Certain Studies Relating to Transmuted Distributions

SREEMATHI V

(17PMA020)

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women

Coimbatore - 641043

In Partial Fulfilment of the Requirements for the Degree of

Master of Science in Mathematics

April, 2019

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Signature of the

Head of the Department

A R Sundaramani Ramaswamy 15/4/19.

Signature of the

Supervisor

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CONTENT

Abstract

This dissertation is devoted to the study of certain studies relating to transmuted distributions. The distribution studied in this dissertation are Transmuted Rayleigh distribution, Transmuted geometric-G family of distribution and Transmuted Weibull-G family of distribution.

The first chapter deals with basic concepts of reliability, statistical inference, order statistics, lifetime distributions, notations and review of literature.

The second chapter deals with generalization of the Rayleigh distribution using the quadratic rank transmutation map to develop a transmuted Rayleigh distribution. The comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior has been provided. The usefulness of the transmuted Rayleigh distribution for modeling data is illustrated using real data.

The third chapter deals with the introduction of a new class of transmuted Rayleigh distribution. The estimates of parameters of transmuted Rayleigh distribution were obtained by using new method of moments. A new distribution which contains as a special case is introduced. The characterizing properties of the model are also determined.

The fourth chapter deals with the introduction of a new family of continuous distributions called the transmuted geometric-G family. Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, generating function, Renyi and Shannon entropies, order statistics and probability weighted moments are also derived.. some special models of the new family has been provided. The maximum likelihood method is used for estimating the model parameters. The importance and flexibility of the proposed family are illustrated by two applications of real data sets.

The fifth chapter deals with the introduction of a new family of continuous distributions called the Transmuted Weibull-G family of distributions. The mathematical properties of the new family are studied. The model parameters are estimated by the maximum likelihood estimators in terms of biases and mean squared errors by means of simulation study are assessed. The importance and flexibility of the proposed family are illustrated by two applications of real data sets.

Chapter - 1

Introduction

Section - 1.1

Basic Concepts of Reliability

1.1.1 Introduction

The concepts of reliability has been known for a number of years, but it has assumed greater significance and importance during the past decade, particularly due to the impact of automations, developments in complex missile and space programmes. The manufacture of highly complex equipment has served to focus greater attention on reliability. However, reliability is only one of the tools of management which must be supplemented by the other tools like quality control and design of experiments for the solution of problems of quality and cost. Reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered. Reliability of a product is the measure of the ability of a product to fuction successfully, when required, in the specified environment.

Study of reliability is important because it is related to the quality of a product. Reliability of a product is more important because it is common for a person to think that , what is the use of buying a product that does not satisfy the customer needs and fails within a short period. Thus the effectiveness of a sysytem is understood to mean the suitability of the system for the fulfillment of the intended tasks and the efficiency og utilizing the means put into it. The suitability of performing definite tasks is primarily determined by the quality of the system.

1.1.2 Definition of Reliability

Reliability of a unit is the probability that the unit performs its intended function adequately for a given period of time under the stated operating conditions or environment. By a unit we mean an element, a system or a part of a sysem. If T is the time till the failure of the unit (a random variable) occurs, then the probability that it will not fail in a given environment before time t (or its reliability) is

$$R(t) = P(T>t)$$

Thus, the reliability is always a function of time. It also depends on environmental conditions which may or may not vary with time. Since it is a probability, its numerical value is always between 1 and 0, that is

$$R(0) = 1, R(\infty) = 0$$

and R(t) is a non increasing function between these limits.

1.1.3 Basic Elements of Reliability

- The reliability definition stresses five element mainly
- Numerical value of probability
- Statement defining successful product performance.
- Statement defining the environment in which the equipment must operate.
- Statement of the required operating time.
- The type of distribution likely to be encountered in reliability measurement.

1.1.4 Design for Reliability

Reliability design is an iterative process that begins with the specification of reliability goals consistent with cost and objectives. This requires consideration of the life- cycle costs of the system and the effect that reliability has an overall costs and system effectiveness. Once these reliability goals have been establised, these goals must be translate into individual component, subcomponent, and part specifications. This not necessarily an easy task, and it generally requires reliability block analysis. After indivdual component and part requirements have been determined, various design methods can be applied in order to meet the goals. These methods include the proper selection of parts and materials, stress- strength analysis, simplification, identification of technologies and use of redundancy.

Following completion of a preliminary detailed design along with initial development and protyping, a failure analysis may be performed to determine whether the specifications are being met and also provide a systematic approach for identifying, ranking, and eliminating failure models. This requires the use of reliability testing, including, perhaps a formalizedreliability growth testins program. Once reliabilty goals have been achieved, verification that safety margins are also being met must be made. If either the reliability or safety goals not met, the design process must continue. This may require reallocating reliability goals among the components if it is not possible to achieve a desired component reliability. The effect of design changes should then be verified through continued use of failure analysis and reliability testing.

Although we are considering reliability as an inherent system or component attribute that can largely be determined during design, we cannot ignore the fact that reliability is influenced throughout the product life cycle by factors external to the product itself.

1.1.5 Achievement of Reliability

There are five effective areas for the achievement of reliability of the product. They are

- Design
- Production
- Measurement and testing
- Maintenance
- Field production

Design is very important that the other four areas and a greater percentage of causes of unreliability can be traced out in this areas.

1.1.6 Failure Pattern

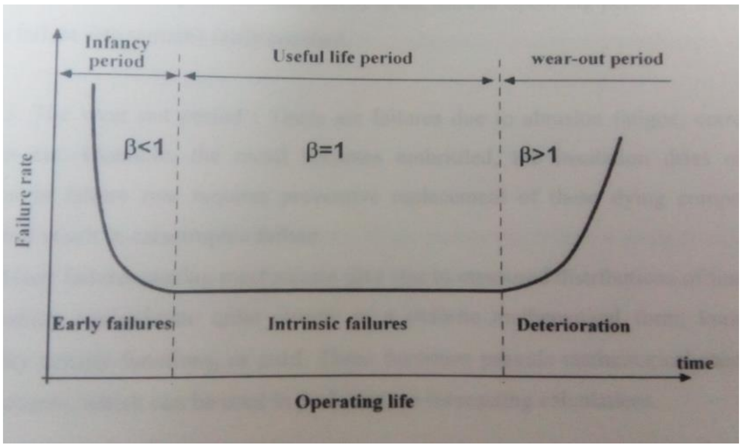


Figure 1.1 : Failure Pattern

The products often follow a familiar failure pattern of failure. When the failure rate (number of failures per unit time) is plotted against a continuous time scale, the resulting chart is known as " bath tub curve ". This curve exhibits three distinct zones. These zones differ from each other in frequency of failure and in the cause of failure pattern. These are as follows :

1. **Infant Morality Period (or born in or the debugging period) :** This is characterized by high failure rates. It begins at the first point during manufacture that total equipment operation is possible and continues for such a period of time as permits (through maintenance and repairs), the elimination of marginal parts initially defective though not inoperative and unrecognizable as such until premature failure. Commonly, these are early failures resulting from defect in manufacturing , or other deficiencies which can be detected by debugging, running on or extended testing.

2. **The constant failure rate period :** upon replacement of all prematurely failing items, the failure rate will have reached a lower value. From this point the failure rate remains fairly constant. These are chance failures which may result from the limitations inherent in the design plus accidents caused by usage or poor maintenance or hidden defects which escape inspection. This period is the normal operating period in which the average failure rate remains fairly constant.

3. **The wear out period :** These are failures due to abrasion fatigue, corrosion, vibration etc., Example , the mental becomes embrittled, the insulation dries out. A reduction in failure rate requires preventive replacement of these dying components before they result in catastrophic failure.

   Mainly failure- causing mechanisms give rise to measured distributions of times- to- failure which approximate density functions, or p.d.f. These functions provide mathematical models of failure pattern, which can be used in performance forecasting calculations.

1.1.7 Methods for improving design reliability

Improving the reliability of a product by changing the design is done by designer himself. The following are some of the approaches used by the designers working jointly with reliability engineers to improve the design.

- Review the index selected to define product reliability to make sure that it reflects customer needs.Question the function of the unreliable parts with a view of eliminating them entirely if the function is found to be unnecessary.
- Review the selection of any parts which are relatively new.
- Conduct a research and development program to increase the reliability of the parts which are contributing most to the unreliability of the equipments.
- Specify corrective replacement items for unreliable parts and replace the parts before they fail.
- Select the parts which will be subjected to stress which are lower than the parts can normally withstand.
- Control the operating environment so that a part will be operating under conditions which yield a lower failure rate.
- Use redundancy so that if one unit fails a redundant unit will be available to do the job.
- Consider possible trade–offs of reliability with functional performance weight or other parameters.

1.1.8 Life testing

Reliability testing refers to the tests conducted to verify that a product will work satisfactorily for a given time period. Reliability testing therefore consists of functional test, environment test and life testing. A functional testing involves a test to determine if the product will function at time zone. An environmental test consists of determining the expected environmental levels and then carrying the functional test under the environments under which the product has to operate. The life of the component is the time period during which it retains its quality characteristic. Life tests are carried out to access the working life of a product, its capabilities and hence to form an idea of its quality level. The life test aims to measure the time or period during which the product will retain its desired quality characteristics. This may apply to either shelf life or life during use or both.

Section 1.2

1.2.1 Statistical inference

Statistical inference is the process of using data analysis to deduce properties of an underlying probability distribution. Inferential statistical analysis infers properties of a population, for example by testing hypotheses and deriving estimates. It is assumed that the observed data set is sampled from a larger population.

1.2.2 Theory of estimation

Estimation theory is a branch of inferential statistics that deals with estimating the values of parameters based on measured empirical data that has a random component. The parameters describe an underlying physical setting in such a way that their value affects the distribution of the measured data. An estimator attempts to approximate the unknown parameters using the measurements.

1.2.2.1 Commonly used method of estimation:

Maximum likelihood estimator.

Bayes estimators.

Method of moments estimators.

Least squares.

Minimum mean squared error (MMSE).

Maximum a posteriori (MAP).

1.2.2.2 Maximum Likelihood Estimation (MLE)

In statistical inference, maximum likelihood estimation (MLE) is a method of estimating the parameters of a statistical model, given observations. The method obtains the parameter estimates by finding the parameter values that maximize the likelihood function. The estimates are called maximum likelihood estimates, which is also abbreviated as MLE.

Let $X_1, X_2,..., X_n$ be a random sample from a distribution that depends on one or more unknown parameters $\theta_1, \theta_2, \dots, \theta_m$ with probability density (or mass) function $f(\, x_i\, ;\ \theta_1, \theta_2, \dots, \theta_m)$. Suppose that $(\theta_1, \theta_2, \dots, \theta_m)$ is restricted to a given parameter space Ω. Then:

(1) When regarded as a function of $\theta_1, \theta_2, \dots, \theta_m$, the joint probability density (or mass) function of $X_1, X_2,..., X_n$:

$$\mathrm{L}(\theta_1, \theta_2, \dots, \theta_m) = \prod_{i=1}^{n} f(\, x_i\, ;\ \theta_1, \theta_2, \dots, \theta_m)$$

($(\theta_1, \theta_2, \dots, \theta_m)$ in Ω) is called the **likelihood function**.

(2)If $[u_1(x_1, x_2,\dots, x_n), u_2(x_1, x_2,\dots, x_n), \dots, u_m(x_1, x_2, \dots, x_n)]$ is the *m*-tuple that the maximum likelihood function, then

$$\hat{\theta}_i = u_i \ (X_1, X_2, \dots, X_n)$$

is the maximum likelihood estimator of θ_i, for *i* = 1, 2, ..., *m*.

(3) The corresponding observed values of the statistics in (2), namely:

$[u_1(x_1, x_2,\dots, x_n), u_2 \ (x_1, x_2,\dots, x_n), \dots, u_m(x_1, x_2, \dots, x_n)]$ are called the maximum likelihood estimates of θ_i, for *i* = 1, 2, ..., *m*.

1.2.2.3 Methods of Moments

In statistical inference , the method of moments is a method of estimation of population parameters. It starts by expressing the population moments (i.e., the expected values of powers of the random variable under consideration) as functions of the parameters of interest. Those expressions are then set equal to the sample moments. Let $x_1, x_2,\dots, x_n$ be a random samples of n observations from a population having probability density function f(x ; $\theta_1, \theta_2, \dots, \theta_k$), where the parameters $\theta_1, \theta_2, \dots, \theta_k$are in the parametric space S. Let the r-th moment about zero of this distribution is μ_r' = E (X^r) and

μ_r' = h $(\theta_1, \theta_2, \dots, \theta_k)$. Here h$(\theta_1, \theta_2, \dots, \theta_k)$ is a function of parameters $\theta_1, \theta_2, \dots, \theta_k$. The sample moment about zero for this population is $m_r' = \frac{1}{n} \sum_{i=1}^{n} x_i^r$. Since μ_r' = h $(\theta_1, \theta_2, \dots, \theta_k)$, we can write θ_r= g $(\mu_1', \mu_2', \dots, \mu_r')$, when r = k.

The method of moments states that m_r' is to be replaced by $\hat{\theta}_r = m_r'$.

Section 1.3: order statistics

In statistics, the *k*th order statistic of a statistical sample is equal to its *k*th-smallest value. The order statistics is the most fundamental tool in non-parametric statistics and inference. Important special cases of the order statistics are the minimum and maximum value of a sample, and the sample median and other sample quantiles. When using probability theory to analyze order statistics of random samples from a continuous distribution, the cumulative distribution function is used to reduce the analysis to the case of order statistics of the uniform distribution.

