Isomorphisms of fuzzy graphs

Kerthika, D

(12PMA008)

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women,

Coimbatore-641 043

In Partial Fulfilment of the Requirements for the

Degree of Master of Science in Mathematics

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Signature of the Head of the Department **Signature of Supervisor**

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Introduction

INTRODUCTION

"Mathematics reveals its secrets only to those who approach it with pure love, for its own beauty."

- *Archimedes*

Graph theory has several interesting applications in system analysis, operations research and economics. Graph representations are widely used for dealing with structural information, in different domains such as networks, psycho-sociology, image interpretation, pattern recognition, etc. One important problem to be solved when using such representations is graph matching. In order to achieve a good correspondence between two graphs, the most used concept is the one of graph isomorphism and a lot of work is dedicated to the search for the best isomorphism between two graphs or sub graphs. Since most of the time the aspects of graph theory problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The concept of fuzzy relation, which has a widespread application in pattern recognition, was introduced by Lotfi A. Zadeh [1965] in his Landmark paper " Fuzzy Sets" . It is the notion of a fuzzy subset of a set as a method for representing the phenomena of uncertainty in real life situation. Since then, the fuzzy set theory had become a wide research area in various disciplines like medicine, social sciences, engineering, statistics, graph theory, management sciences, computer sciences, artificial intelligence, pattern recognition, expert systems, decision making, robotics, signal processing and automata theory.

Based on the definitions of fuzzy sets and fuzzy relations, ten years later Azriel Rosenfeld [1975] defined fuzzy graphs and several fuzzy analogs of graph theoretic concepts like paths, cycles and connectedness were introduced in fuzzy graphs.

In 1987 P. Bhattacharya associated a fuzzy graph with a fuzzy graph in the natural way as an automorphism group. The concepts of weak isomorphism and isomorphism between fuzzy graphs were introduced by K. R. Bhutani [1989]. Operations on fuzzy hypergraphs were introduced by Berge [1970]. In 2007 A. Nagoor Gani and J. Malarvizhi discussed the isomorphism between fuzzy graphs and some properties of self complementary and self weak complementary fuzzy graphs. Mordeson and Permchand Nair [2001] introduced the concept of fuzzy hypergraphs and several fuzzy analogs of hypergraph theory. They also defined the concept of complement of fuzzy graph in 1994 and studied some operations on it. In ”Complement of a Fuzzy graph [2002]” discussed by M. S. Sunitha and A. Vijaya Kumar, the definition of complement of fuzzy graph was modified. Moreover some properties of self-complementary fuzzy graphs and the complement of the operations of union, join and composition of fuzzy graphs that were introduced by J. N. Mordeson in ”Operations on fuzzy graphs” , information sciences [1994] were studied.

Since then, fuzzy graph theory has been finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with differences levels of precision. Fuzzy models are becoming useful because of their aim to reduce the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems.

Yousef Alavi etl., [1987] introduced highly irregular fuzzy graphs and examined several problems relating to the existence and enumeration of highly irregular graphs and also in [1988] extended the concept of highly irregular graphs by introducing k-path irregular graphs, Nagoor Gani and Latha [2012] defined highly irregular fuzzy graphs and discussed some properties on it also they discussed the concept of highly irregular fuzzy graphs.

The main aim of this thesis is to study the various types of Isomorphisms on fuzzy graphs.

The articles taken for our study are the following :

1. “ Isomorphism on fuzzy graphs ” by A. Nagoor Gani and J. Malarvizhi [2007]
2. “ Isomorphism on fuzzy Hypergraphs ” by C. Radhamani and C. Radhika [2012]
3. “ Isomorphic properties of highly irregular fuzzy graph and its complement ” by A. Nagoor Gani and S. R. Latha [2013]

In chapter I we have collected preliminary definitions and results on graph theory, fuzzy graphs , highly irregular fuzzy graphs and fuzzy hypergraphs whose concepts are used in the course of the thesis.

Chapter II deals with the study of Isomorphism- basic properties, the order, the size and the degree of the nodes of the isomorphic fuzzy graphs. Isomorphism between fuzzy graphs is proved to be an equivalence relation. Some properties of self complementary and self weak complementary fuzzy graphs are discussed. The important results proved in this chapter are :

1. For any two isomorphic fuzzy graphs their order and size are same.
2. If G and G' are isomorphic fuzzy graphs then the degrees of their nodes are preserved.
3. Weak isomorphism between fuzzy graphs satisfies the partial order relation.
4. Two fuzzy graphs are isomorphic if and only if their complements are isomorphic.
5. If G is co-weak isomorphic to G', then $\overline{G}$ and $\overline{G'}$ need not be co-weak isomorphic.
6. Let $G:(\sigma,\mu)$ be a self complementary fuzzy graph.

   Then $\sum_{x\neq y}\mu(x,y)=\frac{1}{2}\sum_{x\neq y}(\sigma(x)\wedge\sigma(y))$.

The main aim of chapter III is to discover the order, the size and the degree of the nodes of the isomorphic fuzzy hypergraphs. Isomorphism between fuzzy hypergraphs is proved to be an equivalence relation. Three important theorems proved in this section are:

1. For any two isomorphic fuzzy hypergraphs their order and size are same.
2. If H and H' are isomorphic fuzzy hypergraphs then the degrees of their nodes are preserved.
3. Weak isomorphism between fuzzy hypergraphs satisfies the partial order relation.

Chapter IV is the study of some properties on isomorphism, weak isomorphism and co-weak isomorphism between highly irregular fuzzy graphs and their complements.

Isomorphic properties of μ- complement, self μ- complement and self weak μ- complement of highly irregular fuzzy graphs are established. Also isomorphic properties of busy nodes and free nodes in highly irregular fuzzy graph are discussed. The important results are:

1. The complement of highly irregular fuzzy graph need not be highly irregular.
2. A highly irregular fuzzy graph need not be self complementary.
3. The μ- complement of a highly irregular fuzzy graph need not be highly irregular.
4. Let G be a highly irregular and self μ- complementary fuzzy graph, then $\sum_{u \neq v} \mu(u,v) = \frac{1}{2} \sum_{u \neq v} \sigma(u) \wedge \sigma(v)$.
5. The busy nodes and free nodes for two isomorphic highly irregular fuzzy graphs, are preserved under isomorphism.

Review of Literature

REVIEW OF LITERATURE

Discrete Mathematics is a branch of mathematics dealing with finite or countable processes and elements. Graphs are one of the prime objects of study in discrete mathematics. Graph theory and its applications can be found not only in other branches of mathematics, but also in scientific disciplines such as engineering, computer science and operational research etc…

A mathematical frame work to describe the phenomena of vagueness was suggested by Lotfi A. Zadeh [1965] in his seminar paper entitled "Fuzzy Sets" . The crisp set is defined in such a way as to dichotomize the individuals in some universe of discourse into two groups: members and non members, whose logic relies entirely on the classical Aristotlian one : "A or not A". A sharp, unambiguous distinction exists between the members and non members of the class represented by the crisp set. But, many of the terms that we commonly use, such as 'tall' , 'beauty' etc. which are called 'linguistic variables', do not exhibit this characteristic. Kosko [1993] in his book calls this as Mismatch problem: The world is gray but science is black and white. In fact, the fuzzy principle is that "Everything is a matter of degree". Thus, the membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree. Consequently, the underlying logic is the fuzzy logic: A and Not A.

Application of fuzzy relations are widespread and important; especially in the field of clustering analysis, neural networks, computer networks, pattern recognition, decision making and expert systems. In each of these, the basic mathematical structure is that of a fuzzy graph.

Yeh and Bang's [1975] approach for the study of fuzzy graphs were motivated by its applicability to pattern classification and clustering analysis. They worked more with the fuzzy matrix of fuzzy graph, introduced concepts like vertex connectivity $\Omega(G)$, edge connectivity $\lambda(G)$ and established the fuzzy analogue of Whiteney's theorem. They also proved for any three real numbers a,b,c such that $0 < a \le b \le c$, there exists a fuzzy graph G with $\Omega(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$. Techniques of fuzzy clustering analysis are given by Yeh R.T et al [1975].

Applications of fuzzy graphs to database theory [Kiss, 1991], to problems concerning the group structure [Takeda E, et al, 1976] and also to chemical structure research [Xu J, 1997] are found in literature.

To expand the application base, the notion of fuzzy graphs have been generalized to fuzzy hypergraphs also by Goetschel R. Jr [1995], Goetschel R. Jr et al [1996,1998,1998].

Zimmermann [1996] has discussed some properties of fuzzy graphs. The book by Mordeson and Nair entitled [2000] "Fuzzy graphs and Fuzzy Hypergraphs" is an excellent source for research in fuzzy graphs and fuzzy hypergraphs.

Since the number of papers published on isomorphism in fuzzy graphs is numerous, we present a brief review of literature on some of the important articles published on this topic.

1. SOME REMARKS ON FUZZY GRAPHS

Prabir Bhattacharya [1987]

We show that a fuzzy group can be associated with a fuzzy graph in a natural way. Some properties of fuzzy graphs are considered and we introduce the notions of eccentricity and center. Our examples indicate that results from (crisp) graph theory do not always have analogs for fuzzy graphs.

2. ON AUTOMORPHISMS OF FUZZY GRAPHS

Kiran R. Bhutani [1989]

We introduce some definitions for fuzzy graphs and provide examples to explain various notions introduced. We show that every fuzzy group can be imbedded in a fuzzy group of the group of automorphisms of some fuzzy graph.

3. FUZZY LINE GRAPHS
John N. Mordeson [1993]

The notion of a fuzzy line graph of a fuzzy graph is introduced. We give a necessary and sufficient condition for a fuzzy graph to be isomorphic to its corresponding fuzzy line graph. We examine when an isomorphism between two fuzzy graphs follows from an isomorphism of their corresponding fuzzy line graphs. We give a necessary and sufficient condition for a fuzzy graph to be the fuzzy line graph of some fuzzy graph.

4. OPERATIONS ON FUZZY GRAPHS
John N. Mordeson, Peng Chang-Shyh [1994]

We define the operations of Cartesian product, composition, union, and join on fuzzy sub graphs of graphs G_1 and G_2. If the graph G is formed from G_1 and G_2 by one of these operations, we determine necessary and sufficient conditions for an arbitrary fuzzy subgraph of G also to be formed by the same operation from fuzzy sub graphs of G_1 and G_2.

5. FUZZY MORPHISMS BETWEEN GRAPHS
Michael R. Berthold, Klaus-Peter Huber [2002]

A generic definition of fuzzy morphism between graphs (GFM) is introduced that includes classical graph related problem definitions as sub-cases (such as graph and subgraph isomorphism). The GFM uses a pair of fuzzy relations, one on the vertices and one on the edges. Each relation is a mapping between the elements of two graphs. These two fuzzy relations are linked with constraints derived from the graph structure and the notion of association graph. The theory extends the properties of fuzzy relation to the problem of generic graph correspondence. We introduce two complementary interpretations of GFM from which we derive several interesting properties. The first interpretation is the generalization of the notion of association compatibility. The second is the new notion of edge morphism. One immediate application is the introduction of several composition laws. Each property has a theoretical and a practical interpretation in the problem of graph correspondence that is explained throughout the paper. Special attention is paid to the formulation of a non-algorithmical theory in order to propose a first step towards a unified theoretic framework for graph morphisms.

6. ON REGULAR FUZZY GRAPHS

A. Nagoor Gani and K. Radha [2008]

In this paper, regular fuzzy graphs, total degree and totally regular fuzzy graphs are introduced. Regular fuzzy graphs and totally regular fuzzy graphs are compared through various examples. A necessary and sufficient condition under which they are equivalent is provided. A characterization of regular fuzzy graphs on a cycle is provided. Some properties of regular fuzzy graphs are studied and they are examined for totally regular fuzzy graphs.

7. TYPES OF ARCS IN A FUZZY GRAPH

Sunil Mathew, M. S. Sunitha [2009]

The concept of connectivity plays an important role in both theory and applications of fuzzy graphs. Depending on the strength of an arc, this paper classifies arcs of a fuzzy graph into three types namely α-strong, β-strong and δ-arcs. The advantage of this type of classification is that it helps in understanding the basic structure of a fuzzy graph completely. We analyze the relation between strong paths and strongest paths in a fuzzy graph and obtain characterizations for fuzzy bridges, fuzzy trees and fuzzy cycles using the concept of α-strong, β-strong and δ-arcs. An arc of a fuzzy tree is α-strong if and only if it is an arc of its unique maximum spanning tree. Also we identify different types of arcs in complete fuzzy graphs.

8. FUZZY SET AND FUZZY LOGIC
Y. Leung [2009]

Development and applications of the fuzzy-set and fuzzy-logic approach to geographical research is reviewed in this article. It first points out the paradigm shift in geographical analysis under uncertainty. Instead of solely equating uncertainty with randomness, the fuzzy-set approach addresses uncertainty due to fuzziness/imprecision. It reintroduces value judgment to the analysis of human behaviors in space. It builds a bridge between quantitative and qualitative analysis so that natural languages can be used in geographical investigations with scientific rigor. The article examines the fuzzy-sets approach to spatial characterization, spatial classification, and regionalization that are fundamental in geographical analysis. It compares the ways in which it differs from the conventional approach in terms of philosophy, analytical methods, and results. The discussion then extends to the investigation of fuzziness in spatial economic analysis, spatial optimization, and planning. After that, the focus shifts to the employment of fuzzy logic to spatial reasoning and decision making. It contrasts the reasoning by conventional logic. The application of fuzzy logic in expert systems and spatial decision support systems is also discussed. The article then concludes with an integration of probability, fuzzy set, and rough set to form a unified framework for uncertainty analysis in geography. It also points to the integration of fuzzy set and logic with paradigms such as neural networks and genetic algorithms for the analysis of complex spatial systems

9. ISOMORPHISM ON IRREGULAR FUZZY GRAPHS

A. Nagoor Gani and S. R. Latha [2012]

In this paper a weak isomorphism, co- weak isomorphism and isomorphism of neighbourly irregular fuzzy graphs and highly irregular fuzzy graphs are defined. Some results on order, size and degrees of the nodes between isomorphic neighbourly irregular and isomorphic highly irregular fuzzy graphs are discussed. Isomorphisms between neighborly irregular and highly irregular fuzzy graphs are proved to be an equivalence relation. Also it is proved that weak isomorphism and co-weak isomorphism between neighbourly irregular fuzzy graphs and highly irregular fuzzy graphs satisfy partial order relation.

10.SOME REMARKS ON COMPLEMENT OF FUZZY GRAPHS

K. R. Sandeep Narayan and M. S. Sunitha [2013]

Fuzzy Graphs are having numerous applications in problems like Network analysis, Clustering, Pattern Recognition and Neural Networks. The analysis of properties of fuzzy graphs has facilitated the study of many complicated networks like Internet. In this paper we study the structures of complement of many important fuzzy graphs such as Fuzzy cycles, Blocks etc. The complement of fuzzy graphs with a certain structural property is studied in this paper.

Chapter - I

CHAPTER I

PRELIMINARY DEFINITIONS AND RESULTS

SECTION: 1.1

FUZZY GRAPHS

Definition: 1.1.1

A **graph** $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, ...\}$ called vertices, and another set $E = \{e_1, e_2, ...\}$, whose elements are called edges, ∍ each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k.

Definition: 1.1.2

An edge having the same vertex as both its end vertices is called a **self-loop (simply loop).**

Definition: 1.1.3

If the end vertices of two edges are the same, then the edges are called **parallel edges.**

Definition: 1.1.4

A graph that has neither self-loops nor parallel edges is called a **simple graph.**

Definition: 1.1.5

Two non-parallel edges are said to be **adjacent**, if they are incident on a common vertex.

Definition: 1.1.6

Two vertices are said to be **adjacent** if they are the end vertices of an edge.

Definition: 1.1.7

Let G be a graph. If a vertex v_i is an end vertex of some edge e_j. Then, we say v_i and e_j are **incident** with each other.

Definition: 1.1.8

Two graphs G and G' are said to be **isomorphic** (to each other) if there is a one-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.

Definition: 1.1.9

Let $G = (V, X)$ be a graph. The **complement** $\overline{G}$ of G is defined to be the graph, which has V as its set of points and two points are adjacent in $\overline{G}$ iff they are not adjacent in G.

Definition: 1.1.10

A **fuzzy graph** with S (finite set) as the underlying set is a pair G: (σ, μ) where σ: S $\rightarrow$ [0 1] is a fuzzy subset μ: S×S $\rightarrow$ [0 1] is a fuzzy relation on the fuzzy subset σ, ∋ μ (x, y) $\leq$ σ(x) $\wedge$ σ(y) $\forall$ x,y $\in$ S

Definition: 1.1.11

Let S be a finite set. **Support of** σ is given by

Supp (σ) = {u / σ(u) > 0}.

Definition: 1.1.12

Let S be a finite set. **Support of** μ is given by

Supp (μ) = {(x, y) / μ (x, y) > 0}.

Remark: 1.1.13

i) Supp (σ) is a subset of S.

ii) Supp (μ) is a subset of S.

Definition: 1.1.14

Given a fuzzy graph G: (σ, μ) with the underlying set S, **the order of G** is defined and denoted as $p = \sum_{x\varepsilon s} \sigma(x)$

Definition: 1.1.15

Given a fuzzy graph G: (σ, μ) with the underlying set S, **the size of G** is defined and denoted as $q = \sum_{x,y\varepsilon s} \mu(x, y)$

Definition: 1.1.16

A graph G is called **regular** if vertex is adjacent only to vertices having the same degree.

Definition: 1.1.17

A graph G is said to be **irregular**, if there is a vertex which is adjacent to at least one vertex with distinct degrees.

Definition: 1.1.18

A connected graph G is said to be **highly irregular** if every vertex of G is adjacent only to vertices with distinct degrees.

Definition: 1.1.19

Let G =(σ, μ) be a connected fuzzy graph. G is said to be a **highly irregular fuzzy graph** if every vertex of G is adjacent to vertices with distinct degrees.

Definition: 1.1.20

A fuzzy graph with S, a non empty finite set as the underlying set is a pair

$G: (\sigma, \mu)$ where $\sigma: S \to [0, 1]$ is a fuzzy subset of S, $\mu: S \times S \to [0, 1]$ is a **symmetric fuzzy relation** on the fuzzy subset σ such that

$\mu(x, y) = \min(\sigma(x), \sigma(y)), \forall x, y \in S$

Definition: 1.1.21

A fuzzy relation μ is **symmetric** if $\mu(x, y) = \mu(x, y) \ \forall x, y \in S$

Remark : 1.1.22

The underlying crisp graph of the fuzzy graph $G: (\sigma, \mu)$ is denoted as

$G^*: (\sigma^*, \mu^*)$ where σ^* =$\{x \in S: \sigma(x) > 0\}$ and $\mu^* = \{(x, y) \in S \times S: \mu(x, y) > 0\}$.

If $\mu(x, y) > 0$, then x and y are called **neighbors**.

Note: 1.1.23

For simplicity, an edge (x, y) will be denoted by xy

Definition: 1.1.24

A directed graph (or) diagraph G consist of a set of vertices

$V = \{V_1, V_2, V_3 ....\}$ and a set of edges $E = \{e_1, e_2, e_3 .....\}$, and a mapping that maps

every edge onto some ordered pair of vertices (V_i, V_j) .

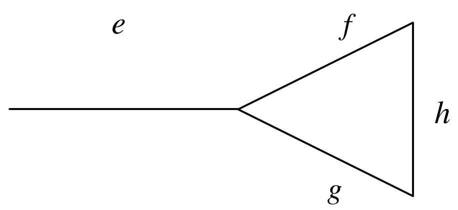
Definition: 1.1.25

A **simple diagraph** is a diagraph in which each ordered pair of vertices occurs atmost once as an edge.

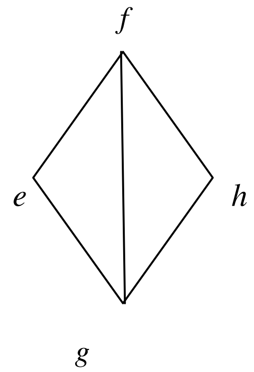
Definition: 1.1.26

The **line graph (or) an edge diagraph** of a graph G,written $L(G)$, is the graph whose vertices are edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G.

