A MATRIX METHOD FOR MARKOVIAN QUEUES

BY

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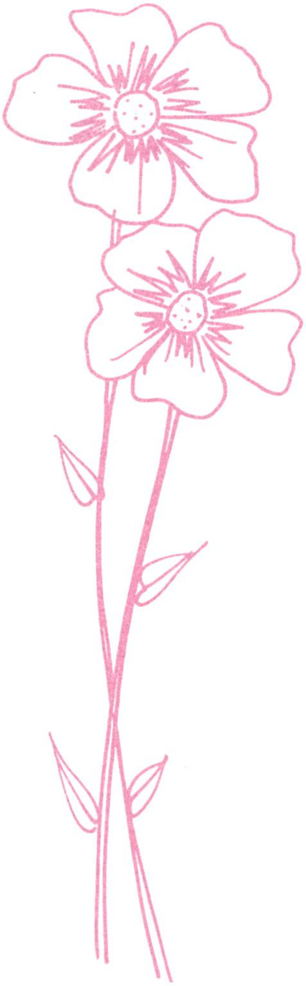

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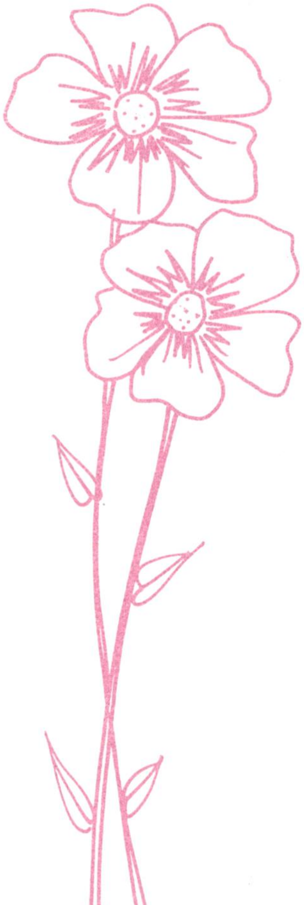

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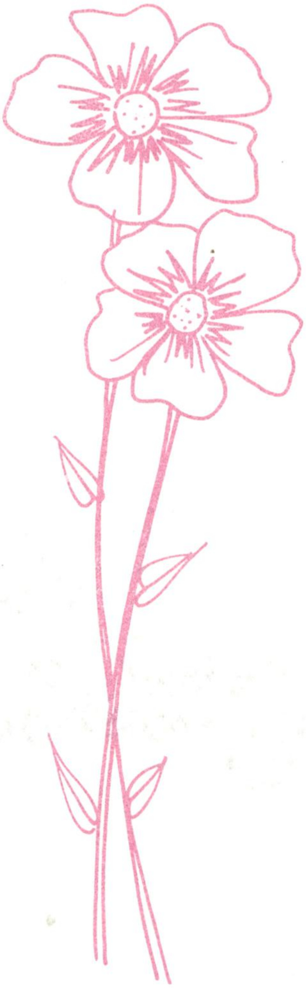


SYNOPSIS

SYNOPSIS

In Chapter I, notations, definitions and some preliminary results which are used in the dissertation are given.

In Chapter II, a computable matrix approach is developed to study and obtain the closed form solutions of a simple homogeneous birth and death process on the finite state space $\{-M, -M+1, \ldots, 0, 1, \ldots N\}$ where M and N are positive integers with the assumption that initially the system is in state 'i'.

In the following sections we have considered the various situations where the simple birth and death model is applicable.

(1) A particular simple birth-death process on finite state space in which the birth and death rate are independent of the population size.

(2) The double-ended queue.

(3) The multi-server Markovian queue.

(4) The queue with heterogeneous servers.

(5) The multi-server queue with balking and reneging.

In all cases analytic expressions for the transient state probabilities are obtained in terms of eigenvalues of a tridiagonal matrix.

In Section 1, a continuous time parameter simple birth and death process on finite state space $\{-M, -M+1, \ldots 0, 1, \ldots N\}$ where M and N are positive integers and λ and μ are the transition rates is considered. The transient solution $p_n(t)$ of the probability that there are n particle in the system at time 't' is obtained.

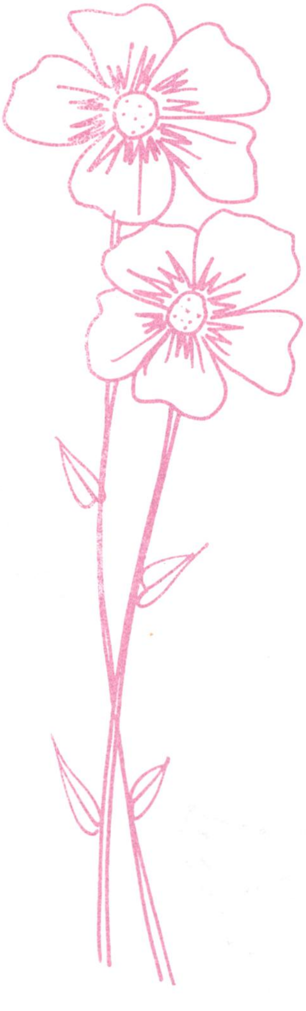
In section 2, the concept of a double-ended queue introduced by Kendall in 1951 is discussed. An example of this type of queueing system is a taxi-stand where at times passengers queue up for taxis and at other times taxis wait for passengers. In this section we obtain a simple algebraic closed-form expression for the transient probabilities using the procedure explained in section 1. In fact such a queueing system is a very simple and straight forward particular case of the model of this chapter. It is shown that the steady state probabilities for M/M/1/N queue can also be obtained by taking $M = 0$ in the corresponding results of this section. Finally we calculate the expected number of units in both the queues for the steady state of this model.

In section 3, we consider a system consisting of C parallel counters. In this system the units arrive in a poisson stream with mean arrival rate λ and the service time distribution at each counter is negative exponential with the same parameter μ. The maximum number of units in the system is restricted to N. Here also we obtain a closed expression for $p(n, t)$ and mention some important parameters that can be obtained from the expression of $p_n(t)$ and check that when $c = 1$, this result coincide with that of M/M/1/N.

In the above sections we have discussed the multiserver queues with identical service rate at each counter.

In section 4, we study the model with two heterogeneous servers. We assume the interarrival and service time of first and second servers to be negative exponential distribution with parameter λ_1, μ_1 and μ_2 respectively with $\mu_1 > \mu_2$ which also implies that we are considering a modified queue discipline, that is the first arriving unit from amongst the initial number of units present at the start of the service joins the first counter for service and thereafter the arriving unit goes to the counter which it finds free.

In section 5, our objective is to find the transient effects of customer impatience upon the development of the waiting time in an M/M/C/N type queue. Impatience commonly takes two forms. The first is balking, that is reluctance of a customer to join a queue upon arrival, and the second is reneging which is reluctance of a customer to remain in the line after joining it and leaving the system without being served. Many researches discussed this problem but have obtained results only in the case of steady state situations. Since the transient solution is more appropriate in the management of real-life problem, here we discuss the transient behaviour after combining reneging and balking and also taking an arbitrary number 'i' of customers being present initially at the time t = 0.



INTRODUCTION

CHAPTER - I

INTRODUCTION

Queueing theory originated when a Danish Mathematician A.K.Erlang published in 1909 his pioneering paper "The theory of probabilities and telephone conversations" on the study of congestion of telephone traffic. His studies are now classics in queueing theory. Until about 1940 the development of this new branch of applied probability was directed by the needs encountered in the design of automatic telephone exchange.

A queueing system may be described as one having a service facility at which units of some kind arrive for service and where, whenever there are more units in the system than the service facility can handle simultaneously, a queue or waiting line is formed. These units take their turn for service according to a preassigned rule and after service they leave the system.

Generally a queueing system is characterized by the following:

(i) The input process
(ii) The queue discipline
(iii) The service mechanism

(i) The input describes the manner in which units arrive (either singly or in a group) and join the system. The interval between two consecutive arrivals is called the interarrival time.

(ii) Service mechanism : The time required for service unit (or a group in case of batch service) is called the service time.

(iii) The Queue discipline indicates the way in which the units form a queue and are served. They are,

(a) First come first served (FCFS)
(b) Last come first served (LCFS)
(c) Random ordering before service

A queueing system is studied under two different situations, namely, transient and steady state. The behaviour of a queueing process indexed by a time parameter generally depends upon time. However, under certain conditions after the system has been in operation for a sufficiently long time it settles down to a behaviour which is independent of time. This is known as steady-state (equilibrium) behaviour and is characterized by the steady-state distribution of queue length, waiting time, etc. The study of the temporal development of the process is important from practical as well as theoretical point of view. The solution obtained in this case is known as the transient solution. In practise, the queueing model may not reach the equilibrium position. Thus study the transient solution is very useful for any model.

For most of the models it is assumed that the system can accommodate any number of units. In practise this may seldom be the case. Thus it is very useful to consider the system with limited waiting space. Thus in our dissertation we concentrate on the transient solution for finite markovian model.

Several authors have analysed the single server queueing systems with finite waiting room.

(i) Giomor, Negri C. and Nobile, A.G. [6] have introduced to solvable model for a finite capacity queueing system.

(ii) Boxma O.J. [3] has determined a queueing model of finite and infinite source interaction. Here the joint distribution of queue length at the M/M/1 queue and finite source customer in the system is determined.

(iii) Lipper E.H. and Sengupta, B. [9] have discussed on assembly like queue with finite capacity, bounds asymptotics and approximations.

(iv) N.Narayan Bhat [11] have discussed the finite capacity assembly - like queues.

B.D.Sivazhan, K.H. Wang, O.P. Sharma and N.Ravichandran [13] have discussed the transient solution with finite calling population. They have analysed the transient solution for the machine repair model.

Most of the researcheses while discussing the queueing models, assume that the queue is empty initially. The initial state system empty or non empty affects the transient study of the system where as the steady state can remains independent of the initial conditiions. O.P. Sharma [15] in his paper starts the queue with arbitrary number of units at time $t = 0$ while discussing a simple homogenous birth and death process on the finite state space $\{-M, -M+1, \dots 0, 1, \dots N\}$ where M and N are positive integers.

Kendell in 1957 [8] have introduced the concept of a double ended queue. In 1982 Srivastava H.M. and Kashyap B.R. [12] have obtained the transient solution of this type of queue interms of summation of the integral of Bessel functions. O.P. Sharma [18] in his paper has given a simple algebraic closed form expression for the transient probabilities by using the procedure explained in his work.

Most of the multiserver waiting time problems talked in the literature assumes the servers to be identical. However this situation is not very realistic and can prevail only when the service process is highly mechanically controlled. In the case of human servers, they cannot be expected to work at the same rate.