The first order statistic (or smallest order statistic) is always the minimum of the sample, that is, $X_{(1)} = min\{X_1, .., X_n\}$ where, following a common convention, Upper-case letters are used to refer the random variables, and Lower-case letters are used to refer their actual observed values. Similarly, for a sample of size *n*, the *n*th order statistic (or largest order statistic) is the maximum, that is, $X_{(n)} = max\{X_1, .., X_n\}$

The sample range is the difference between the maximum and minimum. It is a function of the order statistics: $\{X_1, .., X_n\} = X_{(n)} - X_{(1)}$ .A similar important statistic in exploratory data analysis that is simply related to the order statistics is the sample interquartile range. The sample median may or may not be an order statistic, since there is a single middle value only when the number *n* of observations is odd. More precisely, if *n* = 2*m*+1 for some integer *m*, then the sample median is $X_{(m-1)}$ and so is an order statistic. On the other hand, when *n* is even, *n* = 2*m* and there are two middle values $X_{(m)}\ and\ X_{(m-1)}$, and the sample median is some function of the two (usually the average) and hence not an order statistic. Similar remarks apply to all sample quantiles.

1.3.1 Computing order statistics

The problem of computing the *k*th smallest (or largest) element of a list is called the selection problem and is solved by a selection algorithm. Although this problem is difficult for very large lists, sophisticated selection algorithms have been created that can solve this problem in time proportional to the number of elements in the list, even if the list is totally unordered. If the data is stored in certain specialized data structures, this time can be brought down to O(log *n*).

Section 1:4

Lifetime Distribution

1.4.1 Introduction

The use of parametric distributions complements non-parametric techniques and provides the following advances:

- Parametric models can be described concisely with just few parameters, instead of having to report an entire curve.
- It is possible to use a parametric model to extrapolate (in time) to the lower or upper tail of distribution.
- Parametric models provide smooth estimates of failure-time distributions.

1.4.2 Transmuted Rayleigh Distribution

The transmuted Rayleigh distribution was introduced by Faton Merovci (2013). The transmuted Rayleigh distribution is an extended model to analyze more complex data. The density function (pdf) of a Rayleigh distribution is

$$f(x,\sigma) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right), x > 0, \sigma > 0,$$

and the respective cumulative distribution function is

$$F(x,\sigma) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right), x > 0, \sigma > 0.$$

The transmuted cumulative distribution function is

$$G(x,\sigma,\lambda) = \left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\left(1 + \lambda\exp\left(-\frac{x^2}{2\sigma^2}\right)\right),$$

with transmuted probability density function

$$g(x,\sigma,\lambda) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1 - \lambda + 2\lambda\exp\left(-\frac{x^2}{2\sigma^2}\right)\right).$$

The Rayleigh distribution is clearly a special case for $\lambda = 0$.The following Figure 1.4.2.1 illustrates some of the possible shapes of the probability density function of a transmuted Rayleigh distribution for selected values of the parameters λ and σ.

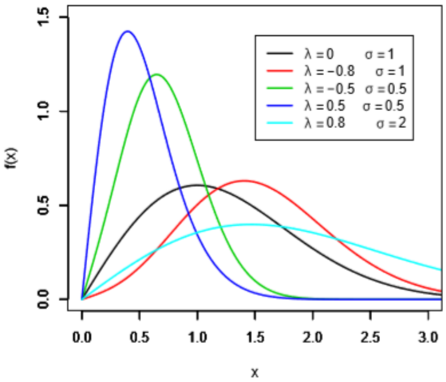


Figure 1.1: The probability density functions of various transmuted Rayleigh distributions.

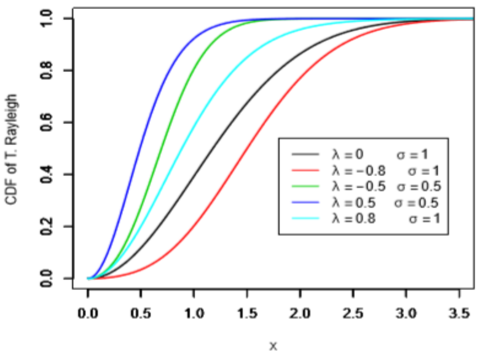


Figure 1.2: The cumulative distribution functions of various transmuted Rayleigh distributions.

1.4.3 The Transmuted Geometric-G Family of Distribution

The transmuted geometric-G family of distributions was introduced by Ahmed, Afify, Morad Alizadeh, Haitham M. Yousof, Gokarna Aryal and Munir Ahmad (2016). The probability density function of the TG-G is given by

$$f(x) = \frac{\theta g(x;\phi)}{[1+(\theta-1)G(x;\phi)]^2}\left[1+\lambda-\frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]$$

For $\lambda = 0$ the geometric-G (GG) family is obtained. A random variable X having density function,

$f(x) = \frac{\theta g(x;\phi)}{[1+(\theta-1)G(x;\phi)]^2}\left[1+\lambda-\frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]$ is denoted by x$\sim TG$.

The reliability function R(x) is given by

$$R(x) = 1 - \frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\left[1+\frac{\lambda\bar{G}(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]$$

And hrf is

$$\tau(x) = \frac{\frac{\theta g(x;\phi)}{[1+(\theta-1)G(x;\phi)]^2}\left[1+\lambda-\frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]}{1-\frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\left[1+\frac{\lambda\bar{G}(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]}$$

Suppose Z_1 and Z_2 be two random variables from $\frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}$. Define

$$X = \begin{cases} Z_{1:2}\ with\ probability\ \frac{1+\lambda}{2} \\ Z_{2:2}\ with\ probability\ \frac{1-\lambda}{2} \end{cases}$$

Where $Z_{1:2} = \min(Z_1, Z_2)\ and\ Z_{2:2} = \max(Z_1, Z_2)$. The TG-G family of distribution appears to be more flexible and could be used for modeling various types of data. It can be seen that the hazard rate could take constant, increasing, decreasing, upside down and bathtub shaped. Therefore, this family of distribution could be used to model diverse nature of data sets.

1.4.4 The Transmuted Weibull-G Family of Distribution

The transmuted Weibull-G Family of Distributions was introduced by Morad Alizadeh, Mahadi Rasekhi, Haitham M. Yousof, Hamedani (2017).Let $h(x;\psi)$ and $H(x;\psi)$ denote the density and cumulative functions of the baseline model with power parameter ψ and consider the Weibull cumulative distribution function $F(x) = 1 - e^{-x^{\alpha}} for\ x > 0$ with positive parameter α. Based on this density, Bourguignon ,Siva and Cordeiro (2014) replaced the argument x by $H(x;\psi)/\bar{H}(x;\psi)$, where $\bar{H}(x;\psi) = 1 - H(x;\psi)$ and defined the cumulative distribution function of their Weibull-G class by

$H(x;\alpha) = \int_0^{\left\{\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right\}} \alpha t^{\alpha-1} \exp(-t^{\alpha})\, dt = 1 - exp\left\{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}\right\}$ Respectively, where $\psi = (\psi_k) = (\psi_1, \psi_2, \ldots)$ is a parameter vector. Based on the TG family and Weibull-G (WG) family, We have $F(x) = \left\{1 - e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}\right\}\left[1 + \lambda e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}\right]$, where $G(x;\psi)$ is the baseline cumulative distribution function, $\alpha > 0\ and\ |\lambda| \leq 1$ are two additional shape parameters. The TW- $G(.;\alpha,\lambda,\psi)$ is a wider class of continuous distributions. It includes the TG family of distributions.

Section 1:5

Notations

X	- Random variable
$\lambda, \sigma, \alpha, \beta$	- parameters
$L(\sigma, \lambda; x)$	- sample likelihood function
$l(\sigma, \lambda; x)$	- sample log-likelihood function
$\hat{\theta}$	- maximum likelihood estimator
$I^{-1}(\theta)$	- limiting variance-covariance matrix of $\hat{\theta}$
Z_α	- upper αth quantile of the standard normal distribution
$I(\theta)$	- matrix
ω	- LR test statistic
$R(t, \sigma, \lambda)$	- reliability function of a transmuted Rayleigh distribution
h(t)	- hazard rate function
$h(t, \sigma, \lambda)$	- hazard rate function for a transmuted Rayleigh distribution
$I(\hat{\theta})^{-1}$	- the variance covariance matrix of the MLE
P(t)	- probability density function
W[G(x)]	- cumulative distribution function
$G(X; \phi)$	- baseline cumulative distribution function
R(x)	- reliability function
$\tau(x)$	- hrf
$\pi_{\alpha(x)}$	- exp-G pdf with power parameter α
$\pi_{\delta(x)}$	- exp-G cdf with power parameter δ
G(x)	- cdf of TG distribution

g(x)	- pdf of TG distribution
U	- uniform variate
(qf)	- quantile function
Y_k	- exp-G distribution with power parameter k
$\phi_s(t)$	- s^{th} incomplete moment
$J_k(x)$	- first incomplete moment of exp-G distribution
$m_n(t)$	- n^{th} moment of residual life
$M_n(t)$	- n^{th} moment of reversed residual life
$I_{\delta(x)}$	- Renyi entropy of the TG-G family
$H_{\delta(x)}$	- δ entropy
$X_{i:n}$	- pdf of ith order statistic

Section 1:6

Review of Literature

Reliability of a unit is the probability that the unit performs its intended function adequatly for a given period of time under the stated operating conditions or environment. By a unit we mean an element, a system or a part of a system. Reliability analysis allows to study the properties of measurement scales and items that compose the scales. The reliability analysis procedure calculates a number of commonly used measures of scale reliability and also provides information about the relationships between individual items in the scale. In statistical inference, maximum likelihood estimation is a method of estimating the parameters of a statistical model, given observations. The method obtains the parameter estimates by finding the parameter values that maximize the likelihood function. The method of maximum likelihood is used with a wide range of statistical analyses. The order statistics is the most fundamental tools in non-parametric statistics and inference. Important special cases of the order statistics are the minimum and maximum value of a sample, and the sample median and other sample quantiles.

Siddiqui (1962) discussed the origin and properties of the Rayleigh distribution. The Rayleigh distribution is a special case of the two parameter Weibull distribution with shape parameter equal to 2 and this model was first introduced by Rayleigh (1980).

Inference for model Rayleigh model has been considered by Sinhaand and Howlader (1993) and Lalitha and Mishra (1996). The Marshall-Olkin-G family (Mo-G) by Marshall and Olkin (1997). Gupta,R.C., Gupta,P.L. and Gupta,R.D (1998) proposed the exponentiated-G class, which consists of raising the cumulative distribution function to a positive power parameter.

Eugene, Lee and Famoye (2002) discussed the beta generalized-G family and its applications. Inference for model Rayleigh model has been considered by Abd Elfattah, Hassanand and Ziedan (2006). The Transmuted family pioneered by Shaw and Buckley (2007). Aryal and Tsokos (2009) studied the transmuted Gumbel distribution and it has been observed that the transmuted Gumbel distribution can be used to model climate data. Shaw and Buckley (2009) studied the quadratic rank transmutation map. Zografos and Balakrishnan (2009) discussed on families of beta and generalized gamma-generated distributions and its associated inference.

The new family of distribution called Kumarasamy-G family (Kw-G) was defined by Cordeiro and de Castro (2011). The Mc Donald-G family (Mc-G) presented by Alexander, Cordeiro, Ortega and Sarabia (2012). A new method for generating families of continuous distributions was proposed by Alzaatreh,

Lee and Famoye (2013). The exponentiated T-X family of distributions by Alzaghal, Famoye and Lee (2013). The exponentiated generalized-G family was discussed by Cordeiro, Ortega and da Cunha (2013). Faton Merovci (2013) generalizes the Rayleigh distribution using the quadratic rank transmutation map. Ahmad, Ahmadand and Ahmed (2014) develops the transmuted inverse Rayleigh distribution and discussed its properties. The Lomax generator of distributions and its properties, minification process and regression model was presented by Cordeiro, Ortega, Popovic and Pescim (2014).

The Kumaraswamy odd log-logistic family of distributions and its properties and applications was discussed by Alizadeh, Emadi, Doostparast, Cordeiro, Ortega and Pescim (2015b).The beta odd log-logistic family of distribution was generalized by Cordeiro, Alizadeh, Tahir, Mansoor, Bourguigon and Hamedani (2015). Many other families could be cited such as generalized transmuted-G family (GT-G) by Nofal, Afify, Yousof and Cordeiro (2015), transmuted exponentiated generalized family of distributions (TEx-G) by Yousof, Afify, Alizadeh, Butt, Hamedani and Ali (2015), beta Marshall-Olkin by Alizadeh, Cordeiro, de Brito and Demetrio(2015a), Kumaraswamy Marshall-Olkin by Alizadeh, Cordeiro, Mansoor, Zubair and Hamedani (2015c), generalized transmuted family of distributions by Alizadeh, Merovci and Hamedani (2015d) and another generalized transmuted family of distributions by Merovci, Alizadeh and Hamedani (2015).

Several generalized families of continuous distributions had been proposed and applied to model various phenomena. However, there is a clear need for extended forms of the well-known distributions by adding one or more shape parameter(s) in order to obtain greater flexibility in modeling various data. Some well-known families are Kumaraswamy transmuted-G family (KwTG) by Afify, Cordeiro, Yousof, Alzaatreh and Nofal(2016,2016b), transmuted geometric-G of distributions by Afify, Alizadeh, Yousof, Aryal and Ahmad (2016a), beta transmuted-H family by Afify, Yousof and Nadarajah (2016c), Zografos –Balakrishnan odd log-logistic family by Cordeiro, Alizadeh, Ortega and Serrano (2016), type I half-logistic family by Cordeiro, Alizadeh and Diniz Marinho (2016) Burr X-G by Yousof, Afify, Hamedani and Aryal (2016), exponentiated transmuted-G family by Merovci, Alizadeh, Yousof and Hamedani (2016), the odd-Burr generalized family by Alizadeh, Cordeiro, Nascimento, Lima and Ortega (2016a) and Complementary generalized transmuted poisson family by Alizadeh, Yousof, Afify, Cordeiro and Mansoor (2016b).

Chapter - 2

Transmuted Rayleigh Distribution

In this first chapter Transmuted Rayleigh Distribution by Faton Merovci [2013] has been reviewed.

This chapter deals with generalization of the Rayleigh distribution using the quadratic rank transmutation map to develop a transmuted Rayleigh distribution. The comprehensive description of the mathematical properties of the subject distribution along with its reliability behaviour has been provided. The usefulness of the transmuted Rayleigh distribution for modeling data is illustrated using real data.

