies that x does not belong to any semi*-open subset of A .

Let F be a semi*-closed set containing $X \setminus A$.

Then $X \setminus F$ is a semi*-open set contained in A .

Therefore $x \notin X \setminus F$ and so $x \in F$. Hence $x \in s^*Cl(X \setminus A)$.

Therefore $X \setminus s^*Int(A) \subseteq s^*Cl(X \setminus A)$.

On the other hand, let $x \in s^*Cl(X \setminus A)$.

Then x belongs to every semi*-closed set containing $X \setminus A$.
Hence x does not belong to any semi*-open subset of A . That is
 $x \notin s^*Int(A)$.
Then $x \in X \setminus s^*Int(A)$.
This proves (i)

- (ii) It can be proved by replacing A by $X \setminus A$ in (i) and using set theoretic properties.

Theorem 2.2.49:

If $A \subseteq X$, then $s^*Cl(A) = A \cup D_s^*[A]$.

Proof:

By definition, $A \subseteq s^*Cl(A)$. If $x \in D_s^*[A]$, then by Theorem 2.2.47 and Definition 2.2.5, $x \in s^*Cl(A)$. Thus $A \cup D_s^*[A] \subseteq s^*Cl(A)$.

On the other hand, let $x \in s^*Cl(A)$.

If $x \in A$, then $x \in A \cup D_s^*[A]$.

Suppose $x \notin A$. We claim that x is a semi*limit point of A .

Let U be a semi*-open set containing x . Then U intersects A .

Otherwise $X \setminus U$ is a semi*-closed set containing A and since $x \in s^*Cl(A)$, we have $x \in X \setminus U$.

This is a contradiction to $x \in U$. Hence $x \in D_s^*[A]$. Therefore $s^*Cl(A) \subseteq A \cup D_s^*[A]$. Hence the theorem.

Corollary 2.2.50:

A subset A of X is semi*-closed if and only if A contains all its semi*-limit points. That is, A is semi*closed if and only if $D_s^*[A] \subseteq A$.

Proof:

Follows from Theorem 2.2.17 and Theorem 2.2.49.

Theorem 2.2.51:

If A is a subset of a topological space X ,

- (i) $s^*Cl(A) = A \cup Int^*(Cl(A)).$
- (ii) $s^*Int(A) = A \cap Cl^*(Int(A)).$

Proof:

$$Int^*(Cl(A \cup Int^*(Cl(A)))) = Int^*(Cl(A) \cup Cl(Int^*(Cl(A)))) \\ = Int^*(Cl(A)) \subseteq A \cup Int^*(Cl(A)).$$

Then by Theorem 2.2.9, $A \cup Int^*(Cl(A))$ is a semi*-closed set containing A.

$$Hence s^*Cl(A) \subseteq A \cup Int^*(Cl(A)).$$

Also since $s^*Cl(A)$ is semi*-closed, by invoking Theorem 2.2.9 again, we get $Int^*(Cl(A)) \subseteq Int^*(Cl(s^*Cl(A))) \subseteq s^*Cl(A).$

Hence $A \cup Int^*(Cl(A)) \subseteq s^*Cl(A)$ which proves (i).

(ii) follows from (i) above by replacing A by $X \setminus A$ and applying Theorem 2.2.48.

Theorem 2.2.52:

If a subset A of a space X is nowhere dense, then $Int(s^*Cl(A)) = \phi.$

Proof:

Follows from $Int(Cl(A)) = \phi$ and the fact $Int(s^*Cl(A)) \subseteq Int(Cl(A)).$

SECTION : 2.3 :SEMI*-CONNECTED AND COMPACT SPACES :

Definition 2.3.1:

A topological space X is said to be **semi*-connected** if X cannot be expressed as the union of two disjoint nonempty semi*-open sets in X.

Definition 2.3.2:

A function $f : X \rightarrow Y$ is said to be **semi*-continuous** if $f^{-1}(V)$ is semi*-open in X for every open set V in Y.

Definition 2.3.3:

A function $f : X \rightarrow Y$ is said to be **semi*-irresolute** if $f^{-1}(V)$ is semi*-open in X for every semi*-open set V in Y.

Definition 2.3.4:

A function $f : X \rightarrow Y$ is said to be **pre semi*-open** if $f(V)$ is semi*-open in Y for every semi*-open set V in X .

Definition 2.3.5:

A function $f : X \rightarrow Y$ is said to be **pre semi*-closed** if $f(V)$ is semi*-closed in Y for every semi*-closed set V in Y .

Definition 2.3.6:

A function $f : X \rightarrow Y$ is said to be **totally semi*-continuous** if $f^{-1}(V)$ is semi*-regular in X for every open set V in Y .

Definition 2.3.7:

A function $f : X \rightarrow Y$ is said to be **strongly semi*-continuous** if $f^{-1}(V)$ is semi*-regular in X for every subset V in Y .

Definition 2.3.8:

A function $f : X \rightarrow Y$ is said to be **contra-semi*-continuous** if $f^{-1}(V)$ is semi*-closed in X for every open set V in Y .

Definition 2.3.9:

A function $f : X \rightarrow Y$ is said to be **contra-semi*-irresolute** if $f^{-1}(V)$ is semi*-closed in X for every semi*-open set V in Y .

Definition 2.3.10:

A space X is said to be **semi*-T₂** if for each pair of distinct points of x and y in X , there exists disjoint semi*-open sets U and V in X such that $x \in U$ and $y \in V$.

Definition 2.3.11:

If A is a subset of X , the **semi*-frontier of A** is defined by $s^*Fr(A) = s^*cl(A) \setminus s^*Int(A)$.

Theorem 2.3.12:

- (i) If a space X is semi*-connected, then it is connected.
- (ii) If a space X is semi-connected, then it is semi*-connected.

Proof:

- (i) Let X be semi*-connected.
 Suppose X is not connected.
 Then there exist disjoint non-empty open sets A and B such that $X=A \cup B$.
 By theorem 2.1.20, A and B are semi*-open sets.
 This is a contradiction to X is semi*-connected.
 This proves (i).
- (ii) Let X be semi-connected.
 Suppose X is not semi*-connected.
 Then there exist disjoint non-empty semi*-open sets A and B such that
 $X=A \cup B$.
 By theorem 2.1.26, A and B are semi-open sets.
 This is a contradiction to X is semi-connected.
 This proves (ii).

Remark 2.3.13:

The converse of the above theorem is not true as shown in the following example.

Example 2.3.14:

Consider the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Clearly, (X, τ) is connected but not semi*-connected.

Example 2.3.15:

It can be verified that the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ is semi*connected but not semi-connected.

Theorem 2.3.16:

A topological space X is semi*-connected if and only if the only semi*- regular subsets of X are ϕ and X itself.

Proof:

Suppose X is a semi*-connected space.

Let A be non-empty proper subset of X that is semi*-regular.

Then A and $X \setminus A$ are non-empty semi*-open sets and $X=A \cup (X \setminus A)$.

This is a contradiction to the assumption that X is semi*connected.

Conversely , Suppose $X=A \cup B$ where A and B are disjoint non-empty semi*-open sets. Then $A=X \setminus B$ is semi*-closed.

Thus A is a non-empty proper subset that is semi*-regular.

This is a contradiction to our assumption.

Theorem 2.3.17:

A topological space X is semi*-connected if and only if every semi*-continuous function of X into a discrete space Y with at least two points is a constant function.

Proof:

Let f be a semi*-continuous function of the semi*-connected space into the discrete space Y.

Then for each $y \in Y$, $f^{-1}(\{y\})$ is a semi*-regular set of X.

Since X is semi*-connected, $f^{-1}(\{y\}) = \emptyset$ or X.

If $f^{-1}(\{y\}) = \emptyset$ for all $y \in Y$, then f ceases to be a function.

Therefore $f^{-1}(\{y_0\}) = X$ for a unique $y_0 \in Y$. This implies $f(X) = \{y_0\}$ and hence f is a constant function.

Conversely , Let U be a semi*- regular set in X. Suppose $U \neq \emptyset$. We claim that $U = X$. Otherwise, choose two fixed points y_1 and y_2 in Y . Define $f: X \rightarrow Y$ by

$$f(x) = \begin{cases} y_1 & \text{if } x \in U \\ y_2 & \text{otherwise} \end{cases}$$

then for every open set V in Y, $f^{-1}(V)$

$$= \begin{cases} U & \text{if } V \text{ contains } y_1 \text{ only} \\ X \setminus U & \text{if } V \text{ contains } y_2 \text{ only} \\ X & \text{if } V \text{ contains both } y_1 \text{ and } y_2 \\ \emptyset & \text{otherwise} \end{cases}$$

In all the cases $f^{-1}(V)$ is semi*-open in X. Hence f is a non-constant semi*-continuous function of X into Y. This is a contradiction to our assumption. This proves that the only semi*- regular subsets of X are \emptyset and X and hence X is semi*connected.

Result 2.3.18:

Let A be a subset of X . Then A is semi*-regular if and only if $s^*Fr(A)=\phi$

Theorem 2.3.19:

A topological space X is semi*-connected if and only if every nonempty proper subset of X has non-empty semi*-frontier.

Proof:

Suppose that a space X is semi*-connected. Let A be a non-empty proper subset of X . We claim that $s^*Fr(A)\neq \phi$.

If possible, let $s^*Fr(A)=\phi$.

Then by above result, A is semi*-regular.

By Theorem 2.3.16, X is not semi*-connected which is a contradiction.

Conversely, suppose that every non-empty proper subset of X has a non-empty semi*-frontier. We claim that X is semi*-connected.

On the contrary, suppose that X is not semi*-connected. By Theorem 2.3.16, X has a non-empty proper subset, say A , which is semi*-regular.

Then by above result, $s^*Fr(A)=\phi$ which is a contradiction to the assumption. Hence X is semi*-connected.

Result 2.3.20:

Let $f : X \rightarrow Y$ be a function then f is semi*-continuous if and only if $f^{-1}(F)$ is semi*-closed in X for every closed set F in Y .