Eg:



G



$L(G)$

Definition: 1.1.27

A **Hypergraph** is a pair (V, E) such that $V \cap E = \phi$ and E is a subset of

$P(V)$, the power set of V, *i.e.*, the set of all subsets of V.

Definition:1.1.28

A Hypergraph is K **-uniform** if its edges all have size K.

Definition: 1.1.29

Let X be a finite set and let ε be a finite family of non trivial fuzzy subsets of X such that $X = \bigcup_{\mu_i} \text{supp}\, \mu_i(x)$ .Then the pair $H : (X, \in)$ is called a **fuzzy hypergraph** and

$\in = \{E_1, E_2, E_3 \ldots. E_n\}$ is called the collection of edge sets of H .

Note: 1.1.30

A fuzzy hypergraph H with underlying set X is $H:(X, \mu_i, \rho)$ where $\mu_i : X \rightarrow [0,1]$ are fuzzy subsets, $\rho : \in \rightarrow [0,1]$ is a fuzzy relation on the fuzzy subsets μ_i such that,

$$\rho(\{x_1, x_2, x_3 \ldots . x_r\}) \le \mu_i(x_1) \wedge \mu_i(x_2) \wedge \mu_i(x_3) \wedge \ldots \ldots . \wedge \mu_i(x_r)$$

Definition: 1.1.31

Given a fuzzy graph $H:(X, \mu_i, \rho)$ with the underlying set X ,**the order of** H is defined and denoted as

$$p = \sum_x \wedge_i \mu_i(x)$$

And **the size of** H is defined and denoted as

$$q = \sum_{\{x_1, x_2, \ldots x_r\} = E_1 \subset X} \rho(x_1, x_2, \ldots . x_r) = \sum_{E_i \subset X} \rho(E_i) \, .$$

Chapter - II

CHAPTER II

ISOMORPHISM ON FUZZY GRAPHS

SECTION: 2.1

ISOMORPHISM BASIC PROPERTIES

Definition: 2.1.1 [Bhutani, 1989]

A **homomorphism** of fuzzy graphs $h : G \to G'$ is a map $h : S \to S'$ which satisfies,

$\sigma(x) \leq \sigma'(h(x)) \quad \forall\, x \in S$ and

$\mu(x, y) \leq \mu'(h(x), h(y)) \quad \forall\, x, y \in S$.

Definition: 2.1.2 [Bhutani, 1989]

A **weak isomorphism** $h : G \to G'$ is a map $h : S \to S'$ which is a bijective homomorphism that satisfies, $\sigma(x) = \sigma'(h(x)) \quad \forall\, x \in S$.

Example: 2.1.3

Let $G : (\sigma, \mu)$ and $G' : (\sigma', \mu')$ be the fuzzy graphs with underlying sets

$S = \{a, b, c\}$ and $S' = \{a', b', c'\}$ where,

$\sigma : S \to [0,1]$, $\mu : S \times S \to [0,1]$

$\sigma' : S' \to [0,1]$, $\mu' : S' \times S' \to [0,1]$ are defined as,

$\sigma(a) = 1/2$, $\sigma(b) = 1/4$, $\sigma(c) = 1/3$;

$\mu(a,b) = 1/5$, $\mu(b,c) = 1/5$, $\mu(a,c) = 1/4$;

$\sigma'(a') = 1/2$, $\sigma'(b') = 1/4$, $\sigma'(c') = 1/3$;

$\mu'(a',b') = 1/4$, $\mu'(b',c') = 1/4$, $\mu'(a',c') = 1/3$;

Defining $h : S \to S'$ as $h(a) = a'$, $h(b) = b'$, $h(c) = c'$

This h is a bijective mapping satisfying ,

$\sigma(a) = 1/2 = \sigma'(a')$ $\quad\quad$ $\mu(a,b) = 1/5 \le \mu'(a',b') = 1/4$

$\sigma(b) = 1/4 = \sigma'(b')$ $\quad\quad$ $\mu(b,c) = 1/5 \le \mu'(b',c') = 1/4$

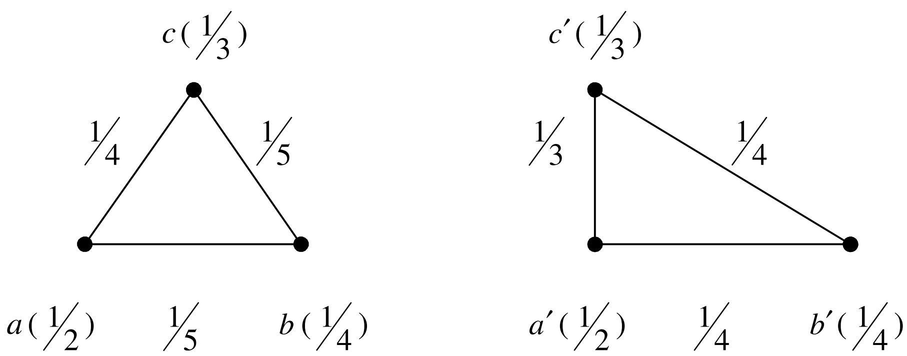
$\sigma(c) = 1/3 = \sigma'(c')$ $\quad\quad$ $\mu(a,c) = 1/4 \le \mu'(a',c') = 1/3$

i.e., $\sigma(x) = \sigma'(h(x)) \quad \forall\, x \in S$ and (1)

$\mu(x,y) \le \mu'(h(x),h(y)) \quad \forall\, x, y \in S$. (2)

$\therefore h : G \to G'$ is a bijective homomorphism and equ (1) is satisfied.

Fig. 1(a) is weak isomorphic to Fig. 1(b)



(a) $G : (\sigma, \mu)$ $\quad\quad$ (b) $G' : (\sigma', \mu')$

Fig. 1 Weak isomorphic

$\therefore$ The map h is a weak isomorphic.

Definition: 2.1.4 [Bhutani, 1989]

A **co-weak isomorphism** $h : G \to G'$ is a map $h : S \to S'$ which is a bijective homomorphism that satisfies, $\mu(x, y) = \mu'(h(x), h(y)) \quad \forall\, x, y \in S$.

Example: 2.1.5

Let $G : (\sigma, \mu)$ and $G' : (\sigma', \mu')$ be the fuzzy graphs with underlying sets S and S' respectively with $S = \{a, b, c\}$ and $S' = \{a', b', c'\}$ where,

$\sigma : S \to [0,1]$, $\mu : S \times S \to [0,1]$

$\sigma' : S' \to [0,1]$, $\mu' : S' \times S' \to [0,1]$ are defined as,

$$\sigma(a) = 1/3 \ , \ \sigma(b) = 1/2 \ , \ \sigma(c) = 1/4 \ ;$$

$$\mu(a,b) = 1/3 \ , \ \mu(b,c) = 1/5 \ , \ \mu(a,c) = 1/4 \ ;$$

$$\sigma'(a') = 1/2 \ , \ \sigma'(b') = 1 \ , \ \sigma'(c') = 1/4 \ ;$$

$$\mu'(a',b') = 1/3 \ , \ \mu'(b',c') = 1/5 \ , \ \mu'(a',c') = 1/4 \ ;$$

Defining $h : S \to S'$ as $h(a) = a'$, $h(b) = b'$, $h(c) = c'$

This h is a bijective mapping satisfying ,

$\sigma(a) = 1/3 \le \sigma'(a') = 1/2$ $\qquad \mu(a,b) = 1/3 = \mu'(a',b')$

$\sigma(b) = 1/2 \le \sigma'(b') = 1$ $\qquad \mu(b,c) = 1/5 = \mu'(b',c')$

$\sigma(c) = 1/4 = \sigma'(c')$ $\qquad \mu(a,c) = 1/4 = \mu'(a',c')$

i.e., $\sigma(x) \le \sigma'(h(x)) \quad \forall\, x \in S$ and (1)

$$\mu(x, y) = \mu'(h(x), h(y)) \quad \forall\, x, y \in S. \qquad (2)$$

$\therefore\ h : G \to G'$ is a bijective homomorphism and equ (2) is satisfied.

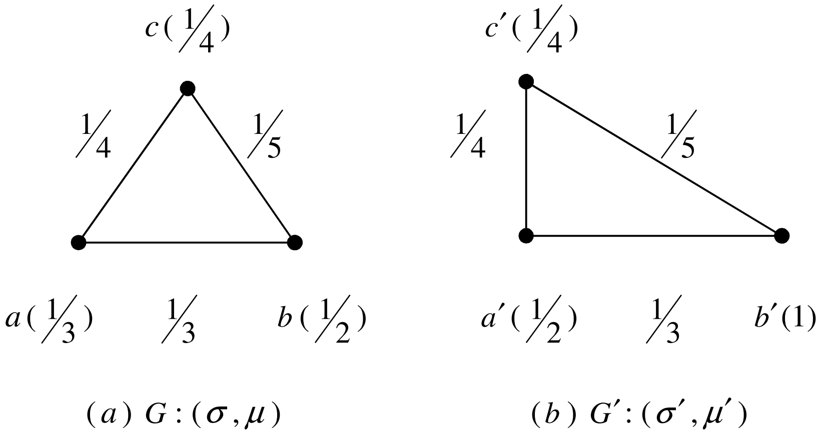


Fig. 2 Co-weak isomorphic

$\therefore$ The map h is a co-weak isomorphic.

Definition: 2.1.6 [Bhutani, 1989]

An **isomorphism** $h : G \to G'$ is a map $h : S \to S'$ which is a bijective that satisfies,

$\sigma(x) = \sigma'(h(x)) \quad \forall\ x \in S$ and

$\mu(x, y) = \mu'(h(x), h(y)) \quad \forall\ x, y \in S$. We denote it as $G \cong G'$.

Definition: 2.1.7

An **endomorphism** of a fuzzy graph $G : (\sigma, \mu)$ is a homomorphism of G to itself.

Definition: 2.1.8

An **automorphism** of a fuzzy graph $G : (\sigma, \mu)$ is a isomorphism of G to itself.

Remark: 2.1.9

(i) A weak isomorphism preserves the weights of the nodes but not necessarily the weights of the edges.

(ii) A co-weak isomorphism preserves the weights of the edges but not necessarily the weights of the nodes.

(iii) An isomorphism preserves the weights of the edges and the weights of the nodes.

(iv) When the two fuzzy graphs G and G' are same the weak isomorphism between them becomes an isomorphism and similarly the co-weak isomorphism between them also become isomorphism.

In crisp graph, when two graphs are isomorphic they are of same size and order. The following theorem is analogous to this.

Theorem: 2.1.10

For any two isomorphic fuzzy graphs their order and size are same.

Proof:

If $h : G \to G'$ is an isomorphism between the fuzzy graphs G and G' with the underlying sets S and S' respectively.

Then by definition of isomorphism,

$$\sigma(x) = \sigma'(h(x)) \quad \forall\, x \in S \quad \text{and}$$

$$\mu(x, y) = \mu'(h(x), h(y)) \quad \forall\, x, y \in S .$$

(i) Order (G) = $\sum_{x \in S} \sigma(x) = \sum_{x \in S} \sigma'(h(x))$ = Order (G')

(ii) Size (G) = $\sum_{x, y \in S} \mu(x, y) = \sum_{x, y \in S} \mu'(h(x), h(y))$ = Size (G').

Converse of the above theorem need not be true. Which can be seen from the following example.

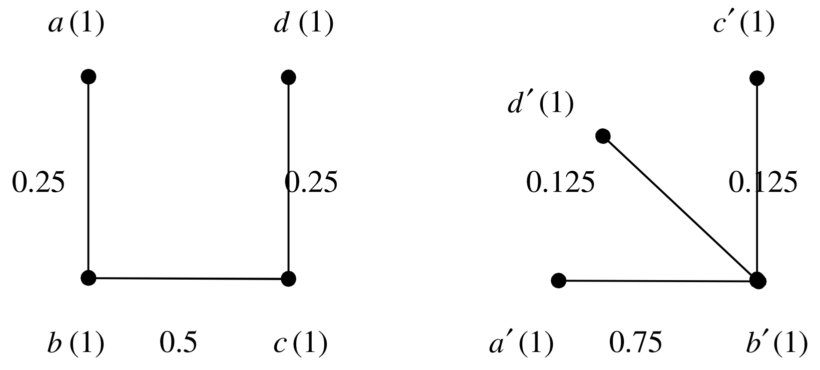
Example: 2.1.11

Consider the fuzzy graphs G and G' with underlying sets S and S' as

$S = \{a, b, c, d\}$, $S' = \{a', b', c', d'\}$ where,

$\sigma(x) = 1 \quad \forall\, x \in S$, and $\mu(a,b) = 0.25$, $\mu(b,c) = 0.5$, $\mu(c,d) = 0.25.$

$\sigma'(x') = 1 \quad \forall\, x' \in S'$, and $\mu'(a',b') = 0.75$, $\mu(b',c') = 0.125$, $\mu(c',d') = 0.125.$



$(a)\ G : (\sigma, \mu)$ $(b)\ G' : (\sigma', \mu')$

Fig. 3 Graphs of same order and size but not isomorphic

Order of $G = \sum_{x \in S} \sigma(x) = \sigma(a) + \sigma(b) + \sigma(c) + \sigma(d)$

$= 1+1+1+1 = 4$

Order of $G' = \sum \sigma'(x') = \sigma'(a') + \sigma'(b') + \sigma'(c') + \sigma'(d')$

$= 1+1+1+1 = 4$

$\therefore$ Order of G = Order of G'

Size $(G) = \sum_{x,y \in S} \mu(x,y) = \mu(a,b) + \mu(b,c) + \mu(c,d)$

$= 0.25 + 0.5 + 0.25 = 1$

Size $(G') = \sum_{x',y' \in S'} \mu'(x',y') = \mu'(a',b') + \mu(b',c') + \mu(c',d')$

$= 0.75 + 0.125 + 0.125 = 1$

$\therefore$ Size (G) = Size (G')

$\therefore$ The given two graphs are of same order and same size.

Here, $\sigma(a) = \sigma'(a')$, $\sigma(b) = \sigma'(b')$, $\sigma(c) = \sigma'(c')$, $\sigma(d) = \sigma'(d')$

i.e., $\sigma(x) = \sigma'(h(x)) \quad \forall\ x \in S$

But, $\mu(a,b) = 0.25 \neq \mu'(a',b') = 0.75$, $\mu(b,c) = 0.5 \neq \mu'(b',c') = 0.125$,

$\mu(c,d) = 0.25 \neq \mu'(c',d') = 0.125.$

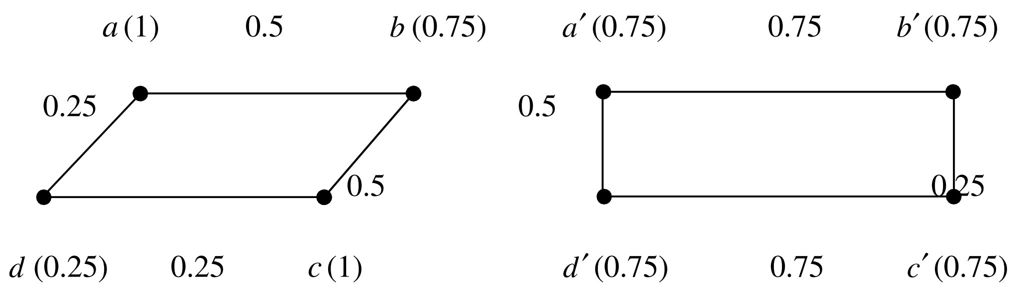
i.e., $\mu(x,y) \neq \mu'(h(x),h(y))$

Even though, the given two graphs are of same order and size but not isomorphic.

Remark: 2.1.12

If the fuzzy graphs are weak isomorphic then their orders are same. But the fuzzy graphs of same order need not be weak isomorphic.

Example: 2.1.13



$(a)\ G : (\sigma, \mu)$ $\qquad$ $(b)\ G' : (\sigma', \mu')$

Fig. 3 Graphs of same order but not weak isomorphic

Order of $G = \sum_{x \in S} \sigma(x) = \sigma(a) + \sigma(b) + \sigma(c) + \sigma(d)$

$= 1 + 0.75 + 0.25 + 1 = 3$

Order of $G' = \sum_{x' \in S} \sigma'(x') = \sigma'(a') + \sigma'(b') + \sigma'(c') + \sigma'(d')$

$= 0.75 + 0.75 + 0.75 + 0.75 = 3$

$\therefore$ Order of the given two graphs G and G' are same . *i.e.*, $p = 3$

But,

$\sigma(a) \neq \sigma'(a')$, $\sigma(b) \neq \sigma'(b')$, $\sigma(c) \neq \sigma'(c')$, $\sigma(d) \neq \sigma'(d')$

i.e., $\sigma(x) \neq \sigma'(h(x))$

$\therefore$ It is not weak isomorphic.

Remark: 2.1.14

If the fuzzy graphs are co-weak isomorphic their sizes are same. But the fuzzy graphs of same size need not be co-weak isomorphic.

Example: 2.1.15

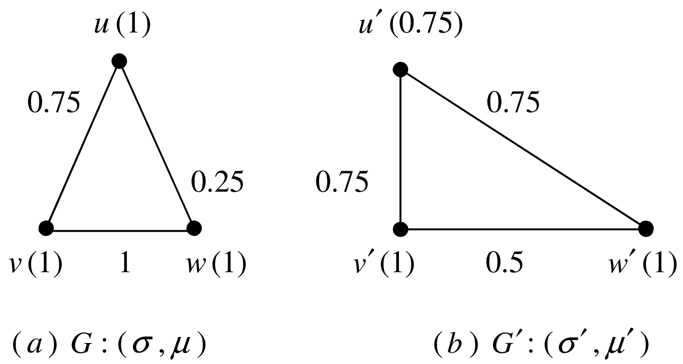


Fig. 3 Graphs of same size but not co-weak isomorphic

Size $(G) = \sum_{x,y \in S} \mu(x,y) = \mu(u,v) + \mu(v,w) + \mu(u,w)$

$= 0.75 + 1 + 0.25 = 2$

Size $(G') = \sum_{x',y' \in S'} \mu'(x',y') = \mu'(u',v') + \mu'(v',w') + \mu'(u',w')$

$= 0.75 + 0.5 + 0.75 = 2$

$\therefore$ Size of the given two graphs G and G' are same . *i.e.*, $q = 3$

But,

$\mu(u,v) = \mu'(u',v')$, $\mu(v,w) \neq \mu'(v',w')$,

$\mu(u,w) \neq \mu'(u',w')$.

i.e., $\mu(x,y) \neq \mu'(h(x),h(y))$

$\therefore$ It is not co-weak isomorphic.

Definition: 2.1.16 [Mordeson, 2001]

Let $G : (\sigma, \mu)$ be a fuzzy graph. The **degree of a vertex '** u **'** is defined as

$d(u) = \sum_{\substack{v \neq u \\ v \in S}} \mu(u,v)$.

Theorem: 2.1.17

If G and G' are isomorphic fuzzy graphs then the degrees of their nodes are preserved.

Proof:

Let $h : S \to S'$ be an isomorphism of G onto G'.

By the definition of isomorphism,

$\mu(x,y) = \mu'(h(x),h(y)) \quad \forall\ x, y \in S$

So, $d(u) = \sum_{\substack{v \neq u \\ v \in S}} \mu(u,v) = \sum_{\substack{v \neq u \\ v \in S}} \mu'(h(u),h(v)) = d(h(u))$.

Converse of the above theorem need not be true. which can be seen,

Consider $G:(\sigma,\mu)$ with the underlying set $S = \{a, b\}$

Where, $\sigma(a) = \frac{1}{2}$, $\sigma(b) = \frac{1}{4}$ and $\mu(a,b) = \frac{1}{4}$;

$G':(\sigma',\mu')$ with the underlying set $S' = \{a', b'\}$

Where, $\sigma'(a') = \frac{1}{2}$, $\sigma'(b') = \frac{3}{4}$ and $\mu'(a',b') = \frac{1}{4}$;

Here, $d(a) = d(b) = d(a') = d(b') = \frac{1}{4}$

But G and G' are not isomorphic, only co-weak isomorphic.

Remark: 2.1.18

The degree of a vertex is measured only by adding the weights of the edges incident with that vertex. But fuzzy graphs preserving the degree of the vertices need not be co-weak isomorphic.