In many applied sciences such as medicine, engineering and finance, amongst others, modeling and analyzing lifetime is crucial. Several lifetime distributions had been used to model such kind of data. The quality of the procedure used in a statistical analysis depends heavily on the assumed probability model or distribution. Because of this, considerable effort has been expended in the development of large classes of standard probability distribution along with relevant statistical methodologies. However, there will remain many important problems where the real data does not follow any of the classical or standard probability models. A new generalization of the Rayleigh distribution called transmuted Rayleigh distribution foe modeling data is provided.

Definition:2.1

A random variable X is said to have a transmuted distribution if its cumulative distribution (cdf) is given by

$$G(x) = (1+\lambda)F(x) - \lambda F^2(x), |\lambda| \leq 1$$

Where F(x) is the cumulative distribution function of the base distribution.

Definition:2.2

The density function (pdf) of a Rayleigh distribution is

$$f(x,\sigma) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right), x > 0, \sigma > 0,$$

and the respective cumulative distribution function is

$$F(x,\sigma) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right), x > 0, \sigma > 0.$$

The transmuted cumulative distribution function is

$$G(x,\sigma,\lambda) = \left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\left(1 + \lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right),$$

with transmuted probability density function

$$g(x,\sigma,\lambda) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right).$$

The transmuted Rayleigh distribution is an extended model to analyze more complex data. The Rayleigh distribution is clearly a special case for $\lambda = 0$.

The following Figure 2.1 and Figure 2.2 illustrates some of the possible shapes of the probability density function and cumulative distribution function of a transmuted Rayleigh distribution for selected values of the parameters λ and σ respectively.

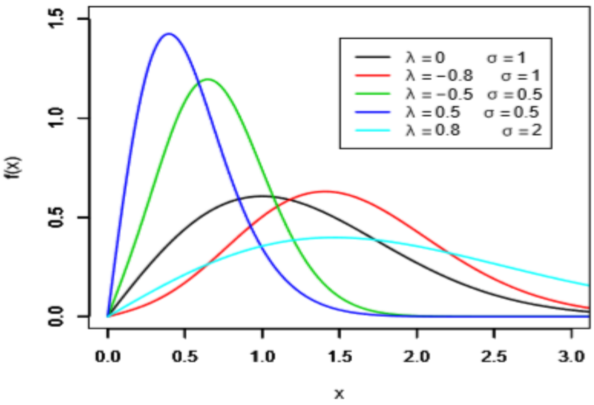


Figure 2.1: The probability density functions of various transmuted Rayleigh distribution.

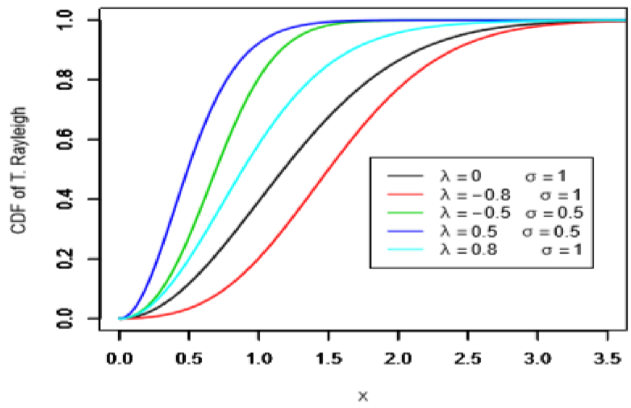


Figure 2.2: The cumulative distribution functions of various transmuted Rayleigh distributions.

2.1 Moments

Theorem 2.1.1: The rth moment $E(X^r)$ of a transmuted Rayleigh distribution random variable X is given as

$$E(X^r) = \frac{1}{2}\sigma^r r\Gamma\left(\frac{r}{2}\right)\left(\lambda + 2^{\frac{r}{2}}(1-\lambda)\right)$$

Especially we have

$$E(X) = \frac{1}{2}\sigma\sqrt{\pi}\left(\lambda + \sqrt{2}(1-\lambda)\right),$$

$$\mathrm{var}(X) = E(X^2) - E^2(X) = \sigma^2\left(2 - \lambda - \frac{\pi}{4}\left(\lambda + \sqrt{2}(1-\lambda)\right)^2\right).$$

Proof.

$$E(X^r) = \int_0^\infty x^r f(x)dx$$

$$=\int_0^\infty \frac{x^{r+1}}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1-\lambda+2\lambda\exp\left(-\frac{x^2}{2\sigma^2}\right)\right)dx$$

$$=\frac{(1-\lambda)}{\sigma^2}\int_0^\infty x^{r+1}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx$$

$$+\frac{2\lambda}{\sigma^2}\int_0^\infty x^{r+1}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx$$

$$=2^{\frac{r-2}{2}}(1-\lambda)\sigma^r\Gamma\left(\frac{r}{2}\right)+\frac{\lambda r}{2}\sigma^r\Gamma\left(\frac{r}{2}\right)$$

$$=\frac{1}{2}\sigma^r r\Gamma\left(\frac{r}{2}\right)\left(\lambda+2^{\frac{r}{2}}(1-\lambda)\right)$$

According to Gradshtein and Ryzhnik(2000)[36] ,

$$\int_0^\infty x^{v-1}\exp(-\mu x^p)\,dx=\frac{1}{p}\mu^{-\frac{v}{p}}\Gamma\left(\frac{v}{p}\right),\quad \text{for } p, v, \mu>0.$$

Theorem 2.1.2:

Let X have a transmuted Rayleigh distribution. Then the moment generating function of X , say $M_X(t)$, is

$$M_x(t)=\sum_{i=0}^{\infty}\frac{t^i}{i!}\frac{1}{2}\sigma^i\Gamma\left(\frac{i}{2}\right)\left(\lambda+2^{\frac{i}{2}}(1-\lambda)\right).$$

Proof

$$M_x(t)=E(e^{tx})=\int_0^\infty \exp(tx)\,f(x)dx$$

$$=\int_0^\infty\left(1+tx+\frac{t^2x^2}{2!}+\cdots+\frac{t^nx^n}{n!}+\cdots\right)f(x)dx$$

$$=\sum_{i=0}^{\infty}\frac{t^iE(X^i)}{i!}$$

$$=\sum_{i=0}^{\infty}\frac{t^i}{i!}\frac{1}{2}\sigma^i\Gamma\left(\frac{i}{2}\right)\left(\lambda+2^{\frac{i}{2}}(1-\lambda)\right)$$

2.2 Parameter Estimators

The maximum likelihood estimation method is used to estimate parameters of the transmuted Rayleigh distribution. The likelihood function under this model is

$$L(\sigma,\lambda;x)=\prod_{i=1}^{n}g(x_i,\sigma,\lambda)$$

$$=\frac{\prod_{i=1}^{n}x_i}{\sigma^{2n}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}{x_i}^2\right)\text{ x}$$

$$\prod_{i=1}^{n}\left(1-\lambda+2\lambda\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)\right) \tag{2.1}$$

The log-likelihood function is

$$l(\sigma,\lambda;x)=\log L(\sigma,\lambda;x)$$

$$=\sum_{i=1}^{n}\log(x_i)-n\log\sigma^2-\frac{1}{2\sigma^2}\sum_{i=1}^{n}{x_i}^2$$

$$+\sum_{i=1}^{n}\left(1-\lambda+2\lambda\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)\right).$$

The first partial derivatives of any function f(x,y) are denoted by f_x and f_y , and its second partial derivatives by f_{xx} , f_{yy} , f_{xy} , and f_{yx}.

Now setting $l_\sigma=0$ and $l_\lambda=0$,

We have $-\frac{2n}{\sigma}+\frac{1}{\sigma^3}\sum_{i=1}^{n}{x_i}^2+\frac{2\lambda}{\sigma^3}\sum_{i=1}^{n}\frac{{x_i}^2\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)}{1-\lambda+2\lambda\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)}=0.$ (2.2)

and $\sum_{i=1}^{n}\frac{2\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)-1}{1-\lambda+2\lambda\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)}=0$ (2.3)

The MLE $\hat{\theta}=(\hat{\sigma},\hat{\lambda})$ of $\theta=(\sigma,\lambda)$ is obtained by solving this nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the sample likelihood function given in (2.1). Applying the usual large sample approximation, the MLE $\hat{\theta}$ can be treated as being approximately bivariate normal with mean θ and variance-covariance matrix equal to the inverse of the expected information matrix, i.e.

$$\sqrt{n}(\hat{\theta}-\theta)\longrightarrow N\left(0,nI^{-1}(\theta)\right),$$

Where $I^{-1}(\theta)$ is the limiting variance-covariance matrix of $\hat{\theta}$. The elements of the 2x2 matrix I(θ) can be estimated by $I_{ij}(\hat{\theta})=-l_{\theta_i\theta_j|\theta=\hat{\theta}}, i,j\in\{1,2\}$.

From (2.2) and (2.3) the second partial derivatives of the log-likelihood function are found to be

$$l_{\sigma\sigma}=\frac{2n}{\sigma^2}-\frac{3}{\sigma^4}\sum_{i=1}^{n}{x_i}^2-\frac{6}{\sigma^4}\sum_{i=1}^{n}\frac{{x_i}^2\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)}{1-\lambda+2\lambda\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)}$$

$$+\frac{2\lambda(1-\lambda)}{\sigma^6}\sum_{i=1}^{n}\frac{{x_i}^4\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)}{\left(1-\lambda+2\lambda\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)\right)^2}$$

$$l_{\sigma\lambda}=\frac{2}{\sigma^3}\sum_{i=1}^{n}\frac{{x_i}^2\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)}{\left(1-\lambda+2\lambda\exp\left(-\frac{{x_i}^2}{2\sigma^2}\right)\right)^2},$$

$$l_{\lambda\lambda} = -\sum_{i=1}^{n}\left(\frac{2\exp\left(-\frac{x_i^2}{2\sigma^2}\right)-1}{1-\lambda+2\lambda\exp\left(-\frac{x_i^2}{2\sigma^2}\right)}\right)^2.$$

Approximate two sided 100(1-α)% confidence intervals for λ and for σ are, respectively, given by

$$\hat{\sigma} \pm Z_{\alpha/2}\sqrt{I_{11}{}^{-1}(\hat{\theta})} \text{ and } \hat{\lambda} \pm Z_{\alpha/2}\sqrt{I_{22}{}^{-1}(\hat{\theta})},$$

where Z_α is the upper α^{th} quantile of the standard normal distribution. Using LR, the Hessian matrix and its inverse can be easily computed and hence the standard errors and asymptotic confidence intervals. The LR statistic is used to check whether the transmuted Rayleigh distribution for a given data set is statistically superior to the Rayleigh distribution. In any case, hypothesis tests of the type $H_0: \theta = \theta_0$ versus $H_0: \theta \neq \theta_0$ can be performed using LR test. In this case, the LR statistic for testing H_0 versus H_1 is $\omega = 2\left(l(\hat{\theta}; x)\right) - l(\hat{\theta}_0; x)$,where $\hat{\theta}$ and $\hat{\theta}_0$ are the MLEs under H_1 and H_0, respectively. The statistic ω is asymptotically (as $n \to \infty$) distributed as x_k^2 , where k is the length of the parameter vector θ of interest. The LR test rejects H_0 if $\omega > x_{k;\gamma}^2$,where $x_{k;\gamma}^2$ denotes the upper 100 γ% quantile of the x_k^2 distribution.

2.4 Reliability Analysis

The reliability function R(t) is defined by R(t) = 1 - F(t). The reliability function of a transmuted Rayleigh distribution is given by $R(t,\sigma,\lambda) = \left(1 - \lambda + \lambda\exp\left(-\frac{t^2}{2\sigma^2}\right)\right)$

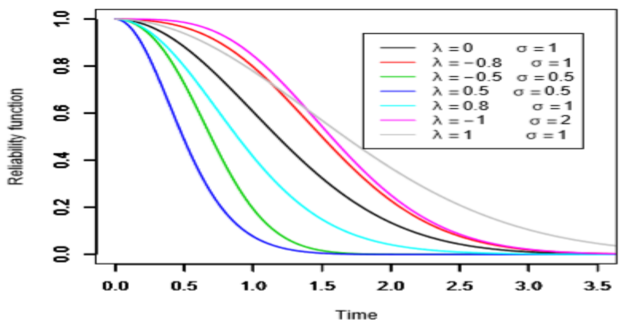


Figure 2.3 illustrates the reliability transmuted Rayleigh distribution for different values of the parameters λ and σ.

2.5 Order Statistics

The kth order statistic of a sample is its kth smallest value. For a sample of size n, the nth order statistic (or largest order statistic) is the maximum, that is $X_{(n)} = \max\{X_1, \dots, X_n\}$. The sample range is the difference between the maximum and minimum. It is clearly a function of the order statistics:

$$\text{range}\{X_1, \dots, X_n\} = X_{(n)} - X_{(1)}$$

we know that if $X_{(1)} \leq \cdots \leq X_{(n)}$ denotes the order statistic of a random sample $X_1, \dots, X_n$ from a continuous population with cumulative distribution function $F_X(x)$ and probability density function $f_X(x)$ then the probability density function of $X_{(j)}$ is given by

$$f_{X_{(j)}} = \frac{n!}{(j-1)!(n-j)!} f_X(x)(F_X(x))^{j-1}(1 - F_X(x))^{n-j} \text{ for } j = 1,\dots,n.$$

The probability density function of the *j*th order statistic for a transmuted Rayleigh distribution is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)$$

$$.\left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\left(1 + \lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)^{j-1}$$

$$.\left(\exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1 - \lambda + \lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\right)^{n-j}$$

Therefore , the probability density function of the largest order statistic $X_{(n)}$ is

$$f_{X_{(n)}}(x) = \frac{nx}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)$$

$$.\left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\left(1 + \lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)^{n-1}$$

and the probability density function of the smallest order statistic $X_{(1)}$ is

$$f_{X_{(n)}}(x) = \frac{nx}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)$$

$$.\left(\exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\left(1 - \lambda + \lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)^{n-1}$$

2.6 Application

A real data set has been considered to show that the transmuted Rayleigh distribution can be a better model than the Rayleigh distribution. Nicotine measurements made from several brands of cigarettes in 1998 were used. The data has been collected by the Federal Trade Commission which is an independent agency of the US government, whose main mission is the promotion of consumer protection.