Theorem 2.3.21:

Let $f : X \rightarrow Y$ be semi*-continuous surjection and X be semi*-connected. Then Y is connected.

Proof:

Let $f : X \rightarrow Y$ be semi*-continuous surjection and X be semi*-connected.

Let V be a open subset of Y .

By definition 2.3.2 and by above result, $f^{-1}(V)$ is semi*-regular in X .

Since X is semi*-connected, $f^{-1}(V) = \emptyset$ or X .

Hence $V = \emptyset$ or Y .

This proves that Y is connected.

Result 2.3.22:

Let $f : X \rightarrow Y$ be a function then f is semi*-irresolute if and only if $f^{-1}(F)$ is semi*-closed in X for every semi*-closed set F in Y .

Theorem 2.3.23:

Let $f : X \rightarrow Y$ be a semi*-irresolute surjection. If X is semi*-connected, so is Y .

Proof:

Let $f : X \rightarrow Y$ be a semi*-irresolute surjection and let X be semi*-connected.

Let V be a subset of Y that is semi*regular in Y .

By definition 2.3.3 and by above result, $f^{-1}(V)$ is semi*-regular in X .

Since X is semi*-connected, $f^{-1}(V) = \emptyset$ or X .

Hence $V = \emptyset$ or Y . This proves that Y is semi*-connected.

Theorem 2.3.24:

Let $f : X \rightarrow Y$ be a pre-semi*-open and pre-semi*-closed injection. If Y is semi*-connected, so is X .

Proof:

Let A be subset of X that is semi*-regular in X .

Since f is both pre-semi*-open and pre-semi*-closed, $f(A)$ is semi*-regular in Y .

Since Y is semi*-connected, $f(A) = \emptyset$ or Y .

Hence $A = \emptyset$ or X . Therefore X is semi*-connected.

Result 2.3.25:

A topological space X is connected if and only if the only clopen subsets of X are ϕ and X .

Theorem 2.3.26:

If $f : X \rightarrow Y$ is a semi*-open and semi*-closed injection and Y is semi*-connected, then X is connected.

Proof:

Let A be a clopen subset of X .

Then $f(A)$ is semi*-regular in Y .

Since Y is semi*-connected, $f(A) = \phi$ or Y .

Hence $A = \phi$ or X .

By above result, X is connected.

Result 2.3.27:

Every open set is semi*-open and every semi*-open is semi-open

Theorem 2.3.28:

If there is a semi*-totally-continuous function from a connected space X onto Y , then Y has the indiscrete topology.

Proof:

Let f be a semi*-totally-continuous function from a connected space X onto Y . Let V be an open set in Y .

Then by above result, V is semi*-open in Y . Since f is semi*-totally-continuous, $f^{-1}(V)$ is clopen in X .

Since X is connected, by result 2.3.25, $f^{-1}(V) = \phi$ or X . This implies $V = \phi$ or Y .

Hence Y has the indiscrete topology.

Theorem 2.3.29:

If there is a totally semi*-continuous function from a semi*-connected space X onto Y , then Y has the indiscrete topology.

Proof:

Let f be a totally semi*-continuous function from a semi*-connected space X onto Y . Let V be an open set in Y .

Since f is totally semi*-continuous, $f^{-1}(V)$ is semi*-regular in X .

Since X is semi*-connected, $f^{-1}(V) = \emptyset$ or X . This implies $V = \emptyset$ or Y .

Thus Y has the indiscrete topology.

Theorem 2.3.30:

If $f : X \rightarrow Y$ is a strongly semi*-continuous bijection and Y is a space with at least two points, then X is not semi*-connected.

Proof:

Let $y \in Y$.

Then $f^{-1}(\{y\})$ is a non-empty proper subset that is semi*-regular in X .

Hence by Theorem 2.3.16, X is not semi*-connected.

Result 2.3.31:

Let $f : X \rightarrow Y$ be a function then f is contra-semi*-continuous if and only if $f^{-1}(F)$ is semi*-open in X for every closed set F in Y .

Theorem 2.3.32:

Let $f : X \rightarrow Y$ be contra-semi*-continuous surjection and X be semi*-connected. Then Y is connected.

Proof:

Let $f : X \rightarrow Y$ be contra-semi*-continuous surjection and X be semi*-connected. Let V be a clopen subset of Y .

By definition 2.3.8 and above result, $f^{-1}(V)$ is semi*-regular in X . Since X is semi*-connected, $f^{-1}(V) = \emptyset$ or X .

Hence $V = \emptyset$ or Y . This proves that Y is connected.

Result 2.3.33:

Let $f : X \rightarrow Y$ be a function then f is contra-semi*-irresolute if and only if $f^{-1}(F)$ is semi*-open in X for every semi*-closed set F in Y .

Theorem 2.3.34:

Let $f : X \rightarrow Y$ be a contra-semi*-irresolute surjection. If X is semi*-connected, so is Y .

Proof:

Let $f : X \rightarrow Y$ be a semi*-irresolute surjection and let X be semi*-connected. Let V be a subset of Y that is semi*regular in Y .

By definition 2.3.9 and above result, $f^{-1}(V)$ is semi*-regular in X .

Since X is semi*-connected, $f^{-1}(V) = \emptyset$ or X .

Hence $V = \emptyset$ or Y . This proves that Y is semi*-connected.

Result 2.3.35:

If (X, τ) is a locally indiscrete space, then $\tau = S^*O(X, \tau) = SO(X, \tau)$

Theorem 2.3.36:

Let X be a locally indiscrete space. Then the following are equivalent:

- (i) X is connected.
- (ii) X is semi*-connected.
- (iii) X is semi-connected.

Proof:

Follows from the above result.

SEMI*-COMPACT AND SEMI*-LINDELÖF SPACES :

Definition 2.4.1:

A collection \mathcal{A} of semi*-open sets in X is called a semi*-open cover of $B \subseteq X$ if $B \subseteq \cup \{U_\alpha : U_\alpha \in \mathcal{A}\}$ holds.

Definition 2.4.2:

A space X is said to be **compact** if every open cover of X has a finite subcover.

Definition 2.4.3 :

A space X is said to be **Lindelöf** if every cover of X by open sets contains a countable subcover.

Definition 2.4.4:

A space X is said to be **semi-compact** if every semi-open cover of X has a finite subcover.

Definition 2.4.5:

A space X is said to be **semi-Lindelöf** if every cover of X by semi-open sets contains a countable subcover.

Definition 2.4.6:

A space X is said to be **semi*-compact** if every semi*-open cover of X has a finite subcover.

Definition 2.4.7:

A subset B of X is said to be **semi*-compact relative to X** if for every semi*-open cover \mathcal{A} of B , there is a finite subcollection of \mathcal{A} that covers B .

Definition 2.4.8:

A space X is said to be **semi*-Lindelöf** if every cover of X by semi*-open sets contains a countable subcover.

Remark 2.4.9:

Every finite space is semi*-compact and every countable space is semi*-Lindelöf.

Theorem 2.4.10:

- (i) Every semi-compact space is semi*-compact.
- (ii) Every semi*-compact space is compact.
- (iii) Every semi-Lindelöf space is semi*-Lindelöf.
- (iv) Every semi*-Lindelöf space is Lindelöf.
- (v) Every semi*-compact space is semi*-Lindelöf

Proof:

(i), (ii), (iii) and (iv) follow from result 2.3.27.

(v) follows from definition 2.4.2, definition 2.4.3, definition 2.4.6 and definition 2.4.8.

Theorem 2.4.11:

Every semi*-closed subset of a semi*-compact space X is semi*-compact relative to X .

Proof:

Let A be a semi*-closed subset of a semi*-compact space X .

Let B be semi*-open cover of A .

Then $B \cup \{X \setminus A\}$ is a semi*-open cover of X .

Since X is semi*-compact, this cover contains a finite subcover of X , namely

$\{B_1, B_2, \dots, B_n, X \setminus A\}$.

Then $\{B_1, B_2, \dots, B_n\}$ is a finite subcollection of B that covers A .

This proves that A is semi*-compact relative to X .

Theorem 2.4.12:

A space X is semi*-compact if and only if every family of semi*-closed sets in X with empty intersection has a finite subfamily with empty intersection.

Proof:

Suppose X is compact and $\{F_\alpha : \alpha \in \Delta\}$ is a family of semi*-closed sets in X such that

$$\bigcap \{F_\alpha : \alpha \in \Delta\} = \phi.$$

Then $\cup \{X \setminus F_\alpha : \alpha \in \Delta\}$ is a semi*-open cover for X .

Since X is semi*-compact, this cover has a finite subcover, say $\{X \setminus F_{\alpha_1}, X \setminus F_{\alpha_2}, \dots, X \setminus F_{\alpha_n}\}$ for X . that is

$X = \bigcup \{ X \setminus F_{\alpha_i} : i = 1, 2, \dots, n \}$. This implies that $\bigcap_{i=1}^n F_{\alpha_i} = \phi$.

Conversely, suppose that every family of semi*-closed sets in X which has empty intersection has a finite subfamily with empty intersection.

Let $\{U_{\alpha} : \alpha \in \Delta\}$ be a semi*-open cover for X . Then $\bigcup \{ U_{\alpha} : \alpha \in \Delta \} = X$. Taking the complements, we get $\bigcap \{ X \setminus U_{\alpha} : \alpha \in \Delta \} = \phi$.

Since $X \setminus U_{\alpha}$ is semi*-closed for each $\alpha \in \Delta$, by the assumption, there is a finite sub family,

$\{ X \setminus U_{\alpha_1}, X \setminus U_{\alpha_2}, \dots, X \setminus U_{\alpha_n} \}$ with empty intersection. That is $\bigcap_{i=1}^n \{ X \setminus U_{\alpha_i} \} = \phi$.

Taking the complements on both sides, we get $\bigcup_{i=1}^n U_{\alpha_i} = X$.

Hence X is semi*-compact.