Example: 2.1.19

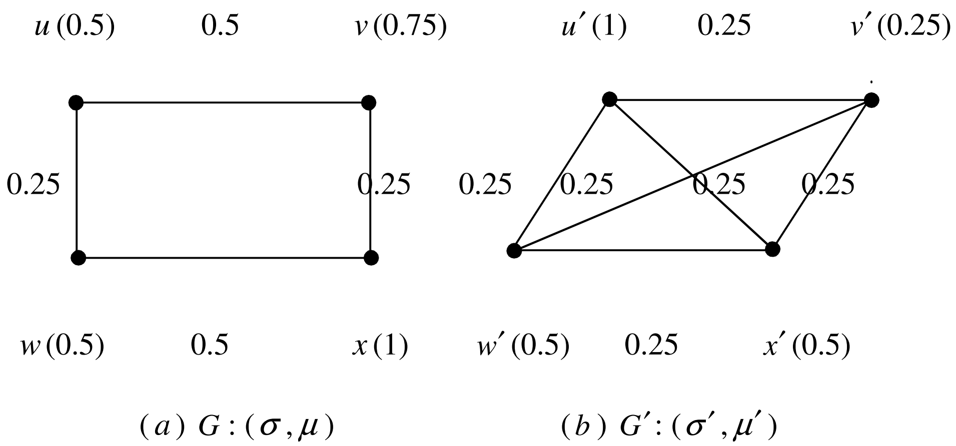


Fig. 3 Graphs preserving the degree of the vertices

In the above two graphs,

$d(u) = d(v) = d(w) = d(x) = d(u') = d(v') = d(w') = d(x') = 0.75$

i.e., each vertex is of degree 0.75

But,

$\sigma(x) \neq \sigma'(h(x)) \quad \forall\, x \in S$ and

$\mu(x, y) \neq \mu'(h(x), h(y)) \quad \forall\, x, y \in S$

Theorem: 2.1.20

Isomorphism between fuzzy graphs is an equivalence relation.

Proof:

Let $G : (\sigma, \mu)$, $G' : (\sigma', \mu')$, $G'' : (\sigma'', \mu'')$ be fuzzy graphs with underlying sets S, S', S'' respectively.

(i) Reflexive:

Consider the identity map $h : S \rightarrow S \ni h(x) = x$ for all x in S.

This h is a bijective map satisfying,

$\sigma(x) = \sigma(h(x)) \quad \forall\, x \in S$ and

$\mu(x, y) = \mu(h(x), h(y)) \quad \forall\, x, y \in S$.

Hence h is an isomorphism of the fuzzy graph to itself.

Therefore it satisfies reflexive relation.

(ii) Symmetric:

Let $h : S \rightarrow S'$ be an isomorphism of G onto G' then h is a bijective map

$h(x) = x'$, $x \in S$ satisfying, (1)

$$\sigma(x) = \sigma'(h(x)) \quad \forall\, x \in S \quad \text{and}$$

$$\mu(x, y) = \mu'(h(x), h(y)) \quad \forall\, x, y \in S. \tag{2}$$

As h is bijective, by (1) $h^{-1}(x') = x \quad \forall\, x' \in S'$

Using (2),

$$\sigma(h^{-1}(x')) = \sigma'(x') \quad \forall\, x' \in S' \quad \text{and}$$

$$\mu(h^{-1}(x'), h^{-1}(y')) = \mu'(x', y') \quad \forall\, x', y' \in S'. \tag{3}$$

Hence we get a 1-1, onto map h^{-1}: $S' \to S$, which is an isomorphism from G' to G

i.e., $G \cong G' \Rightarrow G' \cong G$.

(iii) Transitive:

Let $h : S \to S'$ and $g : S' \to S''$ be isomorphisms of the fuzzy graphs G onto G' and G' onto G'' respectively.

Then $g \circ h$ is a 1-1, onto map from S to S''

Where, $g \circ h(x) = g(h(x)) \quad \forall\, x \in S$

As $h : S \to S'$ is an isomorphism $h(x) = x'$, $x \in S$.

$$\sigma(x) = \sigma'(h(x)) \quad \forall\, x \in S \quad \text{and}$$

$$\mu(x, y) = \mu'(h(x), h(y)) \quad \forall\, x, y \in S. \tag{4}$$

i.e., $\sigma(x) = \sigma'(x') \quad \forall\, x \in S \quad$ and

$$\mu(x, y) = \mu'(x', y') \quad \forall\, x, y \in S. \tag{5}$$

As g is an isomorphism from S' to S'' we have,

$$g(x') = x'', \quad x' \in S' \quad \text{and}$$

$$\sigma'(x') = \sigma''(g(x')) \quad \forall\ x' \in S' \tag{6}$$

$$\mu'(x', y') = \mu''(g(x'), g(y')) \quad \forall\ x', y' \in S' \tag{7}$$

From (4), (6) and using $h(x) = x'$, $x \in S$

$$\sigma(x) = \sigma'(x') = \sigma''(g(x')) \quad \forall\ x' \in S'$$

$$= \sigma''(g(h(x))) \quad \forall\ x \in S .$$

From (5) and (7) we have,

$$\mu(x, y) = \mu'(x', y') \quad \forall\ x, y \in S$$

$$= \mu''(g(x'), g(y')) \quad \forall\ x', y' \in S'$$

$$= \mu''(g(h(x)), g(h(y))) \quad \forall\ x, y \in S .$$

Therefore $g \circ h$ is an isomorphism between G and G''.

Hence isomorphism between fuzzy graphs is an equivalence relation.

Theorem: 2.1.21

Weak isomorphism between fuzzy graphs satisfies the partial order relation.

Proof:

Let $G : (\sigma, \mu)$, $G' : (\sigma', \mu')$, $G'' : (\sigma'', \mu'')$ be fuzzy graphs with underlying sets S, S', S'' respectively.

(i) Reflexive:

Consider the identity map $h : S \to S \ni h(x) = x$ for all x in S.

This h is a bijective map satisfying,

$$\sigma(x) = \sigma(h(x)) \quad \forall\ x \in S \quad \text{and}$$

$$\mu(x, y) = \mu(h(x), h(y)) \quad \forall\, x, y \in S.$$

Hence h is an isomorphism of the fuzzy graph to itself.

Therefore G is weak isomorphic to itself.

(ii) Anti symmetric:

Let h be a weak isomorphism between G and G' and g be a weak isomorphism between G' and G.

i.e., $h : S \to S'$ is a bijective map $h(x) = x'$, $x \in S$ satisfying

$$\sigma(x) = \sigma'(h(x)) \quad \forall\, x \in S \quad \text{and}$$

$$\mu(x, y) \le \mu'(h(x), h(y)) \quad \forall\, x, y \in S. \tag{1}$$

And $g : S' \to S$ is a bijective map satisfying

$$\sigma'(x') = \sigma(g(x')) \quad \forall\, x' \in S' \quad \text{and}$$

$$\mu'(x', y') \le \mu'(g(x'), g(y')) \quad \forall\, x', y' \in S'. \tag{2}$$

The inequalities (1) and (2) hold good for a finite set S and S' only when G and G' have

The same number of edges and the corresponding edges have the same weight.

Hence G and G' are identical.

(iii) Transitive:

Let $h : S \to S'$ and $g : S' \to S''$ be a weak isomorphisms of the fuzzy graphs

G onto G' and G' onto G'' respectively.

Then $g \circ h$ is a 1-1, onto map from S to S''

Where $(g \circ h)(x) = g(h(x)) \quad \forall\, x \in S$

As h is a weak isomorphism, $h(x) = x'$, $\forall\ x \in S$.

$$\sigma(x) = \sigma'(h(x)) \quad \forall\ x \in S \quad \text{and} \tag{3}$$

$$\mu(x, y) \le \mu'(h(x), h(y)) \quad \forall\ x, y \in S \tag{4}$$

As g is a weak isomorphism from S' to S'' we have,

$$g(x') = x'', \quad x' \in S' \quad \text{and}$$

$$\sigma'(x') = \sigma''(g(x')) \quad \forall\ x' \in S' \tag{5}$$

$$\mu'(x', y') \le \mu''(g(x'), g(y')) \quad \forall\ x', y' \in S' \tag{6}$$

From (3) and (5),

$$\begin{aligned}\sigma(x) &= \sigma'(x') \quad \forall\ x \in S\\ &= \sigma''(g(x')) \quad \forall\ x' \in S'\\ &= \sigma''(g(h(x))) \quad \forall\ x \in S\end{aligned}$$

From (3) , (4) , (5) and (6),

$$\begin{aligned}\mu(x, y) &\le \mu'(x', y') \quad \forall\ x, y \in S\\ &\le \mu''(g(x'), g(y')) \quad \forall\ x', y' \in S'\\ &= \mu''(g(h(x)), g(h(y))) \quad \forall\ x, y \in S\end{aligned}$$

Therefore $g \circ h$ is a weak isomorphism between G and G''.

i.e., weak isomorphism satisfies transitivity.

Hence weak isomorphism between fuzzy graphs is a partial order relation.

SECTION: 2.2

ISOMORPHIC GRAPHS AND THEIR COMPLEMENTS

Definition: 2.2.1 [Sunitha, 2002]

Let $G : (\sigma, \mu)$ be a fuzzy graph. The **complement of** $\overline{G}$ is defined as $\overline{G} : (\sigma, \overline{\mu})$

where, $\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \quad \forall\ x, y \in S$.

Theorem: 2.2.2

Two fuzzy graphs are isomorphic if and only if their complements are isomorphic.

Proof:

$G : (\sigma, \mu)$ & $G'(\sigma', \mu')$ be the two fuzzy graphs given.

Assume $G \cong G'$.

There exist a bijective map $h : S \rightarrow S'$ satisfying

$\sigma(x) = \sigma'(h(x))\ \forall\ x \in S$ and

$\mu(x, y) = \mu'(h(x), h(y))\ \forall\ x, y \in S$

By definition,

$\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y)\ \forall\ x, y \in S$.

$\overline{\mu}(x, y) = \sigma'(h(x)) \wedge \sigma'(h(y)) - \mu'(h(x), h(y))$

$= \overline{\mu'}(h(x), h(y))\ \forall\ x, y \in S$

i.e., $\overline{G} \cong \overline{G'}$

Conversely, assume that $\overline{G} \cong \overline{G'}$

i.e., there exist a bijective map $g: S \to S'$ satisfying,

$$\sigma(x) = \sigma'(g(x)) \ \forall\ x \in S \tag{1}$$

$$\overline{\mu}(x, y) = \overline{\mu'}(g(x), g(y)) \ \forall\ x, y \in S \tag{2}$$

Using the definition of complement

$$\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \ \forall\ x,\ y \in S$$

$$\overline{\mu'}(g(x), g(y)) = \sigma'(g(x)) \wedge \sigma'(g(y)) - \mu'(g(x), g(y))$$

Using the above two equations in (2) and by (1)

$$\mu(x, y) = \mu'(g(x), g(y)) \ \forall\ x, y \in S \tag{3}$$

Hence from (2) and (3) $g: S \to S'$ is an isomorphism between G and G'. *i.e.*, $G \cong G'$

Theorem: 2.2.3

If there is a weak isomorphism between G and G' then there is a weak isomorphism between $\overline{G'}$ and $\overline{G}$.

Proof:

If h is a weak isomorphism between G and G'

Then, $h: S \to S'$ is a bijective map that satisfies,

$$h(x) = x', \quad x \in S$$

$$\sigma(x) = \sigma'(h(x)) \quad \forall\ x \in S \tag{1}$$

$$\mu(x, y) \ \leq\ \mu'(h(x), h(y)) \quad \forall\ x, y \in S \tag{2}$$

As $h^{-1}: S' \to S$ is also bijective for every x' in S' there is an $x \in S$ such that

$h^{-1}(x') = x$.

Using this in (1),

$$\sigma'(x') = \sigma(h^{-1}(x')) \quad \forall\, x' \in S' \tag{3}$$

Also, by using (1) & (2) in

$$\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \ \forall\, x, y \in S$$

$$\overline{\mu}(h^{-1}(x'), h^{-1}(y')) \geq \sigma'(h(x)) \wedge \sigma'(h(y)) - \mu'(h(x), h(y))$$

$$= \sigma'(x') \wedge \sigma'(y') - \mu'(x', y') \quad \forall\, x', y' \in S'$$

$$= \overline{\mu'}(x', y') \quad \forall\, x', y' \in S' \tag{4}$$

i.e., $\overline{\mu'}(x', y') \leq \overline{\mu}(h^{-1}(x'), h^{-1}(y'))$

Thus $h^{-1} : S \to S'$ is a bijective map, which is a weak isomorphism between $\overline{G'}$ and $\overline{G}$.

Theorem: 2.2.4

If there is a co-weak isomorphism between G and G' then there can be a homomorphism between $\overline{G}$ and $\overline{G'}$.

Proof:

Let h be co-weak isomorphism between G and G' respectively.

i.e., $h : S \to S'$ is a bijective map that satisfies,

$$\sigma(x) \leq \sigma'(h(x)) \quad \forall\, x \in S \tag{1}$$

$$\mu(x, y) = \mu'(h(x), h(y)) \quad \forall\, x, y \in S \tag{2}$$

In $\overline{G}$, using (1) and (2)

$$\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \quad \forall\, x, y \in S$$

$$\leq \sigma'(h(x)) \wedge \sigma'(h(y)) - \mu'(h(x), h(y))$$

$= \overline{\mu'}\,(h(x),h(y)) \quad \forall\; x, y \in S$

Hence $\overline{\mu}\,(x,y) \leq \overline{\mu'}\,(h(x),h(y)) \quad \forall\; x, y \in S$ (3)

By (1) & (2) h is a bijective homomorphism between $\overline{G}$ and $\overline{G'}$.

Remark: 2.2.5

If G is co-weak isomorphic to G', then $\overline{G}$ and $\overline{G'}$ need not be co-weak isomorphic.

Example: 2.2.6

Let $G:(\sigma,\mu)$ and $G':(\sigma',\mu')$ be the fuzzy graphs with underlying sets S and S' respectively with $S=\{a,b,c\}$ and $S'=\{a',b',c'\}$ where,

$\sigma : S \rightarrow [0,1]$, $\mu : S \times S \rightarrow [0,1]$, $\sigma' : S' \rightarrow [0,1]$, $\mu' : S' \times S' \rightarrow [0,1]$

are defined as, $\sigma(a) = 1/3$, $\sigma(b) = 1/2$, $\sigma(c) = 1/4$;

$$\mu(a,b) = 1/3 \;,\; \mu(b,c) = 1/5 \;,\; \mu(a,c) = 1/4 \;;$$

$$\sigma'(a') = 1/2 \;,\; \sigma'(b') = 1 \;,\; \sigma'(c') = 1/4 \;;$$

$$\mu'(a',b') = 1/3 \;,\; \mu'(b',c') = 1/5 \;,\; \mu'(a',c') = 1/4 \;;$$

Defining $h : S \rightarrow S'$ as $h(a) = a'$, $h(b) = b'$, $h(c) = c'$

This h is a bijective mapping satisfying ,

$\sigma(a) = 1/3 \leq \sigma'(a') = 1/2$ $\qquad \mu(a,b) = 1/3 = \mu'(a',b')$

$\sigma(b) = 1/2 \leq \sigma'(b') = 1$ $\qquad \mu(b,c) = 1/5 = \mu'(b',c')$

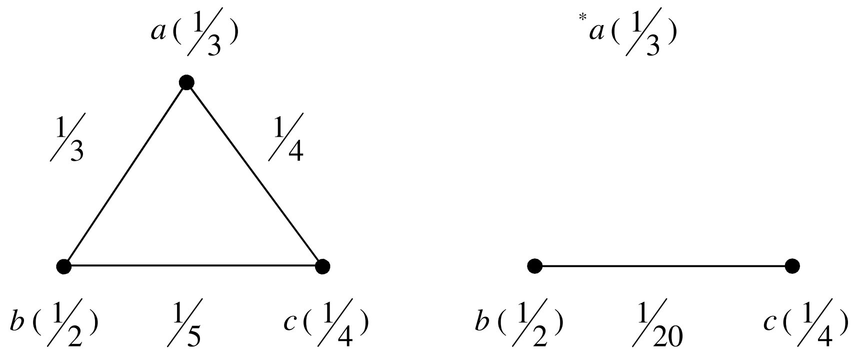
$\sigma(c) = \frac{1}{4} = \sigma'(c')$ $\qquad\qquad$ $\mu(a,c) = \frac{1}{4} = \mu'(a',c')$

i.e., $\sigma(x) \leq \sigma'(h(x)) \quad \forall\, x \in S$ and (1)

$$\mu(x,y) = \mu'(h(x),h(y)) \quad \forall\, x, y \in S. \tag{2}$$

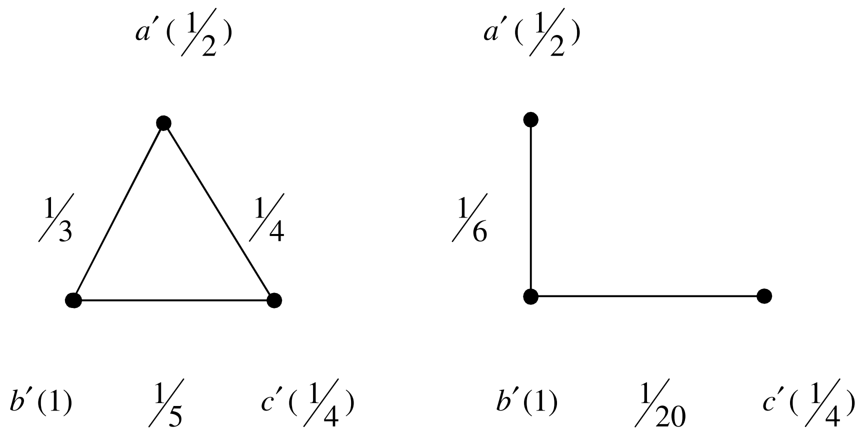
$\therefore\, h : G \to G'$ is a bijective homomorphism and equ (2) is satisfied.

Fig. 7(a) is co-weak isomorphic to Fig. 8(a)



$(a)\ G:(\sigma,\mu)$ $\qquad$ $(b)\ \overline{G}:(\sigma,\overline{\mu})$

Fig. 7 fuzzy graph G and its complement $\overline{G}$



$(a)\ G':(\sigma',\mu')$ $\qquad$ $(b)\ \overline{G'}:(\sigma',\overline{\mu'})$

Fig. 8 fuzzy graph G' and its complement $\overline{G'}$

Here, Fig. 7(a) is co-weak isomorphic to Fig. 8(a).

i.e., $\mu(x, y) = \mu'(h(x),h(y)) \quad \forall\ x, y \in S$

But there is no co-weak isomorphism between Fig. 7(b) and 8(b) as

$\mu(a,b) = 0$

But,

$\mu'(h(a),h(b)) = \mu'(a',b') = 1/2$

But there is a homomorphism between $\overline{G}$ and $\overline{G'}$.

i.e., $\sigma(x) \le \sigma'(h(x)) \ \forall\ x \in S$ and

$\mu(x, y) \le \mu'(h(x),h(y)) \ \forall\ x,y \in S.$

Definition: 2.2.7 [Somasundaram, 1998]

A fuzzy graph $G : (\sigma,\mu)$ with the underlying set S is said to be a **complete fuzzy graph** if, $\mu(x, y) = \sigma(x) \wedge \sigma(y) \quad \forall\ x, y \in S$ and denoted as K_σ.

Theorem: 2.2.8

Any simple fuzzy graph $G : (\sigma, \mu)$ of order 'n' is isomorphic to a partial fuzzy subgraph of the complete fuzzy graphs K_σ.of order 'n'.

Proof:

Let $G : (\sigma, \mu)$ be the given fuzzy graph of order 'n'.

K_σ of order 'n' ,as a fuzzy graph (σ, μ'), can be constructed

Where $\mu'(x, y) = \sigma(x) \wedge \sigma(y) \ \forall\ x, y \in S$ where S is the underlying set of G

.

As in general, $\mu(x, y) \leq \sigma(x) \wedge \sigma(y) \quad \forall\, x, y \in S$

$$\mu(x, y) \leq \mu'(x, y) \qquad \forall\, x, y \in S$$

Hence G is a fuzzy spanning subgraph of K_σ

i.e., G is isomorphic to a partial fuzzy subgraph of the complete fuzzy graph K_σ of order 'n'.