The variance covariance matrix of the MLEs under the transmuted Rayleigh distribution is computed as

$$I(\hat{\theta})^{-1} = \begin{pmatrix} 0.183 \text{ x } 10^{-3} & 0.470 \text{ x } 10^{-3} \\ 0.470 \text{ x } 10^{-3} & 0.529 \text{ x } 10^{-2} \end{pmatrix}.$$

Thus, the variances of the MLE of σ and λ is $\text{var}(\hat{\sigma}) = 0.183 \text{ x } 10^{-3}$ and $\text{var}(\hat{\lambda}) = 0.529 \text{ x } 10^{-2}$. Therefore, 95% confidence intervals for σ and λ are $[-0.528, 0.582]$ and $[-0.914, -0.629]$.

Estimated parameters of the Rayleigh and Transmuted Rayleigh distribution for the nicotine measurements data are presented in Table 2.1.

Table 2.1:

Model	Parameter Estimate	Standard Error	$-\ell(\cdot; x)$
Transmuted	$\hat{\sigma} = 0.5555$	0.0135	121.224
Rayleigh	$\hat{\lambda} = -0.7718$	0.0728	
Rayleigh	$\hat{\sigma} = 0.6475$	0.0175	142.3572

The LR test statistic to test the hypotheses $H_0: \theta = \theta_0$ versus $H_0: \theta \neq \theta_0$ is $= 42.2664 > 3.841 = x^2_{1;0.05}$, so we reject the null hypothesis.

The Empirical, fitted Rayleigh, and transmuted Rayleigh cdf of the nicotine measurements data are presented in the following Figure 2.4

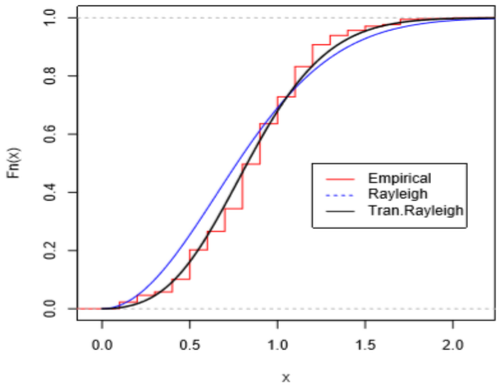


Figure 2.4: Empirical, fitted Rayleigh, and transmuted Rayleigh cdf of the nicotine measurements data.

The Criteria for comparison is given in the following Table 2.2

Table 2.2:

Model	KS	-2ℓ	AIC	AICC	BIC
Rayleigh	0.184	284.714	286.714	285.714	296.407
Transmuted Rayleigh	0.124	242.448	246.448	243.445	265.833

In order to compare the two distribution models, consider criteria like KS (Kolmogorow smirnow), -2*l*, AIC (Akaike information criterion), AICC (corrected Akaike information criterian), and BIC (Bayesian information criterian) for the data set. The better distribution corresponds to smaller KS, -2*l*, AIC, AICC, and BIC values:

AIC = 2k-2*l*, AICC = AIC + $\frac{2k(k-1)}{n-k-1}$, and BIC = klog(n)- 2*l*,

Where k is the number of parameters in the statistical model , n the sample size and *l* is the maximized value of the log-likelihood function under the considered model. Also, here for calculating the values of KS, the sample estimates of λ andσ are used. Table 2.1 shows the MLEs under both distributions, Table 2 shows the values of KS, -2*l*, AIC, AICC, and BIC values. The values in Table 2.2 indicate that the transmuted Rayleigh distribution leads to a better fit than the Rayleigh distribution.

Chapter - 3

Characterization and Estimation of Transmuted Rayleigh Distribution

In this chapter, characterization and estimation of transmuted Rayleigh distribution by Afaq Ahmad, S.P Ahmad and A.Ahmed July (2015) has been reviewed.

This chapter deals with the introduction of a new class of transmuted Rayleigh distribution. The estimates of parameters of transmuted Rayleigh distribution were obtained by using new method of moments. A new distribution which contains as a special case is introduced. The characterizing properties of the model are also derived.

Rayleigh distribution (RD) has been considered to be a very useful life distribution. Rayleigh distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences. One major application of this model is used in analyzing wind speed data. This distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 2. This model was first introduced by Rayleigh in 1980 [38], Siddiqui (1962) [40] discussed the origin and parameters of the Rayleigh distribution. Inference for model Rayleigh model has been considered by Sinha and Howlader (1993) [48], Lalitha and Mishra (1996) [47] and Abd Elfattah, Hassanand, Ziedan (2006) [18]. Faton Merovci (2013)[35] generalizes the Rayleigh distribution using the quadratic rank transmutation map studied by Shaw and Buckley (2009)[50] and named it transmuted Rayleigh distribution. Ahmad, Ahmadand and Ahmed(2014)[1] develops the transmuted inverse distribution and discussed its properties. The probability density function (pdf) of Rayleigh distribution is given as :

$$g(x,\theta) = \frac{x}{\theta^2} exp\left(-\frac{x}{2\theta^2}\right), \; x > 0, \theta > 0 \qquad (3.1)$$

And its corresponding cumulative distribution function (cdf) is given by

$$G(x,\theta) = 1 - exp\left(-\frac{x}{2\theta^2}\right), \; x > 0, \theta > 0 \qquad (3.2)$$

Merovci (2013)[35] used the quadratic rank transmutation map (QRTM) for a pair of distributions F(x) and G(x) where G(x) is a submodel of F(x). Therefore, a random variable X is said to have transmuted probability distribution with cumulative distribution function F(x) if

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x) \quad , |\lambda| \le 1$$

which on differential yields

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)]$$

where G(x) and g(x) is the cumulative distribution function and probability density function of the base distribution. Observe that at $\lambda = 0$, the distribution of the base random variable.

Hence, the probability density function of transmuted Rayleigh distribution with parameters $\theta \ and \ \lambda$ is

$$f(x,\theta,\lambda) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1-\lambda+2\lambda\exp\left(-\frac{x^2}{2\sigma^2}\right)\right) \qquad (3.3)$$

and the corresponding cumulative distribution function is given by

$$F(x,\theta,\lambda) = \left(1-\exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\left(1+\lambda\exp\left(-\frac{x^2}{2\sigma^2}\right)\right) \qquad (3.4)$$

The Rayleigh distribution is clearly a special case for $\lambda = 0$.

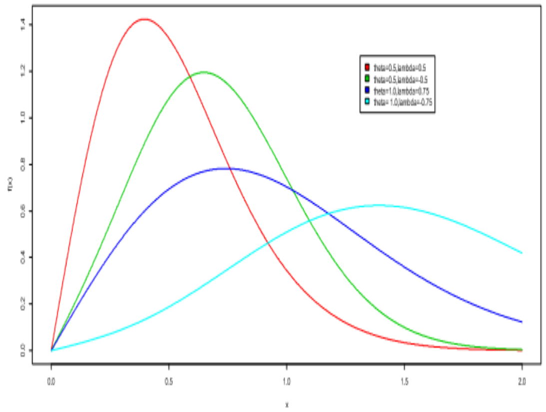


Figure 3.1: The probability density function of various transmuted Rayleigh distributions under various values of parameters.

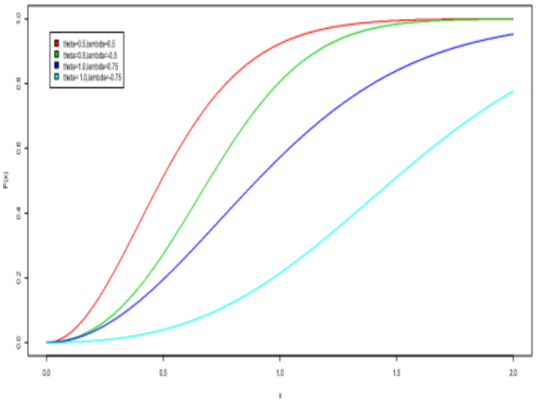


Figure 3.2: The cumulative distribution function of various transmuted Rayleigh distributions under various values of parameters

Figure 3.1 and 3.2 shows some of the possible shapes of the probability density function and cumulative distribution function of transmuted Rayleigh distribution for selected values of parameters θ *and* λ respectively.

3.1 Statistical properties of transmuted Rayleigh distribution (TRD)

The statistical properties of transmuted Rayleigh distribution throughout computing the mean, variance, coefficient of variation, harmonic mean, moments, mode, coefficient of skewness, and coefficient of kurtosis are as follows

3.1.1 Moments of transmuted Rayleigh distribution (TRD)

The rth moment of transmuted Rayleigh distribution is given by

$$E(X^r) = \mu_r' = \int_0^\infty x^r f(x;\theta,\lambda)dx$$

$$= \int_0^\infty x^r \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right)\left(1-\lambda+2\lambda\exp\left(-\frac{x^2}{2\theta^2}\right)\right)dx$$

$$= \frac{(1-\lambda)}{\theta^2}\int_0^\infty x^{r+1} \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right)dx$$

$$+ \frac{2\lambda}{\theta^2}\int_0^\infty x^{r+1} \exp\left(-\frac{x^2}{\theta^2}\right)dx$$

$$= 2^{\frac{r-2}{2}}(1-\lambda)\theta^r r\Gamma\left(\frac{r}{2}\right) + \frac{\lambda r}{2}\theta^r\Gamma\left(\frac{r}{2}\right)$$

$$\Rightarrow \mu_r' = \frac{1}{2}\theta^r r\Gamma\left(\frac{r}{2}\right)\left(\lambda + 2^{\frac{r}{2}}(1-\lambda)\right) \quad (3.1)$$

Put r=1 in (3.1) mean of the transmuted Rayleigh distribution is given by

$$\mu = \mu_1' = \frac{1}{2}\theta\sqrt{\pi}\left(\lambda + \sqrt{2}(1-\lambda)\right) \quad (3.2)$$

Put r=2,3 and 4 in (3.1), second third and fourth moment of transmuted Rayleigh distribution are as follows

$$\mu_2' = \theta^2\left(\lambda + 2(1-\lambda)\right)$$

$$\mu_3' = \frac{3}{4}\theta^3\sqrt{\pi}\left(\lambda + 2^{\frac{3}{2}}(1-\lambda)\right)$$

$$\mu_4' = 2\theta^4\left(\lambda + 4(1-\lambda)\right)$$

3.1.2 variance of transmuted Rayleigh distribution

The variance of transmuted Weibull distribution is given by

$$\mu_2 = \mu_2' - (\mu_1')^2 = \theta^2\left(2 - \lambda - \frac{\pi}{4}\left(\lambda + \sqrt{2}(1-\lambda)\right)^2\right) \quad (3.3)$$

3.1.3 Third and fourth moments of transmuted Rayleigh distribution

$$\mu_3 = \mu_3' - 2\,\mu_2'\,\mu_1' + 2\left(\mu_1'\right)^3$$

$$= \frac{3}{4}\theta^3\sqrt{\pi}\left(\sigma_3 - 2\sigma_2\sigma_1 + \frac{\pi}{3}\sigma_1^3\right)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_2' + 6\mu_2'\mu_1' - 3(\mu_1')^4$$

$$= 2\theta^4\sigma_4 - 3\theta^4\sqrt{\pi}\sigma_3\sigma_2 + 3\theta^3\sqrt{\pi}\sigma_2\sigma_1 - \frac{3}{16}\theta^4\pi^2\sigma_1^4$$

3.1.4 standard deviation

$$\sigma = \sqrt{\mu_2} = \theta\left(2 - \lambda - \frac{\pi}{4}\left(\lambda + \sqrt{2}(1-\lambda)\right)^2\right)^{\frac{1}{2}} \quad (3.4)$$

3.1.5 coefficient of variation

$$C.V = \frac{\sigma}{\mu} = \frac{\left(2-\lambda-\frac{\pi}{4}\left(\lambda+\sqrt{2}(1-\lambda)\right)^2\right)^{\frac{1}{2}}}{\frac{\sqrt{\pi}}{2}\left(\lambda+\sqrt{2}(1-\lambda)\right)} \quad (3.5)$$

3.1.6 Skewness and Kurtosis

The most popular way to measure the skewness and kurtosis of a distribution function rests upon ratios of moments. Lack of symmetry of tails (about mean) of frequency distribution curve is known as skewness. The formula for measure of skewness and kurtosis given by Karl Pearson in terms of moments of frequency distribution is given by

$$\text{skewness} = \beta_1 = \frac{\mu_3^2}{\mu_2^3} \text{ and kurtosis} = \beta_2 = \frac{\mu_4^2}{\mu_2^2}$$

$$\beta_1 = \frac{\left(\frac{3}{4}\theta^3\sqrt{\pi}\left(\sigma_3 - 2\sigma_2\sigma_1 + \frac{\pi}{3}\sigma_1^3\right)\right)^2}{\left(\theta^2\left(2-\lambda-\frac{\pi}{4}\sigma_2^2\right)\right)^3},$$

$$\beta_2 = \frac{2\theta^4\sigma_4 - 3\theta^4\sqrt{\pi}\sigma_3\sigma_2 + 3\theta^3\sqrt{\pi}\sigma_2\sigma_1 - \frac{3}{16}\theta^4\pi^2\sigma_1^4}{\left(\theta^2\left(2-\lambda-\frac{\pi}{4}\sigma_2^2\right)\right)^2}.$$

3.2 New moment Estimator of parameters of Transmuted Rayleigh distribution

The new moment based on moments, using it's characteriazation for estimation of parameters of transmuted Rayleigh distribution is used. The result shows that this new method is easy and more efficient than MLE method in small sample.