Result: A subset A of X is semi*-open if and only if A contains a semi*-open set about each of its points

Theorem 2.4.13:

Let X be a semi*- T_2 space in which $S^*O(X)$ is closed under finite intersection. If A is a semi-compact subset of X , then A is semi*-closed.

Proof:

Suppose X is a semi*- T_2 space in which $S^*O(X)$ is closed under finite intersection.

Let A be a semi*-compact subset of X .

Let $x \in X \setminus A$. Since X is semi*- T_2 , for each $a \in A$, there are disjoint semi*-open sets U_a and V_a containing x and a respectively.

$\{V_a : a \in A\}$ is a semi*-open cover for A . Since A is semi*-compact, this cover has a finite subcover say, $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$. Let $U_x = \bigcap_{i=1}^n U_{\alpha_i}$.

Then by assumption, U_x is a semi*-open set containing x .

Also $U_x \cap A = \phi$ and hence $U_x \subseteq X \setminus A$. Then by above result, $X \setminus A$ is semi*-open and hence A is semi*-closed.

Theorem 2.4.14:

Let $f : X \rightarrow Y$ be a semi*-irresolute surjection and X be semi*-compact. Then Y is semi*-compact.

Proof:

Let $f : X \rightarrow Y$ be a semi*-irresolute surjection and X be semi*-compact.

Let $\{V_\alpha\}$ be a semi*-open cover for Y .

Then $\{f(V_\alpha)\}$ is a cover of X by semi*-open sets.

Since X is semi*-compact, $\{f(V_\alpha)\}$ contains a finite subcover, namely $\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}$.

Then $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$, is a finite subcover for Y .

Thus Y is semi*-compact.

Theorem 2.4.15:

If $f : X \rightarrow Y$ is a pre-semi*-open function and Y is semi*-compact, then X is semi*-compact.

Proof:

Let $\{V_\alpha\}$ be a semi*-open cover for X .

Then $\{f(V_\alpha)\}$ is a cover of Y by semi*-open sets.

Since Y is semi*-compact, $\{f(V_\alpha)\}$ contains a finite subcover, namely $\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}$.

Then $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$ is a finite subcover for X .

Thus X is semi*-compact.

Theorem 2.4.16:

If $f : X \rightarrow Y$ is a semi*-open function and Y is semi*-compact, then X is compact.

Proof:

Let $\{V_\alpha\}$ be an open cover for X .

Then $\{f(V_\alpha)\}$ is a cover of Y by semi*-open sets.

Since Y is semi*-compact, $\{f(V_\alpha)\}$ contains a finite subcover, namely

$$\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}.$$

Then $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$ is a finite subcover for X .

Thus X is compact.

Theorem 2.4.17:

Let $f : X \rightarrow Y$ be a semi*-continuous surjection and X be semi*-compact. Then Y is compact.

Proof:

Let $f : X \rightarrow Y$ be a semi*-continuous surjection and X be semi*-compact.

Let $\{V_\alpha\}$ be an open cover for Y .

Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by semi*-open sets.

Since X is semi*-compact, $\{f^{-1}(V_\alpha)\}$ contains a finite subcover, namely,

$$\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}.$$

Then $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$ is a cover for Y .

Thus Y is compact.

Theorem 2.4.18:

A space X is semi*-Lindelöf if and only if every family of semi*-closed sets in X with empty intersection has a countable subfamily with empty intersection.

Proof:

Suppose X is compact and $\{F_\alpha : \alpha \in \Delta\}$ is a family of semi*-closed sets in X such that $\bigcap \{F_\alpha : \alpha \in \Delta\} = \phi$.

Then $\bigcup \{X \setminus F_\alpha : \alpha \in \Delta\}$ is a semi*-open cover for X .

Since X is semi*-Lindelöf, this cover has a countable sub cover, say $\{X \setminus F_{\alpha_i} : i=1,2,\dots\}$ for X . That is $X = \bigcup \{X \setminus F_{\alpha_i} : i=1,2,\dots\}$.

This implies that $\bigcap_i (X \setminus F_{\alpha_i}) = \phi$.

Conversely, suppose that every family of semi*-closed sets in X which has empty intersection has a countable subfamily with empty intersection.

Let $\{U_\alpha : \alpha \in \Delta\}$ be a semi*-open cover for X. Then $\bigcup \{U_\alpha : \alpha \in \Delta\} = X$. Taking the complements, we get $\bigcap \{X \setminus U_\alpha : \alpha \in \Delta\} = \phi$.

Since $X \setminus U_\alpha$ is semi*-closed for each $\alpha \in \Delta$, by the assumption, there is a countable sub family,

$\{X \setminus U_{\alpha_i} : i=1,2,\dots\}$ with empty intersection.

That is $\bigcap_i (X \setminus F_{\alpha_i}) = \phi$

Taking the complements we get $\bigcup_i U_{\alpha_i} = X$.

Hence X is semi*-Lindelöf.

Theorem 2.4.19:

Let $f: X \rightarrow Y$ be a semi*-continuous surjection and X be semi*-Lindelöf. Then Y is Lindelöf.

Proof:

Let $f: X \rightarrow Y$ be a semi*-continuous surjection and X be semi*-Lindelöf.

Let $\{V_\alpha\}$ be an open cover for Y.

Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by semi*-open sets.

Since X is semi*-Lindelöf, $\{f^{-1}(V_\alpha)\}$ contains a countable subcover, namely $\{f^{-1}(V_{\alpha_n})\}$.

Then $\{V_{\alpha_n}\}$ is a countable subcover for Y.

Thus Y is Lindelöf.

Theorem 2.4.20:

Let $f: X \rightarrow Y$ be a semi*-irresolute surjection and X be semi*-Lindelöf. Then Y is semi*-Lindelöf.

Proof:

Let $f : X \rightarrow Y$ be a semi*-irresolute surjection and X be semi*-Lindelöf.

Let $\{V_\alpha\}$ be a semi*-open cover for Y . Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by semi*-open sets.

Since X is semi*-Lindelöf, $\{f^{-1}(V_\alpha)\}$ contains a countable subcover, namely $\{f^{-1}(V_{\alpha_n})\}$.

Then $\{V_{\alpha_n}\}$ is a countable subcover for Y .

Thus Y is semi*-Lindelöf.

Theorem 2.4.21:

If $f : X \rightarrow Y$ is a pre-semi*-open function and Y is semi*-Lindelöf, then X is semi*-Lindelöf.

Proof:

Let $\{V_\alpha\}$ be a semi*-open cover for X .

Then $\{f(V_\alpha)\}$ is a cover of Y by semi*-open sets.

Since Y is semi*-Lindelöf, $\{f(V_\alpha)\}$ contains a countable subcover, namely $\{f(V_{\alpha_n})\}$.

Then $\{V_{\alpha_n}\}$ is a countable subcover for X .

Thus X is semi*-Lindelöf.

Theorem 2.4.22:

If $f : X \rightarrow Y$ is a semi*-open function and Y is semi*-Lindelöf, then X is Lindelöf.

Proof:

Let $\{V_\alpha\}$ be an open cover for X . Then $\{f(V_\alpha)\}$ is a cover of Y by semi*-open sets.

Since Y is semi*-Lindelöf, $\{f(V_\alpha)\}$ contains a countable subcover, namely $\{f(V_{\alpha_n})\}$.

Then $\{V_{\alpha_n}\}$ is a countable subcover for X . Thus X is Lindelöf.

CHAPTER - III

SEMI*-REGULAR OPEN SET

SECTION 3.1 : SEMI*-REGULAR OPEN SETS

Definition 3.1.1

A subset A of a Topological space (X, τ) is called a **Semi*-regular open set** (briefly s^*r -open) if there exists a regular open set U in X such that $U \subseteq A \subseteq Cl^*(U)$. The Class of all Semi*-regular open sets in (X, τ) is denoted by $S^*RO(X, \tau)$ or simply $S^*RO(X)$.

Example 3.1.2

Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}\}$. In the space (X, τ) , the subsets $\{a, b\}$, $\{b, d\}$, $\{a, b, d\}$ are Semi star regular open sets.

Theorem 3.1.3

For a subset A of a topological space (X, τ) the following are equivalent.

- (i) A is Semi*-regular open
- (ii) $A = Cl^*(r\text{-Int}(A))$
- (iii) $Cl^*(A) = Cl^*(r\text{-Int}(A))$
- (iv) $Cl^*(A) = Cl^*(A \cap \text{Int}(cl(A)))$

Proof:

(i) \Rightarrow (ii):

If A is Semi*-regular open, then there is a regular open set U such that $U \subseteq A \subseteq cl^*(U)$.

Now $U \subseteq A \Rightarrow U = r\text{-Int}(U) \subseteq r\text{-Int}(A)$.

$\Rightarrow A \subseteq Cl^*(U) \subseteq Cl^*(r\text{-int}(A))$ and $A \supseteq Cl^*(r\text{-int}(A))$.

Hence $A = Cl^*(r\text{-int}(A))$.

(ii) \Rightarrow (iii):

By assumption, $A = Cl^*(r\text{-int}(A))$.

Since Cl^* is a Kuratowski Operator, we have

$Cl^*(A) = Cl^*(Cl^*(r\text{-int}(A))) = Cl^*(r\text{-int}(A))$.

Hence $Cl^*(A) = Cl^*(r\text{-int}(A))$

(iii) \Rightarrow (i):

Let $U = r\text{-int}(A)$.

Then U is a regular open set in X .

(ie) $U \subseteq A \subseteq Cl^*(A) = Cl^*(r\text{-int}(A)) = Cl^*(U)$

Hence it is semi*-regular open

(iii) \Leftrightarrow (iv):

Given: $Cl^*(A) = Cl^*(r\text{-int}(A))$

Since A is regular open, if $A = \text{Int}(Cl(A))$,

where $r\text{-Int}(A) = \cup \{ \text{all regular open sets contained in } A \}$

$r\text{-Int}(A) = \text{Int}(Cl(r\text{-Int}(A))) \subseteq \text{Int}(Cl(A))$

Let $x \in r\text{-Int}(A) \Rightarrow x \in U \subseteq A$

$\Rightarrow x \in \text{Int}(Cl(A))$

$x \in A \cap \text{Int}(Cl(A))$

$\therefore r\text{-int}(A) = A \cap (\text{int}(Cl(A)))$

Hence $Cl^*(A) = Cl^*(A \cap \text{int}(cl(A)))$

Theorem 3.1.4

Arbitrary Union of Semi*-regular open sets in X is also Semi*-regular open set in X .