Morse [10] seems to be the first to introduce the concept of heterogeneity in service. Satty [16] further discusses Morse's problem by designing the service rates μ_1 and μ_2 to the two branches.

Ancker C.J. and Gaffarin A.V. [1] have further investigated this problem by putting an upper limit N on the queue size. But they have obtained only steady state results. V.P. Singh [17], Godini, G [17] also discussed the queueing system with balking and two heterogenous servers. O.P. Sharma [18] has discussed a two heterogeneous servers model and obtained the transient solutions with finite waiting room capacity.

Many researchess such as Haight, F.B. [7] Finch, P.O. [5] and Barrer, D.V. [2] have studied the multiserver queue with balking and reneging but they have obtained results only in the case of steady state. O.P. Sharma [14] has discussed the transient behaviour after combining ranging and balking and also taking an arbitrary number 'i' of customers being present initially at the time t = 0.

Notation

A queueing process is described in the form A/B/X/Y/Z where;

A : indicates the inter arrival distribution

B : the service pattern as given by the probability distribution for service time

X : the number of parallel service channels

Y : the capacity of the system and

Z : the queue discipline

In most of the cases if the queue discipline is taken as FIFO, hence in the FIFO case the symbol Z may not be mentioned.

For example M/M/1/$\propto$ denotes the inter arrival and service time distribution are negative exponential distribution with one server and infinite queue capacity.

M/M/1/N denotes the interarrival and service time distribution are negative exponential distribution with one server and finite queue capacity.

Def. 1 : Finite Queue

In some queueing processes there is a physical limitation to the amount of waiting room. So that when the line reaches a certain length, no further customers are allowed to enter until space becomes available by a service completion. These are referred to as finite queue.

Def. 2 : Birth and Death Process

P_r {Number of births between t and t + h is k, given that the number of individuals at epoch t is n} is given by,

$$p(k, h/n, t) = \begin{cases} \lambda_n h + 0(h) & k = 1 \\ 0(h) & k \geq 2 \\ 1 - \lambda_n h + 0(h) & k = 0 \end{cases} \qquad (1)$$

The above holds for all $n \geq 0$; λ_0 may or may not be equal to zero. Here k is a non-negative integer which implies that there can only be an increase by k, that is only births are considered possible . In this case we shall further assume that

P_r { Number of deaths between t and t + h is k,
given that the number of individuals at epoch t is n }

is given by

$$q\ (k,\ h/n,\ t) = \begin{cases} \mu_n h + O(h) & ;\ k = 1 \\ O(h) & ;\ k \geq 2 \\ 1 - \mu_n h + O(h) & ;\ k = 0 \end{cases} \qquad (2)$$

The above holds for $n \geq k$; further $\mu_0 = 0$ with (1) and (2) we have, what is known as a birth and death process.

Def. 3 : Transient State

A system is said to be in transient state if its operating characteristics are dependent on time.

Def. 4 : Steady State

A system is said to be in steady state when its operating characteristics are independent of time. That is the number of arrivals during a certain interval becomes independent of time.

Def. 5 : Multi-Server

Queueing system may have several service channels to provide service. These service channels may be arranged in parallel or in series or as a more complex combination of both depending on the design of the service mechanism. In parallel channels a number of channels provide identical service facilities, so that several customers may be served simultaneously, when the system has a number of parallel servers, it is known as multiserver model.

Def. 6 : Balking

If a customer decides not to enter the queue upon arrival, he is said to have balked.

Result 7

In $M/M/1/\propto$ let K_q be the greatest length at which an arrival does not balk and has the same distribution for all arriving customers, say $F(n) = \Pr\{k_q \leq n\}$, So $F(n-1)$ is the balking distribution in the sense that it is the probability that the arrival refuses to join when n are in the system. An arriving customer joins the queue if $k_q \geq n$; that is

$$\begin{aligned} \Pr\{\text{arrival joins the queue}\} &= \Pr\{k_q \geq n\} \\ &= 1 - \Pr\{k_q < n\} \\ &= 1 - F(n-1) \\ &\qquad (n \geq 0,\ F(-1) \equiv 0) \end{aligned}$$

Let us define 1 - F(n) as G(n). So that G(n-1) is now the probability that an arrival joins the queue when there are n in the system. This process is still birth-death, but the arrival rate must now be adjusted to $\lambda_n = \lambda G(n-1)$.

In section 2.5, we discussed a balking model in which $\lambda_n = \lambda(1 - \frac{n}{N}),\ 0 \leq n \leq N.$

Result 8 : M/M/1 Reneging

Customers who tend to be impatient may not always be discouraged by excessive queue size, but may instead joint the queue to see how long their wait may become, all the time retaining the prenegative to renege if their estimate of their total wait is intolerable. Haight (op.cit) has proposed a single channel birth-death model where both reneging and the simple balking of the previous result exist, which gives rise to a reneging function r(n) defined by

$$r(n) = \lim_{\Delta t \to 0} [\Pr\{\text{unit reneges during } \Delta t \text{ when there are n customers in the system}\} / \Delta t]$$

$$[r(0) \equiv 0 \equiv r(1)]$$

This new process is still birth-death, but the death rate must now be adjusted to $\mu_n = \mu + r(n)$. In section 2.5, we discussed M/M/1 reneging model with

$$\mu_n = n\mu, \quad 0 \le n \le c - 1$$
$$= c\mu + (n - c)\gamma, \quad c \le n \le N$$

Def. 9 : Laplace Transform

Let f(t) be a function of a positive real variable t. Then the Laplace transform of f(t) is defined by

$F(s) = \int_0^\infty \exp(-st)\, f(t)\, dt$ for the range of value of s for which the integral exists.

Inverse Laplace Transform : 10

If F(s) is the Laplace transform of f(t), That is [illegible] $L\{f(t)\} = F(s)$, then f(t) is called the inverse Laplace transform of F(s).

Formula : 11

f(t)	L { f(t) }
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s - a}$

Result : 12

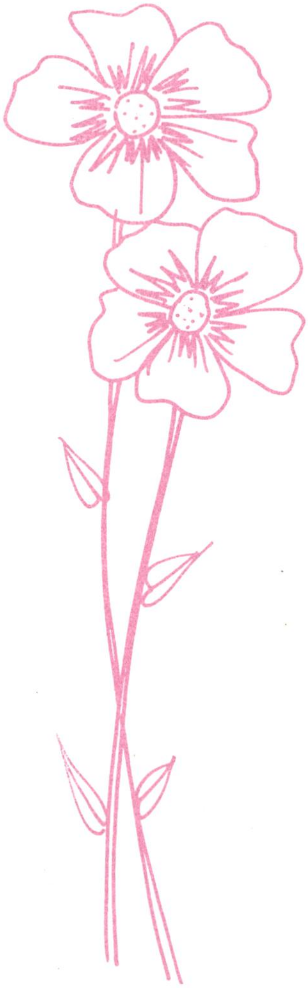
$g_n(\theta)$ be the determinant of the n x n matrix

$$\begin{bmatrix}
\lambda+\theta+\mu & -(\lambda\mu)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 \\
-(\lambda\mu)^{\frac{1}{2}} & \lambda+\theta+\mu & -(\lambda\mu)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\
0 & -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu+\theta & -(\lambda\mu)^{\frac{1}{2}} & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & \lambda+\mu+\theta & -(\lambda\mu)^{\frac{1}{2}} \\
0 & 0 & 0 & 0 & \cdots & -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu+\theta
\end{bmatrix}$$

$$= \theta(\theta+\lambda+\mu) \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \, (n-1-k)\, C_k \, (\lambda\mu)^k$$

$$(\theta+\lambda+\mu)^{n-1-2k} - \lambda\mu \sum_{k=0}^{[\frac{n-2}{2}]} (-1)^k \, (n-2-k)\, C_k$$

$$(\lambda\mu)^k \, (\theta+\lambda+\mu)^{n-2-2k}$$



A MATRIX - METHOD FOR MARKOVIAN QUEUES

CHAPTER - II

In this chapter we find the transient solution of a simple homogeneous birth-death process on the finite state space $\{-M, -M+1, \ldots 0, 1, \ldots N\}$ where M and N are positive integers.

To a particle is at state n, then the probability of the particle moving to the state $n+1$ in a small interval of time Δt is $\lambda_n \Delta t + 0(\Delta t)$ and to the state $n-1$ in the same interval is $\mu_n \Delta t + 0(\Delta t)$. Clearly $\lambda_N = 0$ and $\mu_{-M} = 0$.

Define $p(n, t) = \Pr\{$ the system is in the state n at the time $t > 0\}$,

$$-M \leq n \leq N$$

Then the difference equations are

$$\begin{aligned} p(-M, t+\Delta t) &= (1 - \lambda_{-M} \Delta t)\, p(-M, t) + \mu_{-M+1} \Delta t \\ &\quad p(-M+1, t) + 0(\Delta t) \\ p(n, t+\Delta t) &= (1 - (\lambda_n + \mu_n) \Delta t) + \lambda_{n-1} \Delta t\, p(n-1, t) \\ &\quad + \mu_{n+1} \Delta t\, p(n+1, t) + 0(\Delta t), \\ &\qquad -M+1 \leq n \leq N-1 \\ p(N, t+\Delta t) &= (1 - \mu_N \Delta t)\, p(N_1 t) + \lambda_{N-1} \Delta t\, p(N-1, t) \\ &\quad + 0(\Delta t) \end{aligned} \tag{2.1}$$

The above set of equations yields the following set of differential difference equations,

$$p'(-M, t) = -\lambda_{-M} P(-M, t) + \mu_{-M+1} p(-M+1, t)$$

$$p'(n, t) = -(\lambda_n + \mu_n) p(n, t) + \lambda_{n-1} p(n-1, t) + \mu_{n+1} p(n+1, t) \qquad -M+1 \leq n \leq N-1$$

$$p'(N, t) = -\mu_N p(N, t) + \lambda_{N-1} p(N-1, t)$$

Assuming that initially the particle is at position i, we have $p(i, 0) = 1$ and $p(n, 0) = 0 \quad \forall\, n \neq i$.

