ON KK-ALGEBRAS

BY

SANGAVI S

(19PMA010)

THESIS SUBMITTED TO

AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION

FOR WOMEN, COIMBATORE – 641 043

IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

MAY – 2021

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Signature of the Head of the Department **Signature of the Supervisor**

ACKNOWLEDGEMENT

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First and foremost, I extremely thankful to the **LORD ALMIGHTY** for his graces and blessings showered on me.

I take immense pleasure in thanking **Dr.S.P.THYAGARAJAN**, chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for providing the conductive infrastructure for the conduct of the research study.

I would like to thank **Dr. (Tmt.) PREMAVATHY VIJAYAN**, Vice-Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the encouragement and for providing the opportunity to develop and establish my skills.

I extend my heartfelt thanks to **Dr. (Tmt.) S.KOWSALYA,** Registrar, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the encouragement given by her during the investigation.

I express my sincere thanks to **Dr. (Tmt.) K.UDAYA CHANDRIKA**, Professor of Mathematics, Dean, School of physical sciences and Computational Sciences, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her excellent support, unflinching encouragement and guidance during the course of the investigation.

I express my heartfelt thanks to **Dr. (Tmt.) P. JEYALAKSHMI,** Professor and Head, Department of Mathematics, and my thesis supervisor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her support, inspiring guidance, innovation ideas, critical suggestions and constant encouragement throughout the work and guidance during the course of investigation.

I would like to express my sincere thanks to all the **STAFF MEMBERS OF THE DEPARTMENT OF MATHEMATICS** who were responsible for the good finish of this dissertation.

I own my special thanks to my **BELOVED PARENTS, LOVING SISTERS, BROTHERS, FRIENDS AND ALSO THE GRACEFUL RELATIVES,** who helped me by providing full strength, support and encouragement to complete my project successfully.

CONTENT

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INTRODUCTION

INTRODUCTION

In 1966, Imai and Iseki [13] introduced the notion of BCK-algebras. In the same year Iseki [14] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. A BCC-algebra is an important class of logical algebras introduced by Y.Komari [22] in 1984 and was extensively investigated by many researchers. In 1998 Dudek and Zhang [10] introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences.

In 2009, Prabpayak and Leerawat [25] introduced a new algebraic structure which is called KU-algebra and they gave the concept of homomorphism of KU-algebras and investigated some properties. In 2012, S.Asawasamrit and A.Sudprasert [4] introduced a new algebraic structure called KK-algebras, and described the relation between ideals and congruences.

This thesis is devoted to a study of ideals in KK-algebras, Quotient KK-algebras, KK-isomorphisms of KK-algebras and P-Semisimple KK-algebras.

The following articles are chosen for our discussion:

1) **"A Structure of KK-Algebras and its properties"**, (2012), by S.Asawasamrit and A.Sudprasert [4]

2) **"KK-Isomorphism and its properties"**, (2012), by S.Asawasamrit [2]

3) **"On the special ideals in KK-Algebras"**, (2013), by S.Asawasamrit and A.Sudprasert [5]

4) "On P-Semisimple in KK-Algebras", (2015), by S.Asawasamrit and A.Sudprasert [6]

This thesis is divided into four chapters.

In the first section of first chapter, preliminaries on KK-algebras and its properties are collected from the article [4] due to S.Asawasamrit and A.Sudprasert. In the second section of this chapter, properties of ideals in KK-algebras are discussed.

In this chapter the following interesting results are discussed.

1) Let x ,y, z be any element in a KK-algebra X. Then
   i) $x * (y * z) = y * (x * z)$
   ii) $((x * y) * y) = x * y$

iii) $(x * y) * 0 = (x * 0) * (y * 0)$

2) Let x, y be any element in a KK-algebra X. Then
   (i) $((y*x)*x)^n * x = y^n * x$ for any n ∈ $\mathcal{N}$
   (ii) $(x^n * 0) * 0 = (x*0)^n * 0$ for any n ∈ $\mathcal{N}$

   Where $\mathcal{N}$ is the set of all nonnegative integers.

3) Let A be a closed of KK-algebra X. Then A is an ideal of X if and only if x ∈ A and z ∗ y $\notin$ A imply z ∗ (x ∗ y) $\notin$ A for all x, y ,z ∈ X.
4) If A is an ideal of KK-algebra X and B is an ideal of A, then B is an ideal of X.
5) Let {J$_i$: I ∈ $\mathcal{N}$ } be a family of ideals of a KK-algebra X where $J_n \subseteq J_{n+1}$ for all n ∈ $\mathcal{N}$. Then $\bigcup_{n=1}^{\infty} J_n$ is an ideal of X.

Second chapter deals with the study of Quotient KK-algebras and KK-homomorphisms of KK-algebras. In the first section of second chapter, the ideals and congruences in KK-algebras are studied and some of its properties are investigated. The second section of this chapter deals with the study of isomorphisms of KK-algebras. In this section, some consequences of the relation between Quotient KK-algebras and isomorphisms are discussed. The interesting results studied in this chapter are given as follows.

1) If I is an ideal of KK-algebra X, then $\left(\frac{X}{I};\circ,[0]_I\right)$ is a KK-algebra which is called the quotient KK-algebra.

2) Let I be a closed ideal of KK-algebra X and a, b ∈ X. Then

   (i) $[a]_I = I$ iff $a \in I$
   (ii) $[a]_I = [b]_I$ or $[a]_I \bigcap [b]_I = \phi$

3) Let f be a KK-homomorphism of a KK-algebra (X, $*, 0_X$) onto a KK-algebra (Y, · , 0_Y) and I be an ideal of X contain in ker f. Let g be the natural KK-homomorphism of X onto $\frac{X}{I}$

then there exists a unique KK-homomorphism h of $\frac{X}{I}$ onto Y such that f = h ∘ g. Furthermore, h is an injective if and only if I = ker f.

4) **(First Isomorphism Theorem)** : If f be a KK-homomorphism of a KK-algebra $(X,*,0_X)$ into a KK-algebra $(Y,\cdot,\ 0_Y)$, then the quotient KK-algebra $\frac{X}{\ker(\phi)}$ is isomorphic to $\phi(X)$.

5) **(Second Isomorphism Theorem)** : Let X be a KK-algebra and A, B be ideals of X. If $A \cup B$ is a KK-algebra, then the quotient KK-algebras $\frac{A}{(A \cap B)}$ and $\frac{(A \cup B)}{B}$ are isomorphic.

6) **(Third Isomorphism Theorem)** : Let X be a KK-algebra and A, B be ideals of X, with $A \subseteq B \subseteq X$. Then :

(i) the quotient $\frac{B}{A}$ is an ideal of the quotient $\frac{X}{A}$, and

(ii) the quotient KU-algebra $\frac{\left(\frac{X}{A}\right)}{\left(\frac{B}{A}\right)}$ is isomorphic to $\frac{X}{B}$.

Third chapter deals with the study of q-ideals, a-ideals and p-ideals in KK-algebras. Several characterizations and the extensive theorems for the q-ideal, a-ideal and p-ideal are investigated.

In the first section of this chapter, properties of q-ideals are discussed. In the second section, properties of a-ideals are discussed. Also the relations between a-ideal, q-ideal and p-ideal of KK-algebras are discussed.

In this chapter the following interesting results are discussed.

1) If I is an ideal of KK-algebras X, then the following are equivalent:
    (i) I is an q-ideal of X;
    (ii) for any x, y ∈ X, $(x * 0) * y \in I$ implies $x * y \in I$.
    (iii)for any x, y, z ∈ X, $(x * y) * z \in I$ implies $x * (y * z) \in I$.
2) Any a-ideal of KK-algebra is a p-ideal.

3) A non-empty subset I of KK-algebra X is an a-ideal if and only if it is both a q-ideal and a p-ideal.
4) Let A and I be two ideals of KK-algebra X with I $\subseteq$ A. If I is an a-ideal of X, then so is A.

Fourth chapter deals with the study of p-semisimple in KK-algebras. In the first section of this chapter the notion of p-semisimple in KK-algebras and some related properties are investigated. In the second section, properties of a branchwise commutative KK-algebra are discussed.

The interesting results studied in this chapter are given as follows:

1) Given a KK-algebra X, then the following conditions are equivalent :
   (i) X is a p-semisimple ;
   (ii) for any x, y $\in$ X, $(x* 0) * y = (y* 0) * x$;
   (iii) for any x $\in$ X, $x * 0 = 0$ implies $x = 0$;
   (iv) for any a, x $\in$ X, $(x * a) * a = x$;
   (v) $X= \{x* a : x \in X\}$ for any a $\in$ X.

2) Assume that X is a KK-algebra. Then B and P are closed of X, where B is the set of all positive of X, and P is the set of all minimal.

3) If X is a p-semisimple KK-algebras, then every closed A of X is an ideal of X.

4) Assume that $P = \{x * 0 / x \in X\}$ is a p-semisimple part of KK-algebra X. Then
   (i) $X = \bigcup_{a\in P} V(a)$ and $V(a) \cap V(b) = \emptyset$ whenever $a \neq b$ and a, b $\in$ P;
   (ii) if $x \in V(a)$ and $y \in V(b)$, then $x * y \in V(a * b)$;
   (iii) if $a \in P$, then $x * a \in P$, for any $x \in X$;
   (iv) if $a \in P$ and $x \in V(b)$, then $x * a = b * a$

   where $V(a) = \{x \in X / x * a = 0\}$ is the branch of X generated by a.

5) If X is a branchwise commutative KK-algebra then X is a commutative.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iseki [13, 14] in 1966 and have been extensively investigated by many researchers. Since then a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. The class of all BCK-algebras is a quasi variety. Iseki posted an interesting problem whether the class of BCK-algebras is a variety. In 1983, that problem was solved by Wronski [28], who proved that BCK-algebras do not form a variety. In connection with this problem in 1984, Komari [22] introduced the notion of BCC-algebras, and in 1992, Dudek [8, 9] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komari.

In 1992, S.M.Hong et al. [12] studied the p-semisimple in BCI-algebras and obtained some related properties. In 1998, Dudek and Zhang [10] introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. In 2000, Y.L.Lin and M.Jie [23] introduced the notion of q-ideals and a-ideals in BCI-algebras. The concept of homomorphism of BCI-algebras was studied by H.Yisheng [29] in 2006. In 2009, Prabpayak and Leerawat [25] introduced a new algebraic structure which is called KU-algebra and they gave the concept of homomorphism of KU-algebras and investigated some properties. In 2010, the concepts of isomorphism in binary algebras are discussed by Asawasamrit and Leerawat [3]. In 2012, S.Asawasamrit and A.Sudprasert [4] introduced a new algebraic structure called KK-algebras, and described the relation between ideals and congruences.

Several other authors have also contributed to the study of the concepts mentioned above. We give here a brief survey of some of the articles published on various algebras.

1. An introduction to the theory of BCK-algebras

Iseki.K and Tanaka.S (1978) [16]

In this paper, the definition of BCK-algebra and its fundamental properties are studied. Various ideals in BCK-algebras are discussed in a detailed manner. Also, the homomorphism properties on BCK-algebras are discussed.

2. BCK-algebras with condition(s)

Iseki.K (1979) [15]

In this paper, definition and some properties of BCK-algebras with condition(s) are studied. Also the relationship between this algebra and a positive implicative BCK-algebra are discussed.

3. The class of BCC-algebras is not a variety

Yuichi Komari., (1984) [22]

Kiyoshi Iseki posted an interesting problem whether the class of BCK-algebras is a variety. In connection with the problem, the author introduced a notion of BCC-algebras. In this article, the author showed that the class of BCC-algebras is not a variety.

4. On ideals in BCK-algebras

Jie Meng (1994) [18]

In this note, various ideals in BCK-algebras are discussed in detail. The notion of implicative ideals and commutative ideals are introduced. Their relationships with other ideals are studied. Also the ideal characterizations of several important classes of BCK-algebras are given. In particular, distributive theorems of commutative, implicative and positive implicative ideals are obtained.

5. Ideal theory of BCC-algebras

Jiang Hao., (1998) [17]

The author introduced the concept of ideal in a BCC-algebra and proved some related properties. The author also gave a method for constructing a proper BCC-algebra by the extension of a BCK-algebra with a small atom.

6. On ideals and congruences in BCC-algebras

Wieslaw A.Dudek., and Xiaohong Zhang., (1998) [10]

In this article, the authors introduced a new concept of ideals in BCC-algebras and described connections between such ideals and congruences.