Proof:

Let $\{A_i\}$ be a collection of semi*-regular open sets in X .

Since each A_i is Semi*-regular open set, there is a regular-open set U_i in X such that $U_i \subseteq A_i \subseteq Cl^*(U_i)$.

Then $\cup U_i \subseteq \cup A_i \subseteq \cup Cl^*(U_i) \subseteq Cl^*(\cup U_i)$.

Since $\cup U_i$ is regular-open, by Definition 3.1.1, $\cup A_i$ is Semi*-regular open set.

Remark 3.1.5

The intersection of two Semi*-regular open sets need not be Semi*-regular open as seen from the following example.

Example 3.1.6

Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. In the space (X, τ) , the subsets $\{a, d\}$ and $\{b, d\}$ are

semi*-regular open sets but their intersection $\{d\}$ is not a Semi*-regular open set.

Remark 3.1.7

$S^*RO(X, \tau)$ is a topology if it is closed under finite intersection.

Theorem 3.1.8

If A is Semi*-regular open in X , then A can be expressed as $A=U \cup B$ where

- (i) U is regular open in X
- (ii) B is nowhere dense in X
- (iii) $U \cap B = \phi$.

Proof:

Since A is Semi*-regular open set in X , there is a regular open set U such that $U \subseteq A \subseteq Cl^*(U)$.

$$\text{Then } A=U \cup B \text{ where } B=A \setminus U \Rightarrow U \cap B = \phi$$

$$\text{Then } B \subseteq Cl^*(U) - U \Rightarrow Cl(B) = \phi.$$

$$\Rightarrow Int(Cl(B)) = \phi$$

$$\Rightarrow B \text{ is nowhere dense in } X.$$

Theorem 3.1.9

Let A be a Semi*-regular open set and $B \subseteq X$ such that $A \subseteq B \subseteq Cl^*(A)$. Then B is Semi*-regular open set.

Proof:

Since A is Semi*-regular open set, there is a regular open set U in X such that $U \subseteq A \subseteq Cl^*(U)$.

$$\text{Consider, } U \subseteq A$$

$$\Rightarrow U \subseteq A \subseteq B \subseteq Cl^*(A)$$

$$\therefore U \subseteq A \subseteq B \subseteq Cl^*(A) \subseteq Cl^*(U).$$

$$\text{Hence } B \text{ is Semi*-regular open.}$$

Theorem 3.1.10

- (i) Every Semi*-regular open set is Semi* α -open.
- (ii) Every Semi*-regular open set is Semi*pre-open.

- (iii) Every Semi*-regular open set is Semi*open.
- (iv) Every Semi*-regular open set is Semi open.
- (v) Every Semi*-regular open set is Semi α -open.
- (vi) Every Semi*-regular open set is Semi pre-open.
- (vii) Every Semi*-regular open set is regular generalized open set.
- (viii) Every Semi*-regular open set is generalized pre-regular open set.
- (ix) Every Semi*-regular open is regular weakly generalized open set.

Proof (i):

Let A be a Semi*-regular open set.

Then there is a regular open set U such that $U \subseteq A \subseteq Cl^*(U)$.

Since every regular open set is α open.

Therefore, A is semi* α -open .

Hence (i) is proved

Proof (ii):

Since every semi* α -open set is semi*-preopen, A is semi * pre-open

Hence (ii) is proved.

Proof (iii):

Suppose A is a Semi*-regular open set.

Then $A \subseteq Cl^*(r-int(A))$ and since every regular open set is open,

$A \subseteq Cl^*(int(A))$.

Thus A is Semi*-open.

Proof (iv):

By (iii), Every semi*-open set is semi open

Hence (iv) is proved

Proof (v):

By (iv), Every semi open set is semi α -open.

Hence (v) is proved

Proof (vi):

By (v), Since every α -open set is pre-open.

Hence (vi) is proved

Proof (vii):

Since A is Semi*-regular open set, there is a regular open set U in X such that $U \subseteq A \subseteq Cl^*(U)$.

Consider $A \subseteq U$, then A is a regular open set

If $cl(A) \subseteq U$, every regular open set is regular generalized open set.

Hence (vii) is proved .

Proof (viii):

Since A is Semi*-regular open set, there is a regular open set U in X such that $U \subseteq A \subseteq Cl^*(U)$.

Consider $A \subseteq U$, then A is a regular open set

If $pcl(A) \subseteq U$, every regular open set is generalized pre-regular open set.

Hence (viii) is proved .

Proof (ix):

Since A is Semi*-regular open set, there is a regular open set U in X such that $U \subseteq A \subseteq Cl^*(U)$.

Consider $A \subseteq U$, then A is a regular open set .

If $cl(int(A)) \subseteq U$, every regular open set is a regular weakly generalized open set.

Hence (ix) is proved .

Remark 3.1.11

The Converse of each of the statements in Theorem 3.1.10 is not true as shown in the following examples.

Example 3.1.12

Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. The subset $\{a, b\}$ is Semi*- α -open but it is not Semi*-regular open.

Example 3.1.13

In the topological space (X, τ) where $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}\}$, the subset $\{a, b\}$ is semi*-pre-open but it is not Semi*-regular open.

Example 3.1.14

Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. The subset $\{a, b, c\}$ is semi* open but it is not Semi*-regular open.

Example 3.1.15

Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c, d\}\}$. The subsets $\{a, c\}$ and $\{b, c\}$ are semi open but they are not semi*-regular open set.

Example 3.1.16

Consider the topological space (X, τ) where $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$. The subset $\{a, b, d\}$ is semi α -open but it is not semi*-regular open.

Example 3.1.17

Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$. The subset $\{b, c, d\}$ is semi pre-open but not semi*-regular open.

Example 3.1.18

Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$. The subset $\{a, d\}$ is regular generalized open set but not semi*-regular open.

Example 3.1.19

Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a\}, \{d\}, \{a, b\}, \{a, b, d\}\}$. The subset $\{b, d\}$ is generalized pre-open set but not semi*-regular open.

Example 3.1.20

Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a\}, \{b, c, d\}\}$. The subset $\{a, c, d\}$ is a regular weakly generalized open set but not semi*-regular open.

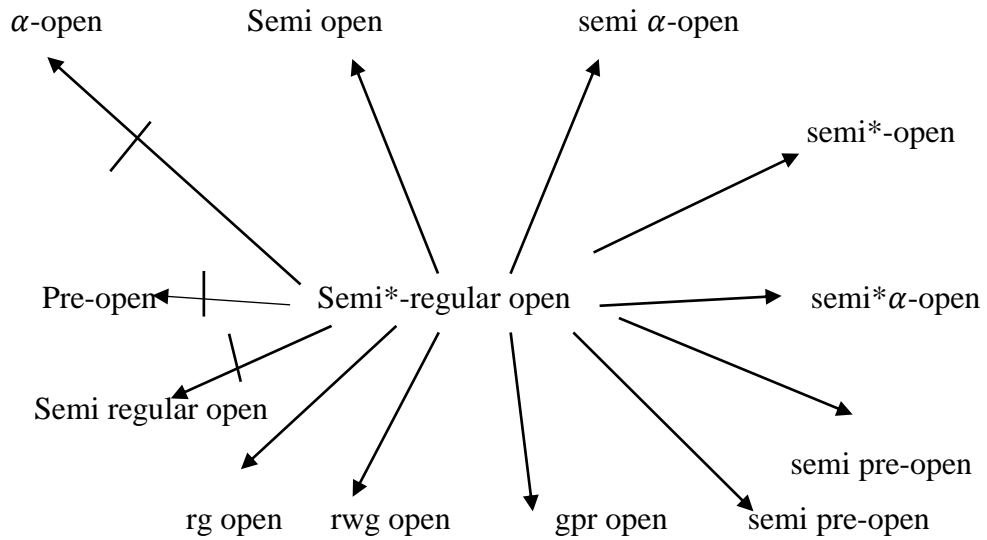
Example 3.1.21

Let $X = \{a, b, c, d, e\}$ be a topological space with the topology $\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\}$. The subset $\{a, e\}$ is semi*-regular open but not α -open.

Example 3.1.22

In the topological space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$, the subset $\{a\}$ is semi*-regular open but not pre-open.

The following figure represents the relation between various types of open sets.



Example 3.1.23

Let (X, τ) be a topological space where $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b, c, d\}\}$.

Here $\tau = \alpha\text{-open} = \text{regular open} = \text{semi open} = \text{semi } \alpha\text{-open} = \text{semi regular open} = \text{semi}^*\text{-open} = \text{semi}^*\alpha\text{-open} = \text{semi}^*\text{-regular open} = \{X,$

$\phi, \{a\}, \{b, c, d\}$ and all the subsets of X are pre-open, semi pre-open and semi*pre-open.

Results 3.1.24

In a topological space (X, τ) ,

- (i) Every semi regular open set is the union of a regular open set and the largest nowhere dense subset of X . Converse is also true.
- (ii) Every semi*-regular open set is the union of regular open set and nowhere dense subset of X . But the converse is not true.

Example 3.1.25

Consider the topological space $X = \{a, b, c, d\}$ with the topology

$\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Semi regular open set = $\{\{a, c, d\}, \{b, c, d\}\}$.

Here $\{a\}, \{b\}$ are regular open sets and $\{c, d\}$ is the largest nowhere dense subset of X .

Example 3.1.26

Consider the topological space $X = \{a, b, c, d, e\}$ with the topology

$\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\}$.

Here Regular open set = $\{a\}$ and $\{e\}$ is nowhere dense subset of X .

$\{a, e\}$ is the semi regular open set. Therefore the converse is true.