Taking Laplace transform of the above equations, we get

$$(\theta + \lambda_{-M}) \psi(-M, \theta) - \mu_{-M+1} \psi(-M+1, \theta) = p(-M,0)$$

$$(\theta + \lambda_n + \mu_n) \psi(n, \theta) - \lambda_{n-1} \psi(n-1, \theta) - \mu_{n+1} \psi(n+1, \theta) = p(n,0)$$

$$(\theta + \mu_N) \psi(n, \theta) - \lambda_{N-1} \psi(N-1, \theta) = p(N, 0)$$

(2.2)

where

$$\psi(n, \theta) = \int_0^\infty e^{-\theta t} p(n, t)\, dt$$

The above system of equations can be written as,

$$
\begin{bmatrix}
\theta + \lambda_{-M} & -\mu_{-M+1} & 0 & 0 & \cdots & 0 & 0 \\
-\lambda - M & \theta + \lambda_{-M+1} + \mu_{-M+1} & -\mu_{-M+2} & 0 & \cdots & 0 & 0 \\
0 & -\lambda_{-M+1} & \theta + \lambda_{-M+2} + \mu_{-M+2} & -\mu_{-M+3} & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & \theta + \lambda_{N-1} + \mu_{N-1} & -\mu_{N} \\
0 & 0 & 0 & 0 & \cdots & -\lambda_{N-1} & \theta + \mu_{N}
\end{bmatrix}
\begin{bmatrix}
\psi(-M, \theta) \\
\psi(-M+1, \theta) \\
\psi(-M+2, \theta) \\
\cdot \\
\cdot \\
\cdot \\
\psi(N-1, \theta) \\
\psi(N, \theta)
\end{bmatrix}
=
\begin{bmatrix}
p(-M, 0) \\
p(-M+1, 0) \\
p(-M+2, 0) \\
\cdot \\
\cdot \\
\cdot \\
p(N-1, 0) \\
p(N, 0)
\end{bmatrix}
\tag{2.3}
$$

which is of the form

$$A\,\psi = \delta$$

The matrix A can be transformed into the symmetric tri-diagonal form by the diagonal matrix

$$M = \text{Diagonal}\,(d_1,\ d_2,\ \ldots\ d_{M+N+1}) \text{ with } d_1 = 1,$$

$$d_r = \prod_{k=1}^{r-1} \left[\frac{\mu_{-M+k}}{\lambda_{-M+k-1}}\right]^{\frac{1}{2}}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{\mu_{-M+1}}{\lambda_{-M}}\right)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \prod\limits_{k=1}^{2} \left[\frac{\mu_{-M+k}}{\lambda_{-M+k-1}}\right]^{\frac{1}{2}} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \prod\limits_{k=1}^{M+N-1} \left[\frac{\mu_{-M+k}}{\lambda_{-M+c-1}}\right]^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \prod\limits_{k=1}^{M+N} \left[\frac{\mu_{-M+k}}{\lambda_{-M+c-1}}\right]^{\frac{1}{2}} \end{bmatrix}$$

$$A = \begin{bmatrix} \theta + \lambda_{-M} & -\mu_{-M+1} & 0 & 0 & \cdots & 0 & 0 \\ -\lambda_{-M} & \theta + \lambda_{-M+1} + \mu_{-M+1} & -\mu_{-M+2} & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_{-M+1} & \theta + \lambda_{-M+2} + \mu_{-M+2} & -\mu_{-M+3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \theta + \lambda_{N-1} + \mu_{N-1} & -\mu_N \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_{N-1} & \theta + \mu_N \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{\lambda_{-M}}{\mu_{-M+1}}\right)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \prod_{k=1}^{2} \left(\frac{\lambda_{-M+k-1}}{\mu_{M+k}}\right)^{\frac{1}{2}} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \prod_{k=1}^{M+N-1} \left(\frac{\lambda_{-M+k-1}}{\mu_{-M+k}}\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \prod_{k=1}^{M+N} \left(\frac{\lambda_{-M+k-1}}{\mu_{-M+k}}\right)^{\frac{1}{2}} \end{bmatrix}$$

$$MAM^{-1} = \begin{bmatrix} \theta + \lambda_{-M} & -(\lambda_{-M}\,\mu_{-M+1})^{\frac{1}{2}} & 0 & \cdots & 0 & \\ -(\lambda_{-M}\,\mu_{-M+1})^{\frac{1}{2}} & \theta + \lambda_{-M+1} + \mu_{-M+1} & -(\lambda_{-M+1}\,\mu_{-M+2})^{\frac{1}{2}} & \cdots & 0 & \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \theta + \lambda_{N-1} + \mu_{N-1} & -(\lambda_{N-1}\,\mu_N)^{\frac{1}{2}} \\ 0 & 0 & 0 & \cdots & -(\lambda_{N-1}\,\mu_N)^{\frac{1}{2}} & \theta + \mu_N \end{bmatrix}$$

which is of the form $\theta I + B$ where,

$$B = \begin{bmatrix} \lambda_{-M} & -(\mu_{-M+1}\lambda_{-M})^{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ -(\lambda_{-M}\mu_{-M+1})^{\frac{1}{2}} & \lambda_{-M+1} + \mu_{-M+1} & -(\lambda_{-M+1}\mu_{-M+1})^{\frac{1}{2}} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_{N-1} + \mu_{N-1} & -(\lambda_{N-1}\mu_N)^{\frac{1}{2}} \\ 0 & 0 & 0 & \cdots & (\lambda_{N-1}\mu_N)^{\frac{1}{2}} & \mu_N \end{bmatrix}$$

Now suppose that $A_r\,\theta$ and $B_r(\theta)$ are the bottom right and top left $(r \times r)$ square matrices, formed form $(\theta I + B)$ with $P_r(\theta)$ and $Q_r(\theta)$ as their respective determinants such that

$$P_{-1}(\theta) = 0 = Q_{-1}(\theta)$$

$$P_0(\theta) = 1 = Q_0(\theta)$$

$$P_1(\theta) = \theta + \mu_N$$

$$Q_1(\theta) = \theta + \lambda_{-M}$$

then $P_r(\theta)$ and $Q_r(\theta)$ satisfies the difference equations given by

$$P_r(\theta) - (\lambda_{N-r+1} + \mu_{N-r+1} + \theta)\, P_{r-1}(\theta) + \lambda_{N-r+1}\, \mu_{N-r+2}\, P_{r-2}(\theta) = 0$$

$$Q_r(\theta) - (\lambda_{-M+r-1} + \mu_{-M+r-1} + \theta)\, Q_{r-1}(\theta) + \lambda_{-M+r-2}\, \mu_{-M+r-1}\, Q_{r-2}(\theta) = 0$$

$$2 \leq r \leq M+N+1 \qquad (2.4)$$

and $|A| = |\theta I + B| = P_{M+N-1}(\theta) = Q_{M+N+1}(\theta)$

Let $(\theta I + B)^{-1} = \dfrac{(C_{ij})}{|\theta I + B|}$

Then the entries of the cofactor matrix (C_{ij}) are given by

$$C_{ij} = \prod_{r=j}^{i-1} (\lambda_{-M+r-1}\, \mu_{-M+r})^{\frac{1}{2}}\, Q_{j-1}(\theta)\, P_{M+N+1-i}(\theta) \qquad \text{if } j < i$$

$$= \prod_{r=1}^{j-1} (\lambda_{-M+r-1}\, \mu_{-M+r})^{\frac{1}{2}}\, Q_{i-1}(\theta)\, P_{M+N+1-j}(\theta), \qquad \text{if } i \leq j \leq M+N+1$$

$$= Q_{i-1}(\theta) + P_{M+N+1-i}(\theta) \qquad \text{if } i = j \qquad (2.5)$$

$$\psi = A^{-1}\, \delta$$

$$= M^{-1} (\theta I + B)^{-1}\, M\, \delta$$

$$= M^{-1}\, \frac{(C_{ij})}{|\theta I + B|}\, M\, \delta$$

Hence,

$$\begin{bmatrix} \psi(-M,\ \theta) \\ \psi(-M+1,\ \theta) \\ \cdot \\ \cdot \\ \cdot \\ \psi(i,\ \theta) \\ \cdot \\ \cdot \\ \cdot \\ \psi(N-1,\ \theta) \\ \psi(N,\ \theta) \end{bmatrix} = M^{-1} \frac{(C_{ij})}{|\theta I + B|} M \begin{bmatrix} p(-M,\ 0) \\ p(-M+1,\ 0) \\ \cdot \\ \cdot \\ \cdot \\ p(i,\ 0) = 1 \\ \cdot \\ \cdot \\ \cdot \\ p(N-1,\ 0) \\ p(N,\ 0) \end{bmatrix} \tag{2.6}$$

Substituting $M^{-1}(C_{ij})M$ in the equation 2.6 we have

$$\begin{bmatrix} \psi(-M,\ \theta) \\ \psi(-M+1,\ \theta) \\ \vdots \\ \psi(i,\ \theta) \\ \vdots \\ \psi(N,\ \theta) \end{bmatrix} = \frac{1}{|\theta I + B|} \begin{bmatrix} C_{1\ M+i+1} \left(\frac{d_{M+i+1}}{d_1}\right) \\ C_{2\ M+i+1} \left(\frac{d_{M+i+1}}{d_2}\right) \\ \vdots \\ C_{M+i+1\ M+i+1} \\ \vdots \\ C_{M+N+1\ M+i+1} \left(\frac{d_{M+i+1}}{d_{M+N+1}}\right) \end{bmatrix} \tag{2.7}$$

For any integers r and t,

$$\frac{d_r}{d_t} = \frac{\prod_{k=1}^{r-1} \left(\frac{\mu_{-M+k}}{\lambda_{-M+k-1}}\right)^{\frac{1}{2}}}{\prod_{k=1}^{t-1} \left(\frac{\mu_{-M+k}}{\lambda_{-M+k-1}}\right)^{\frac{1}{2}}}$$

If $r > t$

$$\frac{d_r}{d_t} = \prod_{k=t}^{r-1} \left(\frac{\mu_{-M+k}}{\lambda_{-M+k-1}}\right)^{\frac{1}{2}}$$

If $r < t$

$$\frac{d_r}{d_t} = \frac{1}{\prod_{k=r}^{t-1} \left(\frac{\mu_{-M+k}}{\lambda_{-M+k-1}}\right)^{\frac{1}{2}}}$$