7. On atoms of BCK-algebras

Dajun Sun., (2001) [7]

Atoms in BCK- algebras are considered. The notions of the star BCK-algebras and the star part of BCK-algebras are introduced. The properties of some substructures which consist of atoms are investigated.

8. Congruences on hyper BCK-algebras

Michiro Kondo., (2001) [24]

In this article, the author introduced the concept of congruences on hyper BCK-algebras and investigated the relationship between hyper BCK-ideals and those congruences.

9. Generalizations of BCK-algebras

Sung Min Hong, Young Bae Jun and Mehmet Ali Ozturk (2003) [27]

As a generalization of positive implicative BCK-algebra, the notion of generalized BCK-algebras is introduced. A method to make BCK-algebra from a quasi-ordered set is provided. The notion of generalized BCK-ideals of generalized BCK-algebras is introduced, and then the connections between such ideals and congruences are considered. Characterizations of generalized BCK-ideals are given. A generalized BCK-ideals generalized by a set is established.

10. On derivations of BCI-algebras

Jun.Y.B. and Xin.X.L. (2004) [20]

The notion of left-right (resp. Right-left) derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular derivation, the authors gave characterizations of a p-semisimple BCI-algebra. They also gave a condition for a derivation to be regular.

11. The role of atoms in BCI-algebras

Jun.Y.B., Xin.X.L. and Roh.E.H. (2004) [21]

In this article, the authors characterized different types of atoms in BCI-algebras. They showed that finite BCI-algebras are generated by I-atoms. They find conditions in order that a

BCI-algebra would be a proper I-branch BCI-algebra. Finally, they gave some properties, of I-atoms and K-atoms in proper I-branch BCI-algebras.

12. A note on BCI-algebras of order six

Farhat Nisar and Bhatti.S.A (2006) [11]

In this article, it is worked out and show that the number of proper BCI-algebras of order six upto isomorphism is 197.

13. On k-algebras and BCI-algebras

Akram.M and Kim.H.S (2007) [1]

In this paper, the author showed that the k-algebra (G, ·, ⊙, e) is equivalent to the p-semisimple BCI-algebra (G, ⊙, e) whenever the group (G,·) is an abelian but not an elementary abelian 2-group.

14. A method to make BCK-algebras

Jun.Y.B., Lee.K.J and Park.C.H (2007) [19]

Using the notion of posets, a method to make BCK-algebras is considered. They showed that if a poset has the least element, then the induced BCK-algebra is bounded.

15. Congruences and quotient algebras of BCI-algebras

Yuzhong Ding and Zhiyong Pang (2007) [30]

In this article, the definition and properties of congruences and quotient algebras are given. Quotient algebras are the basic tools for explaining the structures of BCI-algebras.

16. On ideals and congruences in KU-algebras

Chanwit Prabpayak and Utsanee Leerawat (2009) [25]

In this paper, the authors introduced some kind of algebras which is called KU-algebras. They defined ideals and studied congruences on KU-algebras, and also investigated some related properties.

17. On isomorphisms of KU-algebras

Chanwit Prabpayak and Utsanee Leerawat (2009) [26]

In this paper, the authors studied homomorphisms of KU-algebras and investigated its properties. Moreover, some consequences of the relations between quotient KU-algebras and isomorphisms are shown.

CHAPTER 1

CHAPTER-1

A STRUCTURE OF KK-ALGEBRAS AND ITS PROPERTIES

Section 1.1

Preliminary definitions and results in KK-Algebras

Definition 1.1.1

A non-empty set X with a constant 0 and a binary operation $*$ denoted by $(X\ ; *, 0)$ is called a **BCI-algebra** if for all x, y, z ∈ X the following condition holds :

(**BCI 1**) $((x * y) * (x * z)) * (z * y) = 0$

(**BCI 2**) $\big(x * (x * y)\big) * y = 0$

(**BCI 3**) $x * x = 0$

(**BCI 4**) $x * y = 0\ , y * x = 0 \Rightarrow x = y.$

Definition 1.1.2

A BCI-algebra X satisfying the following conditions $0 * x = 0$ is called a **BCK-algebra.**

Definition 1.1.3

A non-empty set X with a constant 0 and a binary operation $*$ denoted by $(X; *, 0)$ is called a BCC-algebra if it satisfies the following axioms :

(**BCC 1**) $\big((x * y) * (z * y)\big) * (x * z) = 0$

(**BCC 2**) $x * x = 0$

(**BCC 3**) $0 * x = 0$

(**BCC 4**) $x * 0 = 0$

(**BCC 5**) $x * y = 0\ , y * x = 0 \Rightarrow x = y$ for all $x, y, z \in X$.

Definition 1.1.4

A non-empty set X with a constant 0 and a binary operation $*$ denoted by $(X\,;*,0)$ is called a KK-algebra if it satisfies the following axioms : For any $x,y,z \in X$

(KK 1) $(x*y)*\big((y*z)*(x*z)\big)=0$

(KK 2) $0*x=x$

(KK 3) $x*y=0=y*x \Rightarrow x=y$

For brevity, we also call X a KK-algebra.

Example 1.1.5

Let $*$ be defined on an abelian group G by letting $x*y = x^{-1}y$, where x, y in G, with e is unity element of G. Then $(G\,;\cdot\,,e)$ is a KK-algebra.

Example 1.1.6

Let X = {0, 1} and let $*$ be defined by

$*$	0	1
0	0	1
1	1	0

Then $(G\,;*\,,0)$ is a KK-algebra.

Theorem 1.1.7

Let $(X\,;*\,,0)$ be a KK-algebra if and only if it satisfies the following conditions for all x, y, z ∈ X.

(i) $(x*y)*\big((y*z)*(x*z)\big)=0$;

(ii) $x*\big((x*y)*y\big)=0$;

(iii) $x*x=0$;

(iv) $x*y=0$ and $y*x=0$ if and only if x = y.

Proof

Assume that $(X\,;*,0)$ is a KK-algebra.

From definition of KK-algebra, (i) and (iv) holds.

Then, $x*\big((x*y)*y\big)=(0*x)*\big((x*y)*(0*y)\big)=0$

$$x*x=0*(x*x)=(0*0)*\big((0*x)*(0*x)\big)=0$$

Hence (ii) and (iii) holds.

Conversely, to prove $(X;*,0)$ is a KK-algebra,

It is enough to prove KK 2.

By (i),(ii),(iii) $\quad \big((0*x)*x\big)*0=\big((0*x)*x\big)*(0*((0*x)*x))$

$$=\big((0*x)*x\big)*\Big((x*x)*\big((0*x)*x\big)\Big)=0$$

and since $0*((0*x)*x=0$

from (iv) $(0*x)*x=0$

$$x*(0*x)=x*\big((x*x)*x\big)=0$$

Therefore, $0*x=x$

Hence the theorem.

Definition 1.1.8

Define a binary relation $\leq$ on KK-algebra X by letting $x\leq y$ if and only if $y*x=0$.

Note:

If $(X\,;*,0)$ is a KK-algebra, then $(X\,;\,\leq)$ is a partially order set.

Theorem 1.1.9

If $(X; *, 0)$ be a KK-algebra and $x \leq 0$, then $x = 0$, for any $x \in X$. Moreover, 0 is called a **minimal element** in X.

Proof

Let $x \leq 0$, then $0 * x = 0$

By KK 2, $0 * x = x$ and thus $x = 0$.

In general, it is easy to show that the following properties are true for a KK-algebra.

Theorem 1.1.10

Let $(X; *, 0)$ be a KK -algebra if and only if it satisfies the following conditions:

For all $x, y, z \in X$,

(i) $\big((y * z) * (x * z)\big) \leq (x * y)$;
(ii) $\big((x * y) * y\big) \leq x$;
(iii) $x \leq y$ if and only if $y * x = 0$.

Proposition 1.1.11

Let x, y, z be any element in a KK-algebra X.

Then (i) $x \leq y$ implies $y * z \leq x * z$

(ii) $x \leq y$ implies $z * x \leq z * y$.

Proposition 1.1.12

Let x, y, z be any element in a KK–algebra. Then $x * (y * z) = y * (x * z)$.

Proof

By theorem 1.1.10 (ii), $(x * z) * z \leq x$ and by proposition 1.1.11 (i), $x * (y * z) \leq \big((x * z) * z\big) * (y * z)$

Putting $x = y$ and $y = x * z$ in $\big((y * z) * (x * z)\big) \leq (x * y)$

It follows that, $\big((x * z) * z\big) * (y * z) \leq y * (x * y)$

By the transitivity of $\leq$ gives $x * (y * z) \leq y * (x * z) \quad \rightarrow (1)$

Replacing x by y and y by x, we obtain $y * (x * z) \leq x * (y * z) \rightarrow (2)$

By the anti-symmetry of $\leq$, (1) and (2) gives $x * (y * z) = y * (x * z)$

Hence the proof.

Corollary 1.1.13

Let x, y, z be any element in a KK-algebra X. Then

(i) $y * z \leq x$ if and only if $x * z \leq y$

(ii) $(z * x) * (z * y) \leq x * y$

(iii) $x \leq y$ implies $x * z \leq y * z$.

Proposition 1.1.14

Let x, y, z be any element in a KK-algebra X. Then

(i) $\big((x * y) * y\big) = x * y$

(ii) $(x * y) * 0 = (x * 0) * (y * 0)$.

Proof

Let x, y, z be any element in a KK-algebra X.

To prove (i)

From theorem 1.1.7(ii) and theorem 1.1.10 (i), we have

$$(((x * y) * y) * y) * (x * y) \leq (x * (x * y) * y) = 0$$

$$(((x * y) * y) * y) * (x * y) = 0$$

Since, $(x * y) * \Big(\big((x * y) * y\big) * y\Big) = ((x * y) * y) * ((x * y) * y) = 0$

By KK 3, $((x * y) * y) * y = x * y$.

To prove: (ii)

$$\begin{aligned}(x * 0) * (y * 0) &= (x * 0) * (y * ((x * y) * (x * y)))\\ &= (x * 0) * ((x * y) * (y * (x * y)))\\ &= (x * 0) * ((x * y) * (x * (y * y)))\\ &= (x * y) * ((x * 0) * (x * 0))\\ (x * 0) * (y * 0) &= (x * y) * 0\end{aligned}$$

Hence the proof.

Notation:

We will denote $\mathcal{N}$ for the set of all nonnegative integers (ie.,) 0,1,2,…. and $\mathcal{N}^*$ for the set of all natural numbers (ie.,) 1,2,3,… and we will also use the following notation in brevity:

$$y^0 * x = x$$

$$y^n * x = \underbrace{y * (\ldots * (y * (y * x)))}_{ntimes}$$

Where x, y are any elements in a KK-algebra and n ∈ $\mathcal{N}^*$

Proposition 1.1.15

Let x, y be any element in a KK-algebra X. Then

(i) $((y * x) * x)^n * x = y^n * x$ for any n ∈ $\mathcal{N}$

(ii) $(x^n * 0) * 0 = (x * 0)^n * 0$ for any n ∈ $\mathcal{N}$.

Proof

Let X be a KK-algebra and x, y ∈ X and n, m ∈ $\mathcal{N}$

To prove: (i)

Proof follows by induction on n.

Define the statement P(n) as, $((y*x)*x)^n = y^n * x$

We see that P(0) is true, since $((y*x)*x)^0 * x = x = y^0 * x$

Assume that P(k) is true for some arbitrary k ≥ 0

That is, $((y*x)*x)^k * x = y^k * x \rightarrow$ (1)

Since $((y*x)*x)^{k+1} * x = ((y*x)*x)*(((y*x)*x)^k * x)$

$$= ((y*x)*x)*(y^k * x) \qquad \text{by (1)}$$

$$= y^k * (((y*x)*x)*x)$$

$$= y^k * (y*x)$$

$$= y^{k+1} * x$$

This shows that P(k+1) is true and by the principle of mathematical induction, P(n) is true for each n ∈ $\mathcal{N}^*$

To prove: (ii)

Since $(x^n * 0)*0 = (x*(x^{n-1} * 0))*0$

$$= (x*0)*((x^{n-1} * 0)*0)$$

$$= (x*0)*((x*(x^{n-2} * 0))*0)$$

$$= (x*0)*((x*0)*((x^{n-2} * 0)*0))$$

$$= (x*0)^2 * ((x^{n-2} * 0)*0)$$

$$= \ldots\ldots..$$

$$= (x*0)^n * 0.$$

Definition 1.1.16

Given $x \in X$ if it is satisfies $x * 0 = 0$, that is $0 \leq x$, the element x is called a **positive element** of X. By definition, the zero element 0 of X is positive.