Example 3.1.27

Consider the topological space $X = \{a, b, c, d\}$ with the topology

$\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. $\{a, d\}$ is the semi*-regular open set. Here $\{a\}$ is a regular open set and $\{d\}$ is a nowhere dense.

Example 3.1.28

In the above topological space $\{c, d\}$ is nowhere dense set but $\{a, c, d\}$ is not a

semi*-regular open set. Therefore the converse is not true.

Remark 3.1.29

In an extremally disconnected space and in a locally indiscrete space, regular open set = semi regular open set = semi*-regular open set. In a $T_{1/2}$ space, semi regular open set = semi*-regular open set.

Example 3.1.30

Consider $X = \{a, b, c, d\}$ $\tau = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c, d\}\}$. Then (X, τ) is an extremally disconnected space. Regular open set = semi regular open set = semi*-regular open set = $\{\{b\}, \{a, c, d\}\}$.

Example 3.1.31

Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a\}, \{b, c, d\}\}$. Then (X, τ) is a locally indiscrete space. Regular open set = semi regular open set = semi*-regular open set = $\{\{a\}, \{b, c, d\}\}$.

Example 3.1.32

Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a $T_{1/2}$ space, semi regular open set = semi*-regular open set = $\{\{a, c\}, \{b, c\}\}$.

Theorem 3.1.33

Let A be a semi*-regular open set and $B \subseteq X$ such that $r\text{-int}(A) \subseteq B \subseteq Cl^*(A)$. Then B is semi*-regular open.

Proof:

Since A is semi*-regular open

By theorem 3.1.3, we have $Cl^*(A) = Cl^*(r\text{-int}(A))$

Since $r\text{-int}(A) \subseteq B$, $r\text{-int}(A) \subseteq r\text{-int}(B)$ and hence $Cl^*(r\text{-int}(A)) \subseteq Cl^*(r\text{-int}(B))$.

Therefore by assumption, we have $B \subseteq Cl^*(A) = Cl^*(r\text{-int}(A)) \subseteq Cl^*(r\text{-int}(B))$.

Hence $B \subseteq Cl^*(r\text{-int}(B))$ and always $B \supseteq Cl^*(r\text{-int}(B)) \Rightarrow B = Cl^*(r\text{-int}(A))$

Again by using theorem 3.1.3, B is semi*-regular open.

3.1 (a) : SEMI*-REGULAR CLOSED SET

Definition 4.1

A Subset A of a topological space (X, τ) is called a **semi*-regular closed sets** if the complement A^c of A is Semi*-regular open.

Theorem 4.2

If $\{A_\alpha\}$ is a collection of semi*-regular closed sets, then $\cap A$ is also a Semi*-regular closed set.

Proof:

Given $\{A_\alpha\}$ is a collection of semi*-regular closed sets .

Then $X - A_\alpha$ is semi*-regular open $\forall \alpha$

By theorem 3.1.4,

$U(X - A_\alpha)$ is semi*-regular open.

By Demovier's theorem , $U(X - A_\alpha) = X - \cap X_\alpha$

$\therefore X - \cap X_\alpha$ is semi*-regular open.

$\Rightarrow \cap X_\alpha$ is semi*-regular closed.

Remark 4.3

Union of two semi*-regular closed sets need not be semi*-regular closed as seen from the following example.

Example 4.4

Consider the space (X, τ) with $X = \{a, b, c, d\}$ $\tau = \{X, \phi, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}\}$. The subsets $\{a, c\}$ and $\{c, d\}$ are semi*-regular closed sets but $\{a, c, d\}$ is not semi*-regular closed set.

Theorem 4.5

- i. Every Semi*-regular closed set is Semi* α - closed.
- ii. Every Semi*-regular closed set is Semi*pre- closed.
- iii. Every Semi*-regular closed set is Semi*-closed.
- iv. Every Semi*-regular closed set is Semi closed.
- v. Every Semi*-regular closed set is Semi α - closed.
- vi. Every Semi*-regular closed set is Semi pre-closed.
- vii. Every Semi*-regular closed set is regular generalized closed set.

- viii. Every Semi*-regular closed set is generalized pre-regular closed set.
- ix. Every Semi*-regular closed set is regular weakly generalized closed set.

Proof :

By theorem 3.1.10(i), A^c is semi* α -open.

Hence A is semi* α -closed.

Similarly, proof of (ii) to (ix) follows from theorem 3.1.10 (ii) to (ix).

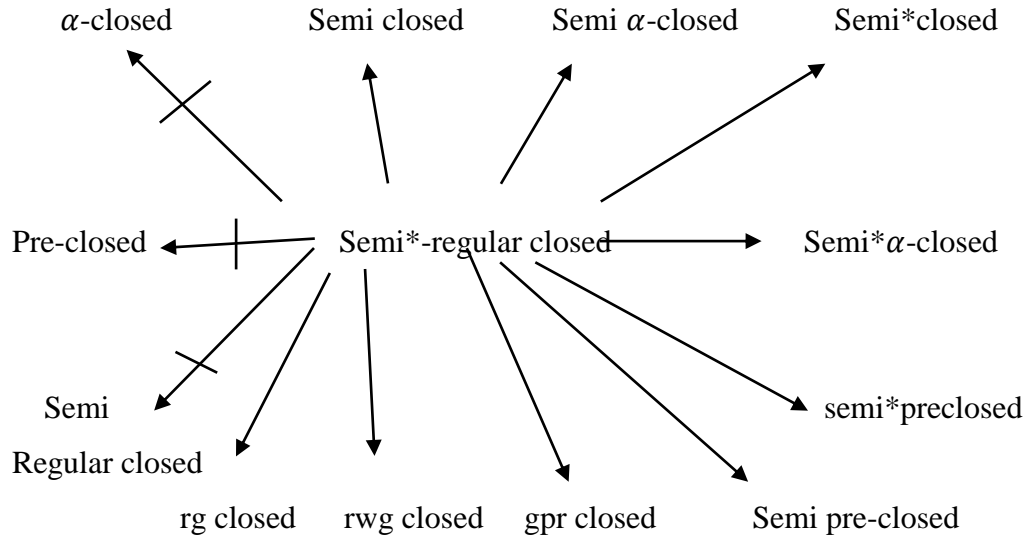
Remark 4.6

In some Topologies, semi*-regular closed set does not imply α -closed, pre-closed and semi regular closed.

Example 4.7

Consider the topological space $X=\{a, b, c, d\}$ with the topology $\tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. The subsets $\{a, c\}$ and $\{b, c\}$ are semi*-regular closed but not α -closed, pre closed and semi -regular closed.

The figure represents the discussions about semi*-regular closed.



Theorem 4.8

A subset A of a topological space (X, τ) is semi*-regular closed set if and only if there is a regular closed set B in (X, τ) such that $\text{Int}^*(B) \subseteq A \subseteq B$ where $\text{Int}^*(B)$ is the generalized interior of B .

Proof:

Suppose A is semi*-regular closed set. Then A^c is semi*-regular open then there exists a regular open set U in X such that $U \subseteq A^c \subseteq \text{Cl}^*(U)$.

(ie) $U^c \supseteq A \supseteq (\text{Cl}^*(U))^c$

Since $(\text{Cl}^*(U))^c = \text{Int}^*(U^c)$, $U^c \supseteq A \supseteq \text{Int}^*(U^c)$ where U^c is regular closed set in X .

Put $U^c = B$ then $\text{Int}^*(B) \subseteq A \subseteq B$.

Conversely suppose there is a regular closed set B in (X, τ) such that $\text{Int}^*(B) \subseteq A \subseteq B$

(ie) $(\text{Int}^*(B))^c \supseteq A^c \supseteq B^c$.

Since $[\text{Int}^*(B)]^c = \text{Cl}^*(B^c)$ and B^c is a regular open set, by the definition 3.1.1, A^c is semi*-regular open.

Hence A is Semi*-regular closed set.

3.1 (b) SEMI*-INTERIOR OF A SET

Definition 5.1

The **semi*-regular-interior of A** is defined as the union of all semi*-regular open sets contained in A and is denoted by $s^*rInt(A)$.

Definition 5.2

Let A be a subset of a topological space (X, τ) . A point x in X is called a **semi*-r-interior point** of A if there is a semi*-regular open subset of A that contains x.

Theorem 5.3

If A is any subset of a topological space (X, τ) , then

- (i) $s^*rInt(A)$ is the largest semi*-regular open set contained in A.
- (ii) A is semi*-regular open set if and only if $s^*rInt(A)=A$
- (iii) $s^*rInt(A)$ is the set of all semi*-r-interior points of A.
- (iv) A is semi*-regular open if and only if every point of A is a semi*-r-interior point of A.

Proof:

- (i) Being the union of all semi*-regular open subsets of A, by theorem 3.1.4, $s^*rInt(A)$ is semi*-regular open and contains every semi*-regular open set contained in A. This proves (i)
- (ii) A is semi*-regular open implies $s^*rInt(A) = A$. On the other hand, Suppose $s^*rInt(A) = A$. By (i), $s^*rInt(A)$ is semi*-regular open and hence A is semi*-regular open.
- (iii) By definition 5.2, $x \in s^*rInt(A)$ if and only if x belongs to some semi*-regular open subset U of A. (ie) if and only if x is a semi*-r-interior point of A.
- (iv) follows from (ii) and (iii)

Theorem 5.4

In any topological space (X, τ) , the following hold

- (i) $s^*rInt(\phi) = \phi$

- (ii) $s^*rInt(X)=X$
- (iii) If A is the subset of X , $s^*rInt(A)\subseteq A$
- (iv) If A and B are subsets of X and $A\subseteq B$ then $s^*rInt(A)\subseteq s^*rInt(B)$
- (v) $s^*rInt(s^*rInt(A))=s^*rInt(A)$
- (vi) $s^*rInt(A)\subseteq s^*int(A)\subseteq s^*\alpha Int(A)\subseteq s^*pInt(A)\subseteq A$
- (vii) $srInt(A)\subseteq s\alpha Int(A)\subseteq spInt(A)\subseteq A$
- (viii) $s^*rInt(A)\cup s^*rInt(B)\subseteq s^*rInt(A\cup B)$
- (ix) $s^*rInt(A\cap B)\subseteq s^*rInt(A)\cap s^*rInt(B)$
- (x) $rInt(s^*rInt(A))=rInt(A)$ only if A is semi*-regular open.