From 2.7

$$\det(\theta I + B)\,\psi(n, \theta) = C_{n+M+1\ M+i+1} \frac{d_{M+i+1}}{d_{n+M+1}}, \quad -M \leq n < i$$

$$= C_{M+i+1\ M+i+1}, \quad n = i$$

$$= C_{n+M+1\ M+i+1} \frac{d_{M+i+1}}{d_{n+M+1}}, \quad i \leq n \leq N$$

Substituting from the above results, we obtain for $-M \leq n < i$,

$$\det(\theta I + B)\,\psi(n, \theta) = C_{n+M+1\ M+i+1} \frac{d_{M+i+1}}{d_{n+M+1}}$$

$$= \prod_{r\,=\,n+M+1}^{M+i} (\lambda_{-M+r-1}\ \mu_{-M+r})^{\frac{1}{2}}\ Q_{M+n}(\theta)\ P_{N-i}(\theta)$$

$$\times \prod_{k\,=\,M+n+1}^{M+i} \left(\frac{\mu_{-M+k}}{\lambda_{-M+k-1}}\right)^{\frac{1}{2}}$$

$$= \prod_{r\,=\,n+1}^{i} \mu_r\ Q_{M+n}(\theta)\ P_{N-i}(\theta)$$

$\therefore\ \psi(n,\ \theta)$

$$= \left(\prod_{r=n}^{i-1} \mu_{r+1}\right) \frac{Q_{M+n}(\theta)\ P_{N-i}(\theta)}{|\theta I + B|} \quad \text{for } -M \leq n < i$$

|||ly

$$\psi(i,\ \theta) = C_{M+i+1\ M+i+1}$$

$$= \frac{Q_{M+i}(\theta)\ P_{N-i}(\theta)}{|\theta I + B|},\ n = i \qquad (2.8)$$

and

$$\psi(n,\ \theta) = \left(\prod_{r=i}^{n-1} \lambda_i\right) \frac{Q_{M+i}(\theta)\ P_{N-n}(\theta)}{|\theta I + B|},\quad i < n \leq N$$

Since B is symmetric tridiagonal matrix, its eigen values are real, non-negative and distinct, consequently the zero's of the polynomial $|\theta I + B|$ are real, distinct and negatives of the eigenvalues of B. $\theta = 0$ is a root of det $A = 0$ and hence a eigen value of B.

Let α_k, $k = 1, 2, \ldots M+N$ be the remaining M+N eigen values of B.

$$\text{Then } \det(\theta I + B) = \theta \prod_{k=1}^{M+N} (\theta + \alpha_k)$$

Hence for $-M \leq n \leq i$ the expression

$\dfrac{Q_{M+N}(\theta)\, P_{N-i}(\theta)}{\theta \prod_{k=1}^{M+N} (\theta + \alpha_k)}$ is a rational function with the degree of the Dinaminaror greater than the degree of the Numerator, so that putting this in partial fraction we get

$$\frac{Q_{M+n}(\theta)\, P_{N-i}(\theta)}{\theta \prod_{k=1}^{M+N} (\theta + \alpha_k)} = \frac{A_n}{\theta} + \sum_{k=1}^{M+N} \frac{A_{nk}}{\theta + \alpha_k}$$

$$\text{Hence, } Q_{M+n}(0)\, P_{N-i}(0) = A_n \prod_{k=1}^{M+n} \alpha_k$$

$$\text{implies } A_n = \frac{Q_{M+n}(0)\, P_{N-i}(0)}{\prod_{k=1}^{M+N} \alpha_k}$$

and

$$Q_{M+n}(-\alpha_k)\, P_{N-i}(-\alpha_k) = \sum_{k=1}^{M+N} (-\alpha_k)\, A_{nk} \prod_{\substack{j=1 \\ j\neq k}}^{N} (-\alpha_k + \alpha_j)$$

implies $$A_{nk} = \frac{Q_{M+n}(-\alpha_k)\ P_{N-i}(-\alpha_k)}{(-\alpha_k) \prod_{\substack{j\neq k\\ j=1}}^{N} (\alpha_j - \alpha_k)}$$

Therefore

$$\psi(n, \theta) = \left(\prod_{r=n}^{i-1} \mu_{r+1}\right) \frac{A_n}{\theta} + \sum_{k=1}^{M+N} \frac{A_{nk}}{\theta + \alpha_k}, \quad -M \leq n < i$$

where $$A_n = \frac{Q_{M+n}(0)\ P_{N-i}(0)}{\prod_{j=1}^{M+N} \alpha_j}, \quad -M \leq n < i$$

$$= \frac{Q_{M+i}(0)\ P_{N-n}(0)}{\prod_{j=1}^{M+N} \alpha_j}, \quad i \leq n \leq N$$

and $$A_{nk} = \frac{Q_{M+n}(-\alpha_k)\ P_{N-i}(-\alpha_k)}{(-\alpha_k) \prod_{\substack{j=1\\ j\neq k}}^{N} (\alpha_j - \alpha_k)}$$

$|||^{\text{ly}}$ we can show that

$$\psi(n, \theta) = \frac{A_n}{\theta} + \sum_{k=1}^{M+N} \frac{B_{nk}}{\theta + \alpha_k}, \quad \text{for } n = i$$

and $$= \left(\prod_{r=i}^{n-1} \lambda_r\right) \left(\frac{A_n}{\theta} + \sum_{k=1}^{M+N} \frac{B_{nk}}{\theta + \alpha_k}\right) \quad \text{for } i \leq n \leq N$$

with
$$B_{nk} = \frac{Q_{M+i}(-\alpha_k)\, P_{N-n}(-\alpha_k)}{(-\alpha_k) \prod_{\substack{j=1 \\ j\neq k}}^{N} (\alpha_j - \alpha_k)}$$

Taking inverse Laplace transform for ψ (n, θ) we get

$$p(n, t) = \left(\prod_{r=1}^{i-1} \mu_{r+1}\right) \left(A_n + \sum_{k=1}^{N} A_{nk}\, e^{-\alpha_k t}\right), \quad -M \leq n < i$$

$$= A_i + \sum_{k=1}^{N} B_{ik}\, e^{-\alpha_k t}, \quad n = i$$

$$= \prod_{r=i}^{n-1} \lambda_r \left(A_n + \sum_{k=1}^{N} B_{nk}\, e^{-\alpha_k t}\right), \quad i < n \leq N$$

In the following sections we consider various situations where the simple birth and death model is applicable.

SECTION 2.1

A PARTICULAR CASE OF SIMPLE BIRTH AND DEATH MODEL

In this section we consider a continuous time parameter simple birth and death process on finite state space $\{ -M, -M+1, -M+2, \ldots N-1, N \}$ where M and N are positive integers and λ and μ are the transition rates and given the transient solution in a more exact form. Let $P_n(t)$ be the probability that there are n particles in the system at the time t and assume that initially there are k particles in the system, so that $p_k(0) = 1$ and $p_n(0) = 0$. Then the Laplace transform of the differential difference equations of this model is obtained by putting $\lambda_r = \lambda$ and $\mu_r = \mu$ $\forall_r$ in equation (2.2),

Thus we get,

$$(\theta + \lambda)\,\psi(-M, \theta) - \mu\psi(-M+1, \theta) = p(-M, 0)$$

$$-\mu\,\psi(n+1, \theta) + (\theta + \lambda + \mu)\,\psi(n, \theta) - \lambda\psi(n-1, \theta) = p(n, 0)$$

$$-M+1 \leq n \leq N-1$$

$$(\theta + \mu)\,\psi(N, \theta) - \lambda\,\psi(N-1), \theta) = p(N, 0) \qquad (2.9)$$

The above system of equations can be written as,

$$\begin{bmatrix} \lambda+\theta & -\mu & 0 & 0 & \cdots & 0 & 0 \\ -\lambda & \theta+\lambda+\mu & -\mu & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \theta+\lambda+\mu & -\mu & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \lambda+\mu+\theta & -\mu \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & \mu+\theta \end{bmatrix} \begin{bmatrix} \psi(-M,\theta) \\ \psi(-M+1,\theta) \\ \psi(-M+2,\theta) \\ \cdot \\ \cdot \\ \cdot \\ \psi(N-1,\theta) \\ \psi(N,\theta) \end{bmatrix} = \begin{bmatrix} P(-M,0) \\ P(-M+1,0) \\ P(-M+2,0) \\ \cdot \\ \cdot \\ \cdot \\ P(N-1,0) \\ P(N,0) \end{bmatrix} \tag{2.9}$$

which is of the form $A\ \psi = \delta$, where A is the tridiagonal matrix of order M+N+1 and

$$\psi = [\psi(-M, \theta), \psi(-M+1, \theta), \ldots \psi(N, \theta)]^1$$

$$\delta = [P(-M, 0), P(-M+1, 0), \ldots P(N, 0)]^1$$

are column vectors of order M+N+1.

The matrix A can be transformed into the symmetric tridiagonal form by the diagonal matrix

$$D = \text{Diag}\ (d_1, d_2, \ldots d_{M+N+1}) \text{ with}$$

$$d_r = (\mu/\lambda)^{\frac{r-1}{2}}$$ and we get

$$B = \begin{bmatrix} \lambda + \theta & -(\lambda\mu)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 \\ -(\lambda\mu)^{\frac{1}{2}} & \lambda + \mu + \theta & -(\lambda\mu)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda\mu)^{\frac{1}{2}} & \lambda + \mu + \theta & -(\lambda\mu)^{\frac{1}{2}} & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & \lambda + \mu + \theta & -(\lambda\mu)^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & \cdots & -(\lambda\mu)^{\frac{1}{2}} & \mu + \theta \end{bmatrix}$$

Proceeding as in the previous section we obtain

$$B^{-1} = \frac{(C_{ij})}{|B|}$$

where

$$C_{ij} = [(\lambda\mu)^{\frac{1}{2}}]^{i-j} \; Q_{j-1}(\theta) \; P_{M+N+1-i}(\theta), \quad j < i,$$

$$= [(\lambda\mu)^{\frac{1}{2}}]^{j-1} \; Q_{i-1}(\theta) \; P_{M+N+1-j}(\theta), \quad i \leq j \leq M+N+1$$

Where $P_r(\theta)$ and $Q_r(\theta)$ are the determinent of the bottom right and top left (r x r) square matrices from B.