Proposition 1.1.17

Let x be any element in a KK-algebra X. Then $((x * 0) * 0) * x$ is a positive element of X for every $x \in X$.

Proof

Since $(((x * 0) * 0) * x) * 0 = (((x * 0) * 0) * 0) * (x * 0)$ by 1.1.14(ii)

$= (x * 0) * (x * 0)$ by 1.1.14(i)

$= 0$

Therefore $((x * 0) * 0) * x$ is a positive element of X.

Section 1.2

Ideals in KK-Algebras

Definition 1.2.1

A non-empty subset A of a KK-algebra X is called a closed of X on condition that $x * y \in A$ whenever $x, y \in A$.

Definition 1.2.2

A non-empty subset A of a KK-algebra X is called an ideal of X if it satisfies the following conditions

(I 1) $0 \in A$

(I 2) for any $x, y \in X$, $x * y \in A$ and $x \in A$ imply $y \in A$.

Example 1.2.3

Let x= {0, 1, 2, 3} and let $*$ be defined by the table

$*$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	3	3	0	0
3	3	2	1	0

Then X is a KK-algebra. And I = {0, 1} and j = {0, 3} are closed ideals of X.

Lemma 1.2.4

Let A be a closed of KK-algebra X. Then A is an ideal of X if and only if $x \in A$ and $z * y \notin A$ imply $z * (x * y) \notin A$ for all $x, y, z \in X$.

Proof

Let A be an ideal of X and let $x \in A$ whereas $z * y \notin A$

Suppose that $z * (x*y) \in A$

By proposition 1.1.12, we see that $x * (z * y) \in A$.

Since A is an ideal of X and $x \in A$, $z * y \in A$ a contradiction. So $z * (x * y) \notin A$.

Conversely, assume that if $x \in A$ and $z * y \notin A$ imply $z * (x * y) \notin A$ for all x, y, $z \in X$

Since A is a closed of X, then there is $x \in A$ which $0 = x * x \in A$. That is, $0 \in A$.

Now, let $x * y \in A$ and $x \in A$.

Assume that $y \notin A$. We have that $0 * y = y \notin A$. It follows that $0 * (x * y) \notin A$.

Hence, $x * y \notin A$, contradiction.

Therefore, A is an ideal of X.

Hence the proof.

Corollary 1.2.5

Let A be a closed of KK-algebra X. Then A is an ideal of X if and only if $x \in A$ and $y \notin A$ imply $x * y \notin A$ for all x, $y \in X$.

Proof

Let X be a KK-algebra

By lemma 1.2.4, we have A is an ideal of X

iff $x \in A$, $z * y \notin A \Rightarrow z * (x * y) \notin A$ for all x, y, $z \in X$

Therefore, A is an ideal of X.

iff $x \in A$, $0 * y \notin A \Rightarrow 0 * (x * y) \notin A$

That is, A is an ideal of X iff $x \in A$, $y \notin A \Rightarrow x * y \in A$ for all x, $y \in X$.

Lemma 1.2.6

Let A be a closed of KK-algebra X. Then A is an ideal of X if and only if $x * (y * z) \in A$ and $x * z \notin A$ imply $y \notin A$ for all $x, y, z \in X$.

Proof

Let A be an ideal of X and let $x * (y * z) \in A$, $x * z \notin A$

Suppose that $y \in A$

By proposition 1.1.12, we have $y * (x * z) \in A$

Since A is an ideal of X, thus $x * z \in A$, contradiction. This shows that $y \notin A$.

Conversely, assume that $x * (y * z) \in A$ and $x * z \notin A$ imply $y \notin A$ for all $x, y, z \in X$ → (1)

Since A is a closed of X, then there is $y \in A$ which $0 = y * y \in A$. Then $0 \in A$ → (2)

Let $y * z \in A$, $y \in A$ and suppose that $z \notin A$

By KK 2, $0 * (y * z) \in A$ and $0 * z \notin A$

By assumption (1), $y \notin A$, which is a contradiction

Hence $y * z \in A$, $y \in A \Rightarrow z \in A$ → (3)

By (2) and (3), A is an ideal of X.

Corollary 1.2.7

Let A be a closed of KK-algebra X. Then A is an ideal of X if and only if $x * y \in A$ and $y \notin A$ imply $x \notin A$ for all $x, y, z \in X$.

Proof :

Proof follows from previous lemma.

The following lemma gives some properties of ideal of KK-algebra.

Lemma 1.2.8

If A is an ideal of KK-algebra X and B is an ideal of A, then B is an ideal of X.

Proof

Let A be an ideal of KK-algebra X.

Since B is an ideal of A, then 0 $\in$ B. Let x, y $\in$ X such that x $*$ y $\in$ B and x $\in$ B.

It follows that x $*$ y $\in$ A and x $\in$ A.

By assumption, A is an ideal of X, so y $\in$ A and x $\in$ B

From B is an ideal of A, so y $\in$ B

Therefore, B is an ideal of X.

Theorem 1.2.9

Let {J_i : i $\in \mathcal{N}$}be a family of ideals of a KK-algebra X where $J_n \subseteq J_{n+1}$ for all n $\in \mathcal{N}$. Then $\bigcup_{n=1}^{\infty} J_n$ is an ideal of X.

Proof

Let {J_i : i $\in \mathcal{N}$} be a family of ideals of X

Then, clearly $\bigcup_{n=1}^{\infty} J_n \subseteq X$

Since J_i is a ideal of X for all i, $0 \in \bigcup_{n=1}^{\infty} J_n$

Let x $*$ y $\in \bigcup_{n=1}^{\infty} J_n$ and x $\in \bigcup_{n=1}^{\infty} J_n$ for any x, y $\in$ X.

It follows that x $*$ y $\in J_j$ for some j $\in \mathcal{N}$ and x $\in J_k$ for some k $\in \mathcal{N}$.

Furthermore, let $J_j \subseteq J_k$

Hence x $*$ y $\in J_k$ and x $\in J_k$

By assumption, J_k is an ideal of X, it follows that y $\in J_k$.

Therefore, y $\in \bigcup_{n=1}^{\infty} J_n$

Hence $\bigcup_{n=1}^{\infty} J_n$ is an ideal of X.

Theorem 1.2.10

Let {J$_i$: i ∈ 𝒩 } be a family of closed ideals of a KK-algebra X where $J_n \subseteq J_{n+1}$ for all n ∈ 𝒩. Then $\bigcup_{n=1}^{\infty} J_n$ is a closed ideal of X.

Proof

Let {J$_i$: i ∈ 𝒩} be a family of closed ideals of X

By theorem 1.2.9, $\bigcup_{n=1}^{\infty} J_n$ is an ideal of X

We will show that $\bigcup_{n=1}^{\infty} J_n$ is a closed of X.

Let x, y $\in \bigcup_{n=1}^{\infty} J_n$

It follows that x ∈ J_j for some j ∈ 𝒩 and y ∈ J_k for some k ∈ 𝒩

Without loss of generality, we can assume that j ≤ k, we obtain $J_j \subseteq J_k$

That is, x ∈ J_k and y ∈ J_{ksss}

Since, J_k is a closed of X, we get x ∗ y$\in J_k \subseteq \bigcup_{n=1}^{\infty} J_n$

This proves that $\bigcup_{n=1}^{\infty} J_n$ is a closed ideal of X.

Theorem 1.2.11

Let {I$_j$: j ∈ 𝒩} be a family of ideals of a KK-algebra X. Then$\bigcap_{j\in 1}^{\infty} I_j$ is an ideal of X.

Proof

Let $\{I_j : j \in \mathcal{N}\}$ be a family of ideals of X.

It is obvious that $\bigcap_{j\in 1}^{\infty} I_j \subseteq X$. Since $0 \in I_j$ for all $j \in \mathcal{N}$ it follows that $0 \in \bigcap_{j\in 1}^{\infty} I_j$

Let $x * y \in \bigcap_{j\in 1}^{\infty} I_j$ and $x \in \bigcap_{j\in 1}^{\infty} I_j$

We get that $x * y \in I_j$ and $x \in I_j$ for all $j \in J$, then $y \in I_j$ for all $j \in N$. Because I_j is an ideal of X

So $y \in \bigcap_{j\in 1}^{\infty} I_j$. Hence the theorem.

Theorem 1.2.12

Let $\{I_j : j \in \mathcal{N}\}$ be a family of closed ideals of a KK-algebra X. Then $\bigcap_{j\in 1}^{\infty} I_j$ is a closed ideal of X.

Proof

Let $\{I_j : j \in \mathcal{N}\}$ be a family of closed ideals of X

By theorem 1.2.11, $\bigcap_{j\in 1}^{\infty} I_j$ is an ideal of X. We will show that $\bigcap_{j\in 1}^{\infty} I_j$ is a closed of X

Let $x, y \in \bigcap_{j\in 1}^{\infty} I_j$

It follows that $x, y \in I_j$ for all $j \in \mathcal{N}$

Since I_j is a closed of X and $x * y \in I_j$ for all $j \in \mathcal{N}$ then $x * y \in \bigcap_{j\in 1}^{\infty} I_j$

This shows that $\bigcap_{j\in 1}^{\infty} I_j$ is a closed ideal of X.

CHAPTER 2

CHAPTER – 2

QUOTIENT KK-ALGEBRAS AND KK-HOMOMORPHISMS OF KK-ALGEBRAS

Section 2.1

Quotient KK-Algebras

In this section, we describe congruence on KK-algebras.

Definition 2.1.1

Let I be an ideal of a KK-algebra X. Define a relation $\sim$ on X by

$x \sim y$ iff $x * y \in I$ and $y * x \in I$.

Theorem 2.1.2

If I is an ideal of a KK-algebra X, then the relation $\sim$ is an equivalence relation on X.

Proof

Let I be an ideal of X and $x, y, z \in X$.

By theorem 1.1.7, $x * x = 0$ and assumption $x * x \in I$.

That is $x \sim x$. Hence $\sim$ is reflective.

Next, suppose that $x \sim y$.

It follows that $x * y \in I$ and $y * x \in I$

Then $y \sim x$, so $\sim$ is symmetric.

Finally, let $x \sim y$ and $y \sim z$

Then, $x * y, y * x, y * z, z * y \in I$ and $(y*x) * ((z * y) * (z * x)) = 0 \in I$

It follows that $(z * y) * (z * x) \in I$, and since $z * y \in I$, so $z * x \in I$.

Similarly, $x * z \in I$. Thus, $\sim$ is transitive.

Therefore $\sim$ is an equivalence relation.

Lemma 2.1.3

Let I be an ideal of KK-algebra X. For any $x, y, u, v \in X$, if $u \sim v$ and $x \sim y$, then $u * x \sim v * y$.

Proof

Assume that $u \sim v$ and $x \sim y$, for any $x, y, u, v \in X$.

Then $u * v, v * u, x * y, y * x \in I$ and by KK 1

We see that $(u * v) * ((v * x) * (u * x)) = 0$ and

$$(v * u) * ((u * x) * (v * x)) = 0.$$

From assumption and I is an ideal of X, these imply that

$$(v * x) * (u * x) \in I \text{ and}$$

$$(u * x) * (v * x) \in I$$

This shows that $v * x \sim u * x$ → (1)

On the other hand, by corollary 1.1.13, we have that

$$(y * x) * ((v * y) * (v * x)) = 0 \text{ and}$$

$$(x * y) * ((v * x) * (v * y)) = 0$$

From assumption and I is an ideal of X, these imply that

$$(v * y) * (v * x) \in I \text{ and}$$

$$(v * x) * (v * y) \in I$$

Thus $(v * x) \sim (v * y)$ → (2)

Since $\sim$ is symmetric

From (1) we get $u * x \sim v * x$ → (3)

since $\sim$ is transitive

From (2) and (3) we get, $u * x \sim v * y$.

Corollary 2.1.4

If I is an ideal of KK -algebra X, then the relation $\sim$ is a congruence relation on X.

Proof

Proof follows by theorem 2.1.2 and lemma 2.1.3.

Definition 2.1.5

Let I be an ideal of a KK-algebra X.