Proof:

Proof of (i),(ii),(iii) and (iv) follow from definition 5.2.

By theorem 5.3(i), $S^*rInt(A)$ is semi*-regular open and by theorem 5.3(ii), $S^*rInt(S^*rInt(A))=S^*rInt(A)$.

Thus (v) is proved and (vi) follows from theorem 3.8, the fact that every semi*-regular open set is semi*open ,every semi*open set is semi* α -open and every semi* α -open set is semi*pre-open.

(vii) follows from the result that every semi regular open set is semi α -open set and every semi α –open set is semi pre-open set. Since $A\subseteq A\cup B$, from (iv) we have $S^*rInt(A)\subseteq S^*rInt(A\cup B)$. Similarly $S^*rInt(B)\subseteq S^*rInt(A\cup B)$.This proves (viii).

In a similar way (ix) is proved.

Since A is semi*-regular open, by theorem 5.3.(ii), $S^*rInt(A) = A$.

This proves (x).

Remark 5.5

The inclusions in (vi), (vii) (viii) and (ix) of Theorem 5.4 may be strict and equality may also hold.

Example 5.6:

Consider the topological space with $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\}$.

Let $A = \{a, b, d, e\}$. Then $s^*rInt(A) = \{a, e\}$, $s^*Int(A)= \{a,e\}$

$s^*\alpha Int(A) = \{a, e\}$, $s^*Int(A) = \{a, b, d, e\}$.

Here $s^*rInt(A) = s^*Int(A) = s^*\alpha Int(A) \subseteq s^*pInt(A)$

Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$.

Let $A = \{a, b, c\}$. Then $s^*rInt(A) = \phi$,

$s^*Int(A) = \{a, b, c\}$,

$s^*\alpha Int(A) = \{a, b, c\}$,

$s^*Int(A) = \{a, b, c\}$

Here $s^*rInt(A) \subseteq s^*Int(A) = s^*\alpha Int(A) = s^*Int(A) = A$

Example 5.7

Let (X, τ) be a topological space where $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}\}$. Let $A = \{a, b, d\}$. Then $srInt(A) = \phi$, $s\alpha Int(A) = spInt(A) = \{a, b, d\}$

Here $srInt(A) \subseteq s\alpha Int(A) = spInt(A) = A$

Let $A = \{b, c, d\}$ Here $srInt(A) = s\alpha Int(A) = spInt(A) = A$

Example 5.8

Consider the topological space in example 5.6.

Let $A = \{a, e\}$, $B = \{b, c, d\}$, $A \cup B = \{a, b, c, d, e\}$ and $A \cap B = \phi$

$s^*rInt(A) = \{a, e\}$

$s^*rInt(B) = \phi$

$s^*rInt(A \cup B) = X$,

$s^*rInt(A \cap B) = \phi$

Here $s^*rInt(A) \cup s^*rInt(B) \subseteq s^*rInt(A \cup B)$ and $s^*rInt(A \cap B)$

$\subseteq s^*rInt(A) \cap s^*rInt(B)$

Example 5.9

Consider the topological space in example 3.27.

Let $A = \{a, d\}$ $B = \{b, d\}$ $A \cup B = \{a, b, d\}$

Then $s^*rInt(A) = \{a, d\}$,

$s^*rInt(B) = \{b, d\}$,

$s^*rInt(A \cup B) = \{a, b, d\}$.

Here $s^*rInt(A \cup B) = s^*rInt(A) \cup s^*rInt(B)$

Let $A = \{a, d\}$, $B = \{a, c, d\}$, $A \cap B = \{a, d\}$

Then $s^*rInt(A) = \{a, d\}$, $s^*rInt(B) = \{a, d\}$, $s^*rInt(A \cap B) = \{a, d\}$

Here $s^*rInt(A \cap B) = s^*rInt(A) \cap s^*rInt(B)$

3.1(c): SEMI REGULAR* CLOSURE OF A SET AND SEMI REGULAR*CONTINUITY AND IRRESOLUTENESS

Definition 6.1

A function $f : X \rightarrow Y$ is said to be

- i. **semi*r-continuous** if $f^{-1}(V)$ is semi*regular open in X for every open set V in Y .
- ii. **semi*r-irresolute** if $f^{-1}(V)$ is semi*regular open in X for every semi*regular open set V in Y .
- iii. **semi*regular open** if $f(V)$ is semi*regular open in Y for every open set V in X .
- iv. **semi*regular closed** if $f(V)$ is semi*regular closed in Y for every closed set V in X .
- v. **pre-semi*regular open** if $f(V)$ is semi*regular open in Y for every semi*regular open set V in X .
- vi. **pre-semi*regular closed** if $f(V)$ is semi*regular closed in Y for every semi*regular closed set V in X .
- vii. **semi*r-totally continuous** if $f^{-1}(V)$ is clopen in X for every semi*regular open set V in Y .
- viii. **contra-semi*r-continuous** if $f^{-1}(V)$ is semi*regular closed in X for every open set V in Y .
- ix. **contra-semi*r-irresolute** if $f^{-1}(V)$ is semi*regular closed in X for every semi*regular open set V in Y .

Theorem 6.2

Let $f : X \rightarrow Y$ be a function. Then

- (i) f is semi*r-continuous if and only if $f^{-1}(F)$ is semi*regular-closed in X for every closed set F in Y .
- (ii) f is semi*r-irresolute if and only if $f^{-1}(F)$ is semi*regular closed in X for every semi*regular closed set F in Y .

- (iii) f is contra-semi*r-continuous if and only if $f^{-1}(F)$ is semi*regular open in X for every closed set F in Y .
- (iv) f is contra-semi*r-irresolute if and only if $f^{-1}(F)$ is semi*regular open in X for every semi*regular closed set F in Y .
- (v) f is semi*r-totally continuous if $f^{-1}(F)$ is clopen in X for every semi*regular closed set F in Y .

Theorem 6.3

If A is a subset of X , then

- (i) $s^*rCl(X \setminus A) = X \setminus s^*rInt(A)$
- (ii) $s^*rInt(X \setminus A) = X \setminus s^*rCl(A)$.
- (iii) A is semi*regular closed if and only if $s^*rCl(A) = A$.

Definition 6.4

Let A be a subset of X . Then the **semi*regular closure** of A is defined as the intersection of all semi*regular closed sets in X containing A and is denoted by $s^*rCl(A)$.

Definition 6.5

- i. Every Semi*regular open set is Semi* α -open.
- ii. Every Semi*regular open set is Semi*pre-open.
- iii. Every Semi*regular open set is Semi*open.
- iv. Every Semi* regular open set is Semi open.
- v. Every Semi*regular open set is Semi α -open.
- vi. Every Semi*regular open set is Semi pre-open.
- vii. Every Semi*regular open set is regular generalized open set.
- viii. Every Semi*regular open set is generalized pre regular open set.

Definition 6.6

A subset A of a topological space (X, τ) is called **s*r regular** if it is both semi*regular -open and semi*regular closed in X .

Section 3.2 : CONNECTED VIA SEMI*-REGULAR OPEN SETS

Definition 3.2.1

A topological space X is said to be **semi*-regular connected** if X cannot be expressed as the union of two disjoint nonempty semi*-regular open sets in X .

Example 3.2.2

Let $X = \{a,b,c,d\}$ and the topology $\tau = \{X, \phi, \{a\}, \{d\}, \{a,d\}, \{a,c,d\}\}$.

$S^*RO(X) = \{\phi, \{a, b\}, \{b, d\}, \{a, b, d\}, X\}$. Then the space X is semi*-regular connected.

Definition 3.2.3

The sets A and B in a topological space X are said to be **semi*-regular separated** if $A \cap s^*rCl(B) = s^*rCl(A) \cap B = \phi$.

Result 3.2.4

If A is subset of X , then $s^*rFr(A) = s^*rCl(A) \cap s^*rCl(X \setminus A)$.
Let A be subset of a space X , then A is semi*-regular iff $s^*rFr(A) = \phi$.

Theorem 3.2.5

For a topological space X , the following statements are equivalent:

- (1). X is semi*-regular connected.
- (2). X cannot be expressed as the union of two disjoint nonempty semi*-regular closed sets in X .
- (3). The only semi*-regular subsets of X are ϕ and X itself.
- (4). Every semi*-regular continuous function of X into a discrete space Y is constant.
- (5). Every nonempty proper subset of X has nonempty semi*-r-frontier.
- (6). X cannot be expressed as the union of two non-empty semi*-regular separated sets.

Proof.

(1) \Rightarrow (2) Let X be a semi*-regular connected space.

Suppose $X = A \cup B$ where A and B are disjoint nonempty semi*-regular closed sets.

Then $A = B^C$ and $B = A^C$ are disjoint non-empty semi*-regular open sets in X . This is a contradiction to the fact that X is semi*-regular connected.

This proves (2).

(2) \Rightarrow (1) Assume that X cannot be expressed as the union of two disjoint nonempty semi*-regular closed sets in X .

Suppose $X = A \cup B$ where A and B are disjoint nonempty semi*-regular open sets.

Then $A = B^C$ and $B = A^C$ are disjoint non-empty semi*-regular closed sets in X .

This is a contradiction to (2).

(1) \Rightarrow (3) Suppose X is a semi*-regular connected space.

Let A be non-empty proper subset of X that is A is semi*r-regular set.

Then A^C is a non-empty semi*-regular open(or semi*-regular closed) and $X = A \cup (A^C)$.

This is a contradiction to X is semi*-regular connected.

(3) \Rightarrow (1) Suppose $X = A \cup B$ where A and B are disjoint non-empty semi*-regular open sets. Then

$A = B^C$ is semi*-regular closed.