Definition: 2.2.9 [Mordeson, 2001]

Let $G:(\sigma, \mu)$ be a fuzzy graph with the underlying graph (V, E). The fuzzy line graph of G is L(G) = (ω, λ) with the underlying graph (Z, W) where the node set Z is

$\{ S_x = \{x\} \cup \{u_x, v_x\} / \; x \in E, \; u_x, v_x \in V, \; x = (u_x, v_x)\}$

$W = \{(S_x, S_y) / S_x \cap S_y \neq \phi, \, x, \, y \in E \,, \; x \neq y\}$

$\omega(S_x) = \mu(x) \;\; \forall \;\; S_x \in Z$ and

$\lambda(S_x, S_y) = \mu(x) \wedge \mu(y) \, \forall (S_x, S_y) \in W.$

Theorem: 2.2.10

If $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ are the two isomorphic fuzzy graphs then their fuzzy line graphs are also isomorphic.

Proof:

Given G_1 and G_2 are the two isomorphic fuzzy graphs with the underlying sets S_1 and S_2 respectively.

i.e., there exists a bijective map $h: S_1 \rightarrow S_2$ satisfying,

$\sigma_1(x) = \sigma_2(h(x)) \quad \forall\ x \in S_1$

$\mu_1(x, y) = \mu_2(h(x), h(y)) \quad \forall\ x, y \in S_1$

Let $L(G_1) : (\omega_1, \lambda_1)$ and $L(G_2) : (\omega_2, \lambda_2)$ be the line graphs of G_1 and G_2 respectively.

Let $x = (u_x, v_x)$ for $x \in E_1$

As $h : S_1 \rightarrow S_2$ is 1-1, onto, $h(x) = (h(u_x), h(v_x)) \in E_2$.

Define $g: Z_1 \rightarrow Z_2$ as $g(S_x) = S_{h(x)}$

As h is one – one onto, g is well defined and one to one onto mapping.

Consider, $\omega_1(S_x) = \mu_1(x) \ \forall\ S_x \in Z_1$ (by definition of line graphs)

$$\begin{aligned}\omega_1(S_x) &= \mu_1(u_x, v_x)\\ &= \mu_2(h(u_x), h(v_x))\\ &= \mu_2(h(x))\end{aligned}$$

$$\omega_1(S_x) = \omega_2(S_{h(x)}) = \omega_2(g(S_x)) \quad \forall\ S_x \in Z_1 \tag{1}$$

$$\begin{aligned}\lambda_1(S_x, S_y) &= \mu_1(x) \wedge \mu_1(y) \quad \forall\ (S_x, S_y) \in W_1\\ &= \mu_1(u_x, v_x) \wedge \mu_1(u_y, v_y)\\ &= \mu_2(h(u_x), h(v_x)) \wedge \mu_2(h(u_y), h(v_y))\\ &= \mu_2(h(x)) \wedge \mu_2(h(y))\\ &= \lambda_2(S_{h(x)}, S_{h(y)})\end{aligned}$$

$$= \lambda_2(g(S_x), g(S_y)) \quad \forall\ x, y \in E_1$$

$$\lambda_1(S_x, S_y) = \lambda_2(g(S_x), g(S_y)) \quad \forall\ S_x, S_y \in Z_1 \qquad (2)$$

From equations (1) and (2) $L(G_1):(\omega_1, \lambda_1)$ and $L(G_2):(\omega_2, \lambda_2)$ are isomorphic fuzzy line graphs when G_1 and G_2 are the two isomorphic fuzzy graphs.

Remark: 2.2.11

In crisp graph we have $L(C_n) \cong C_n$, C_n denotes cycle of length n. But this does not hold good in fuzzy graph.

Example: 2.2.12

Consider the fuzzy cycle $G:(\sigma, \mu)$ with the underlying graph (V, E), represented as $V = \{u, v, w\}$ $E = \{(u,v),(v,w),(w,u)\}$ with

$\sigma(u) = 3/4 \qquad \mu(u,v) = 1/2$

$\sigma(v) = 1 \qquad \mu(u,w) = 1/2$

$\sigma(w) = 1 \qquad \mu(v,w) = 1$

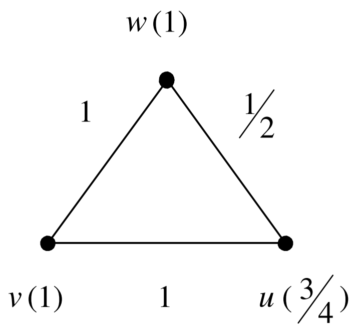


Fig.9 $G:(\sigma, \mu)$

The fuzzy line graph of the fuzzy cycle $G:(\sigma,\mu)$ is $L(G):(\omega,\lambda)$ with the underlying graph (Z,W) where,

$Z=\{S_1,S_2,S_3\}$ &

$S_1=\{(w,u),\{w\},(u)\}$,

$S_2=\{(u,v),\{u\},\{v\}\}$,

$S_3=\{(w,v),\{w\},\{v\}\}$

is in the following figure with $W=\{(S_x,S_y)/S_x\cap S_y\neq\phi,x,y\in E,x\neq y\}$

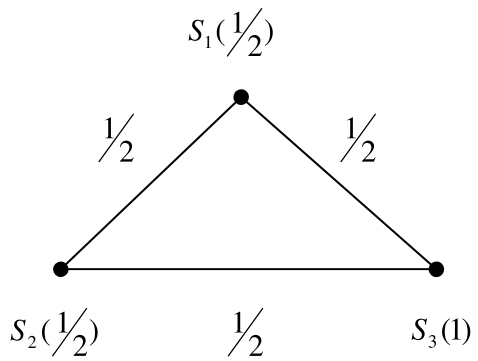


Fig. 10 $L(G):(\omega,\lambda)$

As a map from $V\rightarrow Z$ preserving the weight of the vertices and weight of the arcs cannot be defined, $L(G)\ncong G$.

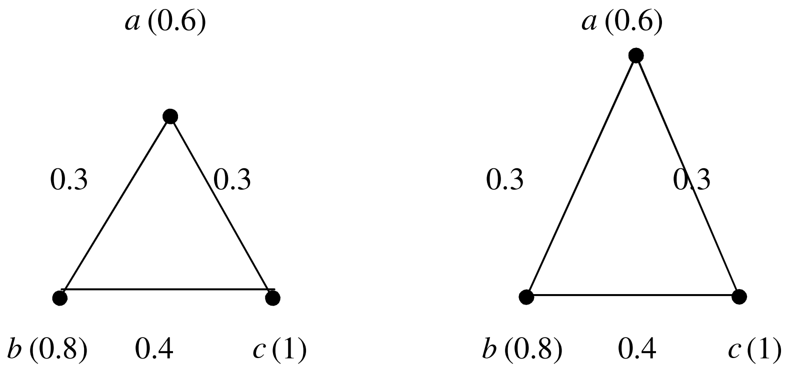
SECTION: 2.3

SELF COMPLEMENTARY FUZZY GRAPHS

Definition: 2.3.1 [Sunitha, 2002]

A fuzzy graph is said to be **self complementary** if $G\cong G'$.

Example: 2.3.2



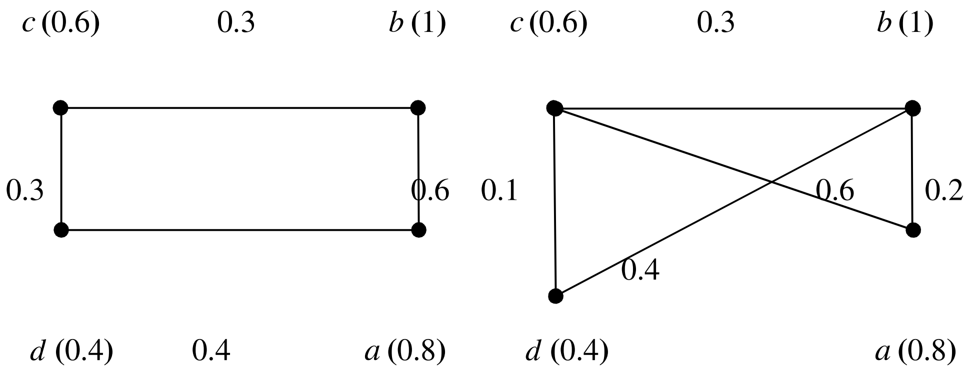
$(a)\ G:(\sigma,\mu)$ $(b)\ \overline{G}:(\sigma,\overline{\mu})$

Fig. 11 self complementary fuzzy graph

Here , $\sigma(x) = \sigma(h(x))$ and $\mu(x, y) = \mu(h(x), h(y))$

$\therefore\ G \cong \overline{G}$.

Example: 2.3.3



$(a)\ G:(\sigma,\mu)$ $(b)\ \overline{G}:(\sigma,\overline{\mu})$

Fig. 13 Fuzzy graph not self complementary.

Here, $\sigma(x) = \sigma'(h(x))$ but, $\mu(x, y) \neq \mu'(h(x), h(y))$

$\therefore G \ncong \overline{G}$.

Theorem: 2.3.4 [Sunitha, 2002]

Let $G:(\sigma,\mu)$ be a self complementary fuzzy graph. Then

$$\sum_{x\neq y}\mu(x,y)=\frac{1}{2}\sum_{x\neq y}\sigma(x)\wedge\sigma(y).$$

Remark: 2.3.5

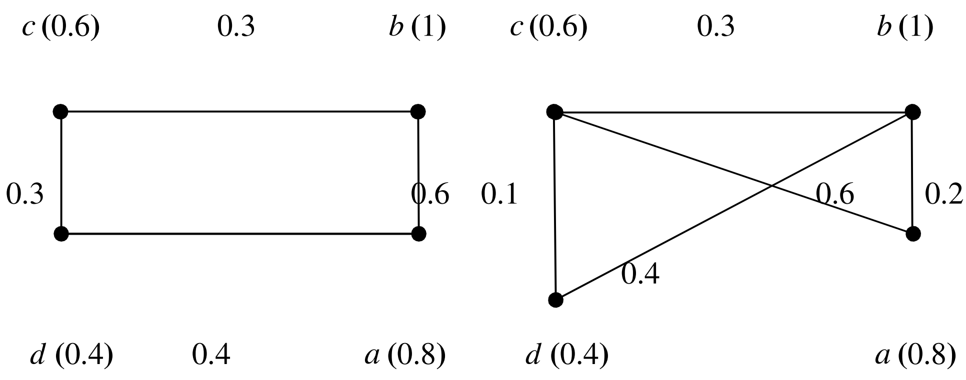
As a consequence of the above theorem if G is a self complementary fuzzy graph, then Size (G) = $q=\frac{1}{2}\sum_{x\neq y}\sigma(x)\wedge\sigma(y)$.

Remark: 2.3.6

But the converse of the ahbove theorem is not true.

Proof:

Consider the graph given below,



(a) $G:(\sigma,\mu)$ (b) $\overline{G}:(\sigma,\overline{\mu})$

Fig. 13 Fuzzy graph not self complementary.

Size (G) = $\sum_{x\in S}\sigma(x)=1.6$ and also

Size ($\overline{G}$) = $\frac{1}{2}\sum_{x \neq y} \sigma(x) \wedge \sigma(y) = 1.6$

Here, $\sigma(x) = \sigma'(h(x))$ but

$\mu(x, y) \neq \mu'(h(x), h(y))$

$\therefore G \ncong \overline{G}$.

SECTION: 2.4

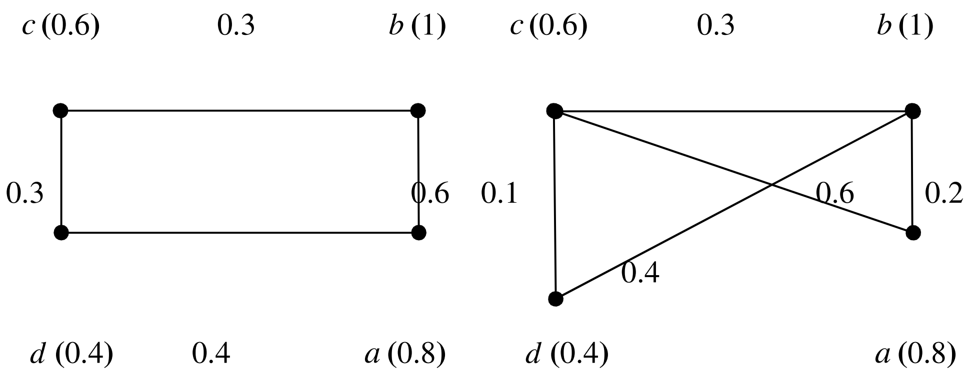
SELF WEAK COMPLEMENTARY FUZZY GRAPHS

Definition: 2.4.1

A fuzzy graph G is said to be **self weak complementary fuzzy graph** if G is weak isomorphic with its complement $\overline{G}$.

Remark: 2.4.2

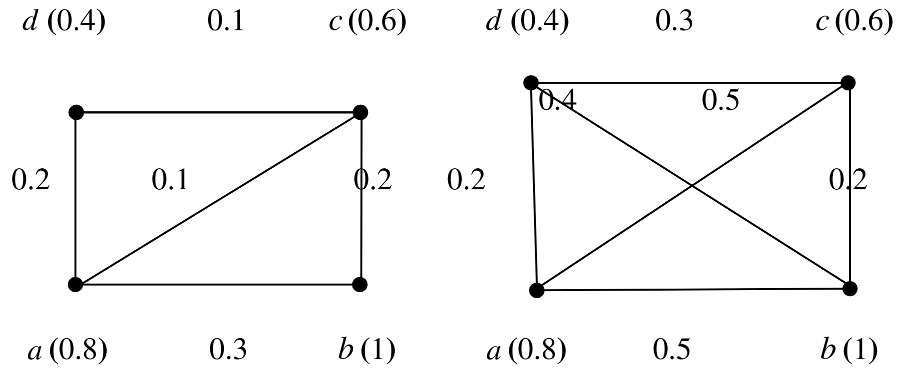
Consider the graphs given below,



(a) $G : (\sigma, \mu)$ (b) $\overline{G} : (\sigma, \overline{\mu})$

Fig. 14 Fuzzy graph not self complementary

Example: 2.4.3



$(a)\ G:(\sigma,\mu)$ $\qquad$ $(b)\ \overline{G}:(\sigma,\overline{\mu})$

Fig. 14 self weak complementary fuzzy graph

Theorem: 2.4.4

Let G be a self weak complementary fuzzy graph then

$$\sum_{x\neq y}\mu(x.y) \leq \frac{1}{2}\sum_{x\neq y}(\sigma(x)\wedge\sigma(y)).$$

Proof:

G is self weak complementary fuzzy graph.

Hence G is weak-isomorphic with $\overline{G}$.

So, there exists a weak isomorphism, $h : S \to S$, a bijective mapping satisfying,

$\sigma(x) = \sigma(h(x)) \quad \forall\ x \in S$ and

$\mu(x, y) = \overline{\mu}(h(x), h(y)) \quad \forall\ x, y \in S$.

Using the definition of complement, in the above inequality,

$\mu(x,y) \leq \sigma(h(x)) \wedge \sigma(h(y)) - \mu(h(x),h(y))$

$$= \sigma(x) \wedge \sigma(y) - \mu(x,y)$$

$$\mu(x,y) + \mu(x,y) \leq \sigma(x) \wedge \sigma(y)$$

Taking summation,

$$\sum_{x \neq y} \mu(x,y) + \sum_{x \neq y} \mu(x,y) \leq \sum (\sigma(x) \wedge \sigma(y))$$

$$2\sum_{x \neq y} \mu(x,y) \leq \sum_{x \neq y} (\sigma(x) \wedge \sigma(y)) \text{ (since } S \text{ is a finite set)}$$

Hence, $\sum_{x \neq y} \mu(x,y) \leq \frac{1}{2} \sum_{x \neq y} (\sigma(x) \wedge \sigma(y))$.

Theorem: 2.4.5

Let G be a fuzzy graph. If $\mu(x,y) \leq \frac{1}{2}(\sigma(x) \wedge \sigma(y))$ for all x, y in S then G is a self weak complementary fuzzy graph.

Proof:

Consider the identity map $h : S \rightarrow S$, $\sigma(x) = \sigma(h(x)) \quad \forall \ x \in S$

By definition of $\overline{\mu}$,

$$\overline{\mu}(x,y) = \sigma(x) \wedge \sigma(y) - \mu(x,y) \quad \forall \ x, y \in S$$

$$\overline{\mu}(x,y) \geq \sigma(x) \wedge \sigma(y) - \frac{1}{2}(\sigma(x) \wedge \sigma(y))$$

$$= \frac{1}{2}(\sigma(x) \wedge \sigma(y))$$

$$\geq \mu(x,y) \quad \forall \ x, y \in S$$

i.e., $\mu(x,y) \leq \overline{\mu}(h(x), h(y)) \quad \forall \ x, y \in S$

Hence G is weak isomorphic with $\overline{G}$.

Therefore G is a self weak complementary fuzzy graph.

Remark: 2.4.6

(i) When G is co-weak isomorphic with $\overline{G}$, then G is self complementary fuzzy graph.

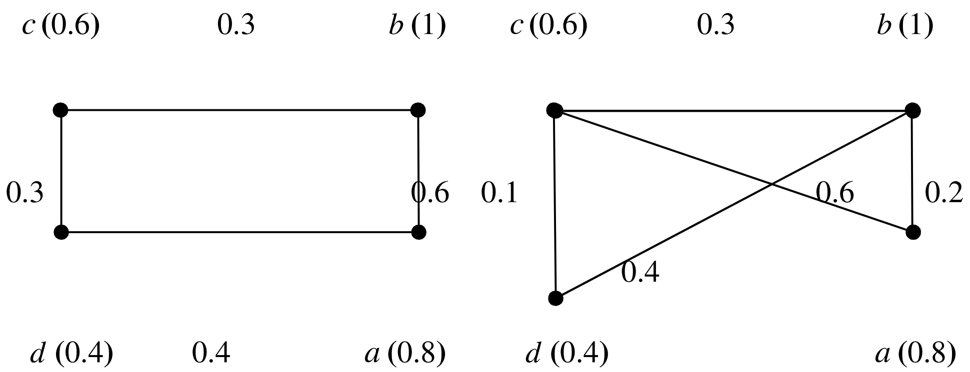
(ii) When G is a self weak complementary fuzzy graph

Order (G) = Order ($\overline{G}$) and

Size (G) = Size ($\overline{G}$).

But the converse of above is not true.

Consider the graph given below,



$(a)\ G:(\sigma,\mu)$ $(b)\ \overline{G}:(\sigma,\overline{\mu})$

Fig. 13 Fuzzy graph not self complementary.

Here we have,

Order (G) = Order ($\overline{G}$) = 2.8

Size (G) = Size ($\overline{G}$) = 1.6

But G is not a self weak complementary fuzzy graph.

Chapter - III

CHAPTER III

ISOMORPHISM ON FUZZY HYPERGRAPHS

SECTION: 3.1

HYPERGRAPHS - BASIC PROPERTIES

Definition: 3.1.1

A **homomorphism** of fuzzy hypergraphs $h : H \to H'$ is a map $h : X \to X'$ which satisfies,

$$\wedge \sigma_i(x) \leq \wedge \sigma_i'(h(x)) \quad \forall x \in X \text{ and}$$

$$\mu(x_1, x_2, x_3, \ldots.. x_r) \leq \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X .$$

Definition: 3.1.2

A **weak isomorphism** $h : H \to H'$ is a map, $h : X \to X'$ which is a bijective homomorphism that satisfies, $\wedge \sigma_i(x) = \wedge \sigma_i'(h(x)) \quad \forall x \in X$.