Thus solving the system $A\,\psi = \delta$ and simplifying as before we get

$$\psi(n, \theta) = \frac{\mu^{k-n} \; Q_{M+n}(\theta) \; P_{N-k}(\theta)}{|B|}, \quad M \leq n \leq k$$

$$\psi(n, \theta) = \lambda^{n-k} \frac{Q_{M+k}(\theta) \; P_{N-1}(\theta)}{|B|}, \quad k < n \leq N \qquad (2.10)$$

Now we find the transient probabilities interms of eigen values. Let $g_r(\theta)$ be the determinant of the (r x r) matrix.

$$\begin{bmatrix} \lambda+\mu+\theta & -(\lambda\mu)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 \\ -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu+\theta & -(\lambda\mu)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu+\theta & -(\lambda\mu)^{\frac{1}{2}} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \lambda+\mu+\theta & -(\lambda\mu)^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & \cdots & -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu+\theta \end{bmatrix}$$

This $g_r(\theta)$ is the shifted Chebychev's polynomial of the second kind of degree r in θ. Applying some elementary row and column transforms, easily seen that

$$\begin{aligned} |B| &= \theta\, g_{M+N}(\theta) \\ &= \theta\, \left|(\lambda+\mu+\theta)\, I_{M+N} + (\lambda\mu)^{\frac{1}{2}}\, D_{M+N}\right| \end{aligned}$$

where I_{M+N} is the identity matrix of order M+N and D_{M+N} is the tridiagonal matrix of order M+N whose diagonal entries are zeros and off diagonal entries are -1 each. The eigenvalues of D_{M+N} are real and distinct and are given by,

$$a_r = 2\cos\frac{r\pi}{M+N+1}, \quad r = 1, 2, \ldots M+N$$

so that

$$|B| = \theta \mathop{\pi}\limits_{r=1}^{M+N} (\theta + \lambda + \mu + (\lambda\mu)^{\frac{1}{2}} a_r)$$

From the def. of $Q_r(\theta)$, we have

$$Q_r(\theta) = \begin{bmatrix} \lambda+\theta & -(\lambda\mu)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 \\ -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu+\theta & -(\lambda\mu)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu+\theta & -(\lambda\mu)^{\frac{1}{2}} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \lambda+\mu+\theta & -(\lambda\mu)^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & \cdots & -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu+\theta \end{bmatrix}$$

Therefore,

$$Q_r\,(-\lambda-\mu-(\lambda\mu)^{\frac{1}{2}}\,a_1 = \begin{bmatrix} -\mu-(\lambda\mu)^{\frac{1}{2}}\,a_1 & -(\lambda\mu)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 \\ -(\lambda\mu)^{\frac{1}{2}} & -(\lambda\mu)^{\frac{1}{2}}\,a_1 & -(\lambda\mu)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda\mu)^{\frac{1}{2}} & -(\lambda\mu)^{\frac{1}{2}}\,a_1 & -(\lambda\mu)^{\frac{1}{2}} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -(\lambda\mu)^{\frac{1}{2}}\,a_1 & -(\lambda\mu)^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & \cdots & -(\lambda\mu)^{\frac{1}{2}} & -(\lambda\mu)^{\frac{1}{2}}\,a_1 \end{bmatrix}_{(r\mathrm{x}r)}$$

$$= \quad (-(\lambda\mu)^{\frac{1}{2}})^r\;\overline{Q}_r(a_1)$$

Where

$$\overline{Q_r}\,(a_i) = \begin{bmatrix} (\frac{1}{\rho})^{\frac{1}{2}} + a_i & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & a_i & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & a_i & 1 & \dots & 0 & 0 \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & a_i & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & a_i \end{bmatrix}_{(rxr)}$$

$|||^{ly}$ $P_r(-\lambda - \mu - (\lambda\mu)^{\frac{1}{2}} a_i) = (-(\lambda\mu)^{\frac{1}{2}})^r\,\overline{P_r}(a_i)$,

where

$$\overline{P_r}(a_i) = \begin{bmatrix} a_i & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & a_i & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & a_i & 1 & \dots & 0 & 0 \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & a_i & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & a_i + (\rho)^{\frac{1}{2}} \end{bmatrix}_{(rxr)}$$

From 2.10 we have,

$$\psi(n, \theta) = \mu^{k-n} \frac{Q_{M+n}(\theta)\, P_{N-k}(\theta)}{|B|}, \quad -M \leq n \leq k$$

$$= \lambda^{n-k} \frac{Q_{M+k}(\theta)\, P_{N-1}(\theta)}{|B|}, \quad k < n \leq N$$

$|B|$ is of degree M+N+1 and that of the Numerator is M+N+n-k when $n-k \leq 0$. Hence putting into the partial fraction,

$$\psi(n, \theta) = \frac{\pi_n}{\theta} + \mu^{k-n} \sum_{i=1}^{M+N} \frac{B_{ni}}{(\theta + \lambda + \mu + (\lambda\mu)^{\frac{1}{2}} a_i)}, \quad -M \leq n \leq k$$

$$= \frac{\pi_n}{\theta} + \lambda^{n-k} \sum_{i=1}^{M+N} \frac{A_{ni}}{(\theta + \lambda + \mu + (\lambda\mu)^{\frac{1}{2}} a_i)} \quad k \leq n \leq N$$

Where $$A_{ni} = \frac{(-1)^{k-n}\, [(\lambda\mu)^{\frac{1}{2}}]^{k-n+1}\, \overline{Q}_{M+k}(a_j)\, \overline{P}_{N-n}(a_i)}{(\lambda + \mu + (\lambda\mu)^{\frac{1}{2}} a_i) \prod_{\substack{j=1 \\ j \neq i}}^{M+N} (a_i - a_j)}$$

$$B_{ni} = \frac{(-1)^{n-k}\, [(\lambda\mu)^{\frac{1}{2}}]^{n-k+1}\, \overline{Q}_{M+n}(a_i)\, \overline{P}_{N-k}(a_i)}{(\lambda + \mu + (\lambda\mu)^{\frac{1}{2}} a_i) \prod_{\substack{j=1 \\ j \neq i}}^{M+N} (a_i - a_j)}$$

and

$$\pi_n = \frac{(1 - \rho)\;{}^{M+n}}{1 - \rho^{M+N+1}}, \quad M \leq n \leq N \qquad \text{for } \rho \leq 1$$

$$= \frac{1}{M+N+1}, \qquad \text{for } \rho = 1$$

Taking the inverse Laplace transform for $\psi(n, \theta)$, we obtain,

$$p_n(t) = \pi_n + \mu^{k-n} e^{-(\lambda + \mu)t} \sum_{i=1}^{M+N} B_{ni}\, e^{-(\lambda\mu)^{\frac{1}{2}} a_i t}, \qquad -M \leq n \leq k \tag{2.11}$$

$$= \pi_n + \lambda^{n-k} e^{-(\lambda + \mu)t} \sum_{i=1}^{M+N} A_{ni}\, e^{-(\lambda\mu)^{\frac{1}{2}} a_i t}, \qquad k < n \leq N$$

Where π_n $(-M \leq n \leq N)$ are the equilibrium probabilities

As $t \to \propto$, $p_n(t) \to p_n = \pi_n$, $-M \leq n \leq N$.

SECTION 2.2

THE DOUBLE - ENDED QUEUE

In this section we derive the transient solution for the double ended queue as a particular case of this chapter.

Consider the system with state space $\{-M, -M+1, \ldots 0, 1, 2, \ldots N\}$ in which the numbers on the negative axis represent one queue whereas the numbers on the positive axis denote another queue. M and N are the maximum numbers of the first and second queue respectively (including the ones being served). Assume that the arrival rate of first queue is λ and that of the second queue is μ.

The system of differential-difference equations given by,

$$p'(-M, t) = -\lambda\ p(-M, t) + \mu\ p(-M+1, t)$$

$$p'(n, t) = \lambda\ p(n-1, t) - (\lambda + \mu)\ p(n, t) + \mu\ p(n+1, t), \quad -M+1 \leq n \leq N-1$$

$$p'(N, t) = -\mu\ (N, t) + \lambda\ p(N-1, t)$$

The above set of equations are same as in Sec. 1. Hence the results are also one and the same. When $-M \leq n \leq 0$, the $p_n(t)$ of (2.11) gives the probability mass function of the number of units in the first queue and when $0 \leq n \leq N$ it gives that of the second queue the expression of π_n in 2.11 gives the corresponding steady state probabilities of this model. The steady tate probabilities results of this double ended queue model also coincide with that of M/M/1/N model by taking $M = 0$ in

$$\pi_n = \frac{(1 - \rho)\,\rho^{M+n}}{1 - \rho^{M+N+1}}, \quad -M \leq n \leq N \qquad \text{for } \rho < 1$$

$$= \frac{1}{M+N+1}, \qquad \text{for } \rho = 1$$

If L and Q are the random variables representing the number of units in the first and second queue respectively, then for the steady-state case, we get

$$E[L] = \sum_{n=0}^{N} n\, p_n$$

$$= \sum_{n=0}^{N} \frac{n[(1 - \frac{\lambda}{\mu})\,(\frac{\lambda}{\mu})^{M+n}]}{1 - (\frac{\lambda}{\mu})^{M+N+1}}$$

$$= \frac{(1 - \frac{\lambda}{\mu})\,(\frac{\lambda}{\mu})^{M+1}}{1 - (\frac{\lambda}{\mu})^{M+N+1}} \sum_{n=0}^{N} n\,(\frac{\lambda}{\mu})^{n-1}$$

$$= \frac{(1 - \frac{\lambda}{\mu})\,(\frac{\lambda}{\mu})^{M+1}}{1 - (\frac{\lambda}{\mu})^{M+N+1}} \sum_{n=0}^{N} \frac{d}{d\frac{\lambda}{\mu}}\,(\frac{\lambda}{\mu})^{n}$$

$$= \frac{(\frac{\lambda}{\mu})^{M+1}}{1 - (\frac{\lambda}{\mu})^{M+N+1} \quad (1 - \frac{\lambda}{\mu})} [1 - (\frac{\lambda}{\mu})^{N} \quad [N+1] \quad + \quad N(\frac{\lambda}{\mu})^{N+1}]$$

$$\lambda \neq \mu$$

If $\lambda = \mu$,

$$E[L] = \sum_{n=0}^{N} n \frac{1}{M+N+1}$$

$$= \frac{N(N+1)}{2(M+N+1)}$$

$$E[Q] = \sum_{r=0}^{M} (M-r) \pi_{-M+r}$$

$$= \sum_{r=0}^{M} \frac{(M-r) \quad (1 - \frac{\lambda}{\mu}) \quad (\frac{\lambda}{\mu})^{r}}{1 - (\frac{\lambda}{\mu})^{M+N+1}}$$

$$= \frac{(1 - \frac{\lambda}{\mu})}{1 - (\frac{\lambda}{\mu})^{M+N+1}} [\sum_{k=0}^{M} k \quad (\frac{\lambda}{\mu})^{M-k}]$$

$$= \frac{(1 - \frac{\lambda}{\mu}) \quad (\frac{\lambda}{\mu})^{M-1}}{1 - (\frac{\lambda}{\mu})^{M+N+1}} [\sum_{k=0}^{M} \frac{d}{d\frac{\mu}{\lambda}} (\frac{\mu}{\lambda})^{\frac{1}{2}}]$$

$$\frac{(1 - \frac{\lambda}{\mu}) \quad (\frac{\lambda}{\mu})^{M-1}}{1 - (\frac{\lambda}{\mu})^{M+N+1}} [\frac{(1 - \frac{\mu}{\lambda}) \quad (-(M+1) \quad (\frac{\mu}{\lambda})^{M}) \quad + \quad (1 - (\frac{\mu}{\lambda})^{M+1})}{(1 - \frac{\mu}{\lambda})^{2}}]$$

$$= \frac{(1 - \frac{\lambda}{\mu})\ (\frac{\lambda}{\mu})^{M-1}}{(\frac{\lambda}{\mu} - 1)^2\ \frac{\mu^2}{\lambda^2}\ (1 - (\frac{\lambda}{\mu})^{M+N+1})} [-(M+1)\ (\frac{\mu}{\lambda})^M + M(\frac{\mu}{\lambda})^{M+1} + 1]$$

$$= \frac{(\frac{\lambda}{\mu})^{M+1} - (M+1)\ (\frac{\lambda}{\mu}) + M}{(1 - \frac{\lambda}{\mu})\ (1 - (\frac{\lambda}{\mu})^{M+N+1})}$$