Given x ∈ X, the equivalence class $[x]_I$ of x is defined as the set of all element of X that are equivalent to x, that is

$$[x]_I = \{y \in X : x \sim y\} = \{y \in X \mid x * y \in I \;\&\; y * x \in I\}$$

We define the set $\frac{X}{I} = \{[x]_I : x \in X\}$ and a binary operation $\circ$ on $\frac{X}{I}$ by

$$[x]_I \circ [y]_I = [x * y]_I$$

Note:

$[0]_I = \{x \in X : x \sim 0\}$ is an ideal of X.

Theorem 2.1.6

If I is an ideal of KK-algebra X with $\frac{X}{I} = \{[x]_I : x \in X\}$ where a binary operation $\circ$ on a set $\frac{X}{I}$ is defined by $[x]_I \circ [y]_I = [x * y]_I$, then the binary operation $\circ$ is a mapping from $\frac{X}{I} \times \frac{X}{I}$ to $\frac{X}{I}$.

Proof

Let $[x_1]_I$, $[x_2]_I, [y_1]_I, [y_2]_I \in \frac{X}{I}$ such that $[x_1]_I = [x_2]_I$ and $[y_1]_I = [y_2]_I$

It follows that $x_1 \sim x_2$ and $y_1 \sim y_2$

By lemma 2.1.3, $x_1 * y_1 \sim x_2 * y_2$. This implies $[x_1 * y_1]_I = [x_2 * y_2]_I$.

Theorem 2.1.7

If I is an ideal of KK-algebra X, then $\left(\frac{X}{I};\text{o},[0]_I\right)$ is a KK-algebra. Moreover, the set $\frac{X}{I}$ is called the quotient KK-algebra.

Proof

Let $[x]_I,[y]_I,[z]_I \in \frac{X}{I}$

Then $\left([z]_I\circ[x]_I\right)\circ\left(\left([x]_I\circ[y]_I\right)\circ\left([z]_I\circ[y]_I\right)\right) = [z*x]_I \text{ o} ([x*y]_I \text{ o} [z*y]_I)$

$$= [z*x]_I \text{ o} \left[(x*y)*(z*y)\right]_I$$

$$= \left[(z*x)*\left((x*y)*(z*y)\right)\right]_I = [0]_I$$

It is clear that $[0]_I\circ[x]_I = [0*x]_I = [x]_I$

Now, let $[x]_I\circ[y]_I = [0]_I$ and $[y]_I\circ[x]_I = [0]_I$

It follows that x ∗ y ~ 0 and y ∗ x ~ 0, that is 0 ∗ (x∗ y), 0 ∗ (y ∗ x) ∈ I

Since I is an ideal of X and 0 ∈ I, we get that x ∗ y, y ∗ x ∈ I.

Consequently, x ~ y, proving that $[x]_I = [y]_I$

Therefore, $\left(\frac{X}{I};\circ,[0]_I\right)$ is a KK-algebra.

Example 2.1.8

Consider the KK-algebra X and the ideal I as in example 1.2.3. Then $\frac{X}{I} = \{[0]_I,[2]_I\}$, where $[0]_I = [1]_I = \{0,1\}$ and $[2]_I = [3]_I = \{2,3\}$.

Let $\circ$ be defined on $\frac{X}{I}$ by

$\circ$	$[0]_I$	$[2]_I$
$[0]_I$	$[0]_I$	$[2]_I$
$[2]_I$	$[2]_I$	$[0]_I$

Then $\left(\frac{X}{I};\circ,[0]_I\right)$ is a KK-algebra.

Lemma 2.1.9

Let X be a KK-algebra and I, J be any sets such that $I \subseteq J \subseteq X$. Suppose that I is an ideal of X, then J is an ideal of X if and only if $\frac{J}{I}$ is an ideal of $\frac{X}{I}$.

Proof

Let I be an ideal of X with $I \subseteq J \subseteq X$.

Suppose firstly that J is an ideal of X, then

$$\frac{J}{I} = \{[x]_I : x \in J\}, \text{ where } [x]_I = \{y \in J : x \sim y\} \text{ and}$$

$$\frac{X}{I} = \{[x]_I : x \in X\}, \text{ where } [x]_I = \{y \in X : x \sim y\}$$

Obviously, $\frac{J}{I} \subseteq \frac{X}{I}$ and $[0]_I \in \frac{J}{I}$

Now, let $[x]_I \circ [y]_I \in \frac{J}{I}$ and $[y]_I \in \frac{J}{I}$

Then $[x * y]_I = [x]_I \circ [y]_I \in \frac{J}{I}$, it follows that $x * y \in J$ and $x \in J$

By assumption, $y \in J$

Accordingly, $[y]_I \in \frac{J}{I}$, this shows that $\frac{J}{I}$ is an ideal of $\frac{X}{I}$.

On the other hand, suppose that $\frac{J}{I}$ is an ideal of $\frac{X}{I}$ and I is an ideal of X with $I \subseteq J \subseteq X$.

Thus $0 \in J$.

Let $x * y \in J$ and $x \in J$

It follows that $[x * y]_I, [x]_I \in \frac{J}{I}$

Since $[x * y]_I = [x]_I \circ [y]_I$, so $[x]_I \circ [y]_I \in \frac{J}{I}$

By hypothesis, $[y]_I \in \frac{J}{I}$ implies $y \in J$, proving our lemma.

Lemma 2.1.10

Let X be a KK-algebra and I, J be any sets such that $I \subseteq J \subseteq X$. Suppose that I is a closed ideal of X. Then J is a closed ideal of X if and only if $\frac{J}{I}$ is a closed ideal of $\frac{X}{I}$.

Proof

Similar to that of lemma 2.1.9.

Next, the basic properties of equivalence classes are considered are as the following theorem.

Theorem 2.1.11

Let I be a closed ideal of KK-algebra X and a, b ∈ X. Then

(i) $[a]_I = I$ iff $a \in I$

(ii) $[a]_I = [b]_I$ or $[a]_I \bigcap [b]_I = \phi$

Proof

Let I be a closed ideal of X and a, b ∈ X.

To prove: (i)

It is clear, due to the fact that a ~ a for all a ∈ X and a $*$ a = 0 ∈ I, so we get that $a \in [a]_I = I$

Conversely, let $x \in [a]_I$

Then x ~ a, it follows that x $*$ a, a $*$ x ∈ I.

By hypothesis, $x \in I$. Hence, $[a]_I \subseteq I$.

To show that $I \subseteq [a]_I$, choose $x \in I$.

Since I is a closed of X, we have x ∗ a, a ∗ x ∈ I

Thus, x ~ a, this means that $x \in [a]_I$ and shows that $I \subseteq [a]_I$.

Consequently, $[a]_I \subseteq I$

To prove: (ii)

Assume that $[a]_I \text{I} [b]_I \neq \phi$

Then there is $x \in [a]_I \bigcap [b]_I$ such that $x \in [a]_I$ and $y \in [a]_I$

It follows that x ~ a and x ~ b, $\Rightarrow$ a ~ x and x ~ b by symmetric property

$\Rightarrow$ a ~ b by transitive property.

Thus $[a]_I = [b]_I$.

Theorem 2.1.12

If I is a closed ideal of a KK-algebra X and $y \in I$, then $[y]_I$ is closed ideal of X.

Proof

Let I be a closed ideal of X and $y \in I$

It is clear that $0 \in [y]_I$

Now, suppose that $a * b \in [y]_I$ and $a \in [y]_I$.

We will show that $b \in [y]_I$

Then a ∗ b ~ y and a ~ y, it follows that y ∗ (a ∗ b) ∈ I and y ∗ a ∈ I.

By assumption, a ∈ I.

From proposition 1.1.12, a ∗ (y ∗ b) = y ∗ (a ∗ b) ∈ I and I is a closed ideal of X and a ∈ I, therefore y ∗ b ∈ I.

By properties of X, we get that

$$(a*0)*\big(((a*b)*y)*(b*y)\big)=(a*(b*b))*\big(((a*b)*y)*(b*y)\big)$$

$$=(b*(a*b))*\big(((a*b)*y)*(b*y)\big)=0$$

By hypothesis, $(a*0)*\big(((a*b)*y)*(b*y)\big)\in I$, where I is closed and a ∈ I. Then $a * 0 \in I$.

Thus $\big((a*b)*y\big)*(b*y)\in I$.

From $a * b \sim y$ and I is closed, then $b * y \in I$. Hence, $b \sim y$, this means $a\in[y]_I$

Hence $[y]_I$ is an ideal of X.

Finally, let $a,b\in[y]_I$

Then $a \sim y$ and $b \sim y$, By lemma 2.1.3, $a * b \sim y * y$.

By theorem 1.1.7, it follows that $a * b \sim 0$

Thus $a*b\in[0]_I$

Now, we have $0,y\in I$ and I is closed. So $0*y\in I$ and $y*0\in I$. That is, $0 \sim y$.

Hence, $[0]_I=[y]_I$

By transitive, $a*b\in[y]_I$.

Hence the theorem.

Section 2.2

KK-homomorphisms of KK-Algebras

Definition 2.2.1

Let $(X,*,0_X)$ and $(Y,\circ,0_Y)$ be KK-algebras. A KK-homomorphism is a map f : X → Y satisfying $f(x*y)=f(x)\circ f(y)$ for all x, y ∈ X.

Definition 2.2.2

Let $f:(X,*,0_X)\to(Y,\text{o},0_Y)$ be a KK-homomorphism of KK-algebras

i) f is called injective KK-homomorphism (KK-monomorphism) if f is one-to-one.

ii) f is called surjective KK-homomorphism (KK-epimorphism) if f is onto.

iii) f is called bijective KK-homomorphism (KK-isomorphism) if f is one-to-one onto.

Moreover we say that X is isomorphic to Y symbolically X ≅ Y

iv) The kernel of the KK-homomorphism f, denoted by ker f, is the set of elements of X that map to 0_Y in Y.

That is, ker f $=\{x\in X \mid f(x)=0_Y\}$

Definition 2.2.3

Let f be a mapping of a KK-algebra X into a KK-algebra Y. Let I$\subseteq X$ and A $\subseteq Y$ .The image of I in X under f is

$$\text{f (I)} =\{f(x)/\ x\in I\}$$

and the inverse image of A in Y is

$$f^{-1}(A)=\{x\in X\ /\ f(x)\in A\}$$

Next, the basic properties of KK-homomorphism are considered as the following theorem.

Theorem 2.2.4

Let f be a KK-homomorphism of a KK-algebra $(X,*,0_X)$ into a KK-algebra $(Y,\circ,0_Y)$. Then:

i) $f(0_X) = 0_Y$

ii) If 0_X is the identity in X, then $f(0_X)$ is the identity in Y.

iii) f is injective KK-homomorphism if and only if ker f = $\{0_X\}$.

iv) $x \leq_X y$ implies $f(x) \leq_Y f(y)$.

Proof

Assume that f : X → Y is a KK-homomorphism.

To prove: (i)

since $0_X * 0_X = 0_X$, then $f(0_X) = f(0_X * 0_X) = f(0_X) \circ f(0_X) = 0_Y$

To prove: (ii)

Assume that 0_X is the identity in X and 0_Y is the identity in Y. From KK 2, $0_Y \circ f(0_X) = 0_Y$

$$\begin{aligned} f(0_X) \circ 0_Y &= f(0_X) \circ [f(0_X) \circ f(0_X)] \\ &= f(0_X) \circ f(0_X * 0_X) \\ &= f(0_X) \circ f(0_X) = 0_Y \end{aligned}$$

By KK 3, we get that $f(0_X) = 0_Y$. This shows that $f(0_X)$ is the identity in Y.

To prove: (iii)

Suppose that f is injective KK-homomorphism and x ∈ ker f

It follows that $f(x) = 0_Y$

Since $f(0_X) = 0_Y$, so $f(x) = f(0_X)$

By assumption, $x = 0_X$

Thus ker f = $\{0_X\}$

Conversely, suppose that ker f = $\{0_X\}$

Let x, y ∈ X be such that f(x) = f(y). Then we get that

$f(x * y) = f(x) \circ f(y) = 0_Y$ and

$f(y * x) = f(y) \circ f(x) = 0_Y$. Thus x ∗ y, y ∗ x ∈ ker f

This means that x ∗ y = 0_X = y ∗ x.

From KK 3, x = y. Hence f is injective KK-homomorphism.