Theorem 3.1:

Let n≥3 and let $X_1, X_2, X_3 \ldots X_n$ be an positive identical independently distributed random variables having a probability density function f(x). Then the independence of the sample mean $\bar{X}_n$and the sample coefficient of variation $V_n = \frac{S_n}{\bar{X}_n}$ is equivalent to that f(x) is a transmuted Rayleigh density where S_n is the sample standard deviation.

The next result and Theorem 3.1 are useful in deriving the expectation and variance of

${V_n}^2 = \left(\frac{S_n}{\bar{X}_n}\right)^2$, where $\bar{X}_n$ and S_n are respectively the sample mean and the sample standard deviation.

Theorem 3.2:

Let $n \geq 3$ and let $X_1, X_2, X_3 \ldots X_n$ be an positive identical independently distributed random samples drawn from a population having a transmuted Rayleigh density

$$f(x, \theta, \lambda) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)$$

then

$$E\left(\overline{X}_n{}^2\right) = \frac{\theta^2}{n}\left(2 - \lambda - \frac{\pi}{4}\left(\lambda + \sqrt{2}(1-\lambda)\right)^2 (1-n)\right)$$

$$E\left(S_n{}^2\right) = \theta^2\left(2 - \lambda - \frac{\pi}{4}\left(\lambda + \sqrt{2}(1-\lambda)\right)^2\right)$$

Where $\bar{X}_n$ and $S_n{}^2$ are respectively their sample mean and sample variance.

Proof: It is easy to prove that

$$E(X) = \frac{1}{2}\theta\sqrt{\pi}\left(\lambda + \sqrt{2}(1-\lambda)\right),$$

$$Var(X) = \theta^2\left(2 - \lambda - \frac{\pi}{4}\left(\lambda + \sqrt{2}(1-\lambda)\right)^2\right)$$

$$E(X^m) = \frac{1}{2}\theta^m m\Gamma\left(\frac{m}{2}\right)\left(\lambda + 2^{\frac{m}{2}}(1-\lambda)\right)$$

$$Var(\overline{X}_n) = \frac{\theta^2\left(2 - \lambda - \frac{\pi}{4}\left(\lambda + \sqrt{2}(1-\lambda)\right)^2\right)}{n}$$

$$E\left(\overline{X}_n{}^2\right) = \frac{\theta^2\left(2 - \lambda - \frac{\pi}{4}\left(\lambda + \sqrt{2}(1-\lambda)\right)^2 (n-1)\right)}{n} \qquad (3.6)$$

Now, $E\left(S_n{}^2\right) = nVar(\overline{X}_n)$

$$E\left(S_n{}^2\right) = \theta^2\left(2 - \lambda - \frac{\pi}{4}\left(\lambda + \sqrt{2}(1-\lambda)\right)^2\right) \qquad (3.7)$$

Theorem 3.3:

Let $n \geq 3$ and let $X_1, X_2, X_3 \ldots X_n$ be an positive identical independently distributed random samples drawn from a population having a transmuted Rayleigh density

$$f(x, \theta, \lambda) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)$$

then $E\left(\frac{S_n{}^2}{\bar{X}_n{}^2}\right) = \frac{n\left(2-\lambda-\frac{\pi}{4}\left(\lambda+\sqrt{2}(1-\lambda)\right)^2\right)}{\left(2-\lambda\frac{\pi}{4}\left(\lambda+\sqrt{2}(1-\lambda)\right)^2(n-1)\right)}$

Where $\bar{X}_n$ and $S_n{}^2$ are respectively their sample mean and sample variance.

Proof: By Theorem 3.1,we have

$$E\left(S_n{}^2\right) = E\left(\frac{S_n{}^2}{\bar{X}_n{}^2}\bar{X}_n{}^2\right) E\left(\frac{S_n{}^2}{\bar{X}_n{}^2}\right) E(\bar{X}_n{}^2)$$

and hence

$$E\left(\frac{S_n{}^2}{\bar{X}_n{}^2}\right) = \frac{E\left(S_n{}^2\right)}{E(\bar{X}_n{}^2)}$$

Applying theorem 3.2 to the above identity yields that

$$E\left(\frac{S_n{}^2}{\bar{X}_n{}^2}\right) = \frac{n\left(2-\lambda-\frac{\pi}{4}\left(\lambda+\sqrt{2}(1-\lambda)\right)^2\right)}{\left(2-\lambda\frac{\pi}{4}\left(\lambda+\sqrt{2}(1-\lambda)\right)^2(n-1)\right)} \quad (3.8)$$

The Theorem 3.3 is established.

Note that $E\left(\frac{S_n{}^2}{\bar{X}_n{}^2}\right) \to \frac{\left(2-\lambda-\frac{\pi}{4}\left(\lambda+\sqrt{2}(1-\lambda)\right)^2\right)}{\frac{\pi}{4}\left(\lambda+\sqrt{2}(1-\lambda)\right)^2}$ as $n \to \infty$ and that this limit is the square of the coefficient of variation. Thus $\left(\frac{S_n{}^2}{\bar{X}_n{}^2}\right)$ is an asymptotically unbiased estimator of the square of the coefficient of variation.

Based on Theorems 3.2 and 3.3 and by using moment estimation approach, two equations for finding two estimators $(\hat{\theta}, \hat{\lambda})$ of parameters (θ, λ) respectively as follows:

$$\frac{\sum_{i=1}^{n} x_i}{n} = \frac{1}{2}\theta\sqrt{\pi}\left(\lambda + \sqrt{2}(1-\lambda)\right) \quad (3.9)$$

$$\frac{S_n{}^2}{n\bar{X}_n{}^2} = \frac{\left(2-\lambda-\frac{\pi}{4}\left(\lambda+\sqrt{2}(1-\lambda)\right)^2\right)}{\left(2-\lambda\frac{\pi}{4}\left(\lambda+\sqrt{2}(1-\lambda)\right)^2(n-1)\right)} \quad (3.10)$$

Thus the solutions of $(\theta, \lambda)$are obtained by solving the two equations (3.9) and (3.10) simultaneously are proposed for their estimators. The estimation of parameters of transmuted Rayleigh distribution were obtained by using new method of moments. A new distribution contains as a special case was introduced. The characterizing properties of the model were also determined.

Chapter - 4

The Transmuted Geometric-G Family of Distributions: Theory and Applications

In this chapter, The Transmuted Geometric-G family of distributions: Theory and applications by Ahmed, Afify, Morad Alizadeh, Haitham M. Yousof, Gokarna Aryal and Munir Ahmad [2016] has been reviewed.

This chapter deals with the introduction of a new family of continuous distributions called the transmuted geometric-G family. Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, generating function, Renyi and Shannon entropies, order statistics and probability weighted moments are also derived. Some special models of the new family has been provided. The maximum likelihood method is used for estimating the model parameters. The importance and flexibility of the proposed family are illustrated by two applications to real data sets.

Recently, several generalized families of continuous distributions had been proposed and applied to model various phenomena. However, there is a clear need for extended forms of well-known distributions by adding one or more shape parameter in order to obtain greater flexibility in modeling various data.

Some well known families are the Marshall-Olkin-G (MO-G) by Marshall and Olkin, (1997)[41], the beta-G (B-G) by Eugene, Lee and Famoye, (2002)[34], the transmuted-G (T-G) by Shaw and Buckley, (2007)[49], the kumaraswamy-G (Kw-G) by Cordeiro and de Castro, (2011)[22], the McDonald-G (Mc-G) by Alexandar, Cordeiro, Ortega and Sarabia, (2012)[9], the gamma-G by Zografos and Balakrishnan, (2009)[56], the Kumaraswamy odd log-logistic-G (KwOLL-G) by Alizadeh, Emadi, Doostparast, Cordeiro, Ortega and Pescim, (2015)[13], the beta odd log-logistic generalized by Cordeiro, Alizadeh, Tahir, Mansoor, Bourguignon and Hamedani, (2015)[25], the generalized transmuted-G (GT-G) by Nofal, Afify, Yousof and Cordeiro, (2015)[45], the transmuted exponentiated generalized-G (TExG-G) by Yousof, Afify, Alizadeh, Butt, Hamedani and Ali, (2015)[54] and the Kumaraswamy transmuted-G family (Kw-TG) by Afify, Cordeiro, Yousof, Alzaatreh and Nofal, (2016)[3].

Let P(t) be the probability density function (pdf) of a random variable $T \in [a, b]$ $for -\infty < a < b < \infty$ and W[G(x)] be a function of the cumulative distribution function (cdf) of a random variable X such that W[G(x)] satisfies the following condition

I. $\mathrm{W[G(x)]} \in [a, b]$,
II. W[G(x)] is differentiable and monotonically nondecreasing and
III. $\mathrm{W[G(x)]} \rightarrow$ a as x → ∞ and $\mathrm{W[G(x)]} \rightarrow$ b as x → ∞

(4.1)

Recently, Alzaatreh, Lee and Famoye (2013)[16] defined the T-X family of distributions by

$$F(x) = \int_a^{\mathrm{W[G(x)]}} p(t)dt \tag{4.2}$$

Where W[G(x)] satisfies conditions (4.1). The probability density function corresponding to (4.2) is given by

$$f(x) = \left\{\frac{d}{dx}\mathrm{W[G(x)]}\right\} p\{\mathrm{W[G(x)]}\} \tag{4.3}$$

The objective of this study is to define a new family of distributions called the transmuted geometric-G (TG-G for short) family of distributions and study its mathematical properties.

Based on the T-X family, construct a new generator by taking

$$\mathrm{W[G(x)]} = \frac{\theta G(x)}{1+(\theta-1)G(x)} \; and \; p(t) = 1 + \lambda - 2\lambda t \, , 0 < t < 1.$$

Then, the cumulative distribution function of the TG-G family is given by

$$F(x) = \int_0^{\frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}} (1 + \lambda - 2\lambda t\,)dt$$

$$= \frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)} \left[1 + \frac{\lambda \bar{G}(x;\phi)}{1+(\theta-1)G(x;\phi)}\right] \tag{4.4}$$

where $G(x; \phi)$ is the baseline cumulative distribution function and $\theta > 0$ and $|\lambda| \leq 1$ are two additional shape parameters. The TG-G is a wider class of continuous distributions. It includes the transmuted-G family of distributions and geometric-G family.

4.1 The TG-G Family

The probability density function of (4.4) is given by

$$f(x) = \frac{\theta g(x;\phi)}{[1+(\theta-1)G(x;\phi)]^2}\left[1+\lambda-\frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right] \qquad (4.5)$$

For $\lambda = 0$ the geometric-G (GG) family is obtained. A random variable X having density function $f(x) = \frac{\theta g(x;\phi)}{[1+(\theta-1)G(x;\phi)]^2}\left[1+\lambda-\frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]$ is denoted by x~TG.

The reliability function R(x) is given by

$$R(x) = 1-\frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\left[1+\frac{\lambda\bar{G}(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]$$

and hrf is

$$\tau(x) = \frac{\frac{\theta g(x;\phi)}{[1+(\theta-1)G(x;\phi)]^2}\left[1+\lambda-\frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]}{1-\frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\left[1+\frac{\lambda\bar{G}(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]}$$

Suppose Z_1 and Z_2 be two random variables from $\frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}$. Define

$$X = \begin{cases} Z_{1:2} \text{ with probability } \frac{1+\lambda}{2} \\ Z_{2:2} \text{ with probability } \frac{1-\lambda}{2} \end{cases}$$

Where $Z_{1:2} = \min(Z_1, Z_2)$ *and* $Z_{2:2} = \max(Z_1, Z_2)$. Then the cumulative distribution function 0f X is given by (4.4).

The TG-G family of distribution appears to be more flexible and could be used for modeling various types of data.pdf and hrf of some special models of this family are provided in Figures 4.1 and 4.2. It can be seen that the hazard rate could take constant, increasing, decreasing, upside down and bathtub shaped. Therefore, this family of distribution could be used to model diverse nature of data sets.

Let $g(x) = g(x;\phi)$ *and* $G(x) = G(x;\phi)$.

4.2 Mixture Representation

The probability density function of (5) can be written as

$$f(x) = \frac{\theta(1+\lambda)g(x)}{[1+(\theta-1)G(x)]^2} - \frac{2\lambda\theta^2 g(x)G(x)}{[1+(\theta-1)G(x)]^3} \tag{4.6}$$

Then, the probability density function (4.6) can be rewritten as

$$f(x) = \left[(1+\lambda)\theta g(x)\sum_{k=0}^{\infty}(\theta-1)^k \binom{-2}{k} G^k(x)\right]$$

$$-\left[2\lambda\theta^2 g(x)\sum_{k=0}^{\infty}(\theta-1)^k \binom{-3}{k} G^{k+1}(x)\right] \tag{4.7}$$

The probability density function (4.7) can be expressed as a mixture of exp-G densities

$$f(x) = \sum_{k=0}^{\infty}[a_k\pi_{k+1}(x) - b_k\pi_{k+2}(x)]. \tag{4.8}$$

But $\binom{-2}{k} = (-1)^k(k+1)$ *and* $\binom{-3}{k} = \frac{(-1)^k(k+1)(k+2)}{2}$

Where $\pi_\alpha(x) = \alpha g(x)G^{(\alpha-1)}(x)$ is the exp-G probability density function with power parameter $\alpha > 0$,

$a_k = \theta(1+\lambda)(1-\theta)^k$ *and* $b_k = \lambda\theta^2(k+1)(1-\theta)^k$.

Thus several mathematical properties of the TG-G family can be obtained simply from those properties of the exp-G family. Equation (4.8) is the main result. The cumulative distribution function of the TG-G family can also be expressed as a mixture of exp-G densities. By integrating (4.8), the same mixture representation is obtained.

$$F(x) = \sum_{k=0}^{\infty}[a_k\pi_{k+1}(x) - b_k\pi_{k+2}(x)].$$

Where $\pi_\delta(x)$ is the cumulative distribution function of the exp-G family with power parameter δ.

4.3 Special models

The two special models of the TG-G family are correspond to the baseline Weibull and the Burr X distributions. These special models generalize some well-known distributions in the literature.