If $\lambda = \mu$

$$E(Q) = \sum_{n=0}^{M} n \frac{1}{M+N+1}$$

$$= \frac{M(M+1)}{2(M+N+1)}$$

SECTION 2.3

THE MULTI-SERVER MARKOVIAN QUEUE

We take here a system consisting of c parallel counters. The units arrive at the system in a poisson stream with mean arrival rate λ and the service time distribution at each counter is negative exponential with the same parameter μ. The waiting room capacity is limited to N-c places. That is the maximum number of units in the system is restricted to N. We also assume that there are k units in the system at the time t = 0 when the service starts and the traffic intensity is $\rho = \frac{\lambda}{c\mu}$. The system can be derived from the model in 2.3 by taking

$$M = 0, \quad \lambda_r = \lambda, \quad r = 0, 1, \ldots N-1; \quad \lambda_N = 0$$

$$\begin{aligned} \mu_r &= r\mu, & r &\leq c-1 \\ &= c\mu, & c &\leq r \leq N \quad ; \quad \mu_0 = 0 \end{aligned}$$

Consider an N x N symmetric tridiagonal matrix

$$
C = \begin{bmatrix}
\lambda + \mu & -(\lambda\mu)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
-(\lambda\mu)^{\frac{1}{2}} & \lambda + 2\mu & -(2\lambda\mu)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & \lambda + c\mu & -(c\mu\lambda)^{\frac{1}{2}} & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda + c\mu & -(c\lambda\mu)^{\frac{1}{2}} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -(c\lambda\mu)^{\frac{1}{2}} & \lambda + c\mu
\end{bmatrix}
$$

If $g_r(\theta)$ be the determinant of the r x r square matrix formed at the bottom right of the matrix $(\theta I + C)$ then $g_r(\theta)$ satisfies the recurrence relation

$$g_r(\theta) \quad = \quad (\theta + \lambda + \mu_{N-r+1})\, g_{r-1}(\theta) + \lambda\mu_{N-r+1}\, g_{r-2}(\theta)$$

$= 0$ with $g_{-1}(\theta) = 0$, $g_0(\theta) = 1$, $g_1(\theta) = \theta + \lambda + c\mu$

Furthermore for $r < N - c + 1$

$$g_r(\theta) \quad = \quad (\theta + \lambda + c\mu)\, g_{r-1}(\theta) + c\lambda\mu\, g_{r-2}(\theta) \quad = \quad 0$$

which means $g_r(\theta)$ is a shifted chebycheve polynomial of the second kind of degree r in θ.

The system explained above can be written as $A\,\psi = \delta$ where,

$$
A = \begin{bmatrix}
\theta + \lambda & -\mu & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
-\lambda & \theta + \lambda + \mu & -2\mu & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & -\lambda & \theta + \lambda + 2\mu & -3\mu & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & -\lambda & \theta + \lambda + c\mu & -c\mu & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \theta + \lambda + c\mu & -c\mu \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\lambda & \theta + c\mu
\end{bmatrix}
$$

$$\psi = [\psi(0, \theta), \quad \psi(1, \theta) \ \dots \ \psi(N, \theta)]$$

$$\delta' = [p(0, 0), \quad p(1, 0) \ \dots \ p(N, 0)]$$

This is obtained by the suitable substusion of λ_r and μ_r in 2.3 and taking M=0. As before $|A| = |\theta I + B| = \theta \ g_N(\bar{\theta})$ and apart from satisfying the relation (2.4) $p_r(\theta)$ in this case also satisfies $p_r(\theta) - g_r(\theta) = \lambda g_{r-1}(\theta)$. The solution of the linear system $A \psi = \delta$ is now obtained from 2.8 with suitable substitution as follows.

If $0 \leq i \leq c - 1$

$$\psi(n, \theta) = \prod_{r=n}^{i-1} (r+1)\mu \ \frac{Q_n(\theta) \ P_{N-i}(\theta)}{\theta \, g_N(\theta)}$$

$$= (n+1)\ (n+2) \ \dots \ i \ \mu^{i-n} \ \frac{Q_n(\theta) \ P_{N-i}(\theta)}{\theta \, g_N(\theta)}$$

$$= \frac{i!}{n!} \ {}^{i-n} \ \frac{Q_n(\theta) \ P_{N-i}(\theta)}{\theta \, g_N(\theta)}$$

ly

$$\psi(n, \theta) = \lambda^{n-i} \ \frac{P_{N-n}(\theta) \ Q_i(\theta)}{\theta \, g_N(\theta)}, \quad i+1 \leq n \leq N$$

and if $c \leq i \leq N$,

$$\psi(n, \theta) = \frac{c!}{n!} \ \mu^{i-n} \ c^{i-n} \ \frac{P_{N-i}(\theta) \ Q_n(\theta)}{\theta \, g_N(\theta)}, \quad 0 \leq n \leq c-1$$

$$= \frac{(c\mu)^{i-n}}{n!}\, n!\, \frac{P_{N-i}(\theta)\ Q_n(\theta)}{\theta\ g_N(\theta)} \quad , \ c \leq n \leq i \qquad (2.12)$$

$$= \lambda^{n-i}\ \frac{P_{N-n}(\theta)\ Q_i(\theta)}{\theta\ g_N(\theta)} \quad , \ i+1 \leq n \leq N$$

Now, C is also positive definite matrix and has therefore real, positive and distinct eigen values consequently, the zeros of the polynomial $g_N(\theta)$ are real distinct and negatives of the eigen values of C from $|\theta I + B| = \theta g_N(\theta)$

Let $-(\lambda + c\mu + \alpha_{Nk}(c\lambda\mu)^{\frac{1}{2}})$ be the k^{th} root of $g_N(\theta)$ then $\lambda + c\mu + \alpha_{Nk}(c\lambda\mu)^{\frac{1}{2}}$ are the eigen values of C hence,

$$\begin{vmatrix}
\mu - c\mu - \alpha_{Nk} (c\lambda\mu)^{\frac{1}{2}} & -(\lambda\mu)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
-(\lambda\mu)^{\frac{1}{2}} & 2\mu - c\mu - \alpha_{Nk} (c\lambda\mu)^{\frac{1}{2}} & -(2\lambda\mu)^{\frac{1}{2}} & \cdots & 0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots & -\alpha_{Nk} (c\lambda\mu)^{\frac{1}{2}} & -(c\lambda\mu)^{\frac{1}{2}} & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & -(c\lambda\mu)^{\frac{1}{2}} \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{vmatrix}$$

$$= 0$$

which implies $\alpha_{Nk}(c\lambda\mu)^{\frac{1}{2}}$ are the eigen vlaues of the matrix

$$\bar{C} = \begin{bmatrix}
\mu - c\mu & -(\lambda\mu)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
-(\lambda\mu)^{\frac{1}{2}} & 2\mu - c\mu & -(2\lambda\mu)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & [(c-1)\lambda\mu]^{\frac{1}{2}} & 0 & -(c\lambda\mu)^{\frac{1}{2}} & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -(c\lambda\mu)^{\frac{1}{2}} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -(c\lambda\mu)^{\frac{1}{2}} & 0
\end{bmatrix}$$

which is obtained from c by putting $\lambda + c\mu = 0$ in each of its leading diagonal elements.

$$\bar{C} = (\lambda c \mu)^{\frac{1}{2}} \begin{bmatrix} \frac{\mu - c\mu}{(\lambda c \mu)^{\frac{1}{2}}} & -\frac{1}{(c)^{\frac{1}{2}}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{(c)^{\frac{1}{2}}} & \frac{2\mu - c\mu}{(\lambda c \mu)^{\frac{1}{2}}} & -(\frac{2}{c})^{\frac{1}{2}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & (\frac{c-1}{c})^{\frac{1}{2}} & 0 & -1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}$$

$\overline{C} = (\lambda c \mu)^{\frac{1}{2}} D$ so that α_{Nk}, $k = 1, 2, \ldots N$ are the eigen values of the matrix D. Proceeding as in section 1 we can here also prove that

$$\overline{g}_N(\xi) = [\frac{-1}{(c\lambda\mu)^{\frac{1}{2}}}]^N g_N [- \lambda - c\mu - \xi(c\lambda\mu)^{\frac{1}{2}}] \qquad (2.13)$$

$$\overline{P}_r(\xi) = [- \frac{1}{(c\lambda\mu)^{\frac{1}{2}}}]^r P_r [- \lambda - c\mu - \xi(c\lambda\mu)^{\frac{1}{2}}]$$

$$= \overline{g}_r(\xi) + \rho^{\frac{1}{2}} \overline{g}_{r-1}(\xi) \qquad (2.14)$$

$$\overline{Q}_r(\xi) = [- \frac{1}{(c\lambda\mu)^{\frac{1}{2}}}]^r Q_r [- \lambda - c\mu - \xi(c\lambda\mu)^{\frac{1}{2}}] \qquad (2.15)$$

Using partial fraction and taking inverse Laplace transform for $\psi(n, \theta)$ we get,

$$p(n, t) = \pi_n + e^{-(\lambda + c\mu)t} \sum_{k=1}^{N} A_{nk} e^{-\alpha_{Nk} t_N (c\lambda\mu)} \qquad \rho \neq 1$$