To prove: (iv)

Let $x \leq_X y$. It follows that y ∗ x = 0_X

So (i) implies $f(y) \circ f(x) = f(y * x) = f(0_X) = 0_Y$

Hence $f(x) \leq_Y f(y)$.

Theorem 2.2.5

Let f : X → Y be a KK-homomorphism. Then:

(i) If I is an ideal of X, then f(I) is an ideal of Y

(ii) If I is a closed of X, then f(I) is a closed of Y

(iii) If A is an ideal in Y, then $f^{-1}(A)$ is an ideal in X

(iv) If A is a closed of Y, then $f^{-1}(A)$ is a closed of X

(v) Ker f is a closed ideal of X

(vi) Im f is a closed of Y.

Proof

Assume that f: X → Y is a KK-homomorphism.

To prove: (i)

Let I be an ideal of X

We see that 0_X ∈ I, and by theorem 2.2.4(i),

$0_Y = f(0_X) \in f(I)$, so $0_Y \in f(I)$

Now, assume that $f(x) \circ f(y) \in f(I)$ and $f(x) \in f(I)$.

It follows that $f(x*y) \in f(I)$, so $x*y, x \in I$

Since I is an ideal of x, y ∈ I, it follows that $f(y) \in f(I)$.

Hence f(I) is an ideal of Y.

To prove: (ii)

Let I be a closed of X and x, y ∈ f(I)

Then there exist a, b ∈I such that x = f(a) and y = f(b)

Since $x \circ y = f(a) \circ f(b) = f(a*b) \in f(I)$

Thus f (I) is a closed of Y

To prove: (iii)

Let A be an ideal in Y

Then 0_Y ∈ A, we get that $0_X = f^{-1}(0_Y) \in f^{-1}(A)$

For any x, y ∈ X, let $x*y \in f^{-1}(A)$ and $x \in f^{-1}(A)$

It follows that $f(x) \circ f(y) = f(x*y) \in A$ and $f(x) \in A$

Since A is an ideal of Y, we obtain that f(y) ∈ A

Consequently y ∈ $f^{-1}(A)$. Hence $f^{-1}(A)$ is an ideal of X.

To prove: (iv)

Let A be a closed of Y and x, y ∈ $f^{-1}(A)$

Then f(x) = a and f(y) =b for some a, b ∈A

Thus $f(x*y) = f(x) \circ f(y) = a*b \in A$, as A is closed.

Hence $x*y \in f^{-1}(A)$.

To prove (v)

It is clear that ker f$\subseteq$X.

Since $f(0_X) = 0_Y$, so $0_X \in$ ker f

It follows that ker f$\neq \phi$

Let x ∗ y ∈ ker f and x ∈ ker f

We get that $f(y) = 0_Y \circ f(y) = f(x) \circ f(y) = f(x * y) = 0_Y$

Thus y ∈ ker f

Now, we will show ker f is closed of X

Let x, y ∈ ker f

Then $f(x * y) = f(x) \circ f(y) = 0_Y \circ 0_Y = 0_Y$. These imply that x ∗ y ∈ ker f

Therefore ker f is a closed ideal of X.

To prove (vi)

Let a, b ∈ Im(f). Then there exist x,y ∈X such that a = f(x) and b = f(y), so

$a \circ b = f(x) \circ f(y) = f(x * y) \in \mathrm{Im}(f)$

This proves that Im(f) is a closed of Y

Note :

In general, Im(f) may not be an ideal which can be seen in the following example.

Example 2.2.6

Let X = {0,1,2}. Define an operation ∗ on X by

∗	0	1	2
0	0	1	2
1	0	0	2
2	0	0	0

Then (X, $*$,0) is a KK-algebra.

Now, let f be the mapping from X to itself such that f(0) = 0, f(1) = 0 and f(2) = 2, then we see that Im(f) = {0,2}. So Im(f) is not an ideal of X, since 2 ∈ Im(f) and 2 $*$ 1 = 0 ∈ Im(f), but 1 ∉ Im(f).

Proposition 2.2.7

Let f be a KK-homomorphism from a KK-algebra X to a KK-algebra Y. Then:

(i) f is an KK-epimorphism if and only if Im(f) = Y.

(ii) f is an KK-isomorphism if and only if the inverse mapping f^{-1} is an KK-isomorphism.

Proof:

Obvious.

Theorem 2.2.8

Let I be a closed ideal of KK-algebra X. Define the map $f : X \to \frac{X}{I}$ by $f(x) = [x]_I$, for all x ∈ X. Then f is KK-epimorphism, we call f is the natural KK-homomorphism of X onto $\frac{X}{I}$. Furthermore, ker f = I

Proof

Let I be a closed ideal of KK-algebra X and x, y ∈ X

Since $f(x * y) = [x * y]_I = [x]_I \circ [y]_I = f(x) \circ f(y)$, Proving that f is a KK-homomorphism.

Next we will show f is surjective

Let $[x]_I \in \frac{X}{I}$ and x ∈ X

Then $f(x) = [x]_I$, so f is surjective KK-homomorphism

Hence f is KK-epimorphism.

Finally, to show that ker f = I

Let x ∈ ker f

We get that $[x]_I = f(x) = [0]_I$, then x ~ 0

It follows that x ∗ 0 ∈ I and 0 ∗ x ∈ I. By hypothesis, 0 ∈ I

Hence, x ∈ I, this mean ker f ⊆ I

To show that I ⊆ ker f

Let x ∈ I

Since I is a closed ideal of X, we have 0 ∈ I. Thus x ∗ 0 ∈ I and 0 ∗ x ∈ I

It follows that x ~ 0, so $[x]_I = [0]_I$,

Since $f(x) = [x]_I = [0]_I$, then x ∈ ker f

Hence, ker f = I.

Theorem 2.2.9

Let f be a KK-homomorphism of a KK-algebra $(X, *, 0_X)$ onto a KK-algebra $(Y, \cdot, 0_Y)$ and I be an ideal of X contain in ker f. Let g be the natural KK-homomorphism of X onto $\frac{X}{I}$ then there exists a unique KK-homomorphism h of $\frac{X}{I}$ onto Y such that f = h ∘ g. Furthermore, h is an injective KK-homomorphism if and only if I = ker f.

Proof

Define the map $h : \frac{X}{I} \to Y$ by $h([a]_I) = f(a)$ for all $[a]_I \in \frac{X}{I}$

We first show that, h is well defined

Let $[a]_I, [b]_I \in \frac{X}{I}$ be such that $[a]_I = [b]_I$

We get that a ~ b, so a ∗ b ∈ I and b ∗ a ∈ I. Since I ⊆ ker f, a ∗ b ∈ Ker f and b ∗ a ∈ ker f

Thus $f(a) \cdot f(b) = f(a * b) = 0_Y$ and $f(b) \cdot f(a) = f(b * a) = 0_Y$

From KK 3, f(a) = f(b).

Hence h is well defined.

We will show that h is Kk-homomorphism

Let $[a]_I, [b]_I \in \frac{X}{I}$

Then $h([a]_I \circ [b]_I) = h([a * b]_I) = f(a * b) = f(a) \cdot f(b) = h([a]_I) \cdot h([b]_I)$

Hence h is a KK-homomorphism.

Next to show that f = h $\circ$ g

For any a ∈ X, then $(h \circ g)(a) = h(g(a)) = h([a]_I) = f(a)$

Hence h $\circ$ g = f

Finally, if $h' : \frac{X}{I} \to Y$ is another function such that $f = h' \circ g$

Let $[a]_I \in \frac{X}{I}$

Then $h([a]_I) = f(a) = (h' \circ g) = h'(g(a)) = h'([a]_I)$

Thus $h([a]_I) = h'([a]_I)$ for all $[a]_I \in \frac{X}{I}$

Hence h is a unique KK-homomorphism.

Now, we will show that h is injective KK-homomorphism iff I = ker f.

Suppose firstly that h is injective KK-homomorphism and a ∈ ker f

Then $h([0_X]_I) = 0_Y = f(a) = h([a]_I)$.

But h is an injective KK-homomorphism. Therefore $[0_X]_I = [a]_I$.

It follows that $0_X \sim$ a, then $0_X *$ a ∈ I and a $* 0_X$ ∈ I

By hypothesis, 0_X ∈ I

Hence, a ∈ I, this means that ker f $\subseteq$ I. This shows that ker f = I

On the other hand, suppose that ker f = I and $[a]_I, [b]_I \in \frac{X}{I}$ such that $h([a]_I) = h([b]_I)$

Then f(a) = f(b), it follows that f(a ∗ b) = f(a) · f(b) = 0_Y

Thus a ∗ b ∈ ker f. Since ker f = I, so a ∗ b ∈ I

Similarly, b ∗ a ∈ I

Hence, a ~ b proving that $[a]_I = [b]_I$

This shows that h is injective KK-homomorphism.

Hence the proof.

Next, we state the first isomorphism of KK-algebras as the following theorem.

Theorem 2.2.10 (First Isomorphism Theorem)

If f be a KK-homomorphism of a KK-algebra (X,∗,0_X) into a KK-algebra (Y,·, 0_Y), then the quotient KK-algebra $\frac{X}{\ker(\phi)}$ is isomorphic to $\phi(X)$.

Proof

Let $\phi:(X,*,0_X)\to(Y,\cdot,0_Y)$ be a KK-homomorphism and let

$K = \ker(\phi) = \{a \in X : \phi(a) = 0_Y\}$

We get that $\frac{X}{K} = \{[a]_K : a \in X\}$, where $[a_K] = \{b \in X : a \sim b\}$

From theorem 2.2.5(v), we have ker (ϕ) is an ideal of X

Thus $\left(\frac{X}{K}, \circ, [o]_K\right)$ is a KK-algebra and $\phi(X) = \{\phi(a) : a \in X\}$

Assume that $f : \frac{X}{K} \to \phi(X)$ defined by $f([a]_K) = \phi(a)$, where $[a]_K \in \frac{X}{K}$

Let $[a]_K \cdot [b]_K \in \dfrac{X}{K}$ be such that $[a]_K = [b]_K$.

Then a ∼ b. It follows that a ∗ b ∈ K and b ∗ a ∈ K

Thus $\phi(a) \cdot \phi(b) = 0_Y = \phi(b) \cdot \phi(a)$

By KK 3, we get that $\phi(a) = \phi(b)$

Hence f is well defined.

Let $[a]_K, [b]_K \in \dfrac{X}{K}$

We get that $f\left([a]_K \circ [b]_K\right) = f\left([a * b]_K\right) = \phi(a * b)$

$$\begin{aligned} &= \phi(a) \cdot \phi(b) \\ &= f\left([a]_K\right) \cdot f\left([b]_K\right) \end{aligned}$$

This shows that f is a KK-homomorphism.

Let $[a]_K \cdot [b]_K \in \dfrac{X}{K}$ be such that $f\left([a]_K\right) = f\left([b]_K\right)$

Then $\phi(a) = \phi(b)$. It follows that $\phi(a * b) = \phi(a) \cdot \phi(b) = 0_Y$

Thus $a * b \in \ker(\phi) = K$

Similarly, b ∗ a ∈ K. We see that a ∼ b, this means that $[a]_K = [b]_K$

Hence f is an injective KK-homomorphism.

Let $a \in \phi(X)$

Then there exists b ∈ X such that $a = \phi(b)$ and $[b]_K \in \dfrac{X}{K}$

Thus $f\left([b]_K\right) = \phi(b) = a$

Therefore f is a surjective KK-homomorphism.

Hence the proof.

Note:

If A is an ideal of KK-algebra X and B is an ideal of A, then B is an ideal of X.

So, it follows that B is an ideal of $A \cup B$ and $A \cap B$ is an ideal of A.

Theorem 2.2.11 (Second Isomorphism Theorem)

Let X be a KK-algebra and A, B be ideals of X. If $A \cup B$ is a KK-algebra, then the quotient KK-algebras $\frac{A}{(A \cap B)}$ and $\frac{(A \cup B)}{B}$ are isomorphic.

Proof

Let $\phi : A \to \frac{(A \cup B)}{B}$ be a map by $\phi(x) = [x]_B$ for all x ∈ A

It is obvious that ϕ is well defined

Let $[x]_B \in \frac{(A \cup B)}{B}$

If x ∈ A, then $[x]_B = \phi(x)$. If x ∈ B, then $[x]_B = [0]_B = \phi(a)$

Thus ϕ is onto $\frac{(A \cup B)}{B}$

Consider the equation $\phi(x * y) = [x * y]_B \ = [x]_B \circ [y]_B \ = \phi(x) \circ \phi(y)$

Shows that ϕ is a KK-homomorphism.