4.3.1 The T-G Weibull (TGW) Distribution

The Weibull distribution with positive parameters $\alpha\ and\ \beta$ has cumulative distribution function and probability density function (for x>0) given by $G(x) = 1 - e^{-(\alpha x)^{\beta}}\ and$

$g(x) = \beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}$, respectively. Then, the probability density function of the TGW model is given by

$$f(x) = \frac{\theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}}{\left[1+(\theta-1)\left(1-e^{-(\alpha x)^{\beta}}\right)\right]^{2}}\left[1+\lambda-\frac{2\lambda\theta\left(1-e^{-(\alpha x)^{\beta}}\right)}{\left[1+(\theta-1)\left(1-e^{-(\alpha x)^{\beta}}\right)\right]}\right],$$

Where $\alpha, \beta\ and\ \theta\ are\ positive\ parameters\ and\ |\lambda| \leq 1$.

The TGW distribution includes the transmuted Weibull (TW) distribution introduced by Aryal and Tsokos (2011)[19] when $\theta = 1$.

The plots of the probability density function and hrf of the TGW distribution are displayed in figure 4.1 for selected parameter values.

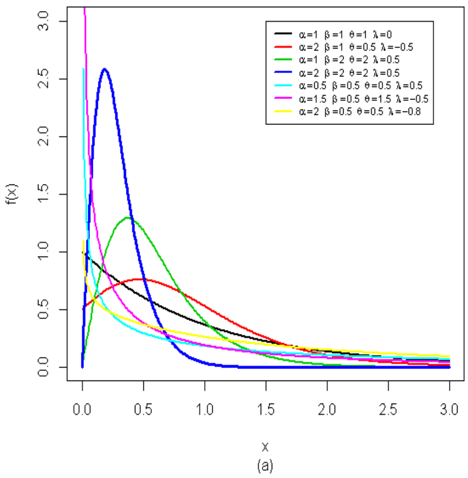


Figure 4.1: (a) probability density function of TGW Distribution

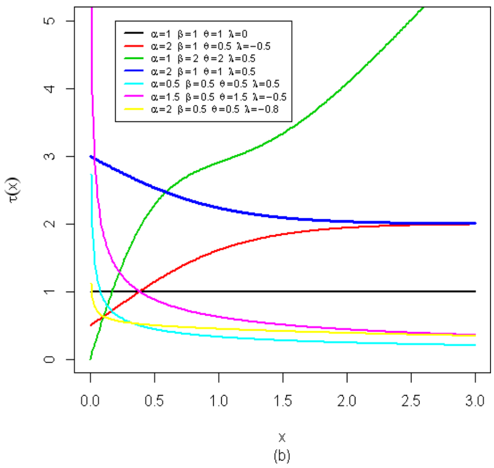


Figure 4.1: (b) hrf of TGW Distribution

4.3.2 The TG-Burr X (TGBrX) Distribution

The Burr X (also known as the generalized Rayleigh) model with positive parameters $\alpha \; and \; \beta$ has cumulative distribution function and probability density function (for x>0) given by $G(x) = \left[1 - e^{-(\beta x)^2}\right]^{\alpha} \; and$

$g(x) = 2\alpha\beta^2 x e^{-(\beta x)^2}\left[\left[1 - e^{-(\beta x)^2}\right]^{\alpha-1}\right]$, respectively. Then, the TGBrX density reduces to

$$f(x) = \frac{2\alpha\beta^2 x e^{-(\beta x)^2}\left[\left[1-e^{-(\beta x)^2}\right]^{\alpha-1}\right]}{\left[1+(\theta-1)\left[1-e^{-(\beta x)^2}\right]^{\alpha}\right]^2}\left[1+\lambda-\frac{2\lambda\theta\left[1-e^{-(\beta x)^2}\right]^{\alpha}}{1+(\theta-1)\left[1-e^{-(\beta x)^2}\right]^{\alpha}}\right],$$

Where $\alpha, \beta \; and \; \theta \; are \; positive \; parameters \; and \; |\lambda| \leq 1.$

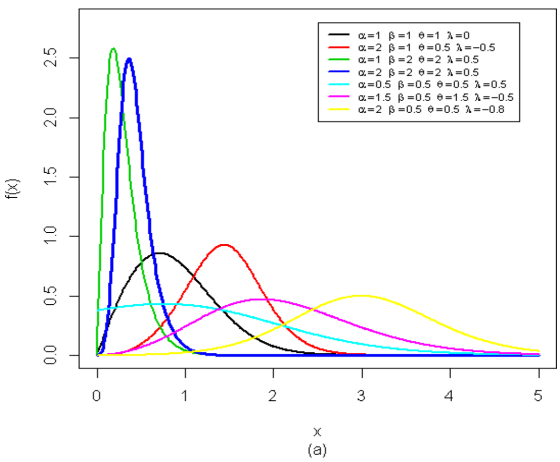


Figure 4.2: (a) probability density function of TGBrX Distribution

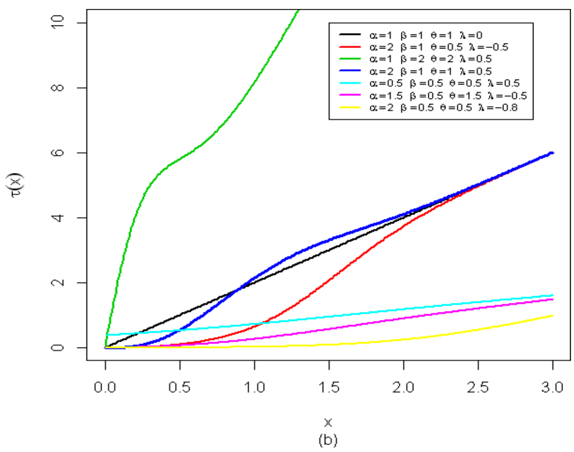


Figure 4.2: (b) hrf of TGBrX Distribution

The plots of the probability density function and hrf of the TGBrX distribution are displayed in Figure 4.2 for selected parameter values.

4.4 Mathematical properties

4.4.1 Quantile Function

The quantile function (qf) of X, where X$\sim TG-G(\lambda,\theta,\phi)$, is obtained by inverting (4). The qf, Q(u), of X is given by

$$Q(u) = F^{(-1)}(u)$$

$$= G^{(-1)}\left\{\frac{\lambda+1-\sqrt{(\lambda+1)^2-4\lambda u}}{2\lambda\theta+(1-\theta)\left[\lambda+1-\sqrt{(\lambda+1)^2-4\lambda u}\right]}\right\},$$

for $0 < u < 1$

$for\ \lambda \neq 0. for\ \ \lambda = 0, we\ have$

$$Q(u) = G^{-1}\left[\frac{u}{\theta+(1-\theta)u}\right].$$

Simulating the TG-G random variable is straightforward. If U is a uniform variate on the unit interval (0,1), then the random variable X=Q(U) follows the TG-G distribution.

4.4.2 Moments

Henceforth, Y_k denotes the exp-G distribution with power parameter k. The rth moment of X, say μ'_r, follows from (9) as

$$\mu'_r = E(X^r) = \sum_{k=0}^{\infty}[a_k E(Y_{k+1}^r) - b_k E(Y_{k+2}^r)].$$

The nth central moment of X, say M_n,is given by

$$M_n = E(X - \mu'_1)^n = \sum_{r=0}^{n}\binom{n}{r}(\mu'_1)^{(n-r)}E(X^r)$$

$$= \sum_{r=0}^{n}\sum_{k=0}^{\infty}(-1)^{(n-r)}\binom{n}{r}(\mu'_r)^{(n-r)}$$

$$[a_k E(Y_{k+1}^r) - b_k E(Y_{k+2}^r)].$$

The cumulants (k_n) of X follow recursively from

$$k_n = \mu'_n - \sum_{r=0}^{n-1}\binom{n-1}{r-1}k_r\mu'_{n-r},$$

where $k_1 = \mu'_1, k_2 = \mu'^2_2 - \mu'^2_1, k_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu'^3_1$, etc. the skewness and kurtosis measures could be calculated from the ordinary moments using well-known relationships.

4.4.3 Generating Function

The two formula for the moment generating function

$M_X(t) = E(e^{tx}) of\ X$ has been provided. Clearly the first equation could be derived from the equation

$f(x) = \sum_{k=0}^{\infty}[a_k\pi_{k+1}(x) - b_k\pi_{k+2}(x)]$ as

$M_X(t) = \sum_{k=0}^{\infty}[a_k M_{k+1}(t) - b_k M_{k+2}(t)]$,

Where $M_X(t)$ is the moment generating function of Y_k. Hence, $M_X(t)$ could be determined from the exp-G generating function.

A second formula for $M_X(t)$ follows from

$f(x) = \sum_{k=0}^{\infty}[a_k\pi_{k+1}(x) - b_k\pi_{k+2}(x)]$ as

$M_X(t) = \sum_{k=0}^{\infty}[a_k\tau(t,k) - b_k\tau(t,k+1)]$, Where $\tau(t,k) = \int_0^1 \exp[tQ_G(u)]u^k du$ and $Q_G(u)$ is the quantile function corresponding to G(x), i.e., $Q_G(u) = G^{-1}(u)$.

4.4.4 Incomplete moments

The sth incomplete moment, say φ_s(t), of X could be expressed from

$f(x) = \sum_{k=0}^{\infty}[a_k\pi_{k+1}(x) - b_k\pi_{k+2}(x)]$ as

$$\varphi_s(\mathrm{t}) = \int_{-\infty}^{t} x^s f(x)dx.$$

$$= \sum_{k=0}^{\infty}\left[a_k \int_{-\infty}^{t} x^s\pi_{k+1}(x)dx - b_k \int_{-\infty}^{t} x^s\pi_{k+2}(x)dx\right]$$

The mean deviations about the mean $[\theta_1 = E(|X - \mu_1'|)]$ and about the median $[\theta_2 = E(|X - M|)]$ of X are given by $\theta_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $\theta_2 = \mu_1' - 2\varphi_1(M)$, respectively,

where $\mu_1' = E(X)$, M = Median(X) = Q(0.5) is the median, $F(\mu_1')$ is calculated from

$$F(x) = \int_0^{\frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}}(1 + \lambda - 2\lambda t\,)dt$$

$= \frac{\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\left[1 + \frac{\lambda \bar{G}(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]$ and φ_1(t) is the first incomplete moment given by

$$\varphi_s(\mathrm{t}) = \int_{-\infty}^{t} x^s f(x)dx$$

$=\sum_{k=0}^{\infty}\left[a_k \int_{-\infty}^{t} x^s\pi_{k+1}(x)dx - b_k \int_{-\infty}^{t} x^s\pi_{k+2}(x)dx\right]$ with s=1.There are two ways to determine θ_1 *and* θ_2. First,a general equation for φ_1(t) can be derived from

$$\varphi_s(\mathrm{t}) = \int_{-\infty}^{t} x^s f(x)dx$$

$$=\sum_{k=0}^{\infty}\left[a_k \int_{-\infty}^{t} x^s\pi_{k+1}(x)dx - b_k \int_{-\infty}^{t} x^s\pi_{k+2}(x)dx\right]$$

as $\varphi_1(\mathrm{t}) = \sum_{k=0}^{\infty}[a_k J_{k+1}(x) - b_k J_{k+2}(x)]$,

where $J_k(x) = \int_{-\infty}^{t} x\pi_k(x)dx$ is the first incomplete moment of the exp-G distribution.

A second general formula for $\varphi_1(\mathrm{t}) = (k+1)\int_0^{G(t)} Q_G(u)u^k du$ can be computed numerically.

These equations for $\varphi_1(\mathrm{t})$ could be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \varphi_1(\mathrm{q})/\ (\pi\mu_1')$ and $L(\pi) = \varphi_1(\mathrm{q})/\mu_1'$ respectively, where $\mu_1' = E(X)\ and\ q = Q(\pi)$ is the qf of X at π.

4.4.5 Residual and Reversed Residual Life Functions

The nth moment of the residual life, say $m_n(t) = E[(X-t)^n|X>t], n = 1,2,\dots,$ is given by

$$m_n(t) = \frac{1}{R(t)}\int_t^\infty (x-t)^n\, dF(x).$$

Therefore

$$m_n(t) = \frac{1}{R(t)}\sum_{r=0}^{n}\binom{n}{r}(-t)^{(n-r)}$$

$$\sum_{k=0}^{\infty}\left[a_k\int_t^\infty x^r\pi_{k+1}(x) - b_k\int_t^\infty x^r\pi_{k+2}(x)\right].$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $m_n(t) = E[(X-t)|X>t]$, which represents the expected additional life length for a unit which is alive at age t. the MRL of X could be obtained by setting n=1 in the last equation.

The nth moment of the reversed residual life, say

$M_n(t) = E[(t-X)^n|X\le t]\ for\ t>0, n = 1,2,\dots,$ is defined by $M_n(t) = \frac{1}{F(t)}\int_0^t (t-x)^n\, dF(x).$

Therefore, the nth moment of the reversed residual life of X becomes

$$M_n(t) = \frac{1}{F(t)}\sum_{r=0}^{n}\binom{n}{r}(t)^{(n-r)}$$

$$\sum_{k=0}^{\infty}\left[a_k\int_0^t x^r\pi_{k+1}(x) - b_k\int_0^t x^r\pi_{k+2}(x)\right]$$

The mean inactivity time (MIT) is defined by $M_1(t) = E[(t-X)|X\le t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in (0,t). The MIT of X could be obtained easily by setting n=1 in the above equation.

4.4.6 Entropies

The Renyi entropy is defined by

$$I_\delta(X) = (1-\delta)^{-1} log\left[\int_{-\infty}^{\infty} f^\delta(x)dx\right], \delta > 0 \ and \ \delta \neq 1.$$

Using the probability density function

$f(x) = \frac{\theta(1+\lambda)g(x)}{[1+(\theta-1)G(x)]^2} - \frac{2\lambda\theta^2 g(x)G(x)}{[1+(\theta-1)G(x)]^3}$. one can write

$$f^\delta(x) = \frac{\theta^\delta g^\delta(x)(1+\lambda)^\delta}{[1+(\theta-1)G(x)]^{2\delta}}\left\{1 - \frac{2\lambda\theta G(x)}{(1+\lambda)[1+(\theta-1)G(x)]}\right\}^\delta .$$

The Taylor series z^β is defined as

$$z^\beta = \sum_{k=0}^{\infty}(\beta)_k \frac{(z-1)^k}{k!},$$

where k is a positive integer

and $(\beta)_k = \beta(\beta-1)\ldots.(\beta-k+1)$ is the descending factorial.