Example: 3.1.3

Let $H : (X, \sigma_i, \mu)$ and $H' : (X', \sigma_i', \mu')$ be two fuzzy hypergraphs with underlying sets $X = \{a, b, c, d\}$ and $X' = \{a', b', c', d'\}$ where $E_1 = \{a, b, c\}$, $E_2 = \{c, d\}$ are the edges of H and $E_1' = \{a', c'\}, E_2' = \{a', b', d'\}$ are the edges of H' defined by

$\mu(E_1) = 1/6 \qquad \mu(E_2) = 1/6;$

$E_1 = \{(a, 1/4), (b, 1/3), (c, 1/5)\}$ $\qquad$ $E_2 = \{(c, 1/4), (d, 1/5)\}$;

$\mu'(E_1') = 1/6$ $\qquad$ $\mu'(E_2') = 1/5$;

$E_1' = \{(a', 1/4), (c', 1/3)\}$ $\qquad$ $E_2' = \{(a', 1/2), (b', 1/3), (d', 1/5)\}$.

Defining $h : X \to X'$ as $h(a) = d', h(b) = b', h(c) = a', h(d) = c'$

Incidence matrices are given as follows:

	E_1	E_2
a	$1/4$	0
b	$1/3$	0
c	$1/5$	$1/4$
d	0	$1/5$

	E_1'	E_2'
a'	$1/4$	$1/2$
b'	0	$1/3$
c'	$1/3$	0
d'	0	$1/5$

Here, $\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(h(x))$

$\therefore$ the map h is a weak isomorphism.

Definition: 3.1.4

A **co-weak isomorphism** $h : H \to H'$ is a map $h : X \to X'$ which is a bijective homomorphism that satisfies,

$$\mu(x_1, x_2, x_3, \ldots.. x_r) = \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X$$

Example: 3.1.5

Let $H:(X,\sigma_i,\mu)$ and $H':(X',\sigma_i',\mu')$ be two fuzzy hypergraphs with underlying sets $X=\{a,b,c,d\}$ and $X'=\{a',b',c',d'\}$ where

$E_1=\{(a,1/4),(b,1/4),(c,1/3)\}$

$E_2=\{(c,1/2),(d,1/3)\}$

$\mu(E_1)=1/5$ $\quad\quad$ $\mu(E_2)=1/6$; and

$E_2'=\{(a',1/3),(d',1/3)\}$

$E_1'=\{(a',1/2),(b',1/3),(c',1/4)\}$

$\mu'(E_1')=1/5$ $\quad\quad$ $\mu'(E_2')=1/6$;

Defining $h:X\to X'$ as $h(a)=c',h(b)=b',h(c)=a',h(d)=d'$

Incidence matrices are given as follows:

	E_1	E_2		E_1'	E_2'
a	$1/4$	0	a'	$1/3$	$1/2$
b	$1/4$	0	b'	0	$1/3$
c	$1/3$	$1/2$	c'	0	$1/4$
d	0	$1/3$	d'	$1/3$	0

Here,

$\mu(x_1, x_2, x_3, \ldots.. x_r) = \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r))$,

$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_1 \subset X$

$\therefore$ the map h is a co-weak isomorphism.

Definition: 3.1.6

An **isomorphism** $h : H \to H'$ is a map $h : X \to X'$ which is a bijective homomorphism that satisfies,

$\wedge \sigma_i(x) = \wedge \sigma_i'(h(x)) \quad \forall x \in X$ and

$\mu(x_1, x_2, x_3, \ldots.. x_r) = \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r))$,

$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X$.

Definition : 3.1.7

An **endomorphism** of a fuzzy hypergraph H is a homomorphism of H to itself .

Definition: 3.1.8

An **automorphism** of a fuzzy hypergraph H is a isomorphism of H to itself .

Remark: 3.1.9

(i) A weak isomorphism of fuzzy hypergraph preserves the weight of nodes but not necessarily the weights of the edges.

(ii) A co- weak isomorphism preserves the weights of edges but not necessarily the weights of the nodes.

(iii) An isomorphism preserves both the weights of the edges and the nodes.

(iv) When the two fuzzy hypergraphs H and H' are same the weak isomorphism between them becomes an isomorphism and similarly the co-weak isomorphism between them also becomes isomorphism.

(v) In crisp hypergraphs when two hypergraphs are isomorphic they are of same order. Also the same is true in the case of fuzzy hypergraphs.

Theorem: 3.1.10

For any two isomorphic fuzzy hypergraphs their order and size are same.

Proof :

If $h : H \rightarrow H'$ is an isomorphism between the fuzzy hypergraphs H & H' with the underlying sets X & X' respectively.

Then, $\wedge \sigma_i(x) = \wedge \sigma_i'(h(x)) \quad \forall\, x \in X$

$$\mu(x_1, x_2, x_3, \ldots.. x_r) = \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X .$$

(i) $p =$ order $(H) = \sum_{x\in X} \wedge_i \sigma_i(x) = \sum_{x\in X} \wedge_i \sigma_i'(h(x)) =$ order (H')

(ii) $q =$ size $(H) = \sum_{E_i \subset X} \mu(E_i) = \sum_{E_i \subset X'} \mu'(E_i') =$ size (H')

Corollary: 3.1.11

Converse of the above theorem need not be true.

We prove this by an example.

Example: 3.1.12

Consider the fuzzy hypergraphs H and H' with the underlying sets X and X' as $X = \{a, b, c, d\}$ and $X' = \{a', b', c', d'\}$ respectively.

Incidence matrices are given as follows

	E_1	E_2	E'_1	E'_1
a	$1/5$	$1/6$	$1/5$	$1/5$
b	0	$1/5$	$1/6$	$1/3$
c	$1/3$	$1/5$	$1/3$	$1/5$
d	$1/2$	0	$1/2$	0

$\mu(E_1) = 1/5$ $\quad\mu(E_2) = 1/6$;

$\mu'(E'_1) = 1/6$ $\quad\mu'(E'_2) = 1/5$

$p = \text{Order}(H) = \sum_{x \in X} \wedge_i \sigma_i(x) = 1/6 + 1/5 + 1/5 + 1/2 = 16/15$

$\text{Order}(H') = \sum_{x \in X} \wedge_i \sigma'_i(h(x)) = 1/5 + 1/6 + 1/5 + 1/2 = 16/15$

$\therefore$ order (H) = order (H')

$q = \text{size}(H) = \sum_{E_i \subset X} \mu(E_i) = 1/5 + 1/6 = 11/30$

$\text{size}(H') = \sum_{E_i \subset X'} \mu'(E'_i) = 1/6 + 1/5 = 11/30$

$\therefore$ size (H) = size (H')

But, $\wedge \sigma_i(x) \neq \wedge \sigma'_i(h(x)) \quad \forall\, x \in X$ and

$$\mu(x_1, x_2, x_3, \ldots.. x_r) \neq \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_1 \subset X.$$

$\therefore$ order and size are same but H is not isomorphic to H'.

Remark: 3.1.13

If the fuzzy hypergraphs are weak isomorphic then their orders are same. But the Fuzzy hypergraphs of same order need not be weak isomorphic.

We prove this by the following example.

Example: 3.1.14

Consider the fuzzy hypergraphs H and H' with the underlying sets X and X' As $X = \{ a,b,c,d \}$ and $X' = \{ a',b',c',d' \}$ respectively.

Incidence matrices are given as follows.

	E_1	E_1	E_1	E_1
a	$1/5$	$1/7$	$1/7$	$1/6$
b	0	$1/5$	0	$1/6$
c	$1/3$	$1/6$	$1/4$	$1/5$
d	$1/2$	0	$1/2$	0

$$p = \text{order}\,(H) = \sum_{x\in X} \wedge_i \sigma_i(x) = 1/7 + 1/5 + 1/6 + 1/2 = 106/105$$

$$\text{order}\,(H') = \sum_{x\in X} \wedge_i \sigma'_i(h(x)) = 1/7 + 1/6 + 1/5 + 1/2 = 106/105$$

$\therefore$ order (H) = order (H')

But,

$$\wedge\, \sigma_i(x) \neq \wedge\, \sigma'_i(h(x)) \quad \forall\, x \in X$$

$\therefore$ The order is same. But they are not weak isomorphic.

Remark: 3.1.15

If the fuzzy hypergraphs are co- weak isomorphic, their sizes are same. But the fuzzy hypergraphs of same size need not be co-weak isomorphic.

We prove this by the following example.

Example: 3.1.16

Consider the fuzzy hypergraphs H and H' with the underlying sets X and X' As $X = \{ a, b, c, d \}$ and $X' = \{ a', b', c', d' \}$ respectively.

Incidence matrices are given as follows

	E_1	E_2
a	$1/5$	$1/7$
b	0	$1/6$
c	$1/3$	$1/5$
d	$1/2$	0

	E_1'	E_1'
a'	0	$1/6$
b'	$1/7$	$1/4$
c'	$1/2$	$1/5$
d'	$1/3$	0

$\mu(E_1) = 1/5$ $\qquad \mu(E_2) = 1/7$;

$\mu'(E_1') = 1/7$ $\qquad \mu'(E_2') = 1/5$

$$q = \text{size}(H) = \sum_{E_i \subset X} \mu(E_i) = 1/5 + 1/7 = 12/35$$

$$\text{size}(H') = \sum_{E_i \subset X'} \mu'(E_i') = 1/7 + 1/5 = 12/35$$

$\therefore$ size (H) = size (H')

But,

$$\mu(x_1, x_2, x_3, \ldots.. x_r) \neq \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_1 \subset X.$$

$\therefore$ The size is same. But they are not co-weak isomorphic.

Definition: 3.1.17

Let $H : (X, \sigma_i, \mu)$ be a fuzzy hypergraphs. The **degree of a vertex** is defined as

$$d(x_1) = \sum \mu(x_1, x_2, \ldots., x_r) \quad \text{for } x_1 \neq x_2 \neq x_3 \ldots\ldots \neq x_r.$$

Theorem: 3.1.18

If H and H' are isomorphic fuzzy hypergraphs then the degrees of their nodes are preserved.

Proof:

Let $h : X \to X'$ be an isomorphism of fuzzy hypergraphs H onto H'.

By the definition of isomorphism,

$$\mu(x_1, x_2, x_3, \ldots.. x_r) = \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X.$$

$$d(x_i) = \sum \mu(x_1, x_2, \ldots., x_r) \text{ for } x_i \neq x_r$$

$$= \sum \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r))$$

$$= d[h(x_i)].$$

Corollary: 3.1.19

Converse of the above theorem need not be true.

Proof:

Consider the following incidency matrices.

	E_1	E_2		E_1'	E_2'
a	$1/2$	$1/3$	a'	$1/3$	$1/2$
b	$1/4$	$1/5$	b'	$1/5$	$1/4$

$d(a) = 1/2 + 1/3$ $\qquad d(a') = 1/3 + 1/2$

$d(b) = 1/4 + 1/5$ $\qquad d(b') = 1/5 + 1/4$

$d(a) = d(a')$ & $d(b) = d(b')$

$\mu(E_1) = 1/4$, $\mu(E_2) = 1/5$; $\mu'(E_1') = 1/5$, $\mu'(E_2') = 1/4$

Here, $\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(h(x))\ \forall\ x \in X$

$\mu(x_1, x_2, x_3, \ldots.. x_r) \neq (h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r))$,

$$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_1 \subset X .$$

Here degrees of the nodes are preserved but edges are not preserved.

Remark:3.1.20

The degree of a vertex is measured only by adding the weights of the edges incident with that vertex. But fuzzy hypergraphs preserving the degree of the vertices need not be co-weak isomorphic.

Example: 3.1.21

Consider the fuzzy hypergraphs H and H' with the underlying sets X and X' As $X = \{ a, b, c, d \}$ and $X' = \{ a', b', c', d' \}$ respectively.

Incidence matrices are given as follows

	E_1	E_1		E_1	E_1
a	$1/6$	$1/6$	a'	0	$1/3$
b	0	$1/3$	b'	$1/3$	0
c	$1/3$	0	c'	$1/3$	0
d	$1/6$	$1/6$	d'	0	$1/3$

$\mu(E_1) = 1/6$ $\qquad \mu(E_2) = 1/6$

$\mu'(E_1') = 1/3$ $\qquad \mu'(E_2') = 1/3$

In the above hypergraphs each vertex is of degree $1/3$, but those hypergraphs are neither co-weak nor weak isomorphic hypergraphs.

SECTION: 3.2

ISOMORPHISMS AND WEAK ISOMORPHISMS OF FUZZY HYPERGRAPHS

Theorem: 3.2.1

Isomorphism between fuzzy hypergraphs is an equivalence relation.

Proof:

Let $H:(X,\sigma_i,\mu)$, $H':(X',\sigma_i',\mu')$, $H'':(X'',\sigma_i'',\mu'')$ be fuzzy hypergraphs with underlying sets X, X' and X'' respectively.

(i)Reflexive

Consider the identity map $h: X \to X$, $h(x) = x \quad \forall\, x \in X$

This h is a bijective map satisfying ,

$\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(h(x)) \quad \forall\, x \in X$ and

$$\mu(x_1, x_2, x_3, \ldots.. x_r) = \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall\, \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X.$$

Hence h is an isomorphism of the fuzzy hypergraph to itself.

Therefore it satisfies reflexive relation.

(ii)Symmetric:

Let $h: X \to X'$ be an isomorphism of H and H',then h is a bijective map

$h(x) = x' \;\; \forall\, x \in X$. (1)

Then h is a bijective map satisfying,

$\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(h(x)) \;\forall\, x \in X$ and

$$\mu(x_1, x_2, x_3, \ldots.. x_r) = \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall\, \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X. \quad (2)$$

Since h is bijective by (1),

$h^{-1}(x') = x \quad \forall\, x' \in X'$

$\wedge [\sigma_i (h^{-1}(x'))] = \wedge \, \sigma_i'(x') \quad \forall \; x' \in X'$ and

$\mu (h^{-1}(x_1'), h^{-1}(x_2'), h^{-1}(x_3'), \ldots . h^{-1}(x_r')) = \mu' (x_1, x_2, x_3, \ldots \ldots x_r)$,

$$\forall \{ x_1, x_2, x_3, \ldots . x_r \} = E_i' \subset X \qquad (3)$$

Hence we get a 1-1, onto map $h^{-1} : X' \to X$ which is an isomorphism from H' to H.

$H \cong H' \Rightarrow H' \cong H$.

(iii) Transitive:

Let $h : X \to X'$ and $g : X' \to X''$ be an isomorphism of fuzzy hypergraphs H onto H' and H' onto H'' respectively.

Then $g \circ h$ is a 1-1, onto map from $X \to X''$ where $(g \circ h)(x) = g(h(x)) \quad \forall \; x \in X$

As $h : X \to X'$ is an isomorphism, $h(x) = x' \quad \forall \; x \in X$

$\wedge \, \sigma_i (x) = \wedge \, \sigma_i' (h(x)) \; \forall \; x \in X$

$\mu (x_1, x_2, x_3, \ldots . x_r) = \mu' (h(x_1), h(x_2), h(x_3), \ldots \ldots h(x_r))$,

$$\forall \{ x_1, x_2, x_3, \ldots . x_r \} = E_i \subset X . \qquad (4)$$

$\wedge \, \sigma_i (x) = \wedge \, \sigma_i' (x') \; \forall \; x \in X$

$\mu (x_1, x_2, x_3, \ldots . x_r) = \mu' (x_1', x_2', x_3', \ldots \ldots x_r')$,

$$\forall \{ x_1, x_2, x_3, \ldots . x_r \} = E_i \subset X . \qquad (5)$$

As $g : X' \to X''$ is an isomorphism, $g(x') = x'' \quad \forall \; x' \in X'$

$$\wedge \, \sigma_i' (x') = \wedge \, \sigma_i'' (g(x')) \; \forall \; x' \in X' \qquad (6)$$

$$\mu'(x_1', x_2', x_3', \ldots\ldots x_r') = \mu''(g(x_1'), g(x_2'), g(x_3'), \ldots\ldots g(x_r'))$$

$$\forall \{x_1', x_2', x_3', \ldots\ldots, x_r'\} = E_i' \subset X' \qquad (7)$$

From (4) & (6) and using $h(x) = x', \quad \forall\, x \in X$

$$\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(x') = \wedge\, \sigma_i''(g(x')) \quad \forall\, x' \in X'$$

$$= \sigma_i''(g(h(x))) \quad \forall\, x \in X$$

From (5) & (7),

$$\mu(x_1, x_2, x_3, \ldots\ldots x_r) = \mu'(x_1', x_2', x_3', \ldots\ldots x_r'),$$

$$\forall \{x_1, x_2, x_3, \ldots\ldots x_r\} = E_i \subset X.$$

$$= \mu''(g(x_1'), g(x_2'), g(x_3'), \ldots\ldots g(x_r'))$$

$$\forall \{x_1', x_2', x_3', \ldots\ldots, x_r'\} = E_i' \subset X'$$

$$= \mu''(g(h(x_1)), g(h(x_2)), g(h(x_3)), \ldots\ldots g(h(x_r)))$$

$$\forall \{x_1, x_2, x_3, \ldots\ldots x_r\} = E_i \subset X.$$

Therefore $g \circ h$ is an isomorphism between H and H''.

Hence isomorphism between fuzzy hypergraphs is an equivalence relation.

Theorem: 3.2.2

Weak isomorphism between fuzzy hypergraphs satisfies the partial order relation.

Proof:

Let $H : (X, \sigma_i, \mu)$, $H' : (X', \sigma_i', \mu')$, $H'' : (X'', \sigma_i'', \mu'')$ be fuzzy hypergraphs with underlying sets X, X' and X'' respectively.

(i) Reflexive:

Consider the identity map $h : X \to X$ such that $h(x) = x \quad \forall\, x \in X$

This h is a bijective map satisfying,

$$\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(h(x))\ \forall\ x \in X \text{ and}$$

$$\mu(x_1, x_2, x_3, \ldots.. x_r) \leq \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall\ \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X.$$

Hence h is a weak isomorphism of the fuzzy hypergraph to itself.

Therefore H is weak isomorphic to itself.

(ii) Anti symmetric:

Let h be a weak isomorphism between H and H' and g be a weak isomorphism between H' and H

i.e., $h: X \to X'$ is a bijective map

$h(x) = x'\ \forall\ x \in X$ satisfying,

$\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(h(x))\ \forall\ x \in X$ and

$$\mu(x_1, x_2, x_3, \ldots.. x_r) \leq \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall\ \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X. \qquad (1)$$

$g: X' \to X$ is a bijective map satisfying

$g(x') = x,\ \forall\ x' \in X'$ satisfying

$\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(g(x'))\ \forall\ x' \in X'$

$$\mu'(x_1', x_2', x_3', \ldots.., x_r') \leq \mu(g(x_1'), g(x_2'), g(x_3'), \ldots.., g(x_r'))$$

$$\forall\ \{x_1', x_2', x_3', \ldots\ldots, x_r'\} = E_i' \subset X' \qquad (2)$$

The inequality (1) and (2) holds good on the finite sets X and X' only when H and H'

Have the same number of edges and the corresponding edges have the same weight.

Hence H and H' are identical.

(ii) Transitive:

Let $h : X \rightarrow X'$ and $g : X' \rightarrow X''$ be weak isomorphism of fuzzy hypergraphs X onto X' and X' onto X'' respectively.

Then $g \circ h$ is a 1-1, onto map from $X \rightarrow X''$ where $(g \circ h)(x) = g(h(x)) \quad \forall\ x \in X$

Given h is a weak isomorphism, $h(x) = x', \quad \forall\ x \in X$ (3)

$$\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(h(x))\ \forall\ x \in X$$

$$\mu(x_1, x_2, x_3, \ldots.. x_r) \leq \mu'(h(x_1), h(x_2), h(x_3), \ldots\ldots h(x_r)),$$

$$\forall\ \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X. \quad (4)$$

Similarly as g is a weak isomorphism from $X' \rightarrow X''$, $g(x') = x''$, $\forall\ x' \in X'$

$$\wedge\, \sigma_i'(x') = \wedge\, \sigma_i''(g(x'))\ \ \forall\ x' \in X' \quad (5)$$

$$\mu'(x_1', x_2', x_3', \ldots.. x_r') \leq \mu''(g(x_1'), g(x_2'), g(x_3'), \ldots\ldots g(x_r')),$$

$$\forall\ \{x_1', x_2', x_3', \ldots.. x_r'\} = E_i' \subset X'. \quad (6)$$

From the above, we have

$$h(x) = x', \quad \forall\ x \in X$$

$$\wedge\, \sigma_i(x) = \wedge\, \sigma_i'(x') = \wedge\, \sigma_i''(g(x')) \quad \forall\ x' \in X'$$

$$= \wedge\, \sigma_i''(g(h(x))) \quad \forall\ x \in X$$

From (3), (4), (5) and (6)

$$\mu(x_1, x_2, x_3, \ldots.. x_r) \leq \mu'(x_1', x_2', x_3', \ldots.. x_r'),$$

$$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X .$$

$$\leq \mu''(g(x_1'), g(x_2'), g(x_3'), \ldots\ldots g(x_r')),$$

$$\forall \{x_1', x_2', x_3', \ldots.. x_r'\} = E_i' \subset X' .$$

$$= \mu''(g(h(x_1)), g(h(x_2)), g(h(x_3)), \ldots\ldots g(h(x_r))),$$

$$\forall \{x_1, x_2, x_3, \ldots.. x_r\} = E_i \subset X .$$

Therefore $g \circ h$ is a weak isomorphism between H and H''.

i.e., weak isomorphism satisfies transitivity.