Thus A is a non-empty proper subset that is semi*r-regular.

This is a contradiction to (3).

(3) \Rightarrow (4) Let f be a semi*-regular continuous function of the semi*-regular connected space X into the discrete space Y . Then for each $y \in Y$, $f^{-1}(\{y\})$ is a semi*r-regular set of X .

Since X is semi* regular connected, $f^{-1}(\{y\}) = \phi$ or X .

If $f^{-1}(\{y\}) = \phi$ for all $y \in Y$, then f fails to be a function.

Therefore $f^{-1}(\{y_0\}) = X$ for a unique $y_0 \in Y$. This implies $f(X) = \{y_0\}$ and hence f is a constant function.

(4) \Rightarrow (3) Let U be a semi*r-regular set in X . Suppose $U \neq \phi$.

We claim that $U = X$. Otherwise, choose two fixed points y_1 and y_2 in Y . Define $f : X \rightarrow Y$ by

$$f(x) = \begin{cases} y_1 & \text{if } x \in U \\ y_2, & \text{otherwise} \end{cases}$$

then for any open set V in Y ,

$$f^{-1}(V) = \begin{cases} U, & \text{if } V \text{ contains } y_1 \text{ but not } y_2 \\ X \setminus U, & \text{if } V \text{ contains } y_2 \text{ but not } y_1 \\ X, & \text{if } V \text{ contains both } y_1 \text{ and } y_2 \\ \phi & \text{Otherwise} \end{cases}$$

In all the cases $f^{-1}(V)$ is semi*-regular open in X . Hence f is a non-constant semi*-regular continuous function of X into Y .

This is a contradiction to our assumption. This proves that the only semi* regular subsets of X are ϕ and X .

(3) \Rightarrow (5) Suppose that a space X is semi*-regular connected. If possible, let A be a non-empty proper subset of X . We claim that $s^*rFr(A) \neq \phi$. If possible, let $s^*rFr(A) = \phi$.

Then by the result 3.2.4, A is semi*r-regular. This is a contradiction.

(5) \Rightarrow (3) Suppose that every non-empty proper subset of X has a non-empty semi*-regular frontier. The only semi*r-regular subsets of X are ϕ and X itself.

On the contrary, suppose that X has a non-empty proper subset A which is semi*r-regular.

By the result 3.2.4, $s^*rFr(A) = \phi$. This contradiction proves (3).

(1) \Rightarrow (6) Suppose $X = A \cup B$ where A and B are disjoint non-empty semi*r-separated sets in X .

Since $A \cap s^*rCl(B) = \phi$, $s^*rCl(B) \subseteq X \cap A = B$ and hence $s^*rCl(B) = B$ and

By theorem 6.3(iii), B is semi*-regular closed.

Therefore A is semi*-regular open.

Similarly, B is semi*-regular open. Hence X is not semi*r-connected.

This is contradiction to (1).

(6) \Rightarrow (1) Suppose X is not semi*r-connected.

Then X can be written as $X = A \cup B$ where A and B are disjoint nonempty semi*-regular open sets.

Now $A = X \cap B$ is semi*-regular closed and hence by theorem 6.3(iii), $s^*rCl(A) = A$ and so $s^*rCl(A) \cap B = \phi$.

Similarly $A \cap s^*rCl(B) = \phi$. Thus A and B are nonempty semi*-regular separated sets. This is a contradiction to (6).

Theorem 3.2.6

- (1). Every semi*-connected space is semi*-regular connected.
- (2). Every semi*pre-connected space is semi*-regular connected.
- (3). Every semi* α -connected space is semi*-regular connected.
- (4). Every semi connected space is semi*-regular connected.
- (5). Every semi α -connected space is semi*-regular connected.
- (6). Every semi pre-connected space is semi*-regular connected.

Proof.

(1). Let X be semi*-connected space. Suppose X is not a semi*-regular connected.

By theorem 3.3(iii) Then there exists a proper nonempty subset B of X which is both semi*-regular open and semi*-regular closed in X .

Since every semi*-regular closed (open) set is semi*closed(open) set then X is not semi*connected.

This proves (1).

In similar manner, (2),(3),(4),(5) and (6) can be proved.

Remark 3.2.7

It can be seen from the following example that the converse of each of the statements in Theorem 3.3 is not true.

Example 3.2.8

Let $X = \{a,b,c,d\}$ and the topology $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$

$$\begin{aligned}
S^*O(X) &= \\
&\{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\} \\
S^*\alpha O(X) &= \\
&\{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\} \\
S^*PO(X) &= \\
&\{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\} \\
S^*RO(X) &= \{X, \phi, \{b,d\}, \{a,c,d\}\}.
\end{aligned}$$

Then X is semi*-regular connected space but not semi*connected, not semi*\(\alpha\)-connected and not semi*pre-connected spaces.

Theorem 3.2.9

Let $f : X \rightarrow Y$ be a semi*-regular continuous bijection and X be semi*-regular connected. Then Y is connected.

Proof.

Let $f : X \rightarrow Y$ be semi*-regular continuous surjection and X be semi*-regular connected.

Let V be a clopen subset of Y .

By definition 6.1(viii), $f^{-1}(V)$ is semi*-regular open and by theorem 6.2(iii), $f^{-1}(V)$ is semi*-regular closed and hence $f^{-1}(V)$ is semi*r-regular in X . Since X is semi*-regular connected, by Theorem 3.2.6, $f^{-1}(V) = \phi$ or X . Hence $V = \phi$ or Y . This proves that Y is connected.

Theorem 3.2.10

Let $f : X \rightarrow Y$ be a semi*r-irresolute bijection. If X is semi*-regular connected, so is Y .

Proof. Let $f : X \rightarrow Y$ be a semi*r-irresolute surjection and let X be semi*-regular connected.

Let V be a subset of Y that is semi*r-regular in Y .

By definition 6.1(ii) and by theorem 6.2(ii), $f^{-1}(V)$ is semi*r-regular in X .

Since X is semi*-regular connected, $f^{-1}(V) = \phi$ or X . Hence $V = \phi$ or Y .

This proves that Y is semi*-regular connected.

Theorem 3.2.11

Let $f : X \rightarrow Y$ be a pre-semi*-regular open and pre-semi*-regular closed bijection. If Y is semi*-regular connected, so is X .

Proof.

Let A be subset of X that is semi*r-regular in X .

Since f is both pre-semi*-regular open and pre-semi*-regular closed, $f(A)$ is semi*r-regular in Y .

Since Y is semi*-regular connected, $f(A) = \phi$ or Y . Hence $A = \phi$ or X .

Therefore by Theorem 3.2.6, X is semi*-regular connected.

Theorem 3.2.12

If $f : X \rightarrow Y$ is a semi*-regular open and semi*-regular closed bijection and Y is semi*-regular connected, then X is connected.

Proof.

Let A be a clopen subset of X .

Since f is semi*-regular open, $f(A)$ is semi*-regular open in Y .

Since f is a semi*-regular closed map, $f(A)$ is semi*-regular closed in Y .

Hence $f(A)$ is semi*r-regular in Y .

Since Y is semi*-regular connected, by Theorem 3.2.7, $f(A) = \phi$ or Y . Hence $A = \phi$ or X . By theorem 1.1.29, X is connected.

Theorem 3.2.13

If there is a semi*r-totally continuous function from a connected space X onto Y , then the only semi*-regular open sets in Y are ϕ and Y .

Proof.

Let f be a semi*r-totally continuous function from a connected space X onto Y . Let V be any open set in Y .

Then by theorem 6.5(ii), V is semi*-regular open in Y .

Since f is semi*-r-totally continuous, $f^{-1}(V)$ is clopen in X .

Since X is connected, by theorem 1.1.29, $f^{-1}(V) = \emptyset$ or X . This implies $V = \emptyset$ or Y .

Theorem 3.2.13

If $f : X \rightarrow Y$ is a strongly semi*-r-continuous bijection and Y is a space with at least two points, then X is not semi*-regular connected.

Proof.

Let $y \in Y$.

Then $f^{-1}(\{y\})$ is a non-empty proper subset that is semi*-r-regular in X .

Hence by Theorem 3.2.7, X is not semi*-regular connected.

Theorem 3.2.14

Let $f : X \rightarrow Y$ be a contra-semi*-regular continuous surjection and X be semi*-regular connected. Then Y is connected.

Proof.

Let $f : X \rightarrow Y$ be a contra-semi*-regular continuous surjection and X be semi*-regular connected. Let V be a clopen subset of Y .

By definition 6.1(viii) and by Theorem 6.2(iii), $f^{-1}(V)$ is semi*-r-regular in X .

Since X is semi*-regular connected, $f^{-1}(V) = \emptyset$ or X . Hence $V = \emptyset$ or Y .

This proves that Y is connected.

Theorem 3.2.15

Let $f : X \rightarrow Y$ be a semi*-r-irresolute bijection. If X is semi*-regular connected, so is Y .

Proof.

Let $f : X \rightarrow Y$ be a semi*-r-irresolute bijection and let X be semi* regular connected.

Let V be a subset of Y that is semi*-r-regular in Y .

By definition 6.1(ii) and by Theorem 6.2(ii), $f^{-1}(V)$ is semi*-r-regular in X .

Since X is semi*-regular connected, $f^{-1}(V) = \phi$ or X . Hence $V = \phi$ or Y .

This proves that Y is semi*-regular connected.

Theorem 3.2.16

Every contra-semi*-r-continuous function from a semi*-regular connected space into a T_1 space is necessarily constant.

Proof.

Let $f : X \rightarrow Y$ be a contra-semi*-r-continuous function and X be semi*-regular connected and Y be T_1 .

Since Y is T_1 , for each $y \in Y$, $\{y\}$ is closed in Y .

Since f is contra-semi*-r-continuous, by theorem 6.2(iii), $f^{-1}(\{y\})$ is semi*-regular open in X . Therefore $\{f^{-1}(\{y\}) : y \in Y\}$ is a collection of pair wise disjoint semi*-regular open sets in X .