Consider $A = \left\{1 - \frac{2\lambda\theta G(x)}{(1+\lambda)[1+(\theta-1)G(x)]}\right\}^\delta$.

Applying the last power series to the quantity A, $f^\delta(x)$ is obtained as

$$f^\delta(x) = \sum_{i=0}^{\infty}(\delta)^i \frac{(-1)^i 2^i \lambda^i \theta^{(\delta+i)}(1+\lambda)^{\delta-i}}{i![1+(\theta-1)G(x)]^{2\delta+i}} g^\delta(x)G^i(x).$$

Then, $f^\delta(x) = \sum_{k=0}^{\infty} m_k g^\delta(x)G^{k+i}(x)$,

where

$$m_k = \sum_{i=0}^{\infty}(\delta)^i \frac{(-1)^i 2^i \lambda^i (1-\theta)^k \theta^{(\delta+i)}(1+\lambda)^{\delta-i}}{k!i!\Gamma(2\delta+i)} \Gamma(2\delta+i+k).$$

Then, the Renyi entropy of the TG-G family is given by

$$I_\delta(X) = (1-\delta)^{-1} log\left[\sum_{k=0}^{\infty} m_k \int_{-\infty}^{\infty} g^\delta(x)G^{k+i}(x)dx\right].$$

The δ-entropy, say $H_\delta(X)$, could be obtained (for $\delta > 0, \delta \neq 1$) as $H_\delta(X) = (\delta-1)^{-1} log\left\{1 - \left[\int_{-\infty}^{\infty} f^\delta(x)dx\right]\right\}$,

which follows from the last equation.

The Shannon entropy of a random variable X, say SI , is a special case of the Renyi entropy when $\delta \uparrow 1$ and it is defined by

$$SI = E\{-[\log f(x)]\},$$

which follows by taking the limit of $I_\delta(X)\ as\ \delta$ tends to 1.

4.5 Order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the TG-G family of distributions. The probability density function of ith order statistic, say $X_{i:n}$,could be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)} \sum_{j=0}^{n-1} (-1)^j \binom{n-i}{j} F^{j+i-1}(x). \qquad (4.10)$$

Then

$$F^{j+i-1}(x) = \frac{\theta^{j+i-1} G^{j+i-1}(x)}{[1+(\theta-1)G(x)]^{j+i-1}} \left[1 + \frac{\lambda \bar{G}(x)}{1+(\theta-1)G(x)}\right]^{j+i-1}$$

$$= \sum_{w=0}^{\infty} (j+i-1)_w \frac{\lambda^w \theta^{j+i-1} [1-G(x)]^w G^{j+i-1}(x)}{w! [1+(\theta-1)G(x)]^{j+i+w-1}} \qquad (4.11)$$

Using Equations (4.5) and (4.11) ,

$$f(x)F^{j+i-1}(x) = \sum_{w=0}^{\infty} (j+i-1)_w \frac{(1+\lambda)\lambda^w \theta^{j+i} g(x)[1-G(x)]^w G^{j+i-1}(x)}{w!\,[1+(\theta-1)G(x)]^{j+i+w-1}}$$

$$- \sum_{w=0}^{\infty} (j+i-1)_w \frac{2\lambda^{w+1} \theta^{j+i+1} g(x)[1-G(x)]^w G^{j+i}(x)}{w!\,[1+(\theta-1)G(x)]^{j+i+w+2}}$$

Then

$$f(x)F^{j+i-1}(x) = \sum_{k=0}^{\infty} \left[\gamma_k \pi_{k+j+i+m}(x) - \Psi_k \pi_{k+j+i+m+1}(x)\right]. \qquad (4.12)$$

Substituting Equation (4.12) in Equation (4.10), the probability density function $X_{i:n}$ could be expressed as

$$f_{i:n}(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} \frac{(-1)^j}{B(i,n-i+1)} \binom{n-i}{j}$$

$$\left[\gamma_k \pi_{k+j+i+m}(x) - \Psi_k \pi_{k+j+i+m+1}(x)\right]$$

Where

$$\gamma_k = \sum_{m,w=0}^{\infty} (j+i-1)_w \frac{(-1)^k (1+\lambda) \lambda^w \theta^{j+i} (1-\theta)^m \Gamma(w+1) \Gamma(j+i+w+m+1)}{w! m! \Gamma(w-k+1) \Gamma(j+i+w+2) [k+j+i+m]},$$

$$\Psi_k = \sum_{m,w=0}^{\infty} (j+i-1)_w \frac{(-1)^k 2\lambda^{w+1}\theta^{j++1i}(1-\theta)^m \Gamma(w+1)\Gamma(j+i+w+m+2)}{w!m!\Gamma(w-k+1)\Gamma(j+i+w+2)[k+j+i+m+1]}$$

and $\pi_k(x)$ is the exp-G density with power parameter k. Then, the density function of the TG-G order statistics is a mixture of exp-G densities. Based on the last equation, one can note that the properties of $X_{i:n}$ follow from those of Y_{a+k}. For example, the moments of $X_{i:n}$ can be expressed as

$$E\left(X^q{}_{(i:n)}\right) = \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} \frac{(-1)^j}{B(i,n-i+1)} \binom{n-i}{j}$$

$$\left[\gamma_k E\left(Y^q{}_{(k+j+i+m)}\right)(x) - \Psi_k E\left(Y^q{}_{(k+j+i+m+1)}\right)(x)\right] \quad (4.13)$$

Based upon the moments in equation (4.13), one can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable TG-G order statistics. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r} \sum_{d=0}^{r-1} (-1)^d \binom{r-1}{d} E(X_{r-d:r}), r \geq 1.$$

4.6. Probability weighted moments

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist.

The (s,r)th PWM of X following the TG-G distribution, say $\rho_{s,r} = E[X^s F^r(x)] = \int_{-\infty}^{\infty} x^s F^r(x) f(x) dx$,

$$F^r(x) = \frac{\theta^r G^r(x)}{[1+(\theta-1)G(x)]^r} \left[1 + \frac{\lambda \bar{G}(x)}{1+(\theta-1)G(x)}\right]^r$$

$$\sum_{w=0}^{\infty} (r)_w \frac{\lambda^w \theta^r [1-G(x)]^w G^r(x)}{w![1+(\theta-1)G(x)]^{r+w}} \quad (4.14)$$

From $f(x) = \frac{\theta g(x;\phi)}{[1+(\theta-1)G(x;\phi)]^2} \left[1 + \lambda - \frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right]$ and the last equation, one can write

$$f(x)F^r(x) = \sum_{k=0}^{\infty} [\gamma_k^* \pi_{k+r+m+1}(x) - \Psi_k^* \pi_{k+r+m+2}(x)],$$

Where

$$\gamma_k^* = \sum_{m,w=0}^{\infty} (j+i-1)_w \frac{(-1)^k (1+\lambda)\lambda^w \theta^{r+1} (1-\theta)^m \Gamma(w+1)\Gamma(r+w+m+2)}{w!m!\Gamma(w-k+1)\Gamma(r+w+2)[k+r+m+1]},$$

$$\Psi_k^* = \sum_{m,w=0}^{\infty} (j+i-1)_w \frac{(-1)^k 2\lambda^{w+1} \theta^{r+2} (1-\theta)^m \Gamma(w+1)\Gamma(r+w+m+3)}{w!m!\Gamma(w-k+1)\Gamma(r+w+3)[k+r+m+2]}$$

Finally, the(s,r)th PWM of X can be obtained from an infinite linear combination of exp-G moments given by

$$\rho_{s,r} = \sum_{k=0}^{\infty} \left[\gamma_k^* E\left({Y^r}_{(k+r+m+1)}\right) - \Psi_k^* E\left({Y^r}_{(k+r+m+2)}\right)\right]$$

4.7. Maximum Likelihood Estimation

The maximum likelihood estimators (MLEs) of the parameters of the TG-G family of distributions are determined from complete samples only. Let $x_1, x_2, \ldots, x_n$ be a random sample from this family with parameter vector $\varphi, where\ \varphi = (\lambda, \theta, \varphi^T)^T$.

Then, the log-likelihood function for $\varphi, say\ l = l(\varphi)$, is given by

$$l = nlog(\theta) + \sum_{i=0}^{n} log[g(x;\phi)] - 2\sum_{i=0}^{n} \log(s_i)$$

$$+\sum_{i=0}^{n} \log(p_i),$$

where

$$s_i = [1 + (\theta - 1)G(x;\phi)]\ and\ P_i = \left[1 + \lambda - \frac{2\lambda\theta G(x;\phi)}{1+(\theta-1)G(x;\phi)}\right].$$

The score vector components, say

$$U(\varphi) = \frac{\partial l}{\partial \varphi} = \left(\frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \theta}, \frac{\partial l}{\partial \phi_k}\right)^T = \left(U_\lambda, U_\theta, U_{\phi_k}\right)^T.$$

$$U_\lambda = \sum_{i=0}^{n} \frac{1}{P_i}\left[1 - \frac{2\theta G(x;\phi)}{S_i}\right],$$

$$U_\theta = \frac{n}{\theta} - 2\sum_{i=o}^{n} \frac{G(x;\phi)}{S_i} - 2\lambda \sum_{i=0}^{n} \frac{[G(x;\phi)S_i - \theta G(x;\phi)^2]}{P_i {S_i}^2}$$

and

$$U_{\phi_k} = \sum_{i=0}^{n} \frac{g'(x_i;\phi)}{g(x;\phi)} - 2\sum_{i=0}^{n} \frac{(\theta-1)G'(x_i;\phi)}{S_i} - 2\lambda\theta \sum_{i=0}^{n} \frac{G'(x_i;\phi)S_i}{P_i {S_i}^2}$$

$$+2\lambda\theta(\theta-1)\sum_{i=0}^{n} \frac{G(x;\phi)G'(x_i;\phi)}{P_i {S_i}^2},$$

where $g'(x_i;\phi) = \partial g(x_i;\phi)/\partial\phi_k$ and

$G'(x_i;\phi) = \partial G(x_i;\phi)/\partial\phi_k$.

Setting the nonlinear system of equations $U_\lambda = U_\theta = U_{\phi_k} = 0$ and solving them simultaneously yields the MLE

$\hat{\varphi} = \left(\hat{\lambda}, \hat{\theta}, \hat{\phi}^T\right)^T of\ \varphi = (\lambda, \theta, \varphi^T)^T$.

4.8 Application

Data set l: The Nicotine Data

The first data set refers to nicotine measurements, made from several brands of cigarettes in 1998, collected by the Federal Trade Commission. The report entitled tar, nicotine, and carbon monoxide of the smoke of 1206 varieties of domestic cigarettes for the year of 1998 consists of the data sets and some information about the source of the data, smokers behavior and beliefs about nicotine, tar and carbon monoxide contents in cigarettes. This data set consists of n=346 observations. These data have been used by Afify, Cordeiro, Butt, Yousof, Alzaatreh and Nofal (2016)[4] to fit the Marshall-Olkin additive Weibull distribution.

We shall compare the fits of the TGW distribution with those of other competitive models, namely: the Kumaraswamy-transmuted exponentiated modified Weibull distribution (Kw-TEMW) (Al-Babtain, Fattah, Ahmed and Merovci, 2015)[8], transmuted exponentiated modified weibull (TEMW) (Eltehiwy and Ashour, 2013)[33], transmuted additive Weibull (TAW))Elbatal and Aryal, 2013)[32], Kumaraswamy modified Weibull (Kw-MW) (Cordeiro, Ortega and Silva, 2014)[30], beta Weibull (BW) (Lee, Famoye and Olumolade, 2007)[39], Kumaraswamy Weibull(Kw-W)(Cordeiro, Ortega, and Nadarajah, 2010)[29], and additive Weibull (AW)(Xie and Lai, 1995)[51] distributions with corresponding densities (for x>0)

Kw-TEMW:

$$f(x) = ab\delta e^{-(\alpha x+\gamma x^\beta)}\left(\alpha x + \gamma\beta x^{\beta-1}\right)\left[1 - e^{-(\alpha x+\gamma x^\beta)}\right]^{a\delta-1}$$

$$x\left\{1 + \lambda - 2\lambda\left[1 - e^{-(\alpha x+\gamma x^\beta)}\right]^\delta\right\} \quad \left\{1 + \lambda - \lambda\left[1 - e^{-(\alpha x+\gamma x^\beta)}\right]^\delta\right\}^{a-1}$$

TEMW:

$$f(\mathrm{x}) = \delta\left(\alpha \mathrm{x} + \gamma\beta \mathrm{x}^{\beta-1}\right)\mathrm{e}^{-\left(\alpha \mathrm{x}+\gamma \mathrm{x}^{\beta}\right)}\left[1-\mathrm{e}^{-\left(\alpha \mathrm{x}+\gamma \mathrm{x}^{\beta}\right)}\right]^{\delta-1}$$

$$\mathrm{x}\left\{1+\lambda-2\lambda\left[1-\mathrm{e}^{-\left(\alpha \mathrm{x}+\gamma \mathrm{x}^{\beta}\right)}\right]^{\delta}\right\} ;$$

TAW:

$$f(\mathrm{x}) = \left(\alpha\theta x^{\theta-1} + \gamma\beta \mathrm{x}^{\beta-1}\right)\mathrm{e}^{-\left(\alpha x^{\theta}+\gamma \mathrm{x}^{\beta}\right)}$$

$$\left\{1+\lambda-2\lambda \mathrm{e}^{-\left(\alpha x^{\theta}+\gamma \mathrm{x}^{\beta}\right)}\right\};$$

Kw-MW:

$$f(\mathrm{x}) = \mathrm{ab}\gamma(\beta+\alpha \mathrm{x})\mathrm{x}^{\beta-1}\mathrm{e}^{\left(\alpha \mathrm{x}-\gamma x^{\beta}\mathrm{e}^{\mathrm{ax}}\right)}\left[1-\mathrm{e}^{-\left(\gamma \mathrm{x}^{\beta}e^{\alpha x}\right)}\right]^{\mathrm{a}-1}$$

$$\left\{1-\left[1-\mathrm{e}^{-\left(\gamma \mathrm{x}^{\beta}e^{\alpha x}\right)}\right]^{\mathrm{a}}\right\}^{b-1};$$

BW:

$$f(x) = \frac{\beta\alpha^{\beta}}{B(a,b)}\mathrm{x}^{\beta-1}\mathrm{e}^{-b(\alpha \mathrm{x})^{\beta}}\left[1-\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}}\right]^{\mathrm{a}-1};$$

Kw-W:

$$f(x) = ab\beta\alpha^{\beta}\mathrm{x}^{\beta-1}\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}}\left[1-\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}}\right]^{\mathrm{a}-1}$$

$$\left\{1-\left[1-\mathrm{e}^{-(\alpha \mathrm{x})^{\beta}}\right]^{\mathrm{a}}\right\}^{b-1};$$

AW:

$$f(\mathrm{x}) = \left(\alpha\theta x^{\theta-1} + \gamma\beta \mathrm{x}^{\beta-1}\right)\mathrm{e}^{-\left(\alpha x^{\theta}+\gamma \mathrm{x}^{\beta}\right)}.$$

The parameters of the above densities are all positive real numbers expect the parameter λ where $|\lambda| \leq 1$.