Hence weak isomorphism between fuzzy hypergraphs is a partial order relation.

Chapter - IV

CHAPTER IV

ISOMORPHISM ON HIGHLY IRREGULAR FUZZY GRAPH

SECTION: 4.1

HIGHLY IRREGULAR FUZZY GRAPHS – BASIC PROPERTIES

Definition: 4.1.1 [Nagoor Gani, 2012]

A **homomorphism** of highly irregular fuzzy graphs G and G' , $h : G \to G'$ is a map $h : V \to V'$ such that $h(u) = u'$, which satisfies

$\sigma(u) \leq \sigma'(h(u)) \quad \forall\, u \in V$ and

$\mu(u, v) \leq \mu'(h(u), h(v)) \quad \forall\, u, v \in V$.

Definition: 4.1.2 [Nagoor Gani, 2012]

A **weak isomorphism** of highly irregular fuzzy graph G and G', $h : G \to G'$ is a map $h : V \to V'$ such that $h(u) = u'$, $\forall\, u \in V$, which is a bijective homomorphism that satisfies, $\sigma(u) = \sigma'(h(u)) \quad \forall\, u \in V$.

Definition: 4.1.3 [Nagoor Gani, 2012]

A **co-weak isomorphism** of highly irregular fuzzy graph G and G', $h : G \to G'$ is a map $h : V \to V'$ such that $h(u) = u'$, $\forall\, u \in V$, which is a bijective homomorphism that satisfies, $\mu(u, v) = \mu'(h(u), h(v)) \quad \forall\, u, v \in V$.

Definition: 4.1.4 [Nagoor Gani, 2012]

An **isomorphism** of highly irregular fuzzy graphs $h : G \to G'$ is a map $h : V \to V'$ which is bijective that satisfies,

$$\sigma(u) = \sigma'(h(u)) \quad \forall\, u \in V \quad \text{and}$$

$$\mu(u,v) = \mu'(h(u),h(v)) \quad \forall\, u,v \in V\,.$$

It is denoted by $G \cong G'$.

Theorem: 4.1.5

The complement of highly irregular fuzzy graph need not be highly irregular.

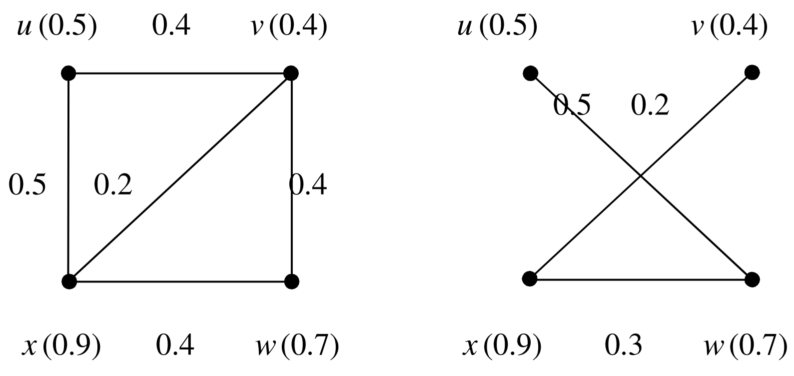
Proof:

To every vertex, the adjacent vertices with distinct degrees or the non-adjacent vertices with distinct or same degrees may happen to be adjacent vertices with same degrees.

This contradicts the definition of highly irregular fuzzy graphs.

The following example illustrates that the complement of highly irregular fuzzy graph need not be highly irregular.

Example: 4.1.6



$G : (\sigma, \mu)$ $\qquad\qquad$ $\overline{G} : (\sigma, \overline{\mu})$

Figure 1

From the Figure 1,

$d(u) = \sum_{u \neq v} \mu(u,v) = 0.9 \qquad d(v) = \sum_{u \neq v} \mu(u,v) = 1$

$d(w) = \sum_{u \neq v} \mu(u,v) = 0.8 \qquad d(x) = \sum_{u \neq v} \mu(u,v) = 1.1$

i.e., G is highly irregular because every vertex of G is adjacent to vertices with distinct degrees.

But,

$d(u) = \sum_{u \neq v} \overline{\mu}(u,v) = 0.5 \qquad d(v) = \sum_{u \neq v} \overline{\mu}(u,v) = 0.2$

$d(w) = \sum_{u \neq v} \overline{\mu}(u,v) = 0.8 \qquad d(x) = \sum_{u \neq v} \overline{\mu}(u,v) = 0.5$

i.e., $\overline{G}$ is not highly irregular because the degrees of the adjacent vertices of w are the same.

Theorem: 4.1.7

Let $G : (\sigma, \mu)$ and $G' : (\sigma', \mu')$ be two highly irregular fuzzy graphs. G and G' are isomorphic if and only if, their complements are isomorphic, but the complements need not be highly irregular.

Proof:

Assume $G \cong G'$

$\Rightarrow$ there exists a bijective map $h : V \to V'$ such that $h(u) = u' \quad \forall\, u \in V$,

satisfying $\sigma(u) = \sigma'(h(u))$ and (1)

$\mu(u,v) = \mu'(h(u), h(v))$

By definition of complement of a fuzzy graph,

$$\overline{\mu}(u,v) = \sigma(u) \wedge \sigma(u) - \mu(u,v) \quad \forall\, u,v \in V$$

$$\overline{\mu}(u,v) = \sigma'(h(u)) \wedge \sigma'(h(v)) - \mu'(h(u),h(v)) \quad \forall\, u,v \in V$$

$$= \overline{\mu'}(h(u),h(v) \quad \forall\, u,v \in V \qquad (2)$$

(1) and (2) $\Rightarrow$ $\overline{G} \cong \overline{G'}$

Conversely, assume that $\overline{G} \cong \overline{G'}$.

(*i.e*) There is a bijective map $g : V \rightarrow V'$ such that $g(u) = u' \quad \forall\, u \in V$

satisfying $\sigma(u) = \sigma'(g(u) \quad \forall\, u \in V$ and (3)

$$\overline{\mu}(u,v) = \overline{\mu'}(g(u), g(v)) \quad \forall\, u,v \in V$$

Using the definition of complement of a fuzzy graph,

$$\sigma(u) \wedge \sigma(u) - \mu(u,v) = \sigma'(g(u)) \wedge \sigma'(g(u)) - \mu'(g(u), g(v))$$

$$\Rightarrow \mu(u,v) = \mu'(g(u), g(v)) \quad \forall\, u,v \in V \qquad (4)$$

From (3) and (4), it is proved that $G \cong G'$.

The following example illustrates the above theorem.

Example: 4.1.8

Let $G : (\sigma, \mu)$ and $G' : (\sigma', \mu')$ be two highly irregular fuzzy graphs defined by

$\sigma(u) = 0.6 = \sigma'(u')$ $\qquad$ $\mu(u,v) = 0.6 = \mu'(u',v')$

$\sigma(v) = 0.8 = \sigma'(v')$ $\qquad$ $\mu(v,w) = 0.6 = \mu'(v',w')$

$\sigma(w) = 1 = \sigma'(w')$ $\qquad$ $\mu(x,w) = 0.5 = \mu'(x',w')$

$\sigma(x) = 1 = \sigma'(x')$ $\quad\quad\quad$ $\mu(u, x) = 0.6 = \mu'(u', x')$

The complements are defined by ,

$\overline{\mu}(u, w) = 0.6 = \overline{\mu'}(u', w')$ $\quad\quad\quad$ $\overline{\mu}(v, x) = 0.8 = \overline{\mu'}(v', x')$

$\overline{\mu}(v, w) = 0.2 = \overline{\mu'}(v', w')$ $\quad\quad\quad$ $\overline{\mu}(x, w) = 0.5 = \overline{\mu'}(x', w')$.

In this example, $G \cong G'$ and $\overline{G} \cong \overline{G'}$.

But, $d(u) = \sum_{u \neq v} \mu(u,v) = 1.2 = d(u')$ $\quad\quad$ $d(v) = \sum_{u \neq v} \mu(u,v) = 1.2 = d(v')$

$d(w) = \sum_{u \neq v} \mu(u,v) = 1.1 = d(w')$ $\quad\quad$ $d(x) = \sum_{u \neq v} \mu(u,v) = 1.1 = d(x')$

i.e., G and G' are highly irregular because every vertex of G and G' are adjacent to vertices with distinct degrees.

But, $d(u) = \sum_{u \neq v} \overline{\mu}(u,v) = 0.6 = d(u')$ $\quad\quad$ $d(v) = \sum_{u \neq v} \overline{\mu}(u,v) = 1 = d(v')$

$d(w) = \sum_{u \neq v} \overline{\mu}(u,v) = 1.3 = d(w')$ $\quad\quad$ $d(x) = \sum_{u \neq v} \overline{\mu}(u,v) = 1.3 = d(x')$

i.e., $\overline{G}$ and $\overline{G'}$ are not highly irregular because the degrees of the adjacent vertices of v and v' are the same.

$\therefore$ The complements are not highly irregular fuzzy graph.

Theorem: 4.1.9

Let G and G' be two highly irregular fuzzy graphs. If G is weak isomorphic with G', then $\overline{G'}$ is weak isomorphic with $\overline{G}$, but the complements need not be highly irregular.

Proof:

Assume that G is weak isomorphic with G'.

Then there is a bijective map $h : V \to V'$ such that $h(u) = u' \quad \forall\, u \in V$,

Satisfying, $\sigma(u) = \sigma'(h(u)) \quad \forall\, u \in V$ and (1)

$$\mu(u,v) \le \mu'(h(u), h(v)) \quad \forall\, u, v \in V \tag{2}$$

Since $h : V \to V'$ is a bijective, $h^{-1} : V' \to V$ exists $\forall\, u' \in V'$.

$\Rightarrow$ there is an $u' \in V'$ such that $h^{-1}(u') = u$ (3)

Using (3) in (1),

$$\sigma(h^{-1}(u')) = \sigma'(u') \quad \forall\, u' \in V' \tag{4}$$

Using the definition of complement of a fuzzy graph,

$$\overline{\mu}(u,v) = \sigma(u) \wedge \sigma(u) - \mu(u,v) \quad \forall\, u, v \in V \tag{5}$$

We have

$$\overline{\mu}(h^{-1}(u'), h^{-1}(v')) = \sigma(h^{-1}(u')) \wedge \sigma(h^{-1}(v')) - \mu(h^{-1}(u'), h^{-1}(v'))$$

$$= \sigma'(u') \wedge \sigma'(v') - \mu(u,v)$$

$$\ge \sigma'(u') \wedge \sigma'(v') - \overline{\mu}(u',v') \quad \forall\, u', v' \in V'$$

i.e., $\overline{\mu'}(u',v') \le \overline{\mu}(h^{-1}(u'), h^{-1}(v')) \quad \forall\, u', v' \in V'$ (6)

From (1) and (6) it is proved that, $\overline{G'}$ is weak isomorphic with $\overline{G}$.

Example: 4.1.10

Let $G : (\sigma, \mu)$ and $G' : (\sigma', \mu')$ be two highly irregular fuzzy graphs defined by

$\sigma(u) = 0.8 \qquad \mu(u,v) = 0.4$

$\sigma(v) = 0.9 \qquad \mu(v,w) = 0.5$

$\sigma(w) = 1 \qquad \mu(x,w) = 0.4$

$\sigma(x) = 0.9$ $\mu(u, x) = 0.8$

$\sigma'(u') = 0.8$ $\mu'(u', v') = 0.5$

$\sigma'(v') = 0.9$ $\mu'(v', w') = 0.6$

$\sigma'(w') = 1$ $\mu'(x', w') = 0.5$

$\sigma'(x') = 0.9$ $\mu'(u', x') = 0.8$

The complements G and G' are defined by ,

$\overline{\mu}(u, v) = 0.4$ $\overline{\mu'}(u', v') = 0.3$

$\overline{\mu}(v, w) = 0.4$ $\overline{\mu'}(v', w') = 0.3$

$\overline{\mu}(u, w) = 0.8$ $\overline{\mu'}(u', w') = 0.8$

$\overline{\mu}(v, x) = 0.9$ $\overline{\mu'}(v', x') = 0.9$

$\overline{\mu}(x, w) = 0.5$ $\overline{\mu'}(x', w') = 0.4$

From the example,

$\sigma(u) = 0.8 = \sigma'(u')$ $\sigma(v) = 0.9 = \sigma'(v')$

$\sigma(w) = 1 = \sigma'(w')$ $\sigma(x) = 0.9 = \sigma'(x')$

i.e., $\sigma(u) = \sigma'(h(u))$ and

$\mu(u, v) = 0.4 \leq \mu'(u', v') = 0.5$ $\mu(v, w) = 0.5 \leq \mu'(v', w') = 0.6$,

$\mu(x, w) = 0.4 \leq \mu'(x', w') = 0.5$ $\mu(u, x) = 0.8 = \mu'(u', x') = 0.8$

i.e., $\mu(u, v) \leq \mu'(h(u), h(v))$ and also,

$\overline{\mu'}(u', v') = 0.3 \leq \overline{\mu}(u, v) = 0.4$ $\overline{\mu'}(v', w') = 0.3 \leq \overline{\mu}(v, w) = 0.4$,

$\overline{\mu'}(u', w') = 0.8 \leq \overline{\mu}(u, w) = 0.8 \quad \overline{\mu'}(v', x') = 0.9 \leq \overline{\mu}(v, x) = 0.9$,

$\overline{\mu'}(x', w') = 0.4 \leq \overline{\mu}(x, w) = 0.5$

i.e., $\overline{\mu'}(u, v) \leq \overline{\mu}(u, v)$

$\therefore G$ is weak isomorphic with G' and $\overline{G'}$ is weak isomorphic with $\overline{G}$.

But ,

$d(u) = \sum_{u \neq v} \overline{\mu}(u, v) = 1.2 \qquad d(v) = \sum_{u \neq v} \overline{\mu}(u, v) = 1.7$

$d(w) = \sum_{u' \neq v'} \overline{\mu}(u, v) = 1.7 \qquad d(x) = \sum_{u' \neq v'} \overline{\mu}(u, v) = 1.4$

$d(u') = \sum_{u \neq v} \overline{\mu}(u', v') = 1.1 \qquad d(v') = \sum_{u \neq v} \overline{\mu}(u', v') = 1.5$

$d(w') = \sum_{u' \neq v'} \overline{\mu}(u', v') = 1.5 \qquad d(x') = \sum_{u \neq v} \overline{\mu}(u', v') = 1.3$

i.e., $\overline{G}$ and $\overline{G'}$ are not highly irregular because the degrees of the adjacent vertices of u and u' are the same.

$\therefore \overline{G}$ and $\overline{G'}$ are not highly irregular.

Theorem: 4.1.11

Let G and G' be two highly irregular fuzzy graphs. If G is co-weak isomorphic with G', then there exists a homomorphism between $\overline{G}$ and $\overline{G'}$, but the complements need not be highly irregular.

Proof:

Assume that G is co-weak isomorphic to G'.

Then $h : V \to V'$ is a bijective map such that $h(u) = u'$ satisfying,

$$\sigma(u) \leq \sigma'(h(u)) \quad \forall\, u \in V \quad \text{and} \tag{1}$$

$$\mu(u,v) = \mu'(h(u), h(v)) \quad \forall\, u, v \in V \tag{2}$$

By the definition of complement of a fuzzy graph,

$$\overline{\mu}(u,v) = \sigma(u) \wedge \sigma(u) - \mu(u,v) \quad \forall\, u, v \in V \tag{3}$$

we get, $\overline{\mu}(u,v) = \sigma'(h(u)) \wedge \sigma'(h(v)) - \mu'(h(u), h(v))$

$$= \overline{\mu'}(h(u), h(v)) \quad \forall\, u, v \in V \tag{4}$$

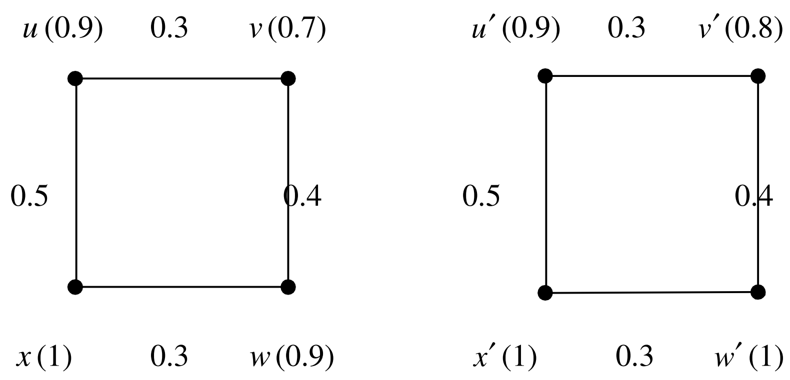
From (1) and (4), h is a bijective homomorphism between $\overline{G}$ and $\overline{G'}$.

Proposition: 4.1.12

If there is a co-weak isomorphism between two highly irregular fuzzy graphs G and G', then there need not be a co-weak isomorphism between $\overline{G}$ and $\overline{G'}$ and the complements need not be highly irregular.

The following example illustrates the above result.

Example: 4.1.13



$G : (\sigma, \mu)$ $\qquad$ $G' : (\sigma', \mu')$

Figure 2(a)

Here it is seen that,

$$\sigma(u) \leq \sigma'(h(u)) \quad \text{and}$$

$$\mu(u,v) = \mu'(h(u),h(v)) \quad \forall\, u,v \in V$$

$\therefore$ G is co-weak isomorphism with G'.

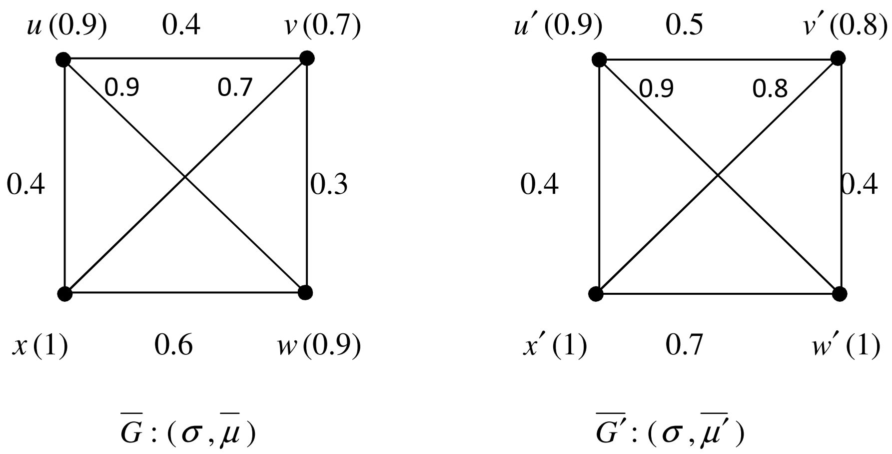


Figure 2(b)

But, $\sigma(u) \leq \sigma'(h(u))$ and

$$\mu(u,v) \neq \mu'(h(u),h(v)) \quad \forall\, u,v \in V$$

$\therefore$ $\overline{G}$ is co-weak isomorphism with $\overline{G'}$.

However there is a homomorphism between $\overline{G}$ and $\overline{G'}$.

i.e., $\sigma(u) \leq \sigma'(h(u))\ \forall\, u \in V$ and $\overline{\mu}(u,v) \leq \overline{\mu'}(h(u),h(v))\ \forall\, u,v \in V$.