Now, let x ∈ ker (ϕ)

Then we get $\phi(x) = [0]_B$, so $[x]_B = [0]_B$

It follows that x ∈ B. Since $\ker(\phi) \subseteq A$, so $x \in A \cap B$

Hence $\ker(\phi) \subseteq A \cap B$

On the other hand, let $x \in A \cap B$

Then x ∈ B

Thus $\phi(x) = [x]_B = [0]_B$, so x ∈ ker (ϕ)

Hence $A \cap B \subseteq \ker(\phi)$

Therefore, $\ker(\phi) = A \cap B$

From theorem 2.2.10, immediately gives us that $\frac{A}{(A \cap B)} \cong \frac{(A \cup B)}{B}$.

Next, we state the third isomorphism theorem of KK-algebra.

Theorem 2.2.12 (Third Isomorphism Theorem)

Let X be a KK-algebra and A, B be ideals of X, with $A \subseteq B \subseteq X$. Then :

(i) the quotient $\frac{B}{A}$ is an ideal of the quotient $\frac{X}{A}$, and

(ii) the quotient KU-algebra $\frac{\left(\frac{X}{A}\right)}{\left(\frac{B}{A}\right)}$ is isomorphic to $\frac{X}{B}$

Proof

To prove (i)

To show that $\frac{B}{A}$ is an ideal of $\frac{X}{A}$

It is clear that $\frac{B}{A} \subseteq \frac{X}{A}$ and $[0]_I \in \frac{B}{A}$

Let $[x]_A \circ [y]_B \in \frac{B}{A}$ and $[x]_A \in \frac{B}{A}$

Then x ∗ y ∈ B and x ∈ B. Since B is an ideal of x, y ∈ B, so $[y]_A \in \frac{B}{A}$

Therefore, $\frac{B}{A}$ is an ideal of $\frac{X}{A}$

To prove (ii)

Let $\phi : \frac{X}{A} \to \frac{X}{B}$ be defined by $\phi([x]_A) = [x]_B$

Assume that $[x]_A = [y]_A$

Then x ∼ y determined by A, that is x ∗ y, y ∗ x ∈ A

Since $A \subseteq B$, x ∗ y, y ∗ x ∈ B

Thus x ∼ y determined by B, and hence $[x]_B = [y]_B$

Then $\phi([x]_A) = \phi([y]_A)$

Therefore, ϕ is well defined.

Next, to show that ϕ is onto $\frac{X}{B}$, let $[x]_B \in \frac{X}{B}$.

If x ∈ X and x $\notin$ B, then $[x]_B = \phi([x]_A)$

If x ∈ B, then $[x]_B = [0]_B = \phi([0]_B)$

Hence ϕ is onto.

Consider $\phi([x]_A \circ [y]_A) = \phi([x * y]_A) = [x * y]_B = [x]_B \circ [y]_B = \phi([x]_A) \circ \phi([y]_A)$

which shows that ϕ is KK-homomorphism.

Finally, to show that $\ker(\phi) = \frac{B}{A}$

Let $[x]_A \in \ker(\phi)$

Then $\phi([x]_A) = [0]_B$, so $[x]_B = [0]_B$

It follows that x ∈ B. Now, we have $[x]_A \in \frac{B}{A}$

Hence $\ker(\phi) \subseteq \frac{B}{A}$

Going the other hand, let $[x]_A \in \frac{B}{A}$

We get that $\phi([x]_A) = [x]_B = [0]_B$, since x ∈ B.

Thus $[x]_A \in \ker(\phi)$ and hence $\frac{A}{B} \subseteq \ker(\phi)$

Consequently, $\ker(\phi) = \frac{B}{A}$

By theorem 2.2.10, $\frac{\left(\frac{X}{A}\right)}{\left(\frac{B}{A}\right)}$ is isomorphic to $\frac{X}{B}$.

Note:

An analogous result of the third isomorphism theorem for groups is also true for KK-algebras.

CHAPTER 3

CHAPTER-3

SPECIAL IDEALS IN KK-ALGEBRAS

Section 3.1

q-ideals in KK-Algebras

Definition 3.1.1

A non-empty subset I of KK-algebras X is said to be a q-ideal of X if it satisfies the following conditions:

(I 1) $0 \in I$

(I 3) for any x, y, z ∈ X, $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$.

Example 3.1.2

Let X = {0,1,2}. Define an operation $*$ on X with the cayley table given by

$*$	0	1	2
0	0	1	2
1	0	0	2
2	2	2	0

Then $(X, *, 0)$ is a KK-algebra and I = {0,1} is a q-ideal.

Next, we give the relation between q-ideal and ideal in the following theorem.

Theorem 3.1.3

A q-ideal is a closed ideal.

Proof

Suppose that I is an q-ideal of KK-algebra and let $x * y \in I$ and $x \in I$.

It follows that $(0 * x) * y \in I$ imply $0 * y \in I$

Thus, $y \in I$, proving (I 2) holds

Combining (I 1), we conclude that I is an ideal of X.

Note

The converse of theorem 3.1.3 is not true can be seen in the following example.

Example 3.1.4

Let X = {0,1,2,3}. Define an operation $*$ on X with the cayley table given by

$*$	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

Then $(X, *, 0)$ is a KK-algebra and $I = \{0\}$ is a closed ideal of X, but not a q-ideal of X. Since $(3 * 0) * 1 = 1 * 1 = 0 \in \{0\}$ and $0 \in \{0\}$ but $3 * 1 = 2 \notin \{0\}$.

The characterization of q-ideal is given in the following theorem.

Theorem 3.1.5

If I is an ideal of KK-algebras X, then the following are equivalent:

(i) I is an q-ideal of X;

(ii) for any $x, y \in X$, $(x * 0) * y \in I$ implies $x * y \in I$.

(iii) for any $x, y, z \in X$, $(x * y) * z \in I$ implies $x * (y * z) \in I$.

Proof

Assume that I is an ideal of KK-algebra X and $x, y, z \in X$.

To prove: (i) $\Rightarrow$ (ii)

Let I be an q-ideal of X and $((x * 0) * y) \in I$

Since $0 \in I$, by (I 3), $x * y \in I$.

To prove: (ii) $\Rightarrow$ (iii)

Suppose that (ii) holds and $(x * y) * z \in I$

By proposition 1.1.12 and theorem 1.1.10(i), we have

$$\begin{aligned}((x*y)*z)*((x*0)*(y*z)) &\leq (x*0)*(((x*y)*z)*(y*z))\\ &\leq (x*0)*(y*(x*y))\\ &= (x*0)*(x*(y*y))\\ &= (x*0)*(x*0)\\ &= 0 \in I\end{aligned}$$

Since I is an ideal of X, ((x ∗ 0) ∗ (y ∗ z)) ∈ I

By (ii), x ∗ (y ∗ z) ∈ I.

To prove: (iii) $\Rightarrow$ (i)

Let (x ∗ y) ∗ z ∈ I and y ∈ I

From (iii), we obtain that x ∗ (y ∗ z) ∈ I. Thus y ∗ (x ∗ z) ∈ I by proposition 1.1.12

Since y ∈ I and I is an ideal, x ∗ z ∈ I,

Hence I is a q-ideal of X.

Theorem 3.1.6

Let A and I be ideals of a KK-algebra X with I $\subseteq$ A. If I is a q-ideal of X, then so is A.

Proof

Let I is a q-ideal of a KK-algebra X and set s = (x ∗ 0) ∗ y ∈ A

Since (x ∗ 0) ∗ (s ∗ y) = s ∗ ((x ∗ 0) ∗ y) = 0 ∈ I.

By 3.1.5 (ii), we get that x ∗ (s ∗ y) ∈ I

And since I is a q-ideal, then s ∗ (x ∗ y) ∈ I

Thus s ∗ (x ∗ y) ∈ A and A is an ideal, so x ∗ y ∈ A.

Therefore A is a q-ideal.

Corollary 3.1.7

If zero ideal $\{0\}$ of KK-algebra X is a q-ideal, then every ideal of X is a q-ideal.

Proof

Proof follows from above theorem.

Theorem 3.1.8

Let I be an ideal of KK-algebra X. If for any $x \in I$ and $y \in X$ imply $x * y \in I$, then I is a q-ideal of X.

Proof

Assume that $(x * y) * z \in I$ and $y \in I$

By hypothesis, we obtain $x * ((x * z) * z) \in I$ and $x * y \in I$

Then $(x * y) * (x * z) \in I$. But I is an ideal of X.

Therefore $x * z \in I$.

Hence I is q-ideal of X.

Lemma 3.1.9

If I is a q-ideal of KK-algebra X, then $x * (x * 0) \in I$ for all $x \in X$.

Proof

Assume that I is a q-ideal of KK-algebra X.

Since $(x * 0) * (x * 0) = 0 \in I$, then it follows that $x * (x * 0) \in I$ by theorem 3.1.5(ii).

Section 3.2

a-ideals in KK-algebras

Definition 3.2.1

A non-empty subset I of KK-algebra X is called an a-ideal of X if it satisfies the following conditions:

(I 1) $0 \in I$

(I 4) for all x, y, z ∈ X, $(x * 0) * (z * y) \in I$ and $z \in I$ imply $y * x \in I$.

Example 3.2.2

Let X = {0,1,2,3}. Define an operation ∗ on X with the cayley table given by

∗	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then (X, ∗,0) is a KK-algebra. Also I = {0,1} is an a-ideal of X.

The following theorem gives the relation between a-ideal and ideals in KK-algebras.

Theorem 3.2.3

If I is an a-ideal of KK-algebra X, then I is a closed ideal.

Proof

Let I be an a-ideal of X. First, we will show that I is an ideal of X.

Assume that $x * y \in I$ and $x \in I$

It follows that $(0 * 0) * (x * y) \in I$ and $x \in I$. By (I 4) we obtain $y * 0 \in I$

Substituting x = 0 = z in (I 4), we get that $(0 * 0) * (0 * y) \in I \ \&\ 0 \in I \Rightarrow y * 0 \in I$

That is, $0 * (y) \in I \ \&\ 0 \in I \Rightarrow y * 0 \in I$. That is $y \in I \ \&\ 0 \in I \Rightarrow y * 0 \in I$

So, it follows that $(y * 0) * 0 \in I$. Putting $y = z = 0$ in (I 4), it follows that

if $(x * 0) * (0 * 0) \in I$ and $0 \in I$, then $0 * x \in I$. That is, $x \in I$

Now $(x * 0) * 0 \in I$, implies that $x \in I$. Since $(y * 0) * 0 \in I$, so $y \in I$.

Hence I is an ideal of X.

Finally, to show that I is a closed

Now, assume that $x \in I$ and $y \in I$. We see that $x * 0 \in I$ and $y * 0 \in I$

Since $x * (y * x) = Y * (x * x) = y * 0 \in I$ and $x \in I$, then $y * x \in I$. Similarly, $x * y \in I$.

Therefore I is closed.

Hence the theorem.

The following theorem gives us some equivalences of a-ideals.

Theorem 3.2.4

Let I be an ideal of KK-algebra X. The following conditions are equivalent:

(i) I is an a-ideal of X;

(ii) $(x * 0) *(z * y) \in I$ implies $(z * y) * x \in I$, for any $x, y, z \in I$

(iii) $(x * 0) *y \in I$ implies $y * x \in I$ for any $x,y \in X$.

Proof

Let I be an ideal of KK-algebra X

To prove (i) $\Rightarrow$(ii)

Assume that I is an a-ideal of X and set $s = (x * 0) * (z * y) \in I$.

We can write $(x * 0) * (s * (z * y)) = s * ((x * 0) * (z * y)) = 0 \in I$

By (I 4) and $s \in I$, thus $(z * y) * x \in I$.

Proving that (ii) holds.

To prove (ii) $\Rightarrow$(iii)

Putting $z = 0$ in (ii), we obtain (iii)

To prove: (iii) $\Rightarrow$(i)

Let $(x * 0) * (z * y) \in I$ and $z \in I$.

Then $((x* 0) * (z * y)) * ((x * 0) *y) \leq (z * y) * y \leq z \in I$

Since I is an ideal of X, thus $((x * 0) * (z * y)) * ((x * 0) *y) \in I$, and imply that $(x * 0) *y \in I$

By (iii), $y * x \in I$, and so I is an a-ideal of X.