Since X is semi*-regular connected, $f^{-1}(\{y_0\}) = X$ for some fixed $y_0 \in Y$.

Hence $f(X) = y_0$.

Thus f is a constant function.

Theorem 3.2.17

Every contra-semi*-r-irresolute function from a semi*-regular connected space into a semi*-r- T_1 space is necessarily constant.

Proof.

Let $f : X \rightarrow Y$ be a contra-semi*-r-irresolute function and X be semi*-regular connected and Y be semi*-r- T_1 .

Since Y is semi*-r- T_1 , for each $y \in Y$, $\{y\}$ is semi*-regular closed in Y .

Since f is contra-semi*-r-continuous, $f^{-1}(\{y\})$ is semi*-regular open in X .

Therefore $\{f^{-1}(\{y\}) : y \in Y\}$ is a collection of pair wise disjoint semi*-regular open sets in X .

Since X is semi*-regular connected, $f^{-1}(\{y_0\}) = X$ for some fixed $y_0 \in Y$.

Hence $f(X) = y_0$. Thus f is constant.

SECTION 3.3: : COMPACTNESS VIA SEMI*-REGULAR OPEN SETS.

Definition 3.3.1

A collection C of semi*-regular open sets in X is called a semi*-regular open cover of a subset B of X if $B \subseteq \cup\{U_\alpha : U_\alpha \in C\}$ holds.

Definition 3.3.2

A space X is said to be **semi*r-compact** if every semi*-regular open cover of X has a finite subcover.

Definition 3.3.3

A subset B of X is said to be **semi*r-compact relative to X** if for every semi*-regular open cover C of B , there is a finite subcollection of C that covers B .

Remark 3.3.4

Every finite topological space is semi*-regular compact.

Theorem 3.3.5

- (i) Every semi-compact space is semi*-regular compact space.
- (ii) Every semi-pre-compact space is semi*-regular compact space.
- (iii) Every semi α -compact space is semi*-regular compact space.
- (iv) Every semi*-compact space is semi*-regular compact space.
- (v) Every semi*pre-compact space is semi*-regular compact space.
- (vi) Every semi* α -compact space is semi*-regular compact space.

Theorem 3.3.6

Every semi*-regular closed subset of a semi*-regular compact space X is semi*-regular compact relative to X .

Definition 3.3.7

Let A be a semi*-regular closed subset of a semi*-regular compact space X . Let B be a semi*-regular open cover of A . Then $B \cup \{X \setminus A\}$ is a semi*-regular open cover of X . Since X is semi*-regular compact, this cover contains a finite subcover of X and hence contains a finite sub-collection of B that covers A . This shows that A is semi*-regular compact relative to X .

Theorem 3.3.8

A space X is semi*-regular compact if and only if for every family of semi*-regular closed sets in X which has empty intersection has a finite subfamily with empty intersection.

Proof.

Suppose X is semi*-regular compact and $\{F_\alpha : \alpha \in \Delta\}$ is a family of semi*-regular closed sets in X such that $\bigcap \{F_\alpha : \alpha \in \Delta\} = \phi$.

Then $\bigcup \{X \setminus F_\alpha : \alpha \in \Delta\}$ is a semi*-regular open cover for X .

Since X is semi*-regular compact, this cover has a finite subcover $\{X \setminus F_{\alpha_1}, X \setminus F_{\alpha_2}, \dots, X \setminus F_{\alpha_n}\}$.

That is, $X = \bigcup \{X \setminus F_{\alpha_i} : i = 1, 2, \dots, n\}$.

On taking the complements on both sides we get $\bigcap_{i=1}^n F_{\alpha_i} = \phi$.

Conversely, suppose that every family of semi*-regular closed sets in X which has empty intersection has a finite subfamily with empty intersection.

Let $\{U_\alpha : \alpha \in \Delta\}$ be a semi*-regular open cover for X .

Then $\bigcup \{U_\alpha : \alpha \in \Delta\} = X$.

Taking the complements, we get $\bigcap \{X \setminus U_\alpha : \alpha \in \Delta\} = \phi$.

Since $X \setminus U_\alpha$ is semi*-regular closed for each $\alpha \in \Delta$, by the assumption, there is a finite sub family, $\{X \setminus U_{\alpha_1}, X \setminus U_{\alpha_2}, \dots, X \setminus U_{\alpha_n}\}$ with empty intersection.

That is $\bigcap_{i=1}^n (X \setminus U_{\alpha_i}) = \phi$.

Taking the complements on both sides, we get $\bigcup_{i=1}^n U_{\alpha_i} = X$.

Hence X is semi*-regular compact.

Theorem 3.3.9

Let $f : X \rightarrow Y$ be a semi*-irresolute bijection. If X is semi*-compact, then so is Y .

Proof.

Let $f : X \rightarrow Y$ be a semi*-irresolute bijection and X be semi*-compact.

Let $\{V_\alpha\}$ be a semi*-regular open cover for Y . Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by semi*-regular open sets.

Since X is semi*-compact, $\{f^{-1}(V_\alpha)\}$ contains a finite subcover, namely $\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}$. Then $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$ is a finite subcover for Y .

Thus Y is semi*-compact.

Theorem 3.3.10

Let $f : X \rightarrow Y$ be a semi*-continuous bijection and X be semi*-compact. Then Y is compact.

Proof.

Let $f : X \rightarrow Y$ be a semi*-continuous bijection and X be semi*-compact.

Let $\{V_\alpha\}$ be an open cover for Y .

Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by semi*-regular open sets.

Since X is semi*-compact, $\{f^{-1}(V_\alpha)\}$ contains a finite sub cover, namely $\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}$. Then $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$ is a cover for Y .

Thus Y is compact.

Theorem 3.3.11.

Let $f : X \rightarrow Y$ be a pre-semi*-regular open injection. If Y is semi*-compact, then so is X .

Proof.

Let $\{V_\alpha\}$ be a semi*-regular open cover for X .

Then $\{f(V_\alpha)\}$ is a cover of Y by semi*-regular open sets.

Since Y is semi*-compact, $\{f(V_\alpha)\}$ contains a finite subcover, namely $\{f(V_{\alpha_1}), f(V_{\alpha_2}), \dots, f(V_{\alpha_n})\}$.

Since f is semi*-regular open injection, $\{V\alpha_1, V\alpha_2, \dots, V\alpha_n\}$ is a finite subcover for X .

Therefore X is semi*r-compact.

Theorem 3.3.12.

If $f : X \rightarrow Y$ is a semi*-regular open injection and Y is semi*r-compact, then X is compact.

Proof.

Let $\{V_\alpha\}$ be an open cover for X .

Then $\{f(V_\alpha)\}$ is a cover of Y by semi*-regular open sets.

Since Y is semi*r compact, $\{f(V_\alpha)\}$ contains a finite subcover, namely $\{f(V\alpha_1), f(V\alpha_2), \dots, f(V\alpha_n)\}$.

Since f is semi*-regular open injection, $\{V\alpha_1, V\alpha_2, \dots, V\alpha_n\}$ is a finite sub cover for X .

Thus X is compact.

Theorem 3.3.13.

Let $f : X \rightarrow Y$ be a contra-semi*r-continuous function and Y be T_1 . If X is semi*r-compact, then the range of f is finite. Further if Y is infinite, f cannot be onto.

Proof.

Since Y is T_1 , for each $y \in Y$, $\{y\}$ is closed in Y .

Since f is contra-semi*r-continuous, by theorem 6.2(iii), $f^{-1}(\{y\})$ is semi*-regular open in X . Therefore $\{f^{-1}(\{y\}) : y \in Y\}$ is a semi*-regular open cover for X .

Since X is semi*r-compact, there are y_1, y_2, \dots, y_n in Y such that $\{f^{-1}(\{y_i\}) : i = 1, 2, \dots, n\}$ is a cover of X by semi*-regular open sets.

Therefore $\cup\{f^{-1}(\{y_i\}) : i = 1, 2, \dots, n\} = X$.

That is, $f^{-1}(\{y_1, y_2, \dots, y_n\}) = X$.

This implies $f(X) = \{y_1, y_2, \dots, y_n\}$. Thus the range of f is finite.

If Y is infinite, $f(X) \neq Y$. Hence f cannot be onto.

Theorem 3.3.14

Let $f : X \rightarrow Y$ be a contra-semi*r-irresolute function and Y be semi*r- T_1 . If X is semi*r-compact, then the range of f is finite. Further if Y is infinite, f cannot be onto.

Proof.

Since Y is semi*r- T_1 , for each $y \in Y$, $\{y\}$ is semi*-regular closed in Y .

Since f is contra-semi*r-continuous, by theorem 6.2(iv), $f^{-1}(\{y\})$ is semi*-regular open in X .

Therefore $\{f^{-1}(\{y\}) : y \in Y\}$ is a semi*-regular open cover for X .

Since X is semi*r-compact, there are y_1, y_2, \dots, y_n in Y such that $\{f^{-1}(\{y_i\}) : i = 1, 2, \dots, n\}$ is a cover of X by semi*-regular open sets.

Therefore $\cup\{f^{-1}(\{y_i\}) : i = 1, 2, \dots, n\} = X$.

That is, $f^{-1}(\{y_1, y_2, \dots, y_n\}) = X$.

This implies $f(X) = \{y_1, y_2, \dots, y_n\}$. Thus the range of f is finite. If Y is infinite, $f(X) \neq Y$. Hence f cannot be onto.

SUMMARY AND CONCLUSION

This dissertation work consists of a study on Semi*-open sets in topological spaces and its properties.

In **Chapter-I**, the preliminaries and definitions which are needed for the study for the work are listed.

In **Chapter-II**, the semi*-open sets, semi*-closed sets and already existing open sets along with their fundamental properties are reviewed and discussed. Semi*-connectedness and semi*-compactness with some basic properties are studied.

In **Chapter III**, the semi*-regular open sets with the connectedness and compactness via semi*regular open sets are discussed.

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