And also,

$$d(u) = \sum_{u \neq v} \overline{\mu}(u,v) = 1.7 \qquad d(v) = \sum_{u \neq v} \overline{\mu}(u,v) = 1.4$$

$$d(w) = \sum_{u' \neq v'} \overline{\mu}(u,v) = 1.8 \qquad d(x) = \sum_{u' \neq v'} \overline{\mu}(u,v) = 1.7$$

$d\,(u') = \sum_{u \neq v} \overline{\mu}(u',v') = 1.8 \qquad d\,(v') = \sum_{u \neq v} \overline{\mu}(u',v') = 1.7$

$d\,(w') = \sum_{u' \neq v'} \overline{\mu}(u',v') = 2 \qquad d\,(x') = \sum_{u \neq v} \overline{\mu}(u',v') = 1.9$

$\therefore \overline{G}$ is not highly irregular because the degrees of the adjacent vertices of v and w are the same.

where $\overline{G'}$ is highly irregular because every vertex of $\overline{G'}$ is adjacent to vertices with distinct degrees.

Definition: 4.1.14 [Nagoor Gani, 2008]

A fuzzy graph G is said to be a **self complementary** if $G \cong \overline{G}$.

Definition: 4.1.15 [Nagoor Gani, 2008]

A fuzzy graph G is said to be a **self weak complementary** if G is weak isomorphic with $\overline{G}$.

Proposition: 4.1.16

A highly irregular fuzzy graph need not be self complementary.

The proof follows from theorem: 4.1.5

Example: 4.1.17

In this example,

$d\,(u) = \sum_{u \neq v} \mu(u,v) = .8 \qquad d\,(v) = \sum_{u \neq v} \mu(u,v) = 1$

$d\,(w) = \sum_{u \neq v} \mu(u,v) = 1 \qquad d\,(x) = \sum_{u \neq v} \mu(u,v) = .8$

i.e., G is highly irregular.

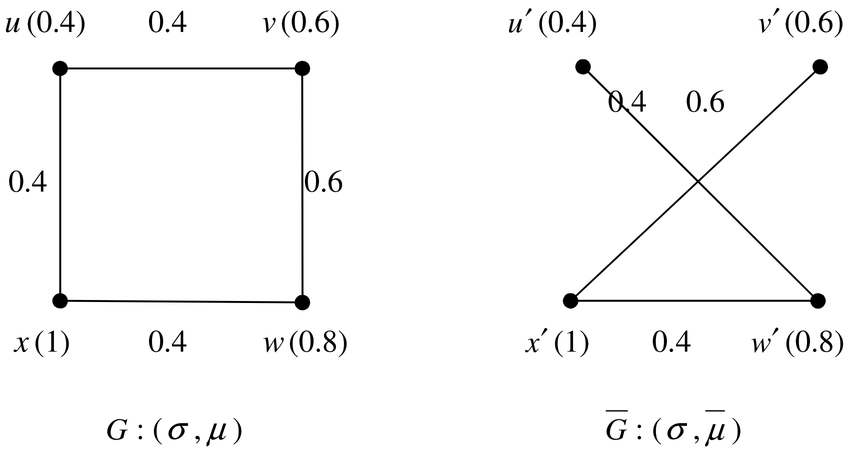


Figure 3

But, $\sigma(u) = \sigma(h(u))\ \forall\ u \in V$, and $\mu(u,v) \neq \overline{\mu}(h(u), h(v))\ \forall\ u, v \in V$.

$\therefore$ G is not a self complementary fuzzy graph.

The following two examples shows that a highly irregular fuzzy graph cannot necessarily be a self complementary fuzzy graph.

Example: 4.1.18

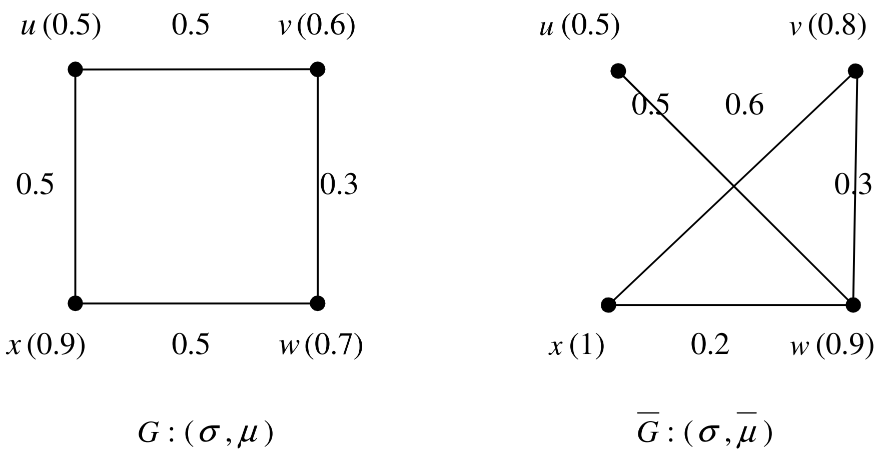


Figure 4

From the above example, we see that

$d(u) = \sum_{u \neq v} \mu(u,v) = 1$ $\qquad d(v) = \sum_{u \neq v} \mu(u,v) = 0.8$

$d(w) = \sum_{u \neq v} \mu(u,v) = 0.8$ $\qquad d(x) = \sum_{u \neq v} \mu(u,v) = 1$ and

$d(u) = \sum_{u \neq v} \overline{\mu}(u,v) = 0.5$ $\qquad d(v) = \sum_{u \neq v} \overline{\mu}(u,v) = 0.9$

$d(w) = \sum_{u' \neq v'} \overline{\mu}(u,v) = 1$ $\qquad d(x) = \sum_{u' \neq v'} \overline{\mu}(u,v) = 0.8$

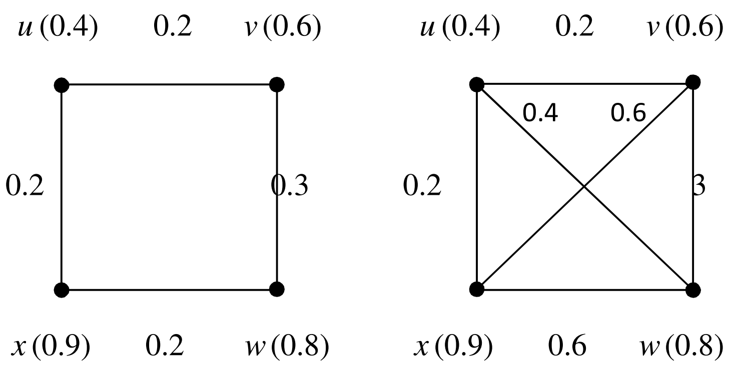
$\therefore$ G and $\overline{G}$ are highly irregular.

And also, $\sigma(u) \neq \sigma(h(u)) \quad \forall\, u \in V$ and

$$\mu(u,v) \neq \overline{\mu}(h(u), h(v)) \quad \forall\, u, v \in V.$$

Hence G is not a self weak complementary highly irregular fuzzy graph.

Example: 4.1.19



$G : (\sigma, \mu)$ $\qquad$ $\overline{G} : (\sigma, \overline{\mu})$

Figure 5

From the above example, we see that

$d(u) = \sum_{u \neq v} \mu(u,v) = 0.4 \qquad d(v) = \sum_{u \neq v} \mu(u,v) = 0.5$

$d(w) = \sum_{u \neq v} \mu(u,v) = 0.5 \qquad d(x) = \sum_{u \neq v} \mu(u,v) = 0.4$ and

$d(u) = \sum_{u \neq v} \overline{\mu}(u,v) = 0.8 \qquad d(v) = \sum_{u \neq v} \overline{\mu}(u,v) = 1.1$

$d(w) = \sum_{u' \neq v'} \overline{\mu}(u,v) = 1.3 \qquad d(x) = \sum_{u' \neq v'} \overline{\mu}(u,v) = 1.4$

$\therefore$ G and $\overline{G}$ are highly irregular.

And also, $\sigma(u) = \sigma(h(u)) \ \forall\, u \in V$ and

$$\mu(u,v) \leq \overline{\mu}(h(u), h(v)) \ \forall\, u, v \in V.$$

Hence G is a self weak complementary highly irregular fuzzy graph.

Theorem: 4.1.20

Let G be a self weak complementary highly irregular fuzzy graph, then

$$\sum_{u \neq v} \mu(u,v) \leq \frac{1}{2} \sum_{u \neq v} \sigma(u) \wedge \sigma(v).$$

Proof:

Let $G : (\sigma, \mu)$ be a self weak complementary and highly irregular fuzzy graph.

$\Rightarrow G$ is weak isomorphic with $\overline{G}$.

Therefore, there exists a bijective map $h : V \rightarrow V(\overline{V})$ with $h(u) = \overline{u} = u$, satisfying

$$\sigma(u) = \sigma(h(u)) \quad \forall\, u \in V \quad \text{and} \tag{1}$$

$$\mu(u,v) \leq \overline{\mu}(h(u), h(v)) \quad \forall\, u, v \in V \tag{2}$$

By the definition of complement of a fuzzy graph,

$$\overline{\mu}(u,v) = \sigma(u) \wedge \sigma(u) - \mu(u,v) \quad \forall\, u,v \in V \tag{3}$$

From (1) and (2),

$$\mu(u,v) \le \overline{\mu}(h(u),h(v)) = \sigma(h(u)) \wedge \sigma(h(v)) - \mu(h(u),h(v)) \quad \forall\, u,v \in V$$

$$\Rightarrow \mu(u,v) \le \sigma(u) \wedge \sigma(u) - \mu(u,v)$$

$$\Rightarrow \mu(u,v) + \mu(u,v) \le \sigma(u) \wedge \sigma(u)$$

$$2\mu(u,v) \le \sigma(u) \wedge \sigma(u)$$

$$2\sum_{u \ne v} \mu(\mathrm{u,v}) \le \sum_{u \ne v} (\sigma(u) \wedge \sigma(v))$$

$$\sum_{u \ne v} \mu(\mathrm{u,v}) \le \frac{1}{2} \sum_{u \ne v} (\sigma(u) \wedge \sigma(v)).$$

SECTION: 4.2

ISOMORPHIC PROPERTIES OF μ- COMPLEMENT OF HIGHLY IRREGULAR FUZZY GRAPH

Definition: 4.2.1 [Nagoor Gani, 2006]

Let $G:(\sigma,\mu)$ be a fuzzy graph. The μ **- complement of** G is defined as $G^{\mu}:(\sigma,\mu^{\mu})$ where

$$\mu^{\mu}(u,v) = \begin{cases} \sigma(u) \wedge \sigma(u) - \mu(u,v), & if \quad \mu(u,v) > 0 \\ 0 & , if \quad \mu(u,v) = 0 \end{cases}$$

Theorem: 4.2.2

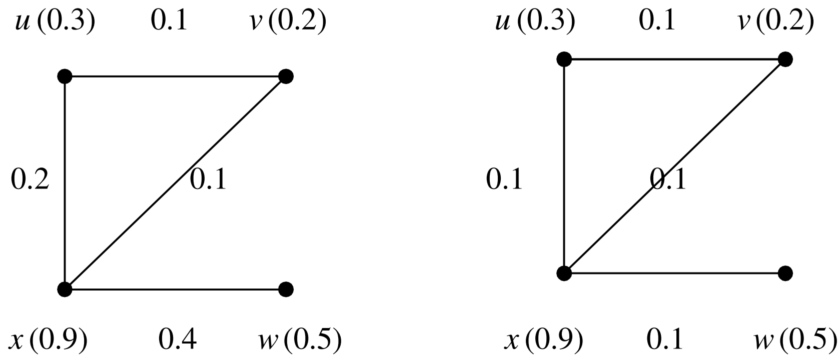
The μ - complement of a highly irregular fuzzy graph need not be highly irregular.

Proof:

The proof is similar to theorem: 4.1.5

The following example illustrates the above theorem.

Example: 4.2.3



$G:(\sigma,\mu)$ $\qquad$ $G^{\mu}:(\sigma,\mu^{\mu})$

Figure 6

$d(u)=\sum_{u\neq v}\mu(u,v)=0.3$ $\qquad$ $d(v)=\sum_{u\neq v}\mu(u,v)=0.2$

$d(w)=\sum_{u\neq v}\mu(u,v)=0.4$ $\qquad$ $d(x)=\sum_{u\neq v}\mu(u,v)=0.7$

Here, every vertex of G is adjacent to vertices with distinct degrees.

$\therefore$ G is a highly irregular fuzzy graph.

But , $d(u)=\sum_{u\neq v}\mu^{\mu}(u,v)=0.2$ $\qquad$ $d(v)=\sum_{u\neq v}\mu^{\mu}(u,v)=0.2$

$d(w)=\sum_{u\neq v}\mu^{\mu}(u,v)=0.1$ $\qquad$ $d(x)=\sum_{u\neq v}\mu^{\mu}(u,v)=0.3$

i.e., G^{μ} is not highly irregular because the degrees of the adjacent vertices of x are the same.

Theorem: 4.2.4

Let G and G' be two highly irregular fuzzy graphs. If G and G' are isomorphic, then μ- complement of G and G' are also isomorphic and vice versa, but the complements need not be highly irregular.

Proof:

The proof is similar to theorem: 4.1.7

Remark: 4.2.5

Let G and G' be two highly irregular fuzzy graphs. If G is weak isomorphic with G', then neither μ- complement of G is weak isomorphic with μ- complement of G' nor μ- complement of G' is weak isomorphic with μ- complement of G.

The following example illustrates the above result.

Example: 4.2.6

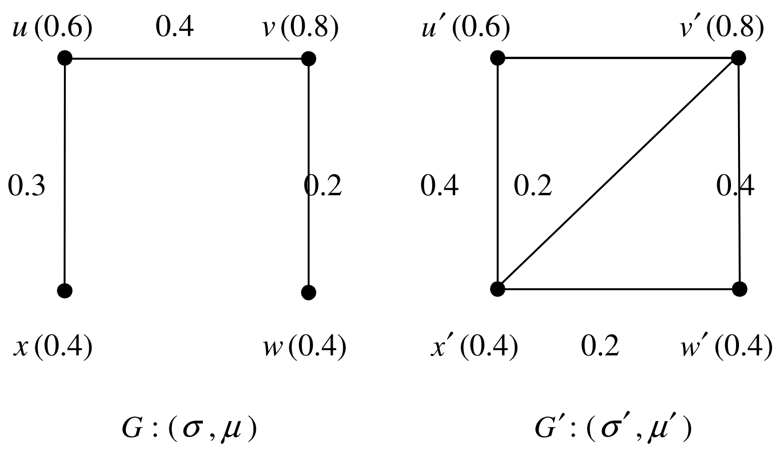


Figure 7

Here,

$$d\,(u) = \sum_{u \neq v} \mu(u,v) = 0.7 \qquad d\,(v) = \sum_{u \neq v} \mu(u,v) = 0.6$$

$$d(w) = \sum_{u \neq v} \mu(u,v) = 0.2 \qquad d(x) = \sum_{u \neq v} \mu(u,v) = 0.3 \quad \text{and also}$$

$$d(u') = \sum_{u \neq v} \mu'(u',v') = 0.9 \qquad d(v') = \sum_{u \neq v} \mu'(u',v') = 1.1$$

$$d(w') = \sum_{u \neq v} \mu'(u',v') = 0.6 \qquad d(x') = \sum_{u \neq v} \mu'(u',v') = 0.8$$

$\therefore$ it is easy to observe that G and G' are highly irregular fuzzy graphs.

And also, $\sigma(u) = \sigma'(h(u))\ \forall\ u \in V$ and

$$\mu(u,v) \leq \mu'(h(u),h(v))\ \forall\ u,v \in V.$$

i.e., G is weak isomorphic with G'.

But, $\sigma(u) = \sigma'(h(u))\quad \forall\ u \in V$ and

$$\mu^{\mu}(u,v) \neq \mu'^{\mu}(h(u),h(v))\quad \forall\ u,v \in V.$$

Also, $\sigma'(h(u)) = \sigma(u)\quad \forall\ u \in V$

$$\mu'^{\mu}(h(u),h(v)) \neq \mu^{\mu}(u,v)\quad \forall\ u,v \in V.$$

i.e., μ - complement of G is not weak isomorphic with μ - complement of G' and μ - complement of G' is not weak isomorphic with μ - complement of G.

Theorem: 4.2.7

If there is a co-weak isomorphism between G and G', then μ - complement of G and G' need not be co-weak isomorphic, but there can be a homomorphism between μ - complement of G and G'.

Proof:

The proof is similar to theorem: 4.1.11

The following example illustrates that there exists a homomorphism between μ-complement of G and G'.

Example: 4.2.8

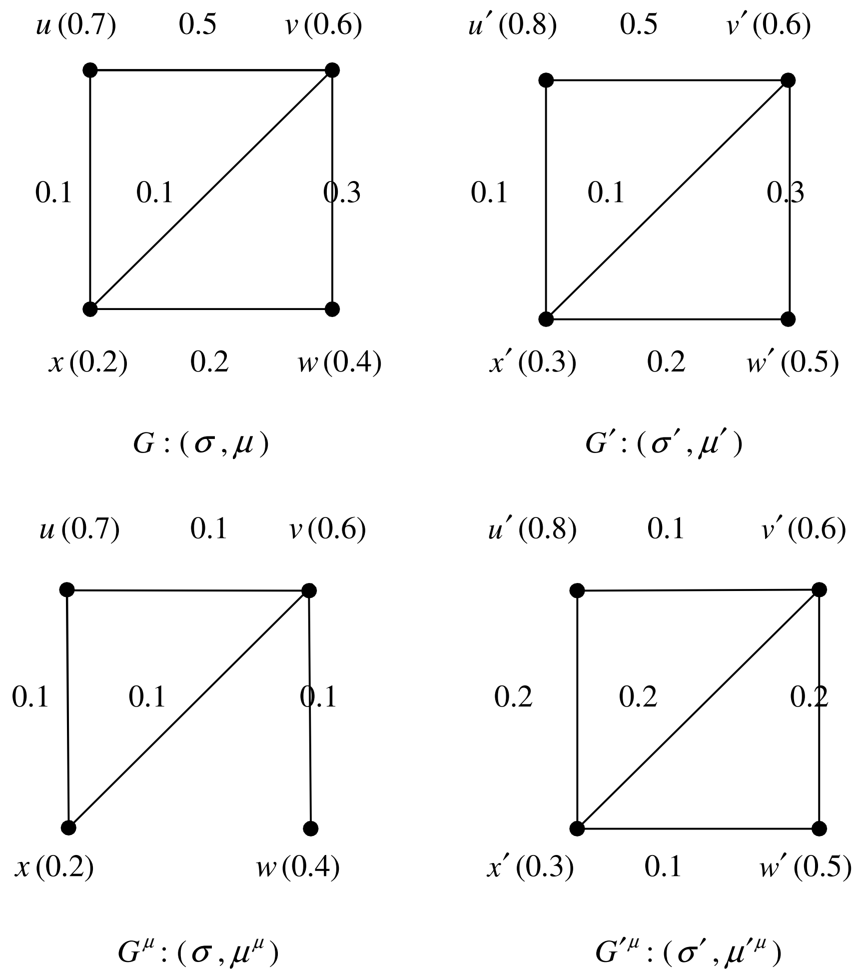


Figure 8

Here, $\sigma(u) \leq \sigma'(h(u)) \ \forall \ u \in V$ and

$$\mu\ (u,v) = \mu'(h(u), h(v)) \ \forall \ u, v \in \ V.$$

$\therefore$ G is co-weak isomorphic with G'.

But, $\sigma(u) \leq \sigma'(h(u)) \ \forall \ u \in V$ and

$$\mu(u,v) \neq \mu'(h(u),h(v)) \ \forall\ u,v \in V.$$

i.e., μ - complement of G is not co-weak isomorphic with μ - complement of G'.

Also, $\sigma(u) \leq \sigma'(h(u)) \ \forall\ u \in V$ and

$$\mu(u,v) \leq \mu'(h(u),h(v)) \ \forall\ u,v \in V.$$

i.e., there is a homomorphism between μ - complement of G and G'.

Definition: 4.2.9 [Nagoor Gani, 2009]

A fuzzy graph $G:(\sigma,\mu)$ is said to be a **self μ - complementary** fuzzy graph if $G \cong G^{\mu}$.