Definition 3.2.5

An ideal I of KK-algebra X is called an p-ideal of X if it satisfies $(x * 0) * 0 \in I$ implies $x \in I$.

Next, we give the relation between a-ideal and p-ideal in the following theorem.

Theorem 3.2.6

Any a-ideal of KK-algebra is a p-ideal.

Proof

Suppose that I is an a-ideal of KK-algebras.

By theorem 3.2.3, it follows that I is an ideal.

Putting y = z = 0 in theorem 3.2.4(ii), we get that $(x * 0) * (0 * 0) \in I \Rightarrow (0 * 0) * x \in I$

That is $(x * 0) * 0 \in I$ implies $x \in I$.

Therefore I is an p-ideal.

Note:

The converse of theorem 3.2.6 is not true can be seen in the following example.

Example 3.2.7

Let X = {0,1,2,3}. Define an operation $*$ on X with the cayley table given by

*	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Then (X,∗,0) is a KK-algebras and I = {0} is a p-ideal of X, but it is not an a-ideal of X. Since $(2 * 0) *(0 * 1) = 1 * 1 = 0 \in \{0\}$ and $0 \in \{0\}$ but $1 * 2 = 1 \notin \{0\}$

The proof is complete.

Theorem 3.2.8

Any a-ideal of KK-algebra is a q-ideal.

Proof

Suppose that I is an a-ideal of KK-algebras. It follows that I is an ideal.

Now, let $(x * 0) * y \in I$. We obtain that $((x*0)*y)*((((y*0)*x)*0)*0)$

$$\begin{aligned} &= ((x*0)*y)*((((y*0)*0)*(x*0))*0) \\ &= ((x*0)*y)*((((y*0)*0)*0)*((x*0)*0)) \\ &= ((x*0)*y)*(y*0)*((x*0)*0)) \\ &\le ((x*0)*y)*((x*0)*y) \\ &= 0 \in I \end{aligned}$$

Since I is an ideal, so $(((y * 0) * x) * 0) * 0) \in I$. By theorem 3.2.6, I is a p-ideal.

Therefore $(y * 0) * x \in I$.

By theorem 3.2.4(iii), we get $x * y \in I$. Hence I is a q-ideal.

Note:

The converse of theorem 3.2.8 is not true. From example 3.1.2 it is clear that I = {0} is not a-ideal of X but I = {0} is a q-ideal.

For the converse part of the above theorem we need the condition that I is also a p-ideal given as follows.

Theorem 3.2.9

A non-empty subset I of KK-algebra X is an a-ideal if and only if it is both a q-ideal and p-ideal.

Proof

Let I be an a-ideal of X.

It is clear that I is both a q-ideal and a p-ideal by theorem 3.2.6 and theorem 3.2.8

On the other hand, suppose that I is both a q-ideal and a p-ideal. It follows that I is a closed.

Now, assume that $(x * 0) * y \in I$

By theorem 3.1.5(ii), we have that $x * (0 * y) \in I$ implies $x * y \in I$. Consider the equation

$$\begin{aligned}(x * y) * ((y * 0) * (x * 0)) &= (y * 0) * ((x * y) * (x * 0)) \\ &= (y * 0) * (x * ((x * y) * 0)) \\ &= (y * 0) * (x * ((x * 0) * (y*0))) \\ &= x * ((y * 0) * ((x * 0) * (y* 0)) \\ &= x * ((x * 0) * ((y * 0) * (y* 0)) \\ &= x * ((x * 0) * 0) \\ &= 0 \in I\end{aligned}$$

From I is an ideal and $x * y \in I$, implies $(y * x) * 0 \in I$

It follows that $((y * x) * 0) * 0 \in I$ because I is a closed.

And since I is a p-ideal, we have $y * x \in I$, and so I is an a-ideal.

The extensive theorem of a-ideal was given as follows:

Theorem 3.2.10

Let A and I be two ideals of KK-algebra X with $I \subseteq A$. If I is an a-ideal of X, then so is A.

Proof

Let I be an a-ideal of X, then I is both a q-ideal and a p-ideal

Now, we need to show A is a q-ideal and a p-ideal of X.

Assume that $s = (x * 0) * y \in A$, then we have $(x * 0) * (s * y) = s * ((x * 0) * y) = 0 \in I$

By theorem 3.1.5(ii), $x * (s * y) \in I$, and it follows $s * (x * y) \in I$.

Thus $s * (x * y) \in A$ and since A is an ideal, so $x * y \in A$

Therefore A is a q-ideal of X

Finally, to show that A is a p-ideal of X and let $t = (x * 0) * 0 \in A$

Then $(x * 0) * (t * 0) = t * ((x * 0) * 0) = 0 \in I$, and we can write

$$((x * 0) * (t * 0)) * (((t * x) * 0) * 0) = ((x * 0) * (t * 0)) * (((t * 0) * 0) * ((x * 0) * 0))$$

$$\leq ((x * 0) * (t * 0)) * ((x * 0) * (t * 0)) = 0 \in I$$

Since I is an ideal, we get that $((t * x) * 0) * 0 \in I$

And since I is a p-ideal, $t * x \in I \subseteq A$.

This implies $x \in A$, since A is an ideal.

Hence A is a p-ideal of X.

CHAPTER 4

CHAPTER – 4

P-SEMISIMPLE IN KK-ALGEBRAS

Section 4.1

Properties of p-semisimple KK-algebras

In this section, we define a p-semisimple in a KK-algebra and prove some of its basic properties.

Definition 4.1.1

(i) A KK-algebra X is called p-semisimple if $(x * 0) * 0 = x$ for all $x \in X$.

(ii) In general in a KK-algebra X is said to be a minimal element if $a * x = 0$ (i.e. $x \leq a$) implied $x = a$ for all $x \in X$.

Proposition 4.1.2

The following conditions are equivalent for a KK-algebra X :

(i) a is a minimal element of X;

(ii) $(a * 0) * 0 = a$;

(iii) there is $x \in X$ such that $a = x * 0$;

(iv) for all $x \in X$, $x * a = (a * 0) * (x * 0)$;

(v) for all $x \in X$, $x * a = (a * x) * 0$.

proof

Proof follows by the definition and properties of KK-algebra.

In the next theorem equivalent conditions of p-semisimple is given.

Theorem 4.1.3

Given a KK-algebra X, then the following conditions are equivalent :

(i) X is a p-semisimple;

(ii) Every element x in X is minimal;

(iii) $X=\{x* 0 : x \in X\}$.

Proof : Obvious.

Theorem 4.1.4

Given a KK-algebra X, then the following conditions are equivalent :

(i) X is a p-semisimple ;

(ii) for any $x, y \in X$, $(x * 0) * y = (y * 0) * x$;

(iii) for any $x \in X$, $x * 0 = 0$ implies $x = 0$;

(iv) for any $a, x \in X$, $(x * a) * a = x$;

(v) $X = \{x * a : x \in X\}$ for any $a \in X$.

Proof

Let X be a KK-algebra and $a, x, y \in X$.

To prove (i) ⟹ (ii)

Let X be a p-semisimple, by theorem 4.1.3, it follows that for any element in X is minimal.

Since $(y * 0) * x = (y * 0) * ((x * 0) * 0) = (x * 0) * ((y * 0) * 0) = (x * 0) * y$.

To prove (ii) ⟹ (iii)

Assume that $x * 0 = 0$, by (ii) it follows that $x = 0 * x = (0 * 0) * x = (x * 0) * 0 = 0$, proving that $x = 0$.

To prove (iii) ⟹ (i)

Suppose that (iii) holds, by proposition 1.1.16

$(((x * 0) * 0) * x) * 0 = 0$ and by hypothesis, we get that $((x * 0) * 0) * x = 0$

Since $(x * 0) * 0$ is a minimal element of X, it follows that $x = (x * 0) * 0$

Therefore X is a p-semisimple.

To prove (i) ⟹ (iv)

Suppose that X be a p-semisimple

It follows that a and x are minimal elements of X

Since $(x * a) * a \leq x$, then $(x * a) * a = x$.

To prove (iv) ⟹ (v)

Suppose that (iv) holds in X, then $x = (x * a) * a = y * a$ whenever $y = x * a$.

To prove (v) ⟹ (i)

Obvious.

Example 4.1.5

Suppose that (G ; · , e) is an abelian group with e is a unity element in G. Define a binary operation $*$ on X by putting $x * y = x^{-1} \cdot y$. Then we get that (G, $*$, 0) is a KK-algebras. And since $(x * e) * e = (x^{-1} \cdot e)^{-1} \cdot e = (x^{-1})^{-1} = x$, for any $x \in X$. Hence (G ; · , e) is a p-semisimple.

Example 4.1.6

Let (X; $*$, 0) be a p-semisimple algebra. Define another binary operation on X as follows : $x \cdot y = (x * 0) * y$. Then (X ; · , e)is an abelian group with 0 as the unity element. In fact, by theorem 4.1.4 $x \cdot y = (x * 0) * y = (y * 0) * x = y \cdot x$, then the operation '·' satisfies the commutative law. Also since

$$x \cdot (y \cdot z) = (x * 0) * ((y * 0) * z) = (y * o) * ((x * 0) * z) = y \cdot (x \cdot z)$$

then the commutative law gives $(x \cdot y) \cdot z = z \cdot (x \cdot y) = x \cdot (z \cdot y) = x \cdot (y \cdot z)$. So, the operation '·' meets the associative law.

Moreover, since $x \cdot 0 = (x * 0) * 0 = x$, as X is a p-semisimple, so 0 is the unit element of X. Finally, the inverse element of any element x in X is $x * 0$. That is because

$(x * 0) \cdot x = ((x * 0) * 0) * x = x * x = 0$ and $x \cdot (x * 0) = (x * 0) * (x * 0) = 0$.

We call the group (X ; · , 0) the adjoint abelian group of (X; $*$, 0).

Theorem 4.1.7

Given a KK-algebra X, then the following conditions are equivalent :

(i) $(x * 0) * 0 = x$ for any $x \in X$;

(ii) $(x * 0) * y = (y * 0) * x$ for any $x, y \in X$;

(iii) $x * 0 = 0$ implies $x = 0$.

Proof

Let X be a KK-algebra and $x, y \in X$

To prove (i) ⟹ (ii)

Suppose that X has the property (i). It follows that

$(x * 0) * y = (x * 0) * ((y * 0) * 0) = (y * 0) * ((x * 0) * 0) = (y * 0) * x$.

To prove (ii) ⟹ (iii)

Suppose that $(x * 0) * y = (y * 0) * x$ such that $x * 0 = 0$

Then $x = (0 * 0) * x = (x * 0) * 0 = 0$.

To prove (iii) ⟹ (i)

Suppose that (iii) holds in X, then $(x * 0) * (x * 0) = 0$

By proposition 1.1.12, $x * ((x * 0) * 0) = 0$, which means $(x * 0) * 0 \leq x$.

And since $0 = x * x \leq ((x * 0) * 0) * x$

Therefore $(((x * 0) * 0) * x) * 0 = 0$, and by hypothesis, it follows that $((x * 0) * 0) * x = 0$

From KK 3, we conclude that $(x * 0) * 0$.

Theorem 4.1.8

Given a KK-algebra X, then the following conditions are equivalent, for all $x, y, z, u \in X$,

(i) X is a p-semisimple;

(ii) $(x * y) * (z * u) = (u * z) * (y * x)$;

(iii) $(x * y) * 0 = y * x$;

(iv) $(x * y) * (z * y) = z * x$;

(v) $x * z = y * z$ implies $x = y$;

(vi) $x * y = 0$ implies $x = y$;

Proof

Let X be a KK-algebra and $x, y, z, u \in X$

To prove (i) ⟹ (ii)

Suppose that X is a p-semisimple

From proposition 4.1.2(v) and proposition 1.1.14(ii), we get

$$\begin{aligned}(x * y) * (z * u) &= ((z * u) *(x * y)) * 0\\ &= ((z * u) * 0) * ((x * y) * 0)\\ &= (u * z) * (y * x).\end{aligned}$$

To prove (ii) ⟹ (iii)

Suppose that (ii) holds, it follows that

$(x * y) * 0 = (x * y) * (0 * 0) = (0 * 0) * (y * x) = y * x$.