(2.16)

$$= \pi_n^1 + e^{-2\lambda t} \sum_{k=1}^{N} (-1)^{n-i} [(2 + \alpha_{nk}) b_{nk}]^{-1} B_{nk}, \qquad \rho = 1$$

where

$$\pi_n = \frac{(c\rho)^n}{n!} \pi_0, \qquad 0 \leq n \leq c-1$$

$$= \frac{c^c}{c!} \rho^n \pi_0', \qquad c \leq n \leq N$$

$$\pi_0 = [\sum_{k=0}^{c-1} \frac{(c\rho)^k}{k!} + \frac{c^c}{c!} (\rho^c - \rho^{N+1}) (1 - \rho)^{-1}]^{-1}$$

$$A_{nk} = (-1)^{n-i} \rho^{(n-i)/2} [(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}} + \alpha_{Nk})b_{NK}]^{-1} B_{nk}$$

$$b_{Nk} = \prod_{\substack{j=1 \\ j \neq k}}^{N} (\alpha_{Nk} - \alpha_{Nj}) = g_N^{-1} (\alpha_{Nk})$$

$$B_{nk} = \frac{i!}{n!} e^{n-i} \overline{P}_{N-i} (\alpha_{Nk}) \overline{Q}_n(\alpha_{Nk}), \quad 0 \leq n \leq i$$

$$0 \leq i \leq c-1$$

If $0 \leq c \leq i$,

$$B_{nk} = \frac{c!}{n!} c^{n-c} \overline{P}_{N-i}(\alpha_{Nk}) \overline{Q}_n(\alpha_{Nk}), \quad 0 \leq n \leq c-1$$

$$= \overline{P}_{N-i} (\alpha_{Nk}) \overline{Q}_n (\alpha_{Nk}), \quad c \leq n \leq i$$

$$= \overline{P}_{N-n}(\alpha_{Nk}) \overline{Q}_i(\alpha_{Nk}), \quad i+1 \leq n \leq N$$

$$\pi_n' = \frac{c! \, c^n}{n!} \pi_0', \quad 0 \leq n \leq c-1$$

$$= c^c \pi_0', \quad c \leq n \leq N$$

$$\pi_0' = (c^c N + c! \sum_{k=0}^{c} c^k - c^{\rho+1})^{-1}$$

Letting $t \to \infty$, we get the steady state distribution π_n. If we further assume that $\rho < 1$ and $N \to \infty$ then the well-known steady state distribution for $M/M/c/\infty$ queue is given by

$$\pi_n = \frac{(c\rho)^n}{n!}\pi_0, \qquad 0 \leq n \leq c-1$$

$$= \frac{c^c}{c!}\rho^n \pi_0, \qquad n \geq c$$

where

$$\pi_0 = [\frac{c^c}{c!}\rho^{c+1} + (1-\rho)\sum_{k=0}^{c}\frac{(c\rho)^k}{k!}]^{-1}$$

Important parameters

Using the closed form expressions for p(n, t) some parameters of the system can be analytically as well as computationally examined. For example the probability that the system state is not less than a given number r is given by

$$p_r = \sum_{n=r}^{N} p(n, t)$$

Taking $r = c$ in the above equation we get the probability that all the servers are busy, so

$$p_b = \sum_{n=c}^{N} p(n, t)$$

If $N(t)$ is the number of customers present in the system at time t, then

$$E[N(t)] = \sum_{n=0}^{N} n\, p(n, t)$$

and if $N_q(t)$ denotes the number of customers present in the queue, then

$$E[N_q(t)] = \sum_{n=c}^{N} (n-c)\, p(n, t)$$

The method gives excellent results for the case c=1. In this case the matrix c becomes

$$c = \begin{bmatrix} \lambda+\mu & -(\lambda\mu)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 \\ -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu & -(\lambda\mu)^{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu & -(\lambda\mu)^{\frac{1}{2}} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \lambda+\mu & -(\lambda\mu)^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & \cdots & -(\lambda\mu)^{\frac{1}{2}} & \lambda+\mu \end{bmatrix}$$

So that the matrix D is given by

$$
\begin{bmatrix}
0 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & -1 & \cdots & 0 & 0 \\
0 & -1 & 0 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 0 & -1 \\
0 & 0 & 0 & \cdots & -1 & 0
\end{bmatrix}
$$

The eigen values of the matrix D are known to be

$$\alpha_{Nk} = 2 \cos \frac{k\pi}{N+1}, \quad k = 1, 2, \ldots N$$

Thus in this case the eigen values are analytically determined. Now putting $\rho = \frac{\lambda}{\mu}$ in 2.16 we obtain the state probabilities which are given by

$$p(n, t) = \pi_n + e^{-(\lambda + \mu)t} \sum_{k=1}^{N} A_{nk} \, e^{-\alpha_{Nk} t (\lambda\mu)^{\frac{1}{2}}} \qquad \rho \neq 1$$

$$= \pi_n' + e^{-2\lambda t} \sum_{k=1}^{N} (-1)^{n-i} \left[(2 + \alpha_{nk}) \, b_{Nk}\right]^{-1} B_{nk}, \qquad \rho = 1$$

where now the constants simplify to,

$$\pi_n = \frac{(1 - \rho) \cdot \rho^n}{1 - \rho^{N+1}}$$

$$\pi_n' = \frac{1}{N + 1}, \quad n = 0, 1, 2, \ldots N$$

$$A_{nk} = (-1)^{n-i} \rho^{(n-i)/2} B_k [\bar{g}_{N-i} (\alpha_{Nk})$$

$$+ \rho^{\frac{1}{2}} \bar{g}_{N-i-1} (\alpha_{Nk})] [\bar{g}_n(\alpha_{Nk} + \rho^{-\frac{1}{2}} \bar{g}_{n-1} (\alpha_{Nk})],$$

$$0 \leq n \leq i$$

$$= (-1)^{n-i} \rho^{(n-i)/2} B_k [\bar{g}_{N-n} (\alpha_{Nk})$$

$$+ \rho^{\frac{1}{2}} \bar{g}_{N-n-1} (\alpha_{Nk})] [\bar{g}_i (\alpha_{Nk}) + \rho^{-\frac{1}{2}} \bar{g}_{i-1} (\alpha_{Nk})],$$

$$i+1 \leq n \leq N$$

$$B_k = [\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}} + \alpha_{Nk}) b_{Nk}]^{-1}$$

$$B_{nk} = (-1)^{n-i} [\bar{g}_{N-i} (\alpha_{Nk}) + \bar{g}_{N-i-1} (\alpha_{Nk})]$$

$$[\bar{g}_n (\alpha_{Nk}) + \bar{g}_{n-1} (\alpha_{Nk})], \quad 0 \leq n \leq i$$

$$= (-1) [\bar{g}_{N-n} (\alpha_{Nk}) + \bar{g}_{N-n-1}(\alpha_{Nk})]$$

$$[\bar{g}_i (\alpha_{Nk} + \bar{g}_{i-1} (\alpha_{Nk})], \quad i+1 \leq n \leq N$$

This result coincide with the corresponding result of M/M/1/N model.

SECTION 2.4

THE QUEUE WITH HETEROGENEOUS SERVERS

We here discuss a two-server model with the interarrival and service times of first and second servers to be negative-exponentially distributed with parameters λ, μ_1 and μ_2 respectively. We take $\mu_1 > \mu_2$, which also implies that we are considering a modified queue discipline. That is the first arriving unit from amongst the initial number of units present at the start of the service joins the first counter for service and there after the arriving unit goes to the counter which is finds free. The waiting room capacity is limited to N-2 places. That is the maximum number of customers in the system is restricted to N. We also assume that there are $i (0 \leq i < N)$ initial units waiting at the time $t = 0$. When the service starts and that the traffic intensity is $\rho = \frac{\lambda}{\mu_1 + \mu_2}$. The model can be easily obtained from the one described in 1st sec. by taking $M = 0$

$$\lambda_r = \lambda, \qquad r = 0, 1, 2, \ldots N-1, \qquad \lambda_M = 0$$

$$\mu_r = \mu_1 + (1 - \delta_{1r})\mu_N ; \quad 1 \leq r \leq N, \quad \mu_0 = 0$$

Now prescribing the same meaning to symbols as used in equation 2.3, the system can be written as

$$A \psi = \delta$$

where A is given by

$$A = \begin{bmatrix} \theta + \lambda & -\mu_1 & 0 & \cdots & 0 & 0 \\ -\lambda & \theta + \lambda + \mu_1 & -(\mu_1 + \mu_2) & \cdots & 0 & 0 \\ 0 & -\lambda & \theta + \lambda + \mu_1 + \mu_2 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \theta + \lambda + \mu_1 + \mu_2 & -(\mu_1 + \mu_2) \\ 0 & 0 & 0 & \cdots & -\lambda & \theta + \mu_1 + \mu_2 \end{bmatrix}$$

From 2.8 we find

$$\psi(n, \theta) = \frac{D_n(\theta)}{|A|}, \quad n = 0, 1, 2, \ldots N$$

where $D_n(\theta)$ is obtained by putting $\mu_1 = \mu_1$ and

$\mu_r = \mu_1 + \mu_2$ in the numerator expressions of 2.8

Thus,

$$D_n(\theta) = [1 - \delta_{i0}] [\mu_1(\mu_1 + \mu_2)^{i-1} P_{N-i}(\theta)] + \delta_{i0} P_N(\theta),$$

$$n = 0$$

$$= \quad (\mu_1 + \mu_2)^{i-n} P_{N-i}(\theta)\, Q_n(\theta), \qquad 1 \le n \le i$$

$$= \quad \lambda^{n-1} P_{N-n}(\theta)\, Q_i(\theta)\ , \qquad i+1 \le n \le N$$

Proceeding as in sec 2.3 we get the transient solution for $p_n(t)$.

SECTION 2.5

THE MULTI-SERVER QUEUE WITH BALKING AND RENEGING

In this section we discuss the transient behaviour of a multi-server model with reneging and balking and also taking an arbitrary number i of customers being present initially at the time $t = 0$.

We assume the initial number of customers c would not renege because of their immediately entrance to the service facilicity, whereas these C customers join the queue with certain balking probability.

We also assume that the mean arrival rate of a customer and the mean service rate of each server are λ and μ respectively, with negative exponential distributions and the queue discipline in first-come first served. An arriving customer balks with probability n/N and therefore joins the system with probability,

$$\beta_n = (1 - n/N), \quad n = 0, 1, 2, \ldots N$$

which indicates that N is a measure of customer willingness to join the queue. A customer reneges after joining the queue if he or she decides that the certain length of waiting time will be larger than can be tolerated.