Example: 4.2.10

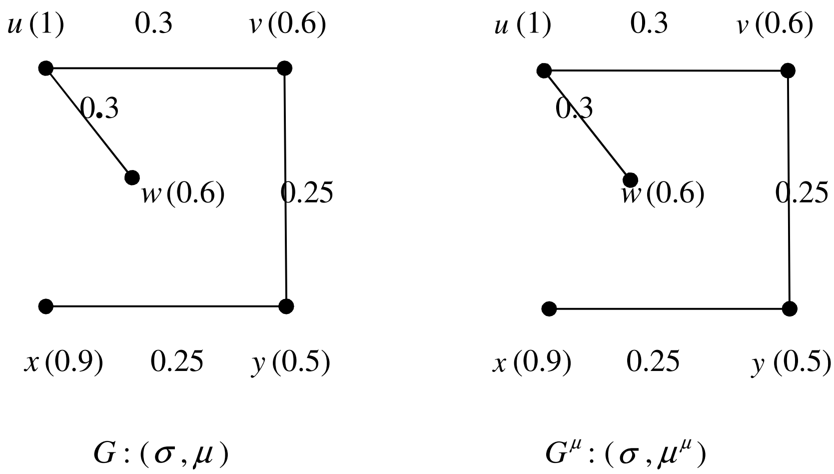


Figure 9

Here degrees of vertices for both the graph G and G^{μ} are given by,

$d(u) = 0.6$, $d(v) = 0.55$, $d(y) = 0.5$, $d(x) = 0.25$, $d(w) = 0.3$

Clearly G and G^{μ} are highly irregular because every very vertex of G and G^{μ} are adjacent to vertices with distinct degrees.

And, $\sigma(u) = \sigma'(h(u)) \ \forall \ u \in V$ and

$$\mu\ (u, v) = \mu'(h(u), h(v)) \ \forall \ u, v \in V.$$

i.e., G is isomorphic with G^{μ}.

Therefore G is self μ - complementary fuzzy graph.

Theorem: 4.2.11

Let G be a highly irregular and self μ - complementary fuzzy graph,

then $\sum_{u \neq v} \mu(u,v) = \frac{1}{2} \sum_{u \neq v} \sigma(u) \wedge \sigma(v)$.

Proof:

Let $G : (\sigma, \mu)$ be a self μ - complementary highly irregular fuzzy graph.

$\Rightarrow G \cong G^{\mu}$.

$\Rightarrow$ there exists a bijective map $h : V \rightarrow V$ such that

$$\sigma(u) = \sigma^{\mu}(h(u)) = \sigma(h(u)) \quad \forall \ u \in V \text{ and} \tag{1}$$

$$\mu(u, v) = \mu^{\mu}(h(u), h(u)) \quad \forall \ u, v \in V \tag{2}$$

If $(u, v) \in \mu^{*}$, then $\mu(u, v) > 0$.

By the definition of μ - complement of a fuzzy graph

$$\mu^{\mu}(u, v) = \begin{cases} \sigma(u) \wedge \sigma(u) - \mu(u, v), & if \quad \mu(u, v) > 0 \\ 0 & , if \quad \mu(u, v) = 0 \end{cases}$$

Therefore, $\mu^{\mu}(h(u), h(u)) = \sigma^{\mu}(h(u)) \wedge \sigma^{\mu}(h(v)) - \mu(h(u), h(u))$

$\mu(u, v) = \sigma(h(u) \wedge \sigma(h(u) - \mu(h(u), h(v))$ (from (1) and (2))

$\mu(u,v) + \mu(h(u), h(v)) = \sigma(u) \wedge \sigma(u)$

$2\,\mu(u,v) = \sigma(u) \wedge \sigma(u).$

Taking summation,

$$2\sum_{u \neq v} \mu(\mathrm{u}, \mathrm{v}) = \sum_{u \neq v} (\sigma(u) \wedge \sigma(v))$$

$$\Rightarrow \sum_{u \neq v} \mu(\mathrm{u}, \mathrm{v}) = \frac{1}{2} \sum_{u \neq v} (\sigma(u) \wedge \sigma(v)).$$

The following example illustrates the above theorem.

Example: 4.2.12

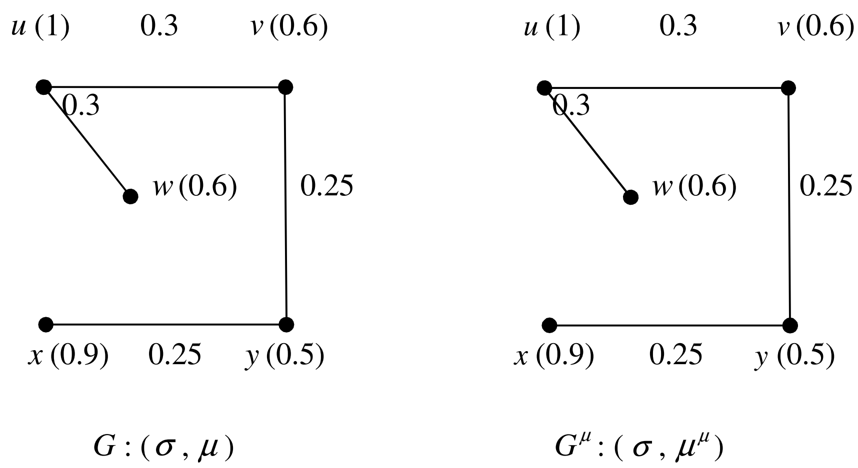


Figure 10

We have,

$$\sum_{u \neq v} \mu(u,v) = S(G) = 0.3 + 0.3 + 0.25 + 0.25 = 1.1$$

$$\frac{1}{2} \sum_{u \neq v} \sigma(u) \wedge \sigma(v) = \frac{1}{2}[0.6 + 0.5 + 0.5 + 0.6] = \frac{2.2}{2} = 1.1$$

Thus, $\sum_{u \neq v} \mu(u,v) = \frac{1}{2} \sum_{u \neq v} [\sigma(u) \wedge \sigma(v)]$.

The following example illustrates that, in a highly irregular fuzzy graph, even if

$\sum_{u \neq v} \mu(u,v) = \frac{1}{2} \sum_{u \neq v} [\sigma(u) \wedge \sigma(v)]$, G need not be self μ - complementary.

Example: 4.2.13

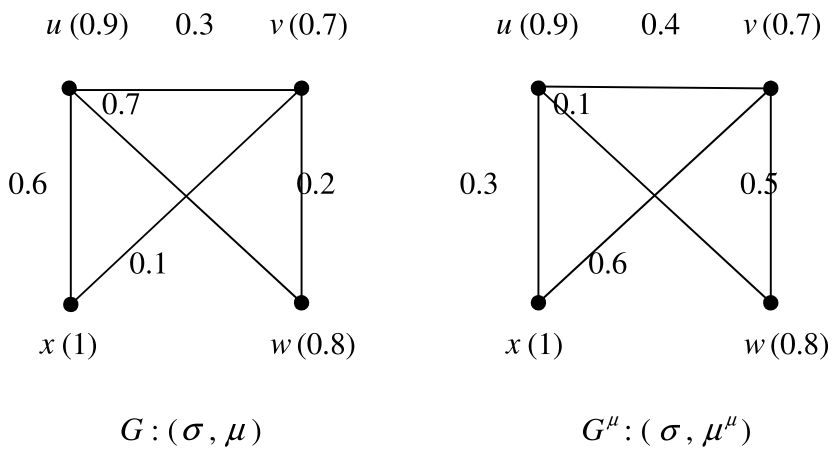


Figure 11

Here,

$d(u) = 1.6$, $d(v) = 1.2$, $d(w) = 0.9$, $d(x) = 0.7$ are the degrees of vertices in G .

And,

$d(u) = 0.8$, $d(v) = 1.5$, $d(w) = 0.6$, $d(x) = 0.9$ are the degrees of vertices in G^{μ}.

$\therefore$ G and G^{μ} are highly irregular fuzzy graphs.

But, $\sigma(u) = \sigma^{\mu}(h(u)) \quad \forall\ u \in V$ and

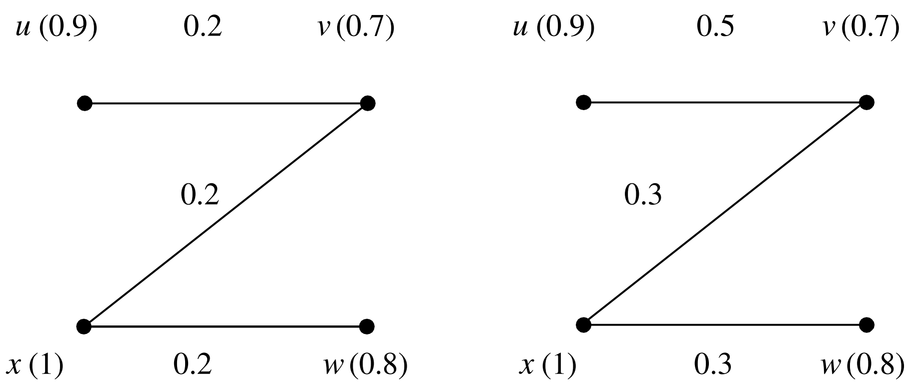
$\mu(u,v) \neq \mu^{\mu}(h(u), h(u)) \quad \forall\ u, v \in V$

i.e., G is not self μ - complement.

Definition: 4.2.14 [Nagoor Gani, 2009]

A fuzzy graph $G : (\sigma, \mu)$ is said to be a **self weak μ - complementary fuzzy graph** if G is weak isomorphic with G^{μ}.

Example: 4.2.15



$G : (\sigma, \mu)$ $\qquad$ $G^{\mu} : (\sigma, \mu^{\mu})$

Figure 12

Here,

$d(u) = 0.2$, $d(v) = 0.4$, $d(w) = 0.2$, $d(x) = 0.4$ are the degrees of vertices in G.

And,

$d(u) = 0.5$, $d(v) = 0.8$, $d(w) = 0.3$, $d(x) = 0.3$ are the degrees of vertices in G^{μ}.

$\therefore$ G and G^{μ} are highly irregular fuzzy graphs.

But, $\sigma(u) = \sigma^{\mu}(h(u)) \quad \forall\ u \in V$ and

$$\mu(u, v) \leq \mu^{\mu}(h(u), h(u)) \quad \forall\ u, v \in V$$

Thus, G is a self weak μ - complementary fuzzy graph.

Theorem: 4.2.16

Let G be highly irregular and self weak μ - complementary fuzzy graph,

then $\sum_{u \neq v} \mu(u,v) \leq \frac{1}{2} \sum_{u \neq v} \sigma(u) \wedge \sigma(v)$.

Proof:

Using the definition of self weak μ - complementary fuzzy graph, the proof is similar to Theorem: 4.2.11.

SECTION: 4.3

ISOMORPHIC PROPERTIES OF BUSY NODES AND FREE NODES IN HIGHLY IRREGULAR FUZZY GRAPH

Definition: 4.3.1 [Nagoor Gani, 2006]

A node "u" in a fuzzy graph is said to be a **busy node** if $\sigma(u) \leq d(u)$. Otherwise it is called as a **free node**.

Theorem: 4.3.2

If $G : (\sigma, \mu)$ and $G' : (\sigma', \mu')$ are two isomorphic highly irregular fuzzy graphs, then the busy nodes and free nodes are preserved under isomorphism.

Proof:

Let $G : (\sigma, \mu)$ and $G' : (\sigma', \mu')$ be two highly irregular fuzzy graphs and

let $h : G \to G'$ be an isomorphism between the highly irregular fuzzy graphs G and G' with the underlying sets V and V' respectively.

Then $\sigma(u) = \sigma'(h(u)) \quad \forall\, u \in V$ and

$\mu(u, v) = \mu'(h(u), h(v)) \quad \forall\, u, v \in V$

By theorem: 3.22 [Nagoor Gani, 2012],

The bijective mapping h preserves the degree of the vertex u.

$(i.e.,)\ d(u) = d(h(u))$.

If u is a busy node in G, then $\sigma(u) \leq d(u) \Rightarrow \sigma'(h(u)) \leq d(h(u))$.

Thus $h(u)$ is a busy node in G'.

If v is a free node in G, then $\sigma(v) > d(v) \Rightarrow \sigma'(h(v)) > d(h(v))$.

Thus $h(v)$ is a free node in G'.

Theorem: 4.3.3

Let $G:(\sigma,\mu)$ and $G':(\sigma',\mu')$ be two highly irregular fuzzy graphs, and let G be co-weak isomorphism with G'. Then the image of a free node in G is also a free node in G'.

Proof:

Let u be a free node in $G \Rightarrow \sigma(u) > d(u)$ (1)

Let $h: G \to G'$ be a co-weak isomorphism between G and G'.

Then, $\sigma(u) \leq \sigma'(h(u)) \quad \forall\ u \in V$ and (2)

$$\mu(u,v) = \mu'(h(u), h(v)) \quad \forall\ u, v \in V \tag{3}$$

From (1) and (2),

$\sigma'(h(u)) \geq \sigma(u) > d(u)$

Hence $\sigma'(h(u)) > d(u) = \sum_{u \neq v} \mu(u,v) = \sum_{u \neq v} \mu'(h(u),h(v)) = d(h(u))$

Therefore, $\sigma'(h(u)) > d(u) \Rightarrow h(u)$ is a free node in G'.

Remark: 4.3.4

Let $G:(\sigma,\mu)$ and $G':(\sigma',\mu')$ be two highly irregular fuzzy graphs. If G is co-weak isomorphism with G', Then the image of a busy node in G need not be a busy node in G'.

The following example illustrates the above result.

Example: 4.3.5

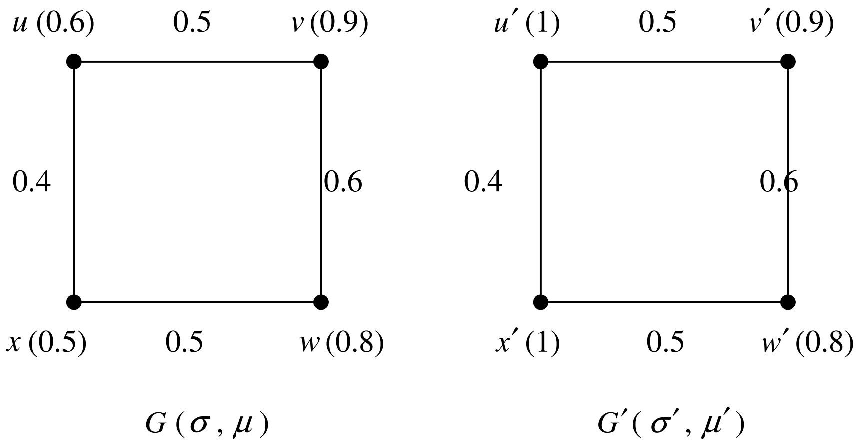


Figure 12

Here,

$$\sigma(u) \le \sigma'(h(u)) \quad \forall\ u \in V \text{ and}$$

$$\mu(u,v) = \mu'(h(u),h(v)) \quad \forall\ u,v \in V.$$

And also,

$$\sigma(u) = 0.6 < d(u) = 0.9 \quad , \quad \sigma(v) = 0.9 < d(v) = 1.1$$

$$\sigma(w) = 0.8 < d(w) = 1.1 \quad , \quad \sigma(x) = 0.5 < d(x) = 0.$$

$$\sigma'(u') = 1 > d(u') = 0.9 \quad , \quad \sigma'(v') = 0.9 < d(v') = 1.1$$

$$\sigma'(w') = 0.8 < d(w') = 1.1 \;,\; \sigma(x') = 1 > d(x') = 0.9$$

$\therefore$ G co-weak isomorphic with G'.

The busy nodes u & x in G are not busy in G'

Theorem: 4.3.6

Let $G:(\sigma,\mu)$ and $G':(\sigma',\mu')$ be two highly irregular fuzzy graphs and let G be weak isomorphism with G'. Then the image of a busy node in G is a busy node in G'.

Proof:

Let $h: G \to G'$ be a weak isomorphism between G and G'.

Then ,

$$\sigma(u) = \sigma'(h(u)) \quad \forall\, u \in V \text{ and} \tag{1}$$

$$\mu(u,v) \le \mu'(h(u), h(v) \quad \forall\, u, v \in V \tag{2}$$

Let $u \in V$ be a busy node in G.

$$(i.e.,)\, \sigma(u) \le d(u) \tag{3}$$

From (1) and (3)

$$\sigma'(h(u)) = \sigma(u) \le d(u) = \sum_{u\neq v}\mu(u,v) \le \sum_{u\neq v}\mu'(h(u),h(v)) = d(h(u))$$

Hence $\sigma'(h(u)) \le d(h(u))$

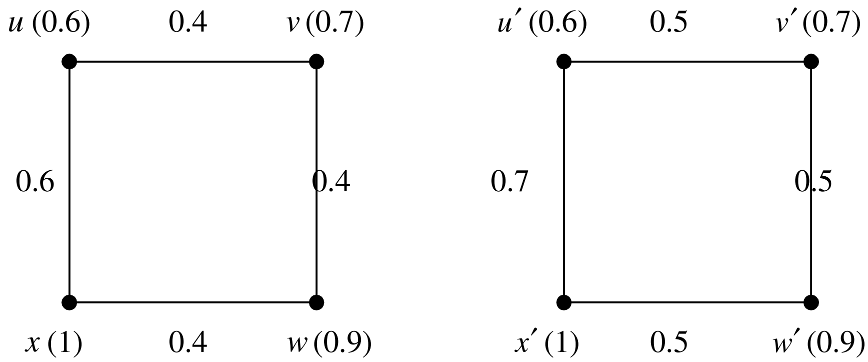
$\Rightarrow h(u)$ is a busy node in G'.

Remark: 4.3.7

Under weak isomorphism the image of a free node in G need not be a free node in G'.

This is illustrated in the following example.

Example: 4.3.8



$G(\sigma, \mu)$ $G'(\sigma', \mu')$

Figure 13

Here, $\sigma(u) = \sigma'(h(u)) \ \forall \ u \in V$ and

$$\mu(u, v) \le \mu'(h(u), h(v)) \ \forall \ u, v \in V.$$

And also, $\sigma(u) = 0.6 < d(u) = 1$ $\sigma(v) = 0.7 < d(v) = 0.8$

$\sigma(w) = 0.9 > d(w) = 0.8$ $\sigma(x) = 1 = d(x) = 1$

$\sigma'(u') = 0.6 < d(u') = 1.2$ $\sigma'(v') = 0.7 < d(v') = 1$

$\sigma'(w') = 0.9 < d(w') = 1$ $\sigma'(x') = 1 < d(x') = 1.2$

Though w is a free node in G, w' is a busy node in G', because $\sigma'(w') < d(w')$.

Summary and Conclusion

SUMMARY AND CONCLUSION

This is an introduction to one new branch of mathematics coined as 'Fuzzy Graph'. It is an outcome of the wedding of two smart branches of mathematics namely Graph Theory and Fuzzy mathematics. Graphs are data structures endowed with such an expressive power to make their use profitable in the most disparate area. The fuzzy graph theory as a generalization of Euler's graph theory was first introduced by Rosenfeld. The concepts of weak isomorphism and isomorphism between fuzzy graphs were introduced by K. R. Bhutani [1989].

The main aim of this work is to discover a road map between a few branches of mathematics by making use of Fuzzy graph. In fuzzy graphs and fuzzy hypergraphs, Isomorphism is proved to be an equivalence relation and weak isomorphism is proved to be a partial order relation. A necessary and then a sufficient condition for a fuzzy graph to be self weak complementary are studied. Some properties on isomorphism, weak isomorphism and co-weak isomorphism between highly irregular fuzzy graphs and their complements are established. Isomorphic properties of μ- complement, self μ- complement and self weak μ- complement of highly irregular fuzzy graph are brought out. Finally, some properties of isomorphism with respect to busy nodes and free nodes in highly irregular fuzzy graphs are also discussed.

For further study, it is expected that co-weak isomorphism between fuzzy graphs and fuzzy hypergraphs can be proved to be a partial order relation. The result in the thesis can be extended to Interval valued Fuzzy Graph, Intuitionistic Fuzzy Graph, Neutrosophic Fuzzy Theory.

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