To prove (iii) ⟹ (iv)

From (KK 1), proposition 1.1.14(ii) and corollary 1.1.13(ii), we can write

$(z * x) * ((x * y) * (z * y)) = 0$ and

$$\begin{aligned}((x * y) * (z * y)) * (z * x) &= (((y * x) * 0) * ((y * z) * 0)) * (z * x)\\ &= (((y * x) * (y * z)) * 0) * ((x * z) * 0)\\ &= (((y * x) * (y * z)) * (x * z)) * 0\\ &= (x * z) * ((y * x) * (y * z))\\ &= 0.\end{aligned}$$

To prove (iv) $\Longrightarrow$ (v)

Assume that $x * z = y * z$

Substituting y for x and z for y and x for z in (iv). We obtain $x * y = (y * z) * (x * z) = 0$

And replacing z by y and y by z in (iv), we have $(y * x) = (x * z) * (y * z) = 0$

Therefore, by (KK 3), $x = y$.

To prove (v) $\Longrightarrow$ (vi)

Assume that $x * y = 0$, it follows that $x * y = y * y$. By (v), we have $x = y$.

To prove (vi) $\Longrightarrow$ (i)

For $x \in X$ by proposition 1.1.14(ii), we get $x * ((x * 0) * 0) = (x * 0) * (x * 0) = 0$,

Using (vi), it yields $x = (x * 0) * 0$, proving that X is a p-semisimple.

Theorem 4.1.9

Let X be a KK-algebra. Then X is the p-semisimple if and only if one of the following conditions holds; for all $x, y, z \in X$

(i) $z * x = z * y$ implies $x = y$
(ii) $(x * y) * (x * z) = y * z$
(iii) $(y * x) * (z * x) = (y * z) * 0$
(iv) $(x * y) * z = (z * y) * x$.

Proof

Let X be a KK-algebra and $x, y, z \in X$

To prove (i)

From corollary 1.1.13 (ii), $(z * x) * (z * y) \leq x * y$

Then we have $0 \leq x * y$, it follows that $(x * y) * 0 = 0$

By theorem 4.1.8, we obtain $y * x = 0$, so $x = y$

Conversely, since $x * x = 0 = x * ((x * 0) * 0)$

From (i), then $x = (x * 0) * 0$, proving that X is the p-semisimple.

To prove (ii)

Now, we will show $(x * y) * (x * z) = y * z$

We see that $(y * z) * ((x * y) * (x * z)) = 0$

and $((x * y) * (x * z)) * (y * z) = ((z * x) * (y * z)) * (y * z) = (y * z) * (y * z) = 0$

These imply that $(x * y) * (x * z) = y * z$

On the other hand, by (ii), we get $(x * 0) * 0 = (x * 0) * (x * x) = 0 * x = x$.

To prove (iii)

By theorem 4.1.8, $(y * x) * (z * x) = z * y = (y * z) * 0$

Conversely, by (iii) we obtain $(x * 0) * 0 = (x * x) * (0 * x) = 0 * (0 * x) = x$

To prove (iv)

By proposition 4.1.2 and proposition 1.1.14 (ii), we obtain

$(x * y) * z = (z * 0) * ((x * y) * 0) = (z * (x * y)) * 0 =(x * (z * y)) = 0$

Then theorem 4.1.8 implies, $(x * y) * z = (z * y) * x$

On the other hand, we see that $(x * 0) * 0 = ((0 * x) * 0) = (0 * 0) * (0 * x) = x$.

Hence the proof.

Proposition 4.1.10

Assume that X is a KK-algebra. Then B and P are closed of X, where B is the set of all positive of X, and P is the set of all minimal.

Proof

Since 0 is a positive, then B is non-empty set.

Let $x, y \in B$, it follows that $x * 0 = 0$ and $y * 0 = 0$

By proposition 1.1.14 (ii), $(x * y) * 0 = (x * 0) * (y * 0) = 0 * 0 = 0$, it means that $x * y \in B$

And similarly, $y * x \in B$, proving that B is a closed of X.

Next we will show that P is closed of X. Since 0 is a minimal, then P is non-empty set.

Let $a, b \in P$ and $x \leq a * b$

Then $b * x \leq b * (a * b) = a * (b * b) = a * 0$

Thus $b * x \leq a * 0$ implies $0 = (a * 0) * (b * x) = b * (a * 0) * x)$

It follows that $(a * 0) * x \leq b$ and since b is minimal, so $(a * 0) * x = b$

We get that $x * b = x * ((a * 0) * x) = (a * 0) * (x * x) = (a * 0) * 0 \leq a$

By corollary 1.1.13(i), $a * b \leq x$ and $x \leq a * b$, so $x = a * b$ (ie.,) $a * b$ is a minimal

Therefore $a * b \in P$, proving that P is a closed.

Definition 4.1.11

The set P of all minimal element of a KK-algebra X is called the p-semisimple part of X.

Proposition 4.1.12

Assume that X is a KK-algebra. Then the p-semisimple part P of X is a p-semisimple closed of X, and $P = \{x * 0 : x \in X\}$.

Proof

Proof follows by theorem 4.1.3 and proposition 4.1.10.

Proposition 4.1.13

If X is a p-semisimple KK-algebras, then every closed A of X is an ideal of X

Proof

Let X be a p-semisimple KK-algebras and A be a closed of X.

Since A is a closed of X, then $0 \in A$

Now, to show that A satisfies (I 2), which assume that $x, y \in X$ such that $x * y \in A$ and $x \in A$

By closeness of A, it follows that $x * 0 \in A$ and $(x * 0) * (x * y) \in A$.

Then
$$\begin{aligned} y * ((x * 0) * (x * y)) &= (x * 0) * (y * (x * y)) \\ &= (x * 0) * (x * (y * y)) = 0 \end{aligned}$$

So, $(x * 0) * (x * y) \leq y$ and since y is a minimal of X, $(x * 0) * (x * y) = y$

Hence $y \in A$, this shows that A is an ideal of X.

Section 4.2

Branchwise Commutative KK-algebra

Definition 4.2.1

Let X be a KK-algebra and ‘a’ be a minimal element of X. Then the set V(a) defined by V(a) = {x ∈ X : x ∗ a = 0}is called the branch of X generated by a.

Proposition 4.2.2

Let P be the set of all minimal elements of a KK-algebra X. For any a ∈ P and x ∈ X, if x ∈ V(a), then x ∗ 0 = a ∗ 0.

Proof

Since x ∈ V(a), it follows that x ∗ a = 0

Then (x ∗ 0) ∗ (a ∗ 0) = (x ∗ a) ∗ 0 = 0. Since a is a minimal, so (a ∗ 0) ∗ (x ∗ 0) = x ∗ a = 0

From (KK 3) proving that x ∗ 0 = a ∗ 0.

Theorem 4.2.3

Let P be the set of all minimal elements of a KK-algebra X. Assume that P is a p-semisimple part of KK-algebra X. Then

(i) $X = \bigcup_{a \in P} V(a)$ and V(a) ∩ V(b) = ∅ whenever a ≠ b and a, b ∈ P;

(ii) if x ∈ V(a) and y ∈ V(b), then x ∗ y ∈ V(a ∗ b);

(iii) if a ∈ P, then x ∗ a ∈ P, for any x ∈ X;

(iv) if a ∈ P and x ∈ V(b), then x ∗ a = b ∗ a.

Proof

Let P be a p-semisimple part of KK-algebra X.

To prove (i)

For any x ∈ X, put a' = (x ∗ 0) ∗ 0, then a' is a minimal element of X, and so a' ∈ P

Since x ∗ a'= x ∗ ((x ∗ 0) ∗0) = 0, it follows that $X \in V(a') \subseteq \bigcup_{a \in P} (a)$

Hence $X = \bigcup_{a \in P} (a)$

Suppose that a, b are minimals of X such that a ≠ b.

Now, assume that V(a) ∩ V(b) ≠ ∅, then there exists x ∈ V(a) ∩ V(b) , it means that, x ∗ a = 0 and x ∗ b =0.

By proposition 4.2.2, we obtain that a ∗ 0 = x ∗ 0 = b ∗ 0. Then b ∗ a = (a ∗ 0) ∗ (b ∗ 0) = 0

Likewise, a ∗ b = 0

Therefore a = b, contradiction with a ≠ b

This shows that V(a) ∩ V(b) = ∅ whenever a ≠ b.

To prove (ii)

Assume that x ∈ V(a), y ∈ V(b) and a, b are minimals of X

Then x ∗ a = 0 = y ∗ b, it follows that x ∗ 0 = a ∗ 0 and y ∗ 0 = b ∗ 0.

Since a ∗ b = (b ∗ 0) ∗ (a ∗ 0) = (y ∗ 0) ∗ (x ∗ 0) ≤ x ∗ y, so a ∗ b ≤ x ∗ y

And since a ∗ b ∈ P, x ∗ y ∈ V(a ∗ b).

To prove (iii)

Assume that a ∈ P and x ∈ X.

By proposition 4.1.2, it follows that x ∗ a = (a ∗ 0) ∗ (x ∗ 0)

Since a ∗ 0, x ∗ 0 ∈ P and P has closeness property, x ∗ a ∈ P.

To prove (iv)

Assume that a ∈ P and x ∈ V(b)

From proposition 4.2.2, we get that b ∗ 0 = x ∗ 0 and

x ∗ a = (a ∗ 0) ∗ (x ∗ 0) = (a ∗ 0) ∗ (b ∗ 0) = b ∗ a

We conclude that if a ∈ P and x ∈ V(b), then x ∗ a = b ∗ a.

Definition 4.2.4

For a KK-algebra X, define a binary operation 'Λ' by $x \Lambda y = (x * y) * y$, for each $x, y \in X$. In particular $a_x = (x * 0) * 0$ and

$$L_P(X) = \{a \in X : a * x = 0 \Longrightarrow a = x, \forall x \in X\}$$

The elements of $L_P(X)$ are called the p-atoms of X.

Note :

For any $a \in X$, $a_x \in L_P(X)$ (ie)., $(a_x * 0) * 0 = a_x$.

Definition 4.2.5

A KK-algebras X is said to be commutative if $x = x \Lambda y$ whenever $x \leq y$ for all $x, y \in X$

Note :

Every p-semisimple KK-algebra is commutative.

Definition 4.2.6

A KK-algebra X is said to be branchwise commutative if $x \Lambda y = y \Lambda x$ for all $x, y \in V(a)$ and all $a \in L_P(X)$.

Proposition 4.2.6

If X is a branchwise commutative KK-algebra, then X is a commutative.

Proof

Assume that a KK-algebra X is a branchwise commutative

We have that $x \Lambda y = y \Lambda x$, for any $x, y \in V(a)$ and $a \in L_P(X)$

Now, let $x, y \in X$ such that $x \leq y$. Since $(x * 0) * 0 \in L_P(X)$ and $(x * 0) * 0 \leq 0 * x \leq y$

Hence $x, y \in V((x * 0) * 0)$, implies that $x \Lambda y = y \Lambda x = (y * x) * x = 0 * x = x$.

Consequently, X is a commutative KK-algebra.

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

In 1966, Y.Imai and K.Iseki [13, 14] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1984, Y.Komari [22] introduced the notion of BCC-algebra. In 2009, C.Prabpayak and U.Leerawat [25] introduced the new algebraic structure called KU-algebra. For the general development of BCK / BCI / BCC-algebras, the ideal theory plays an important role.

In 2012, S.Asawasamrit and A.Sudprasert [4] introduced a new algebraic structure called KK-algebras, and described the relation between ideals and congruences. In this thesis some interesting results on Quotient KK-algebras, KK-isomorphisms, p-semisimple KK-algebras and on various ideals in KK-algebras are discussed.

In the first chapter, we have studied the notion of KK-algebras and their properties. Also ideals in KK-algebras and their properties are discussed due to S.Asawasamrit and A.Sudprasert [4].

In chapter 2, Quotient KK-algebras and its properties are discussed. The relation between ideals and congruences are described. Also, we have made a detailed study of the concepts of isomorphism of KK-algebras due to Asawasamrit [2].

In chapter 3, the notion of q-ideal, a-ideal and p-ideal in KK-algebras and their properties are studied. Also the relations between them are discussed due to Asawasamrit and Sudprasert [5].

In chapter 4, the notions of p-semisimple KK-algebras, branchwise commutative KK-algebras and their properties are discussed due to Asawasamrit and Sudprasert [6].

We hope that a deep study of KK ideals and KK-isomorphism in KK-algebras can be extended to various algebras. So, it provides a lot of scope for further research.

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