This time is taken as a random variable with density function

$$R(t) = \gamma e^{-\gamma t}$$

Since any one of the (n-c) customers in the queue may renege. We find that the density function for the minimum of (n-c) selections from R(t) is

$$R_{n-c}(t) = (n-c)\gamma e^{-(n-c)\gamma t}$$

which is equivalent to the assumption that the reneging function is $R(n) = (n-c)\gamma$

This new process is still birth-death process where the birth and death rates are now adjusted to

$$\lambda_n = \lambda \cdot \beta_n = \frac{\lambda(N-n)}{N}, \qquad 0 \leq n \leq N$$

$$\begin{aligned} \mu_n &= n\mu, & 0 \leq n \leq c-1 \\ &= c\mu + (n-c)\gamma, & c \leq n \leq N \end{aligned}$$

After substituting these rates in (2.1) with M = 0 we get the following equations of the new system.

$$p'(0, t) = -\lambda p(0, t) + \mu p(1, t)$$

$$p'(1, t) = -[(1 - \frac{n}{N})\lambda + n\mu]\, p(n, t) + (1 - \frac{n-1}{N})\lambda\, p(n-1, t)$$

$$+ (n-1)\mu\, p(n+1, t), \qquad 1 \leq n \leq c-1$$

$$p'(n, t) = -[(1 - \frac{n}{N})\lambda + c\mu + (n-c)\gamma]\, p(n, t)$$

$$+ (1 - \frac{n-1}{N})\lambda\, p(n-1, t) + [c\mu + (n-c+1)\gamma]$$

$$p(n+1, t), \qquad c \leq n \leq N-1$$

$$p'(N, t) = [c\mu + (N-c)\gamma]\, p(N, t) + \frac{\lambda}{N}\, p(N-1, t)$$

Now prescribing the same meaning to symbols as used in equation (2.3) the system can be written as $A\psi = \delta$

Here also $|A| = \theta g_N(\theta)$ where $g_n(\theta)$ satisfies the recurrence relation

$$g_n(\theta) - (\theta + \lambda_{N-n} + \mu_{N-n+1})\, g_{n-1}(\theta) + \lambda_{N-n+1}\, \mu_{N-n+1}\, g_{n-2}(\theta)$$

$$= 0, \qquad 1 \leq n \leq N$$

with $g_0(0) = 1$ and $g_{-1}(0) = 0$

The solutions of $A\psi = \delta$ of this model is obtained by putting the suitable values of λ_n and μ_n given above in (2.8).

Case (i) $0 \leq i \leq c-1$

(i) $0 \leq n \leq i$

$$\psi(n,\ \theta) = \prod_{r=n}^{i-1} \frac{(1+r)\mu\, P_{N-i}(\theta)\, Q_n(\theta)}{\theta\, g_N(\theta)}$$

$$= (n+1)\,\mu\,(n+2)\,\mu\ \ldots\ i\,\mu\ \frac{P_{N-i}(\theta)\ Q_n(\theta)}{\theta\, g_N(\theta)}$$

$$= \frac{\lfloor\underline{i}}{\lfloor\underline{n}}\ \mu^{i-n}\ \frac{P_{N-i}(\theta)\ Q_n(\theta)}{\theta\, g_N(\theta)}$$

(ii) $i+1 \leq n \leq N$

$$\psi(n,\ \theta) = \prod_{r=i}^{n-1} (N-r)\,\frac{\lambda}{N}\ \frac{P_{N-n}(\theta)\ Q_i(\theta)}{\theta\, g_N(\theta)}$$

$$= (N-i)\ (N-(i+1))\ \ldots\ (N-n+1)\ \left(\frac{\lambda}{N}\right)^{n-i}$$

$$\frac{P^{N-n}(\theta)\ Q_i(\theta)}{\theta\, g_N(\theta)}$$

$$= \frac{\lfloor\underline{N-i}}{\lfloor\underline{N-n}}\ \left(\frac{\lambda}{N}\right)^{n-i}\ \frac{P_{N-n}(\theta)\ Q_i(\theta)}{\theta\, g_N(\theta)}$$

Case (2) : For $c \leq i \leq N$

(i) $0 \leq n \leq c-1$

$$\psi(n, \theta) = \prod_{r=n}^{c-1} (1+r)\,\mu \; \frac{P_{N-i}(\theta)\, Q_n(\theta)}{\theta\, g_N(\theta)}$$

$$= \frac{\lfloor c-1}{\lfloor n} \; \mu^{c-n-1} \; B_c \; \frac{P_{N-i}(\theta)\, Q_n(\theta)}{\theta\, g_N(\theta)}$$

(ii) $c \leq n \leq i$

$$\psi(n, \theta) = [\delta_{in} + (1 - \delta_{in})\, B_{n+1}] \; \frac{P_{N-i}(\theta)\, Q_n(\theta)}{\theta\, g_N(\theta)}, \quad c \leq n \leq i$$

(iii) $i+1 \leq n \leq N$

$$\psi(n, \theta) = \frac{(N-i)!}{(N-n)!} \left(\frac{\lambda}{N}\right)^{n+i} \frac{P_{N-n}(\theta)\, Q_i(\theta)}{\theta\, g_N(\theta)}$$

where

$$B_m = \prod_{k=1}^{i+1-m} [c\,\mu + (i-c+1-k)\,\gamma]$$

Hence as before the transient solution of this model is given as follows :

Case (i) : $0 \leq i \leq c-1$

$$p(n,\ t) = \pi_n + \sum_{j=1}^{N} A_{nj}\ e^{-\alpha N_j t}, \qquad 0 \leq n \leq i$$

$$= \pi_n{}' + \sum_{j=1}^{n} A_{nj}{}'\ d^{-\alpha N_j t} \qquad i+1 \leq n \leq N$$

$$(2.17)$$

Case (ii) : **For** $c \leq i \leq N$

$$p(n,\ t) = \pi_n + \frac{(c-1)!}{i!}\ \mu^{c-n-1}\ B_c \sum_{j=1}^{N} A_{nj}\ e^{-\alpha N_j t}, \qquad 0 \leq n \leq c-1$$

$$= \pi_n{}' + \frac{n!}{i!}\ \mu^{n-i}\ [\delta_{in} + (1 - \delta_{in})\ B_{n+1}] \sum_{j=1}^{N} A_{nj}\ e^{-\alpha N_j t}, \qquad c \leq n \leq i$$

$$= \pi_n{}' + \frac{(N-i)!}{(N-n)!}\ (\frac{\lambda}{N})^{n-i} \sum_{j=1}^{N} A_{nj}{}'\ e^{-\alpha N_j t}, \quad i+1 \leq n \leq N$$

$$(2.18)$$

Where

$$\pi_n = \begin{bmatrix} N \\ n \end{bmatrix} (\frac{\rho_1}{\rho_2})^n\ \pi_0$$

$$\pi_n' \, \pi_0^{-1} = \begin{bmatrix} N \\ n \end{bmatrix} \left(\frac{\rho_1}{\rho_2}\right)^n \left[1 - \sum_{k=i+1}^{n} \delta_{ik} \right] +$$

$$+ \frac{N! \, \rho_1^{\,n} \, \Gamma(c \, \rho_2))}{(N-n)! \, (c-1)! \, \rho_2^{\,c-1} \, \Gamma(c \quad {}_2 + n - c + 1)} \sum_{k=i+1}^{n} \delta_{ik}$$

$$\pi_n^{-1} = \sum_{r=0}^{c-1} \begin{bmatrix} N \\ r \end{bmatrix} \left(\frac{\rho_1}{\rho_2}\right)^r + \sum_{r=c}^{N} (N-r)! \, (c-1)! \, \rho_2^{\,c-1}$$

$$\Gamma(c\rho_2 + r - c + 1)$$

$$A_{nj} = P_{N-i}(\alpha_{Nj}) \, Q_n(\alpha_{Nj}) \, [- \alpha_{Nj} \prod_{\substack{i=1 \\ i \neq j}}^{N} (\alpha_{Ni} - \alpha_{Nj})]^{-1}$$

$$A'_{nj} = P_{N-n}(\alpha_{Nj}) \, Q_i(\alpha_{Nj}) \, [- \alpha_{Nj} \prod_{\substack{i=1 \\ i \neq j}}^{N} (\alpha_{Ni} - \alpha_{Nj})]^{-1}$$

with $P_1 = (\lambda / N\gamma)$ and $P_2 = (\mu / \gamma)$

Having obtained the transient probabilities in closed form given by (2.17) and (2.18) one can obtain many performance measures of the system involving these probabilities both analytically and numerically as has been shown in earlier sections. Some of the easily obtainable measures are

(i) the mean number of units in the system at the time t is

$$\sum_{n=1}^{N} n\ p(n,\ t)$$

(ii) the mean number of units in the queue at the time t is

$$\sum_{n=1}^{N-c} n\ p(n,\ t)$$

(iii) the probability of these being r or more units in the system at the time t is

$$\sum_{r=r}^{N} p(n,\ t)$$

(iv) the probability of balking at the time of t is

$$\sum_{n=1}^{N} (1 - \beta_n)\ p(n,\ t)$$

(v) the probability of waiting up to the time t in queue by those who join it is

$$\sum_{n=1}^{N-c} \beta_n\ p(n,\ t)$$

(vi) the probability of reneging at the time t is

$$\sum_{n=1}^{N-c} \beta_n\ (1 - \alpha_n)\ p(n,\ t)$$

Where $\alpha_n = \rho_2 / (\rho_2 + n)$ is the probability that a new arrival will survive to be serviced given 'n' in the system and given that it joins the system

(vii) the probability of acquiring service at the time t is

$$\sum_{n=0}^{N-1} \beta_n d_n p(n, t)$$

(viii) the mean reneging rate is

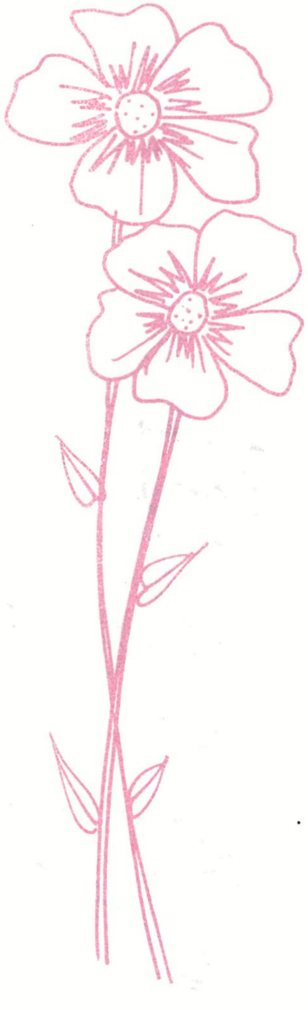
$$\sum_{n=c}^{N} (n-c)\gamma \, p(n, t)$$

(ix) the mean balking rate is

$$\sum_{n=1}^{N} \frac{n\lambda}{N} p(n, t)$$

and

(x) the mean rate of customer loss is the sum of the mean balking and mean reneging rates.



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