Data set II: The Gauge Lengths Data

The second data set (guage lengths of 20 mm) consists of 74 observations. This data set is previously studied by Afify, Cordeiro, Butt, Ortega and Suzuki (2016)[5]t to fit the Kumaraswamy complementary Weibull geometric distribution.

For this data set, we shall compare the fits of the TGBrX distribution with those of other competitive models, namely: the generalized transmuted Burr X (GT-BrX) (Nofal, Afify, Yousof and Cordeiro, 2015)[45], McDonald Weibull (Mc-W) (Cordeiro, Ortega and Silva, 2014)[30], exponentiated transmuted generalized Rayleigh (ETGR) (Afify, Nofal and Ebraheim, 2015)[6], T-BrX and BrX models with corresponding densities (for x>0):

GT-BrX:

$$f(x) = 2\alpha\beta^2 x e^{-(\beta x)^2}\left[1 - \mathrm{e}^{-(\beta x)^2}\right]^{\alpha a - 1}$$

$$\left\{a(1+\lambda) - \lambda(a+b)\left[1 - \mathrm{e}^{-(\beta x)^2}\right]^{\alpha b}\right\};$$

Mc-W:

$$f(x) = \frac{\beta c \alpha^{\beta}}{B\left(\frac{a}{c}, b\right)} x^{\beta-1} \mathrm{e}^{-(\alpha x)^{\beta}} \left[1 - \mathrm{e}^{-(\alpha x)^{\beta}}\right]^{a-1}$$

$$\left\{1 - \left[1 - \mathrm{e}^{-(\alpha x)^{\beta}}\right]^{c}\right\}^{b-1}$$

MBW:

$$f(x) = \frac{\beta \alpha^{-\beta} c^{a}}{B(a,b)} x^{\beta-1} \mathrm{e}^{-b\left(\frac{x}{\alpha}\right)^{\beta}} \left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{a-1}$$

$$\left\{1 - (1-c)\left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]\right\}^{-a-b};$$

ETGR: $f(x) = 2\alpha\delta\beta^2 x e^{-(\beta x)^2}\left[1 - \mathrm{e}^{-(\beta x)^2}\right]^{\alpha\delta - 1}$

$$\mathrm{x} \left\{1 + \lambda - 2\lambda\left[1 - \mathrm{e}^{-(\beta x)^2}\right]^{\alpha}\right\}$$

$$\mathrm{x} \left\{1 + \lambda - \lambda\left[1 - \mathrm{e}^{-(\beta x)^2}\right]^{\alpha}\right\}^{\delta-1};$$

The parameters of the above densities are all positive real numbers expect the parameter λ where $|\lambda| \leq 1$.

In order to compare the fitted models, some goodness-of-fit criteria like the Akaike information criterian (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), consistent Akaike information criterian (CAIC), -2$\hat{l}$, where $\hat{l}$ is the maximized log-likelihood, Anderson-Darling (A^*) and the cramervon Mises (W^*) statistics are considered. The better distribution corresponds to smaller AIC, BIC,HQIC, CAIC, $A^*$and W^* values.Goodness-of-fit statistics for data set I and data set II are presented by Table 4.1 and Table 4.2 respectively.

Table 4.1:

Goodness-of-Fit Statistics for Data Set I

Model	$-2\hat{\ell}$	AIC	BIC	$HQIC$	$CAIC$	W^*	A^*
TGW	212.176	220.176	235.562	226.303	220.293	0.34466	1.89074
Kw-TEMW	215.674	229.674	256.599	240.396	230.005	0.37863	2.08814
TEMW	215.967	225.967	245.199	233.625	226.143	0.38319	2.14169
TAW	217.393	227.393	246.625	235.051	227.569	0.37208	2.08766
Kw-MW	221.938	231.938	251.17	239.596	232.114	0.43426	2.52687
BW	225.173	233.173	248.559	239.3	233.29	0.49664	2.89774
Kw-W	226.184	234.184	249.57	240.311	234.302	0.5325	3.08454
AW	226.581	234.581	249.966	240.707	234.698	0.55222	3.17512

Table 4.2:

Goodness-of-Fit Statistics for Data Set II

Model	$-2\hat{\ell}$	AIC	CAIC	HQIC	BIC	W^*	A^*
TGBrX	104.316	112.316	112.896	115.993	121.533	0.03531	0.25151
GT-BrX	108.055	118.055	118.937	122.65	129.575	0.10458	0.68807
Mc-W	108.784	118.784	119.667	123.38	130.305	0.1196	0.77957
MBW	109.145	119.145	120.028	123.741	130.666	0.12414	0.81141
ETGR	113.4	121.352	121.9	125.029	130.6	0.20714	1.3407
T-BrX	123.61	129.61	129.95	132.376	136.5	0.16923	1.28629
BrX	135.202	139.202	139.371	141.041	143.811	0.13403	0.86836

MLEs and their standard errors for Data set I and Data set II are presented in Table 4.3 and Table 4.4 respectively.

Table 4.3:

MLEs and their Standard Errors for Data Set I

Model	Estimates (Standard Errors)				
TGW	$\alpha = 2.1296(0.67)$	$\beta = 1.523(0.295)$	$\theta = 0.1413(0.142)$	$\lambda = -0.4468(0.332)$	
BW	$\alpha = 0.6686(0.578)$	$\beta = 3.1645(0.426)$	$a = 0.7784(0.163)$	$b = 3.0922(8.174)$	
Kw-W	$\alpha = 0.6157(0.392)$	$\beta = 3.1187(0.698)$	$a = 0.8395(0.233)$	$b = 3.7931(6.921)$	
AW	$\alpha = 1.135(0.062)$	$\beta = 0.3084(0.1)$	$\gamma = 0.0002(0.001369)$	$\theta = 2.7219(0.114)$	
TEMW	$\alpha = 0.6977(0.492)$	$\beta = 2.5908(0.265)$	$\gamma = 1.1925(0.259)$	$\delta = 1.5007(0.487)$	$\lambda = -0.6328(0.228)$
TAW	$\alpha = 1.2252(0.239)$	$\beta = 0.8994(0.091)$	$\gamma = 0.433(0.229)$	$\theta = 2.6404(0.267)$	$\lambda = -0.8831(0.147)$
Kw-MW	$\alpha = 0.6145(0.09)$	$\beta = 0.4466(0.364)$	$\gamma = 0.5622(0.353)$	$a = 4.3285(3.595)$	$b = 6.7039(6.728)$
Kw-TEMW	$\alpha = 0.113(0.22)$ $a = 0.47(0.213)$	$\beta = 2.316(0.62)$ $b = 1.079(1.828)$	$\gamma = 1.436(1.71)$	$\delta = 2.033(1.145)$	$\lambda = -0.902(0.197)$

Table 4.4:

MLEs and their Standard Errors for Data Set II

Model	Estimates (Standard Errors)				
GTBrX	$\alpha = 3.4900(2.084)$	$\beta = 0.6615(0.120)$	$\lambda = 0.0019(0.048)$	$a = 2.5190(1.503)$	$b = 0.0161(0.428)$
Mc-W	$\alpha = 1.4383(1.447)$	$\beta = 0.5832(0.211)$	$a = 83.7204(78.89)$	$b = 14.4281(15.87)$	$c = 3.4606(9.663)$
MBW	$\alpha = 1.7656(1.097)$	$\beta = 1.4265(1.488)$	$a = 36.3366(4.439)$	$b = 3.3618(6.695)$	$c = 3.0967(4.714)$
TG-BrX	$\alpha = 0.7477(0.891)$	$\beta = 0.8516(0.011)$	$\lambda = -0.1444(0.856)$	$\theta = 0.0092(0.018)$	
ETGR	$\alpha = 2.1214(0.315)$	$\beta = 0.6985(0.040)$	$\lambda = 0.3201(0.228)$	$\delta = 7.790(1.727)$	
T-BrX	$\alpha = 5.5052(0.776)$	$\beta = 0.6245(0.017)$	$\lambda = 0.3599(0.253)$		
BrX	$\alpha = 7.784(1.625)$	$\beta = 0.6445(0.024)$			

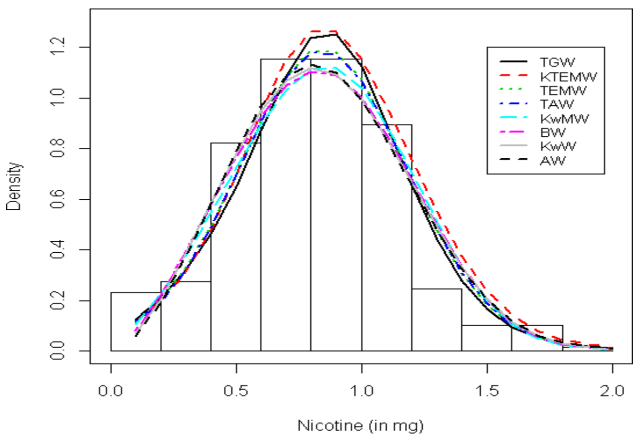


Figure 4.3: Fitted probability density function of TGW Model and other Distributions for Data Set I.

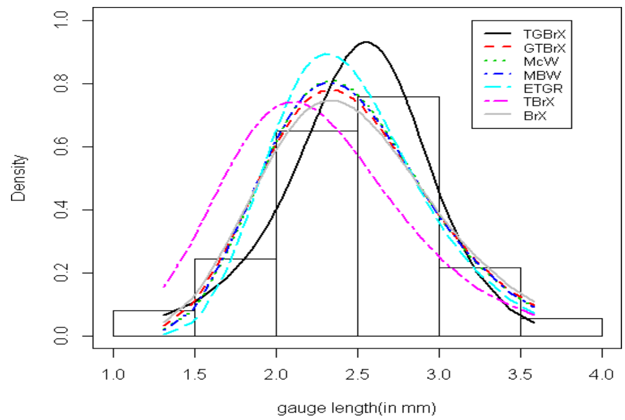


Figure 4.4: Fitted probability density function of TGBrX Model and other Distributions for Data Set.

Tables 4.1 and 4.2 list the values of goodness-of-fit statistics whereas the MLEs of the model parameters and their standard errors are given in Tables 4.3 and 4.4. The histogram of the nicotine data and the estimated densities are displayed in Figure 4.3 and Figure 4.4, displays the histogram of the guage lengths data and the estimated densities.

In Table 4.1, we compare the fit of the TGW model with the Kw-TEMW, TEMW, TAW, Kw-MW, BW, Kw-W and AW distributions. The values in these tables indicate that the TGW model has the lowest goodness-of-fit statistics (for data set I) among the fitted models. So, the TGW model could be chosen as the best model for the subject data. Similarily, in Table 4.2, we compare the fits of the TGBrX model with the GT-BrX, Mc-W, MBW, ETGR, T-BrX and BrX distributions. It is shown that the TGBrX model has the lowest goodness-of-fit statistics values(for Data set II) among all fitted models. So, the TGBrX model can be choosen as the best model for the subject data. It is clear from Tables 4.1 and 4.2 and Figures 4.3 and 4.4 that these special case of TG-G family provide the best fit to both data set. The mathematical properties of the new family including explicit expansions for the ordinary and incomplete moments, generating function, mean deviations, entropies, order statistics and probability weighted moments were provided. It was shown, by means of two real data sets, that special cases of the TG-G class can give a better fit than other models generated by well-known families.

Chapter - 5

The Transmuted Weibull G Family of Distributions

In this chapter, the Transmuted Weibull G Family of Distributions by Morad Alizadeh, Mahadi Rasekhi, Haitham M. Yousof, Hamedani [2017] has been reviewed.

The fifth chapter deals with the introduction of a new family of continuous distributions called the transmuted Weibull-G family of distributions. The mathematical properties of the new family are studied. Some useful characterizations based on the ratio of two truncated moments as well as based on hazard function are presented. The model parameters are estimated by the maximum likelihood estimators in terms of biases and mean squared errors by means of a simulation study are assessed.

Several continuous univariate distributions had been extensively used for modeling data in many areas such as economics, engineering, biological studies and environmental sciences. However, applied areas such as finance, lifetime analysis and insurance clearly require extended forms of these distributions had been constructed by extending common families of continuous distributions. These generalized distributions give more flexibility by adding one or more parameters to the baseline model. They were pioneered by Gupta et al.(1998)[37] who proposed the exponentiated-G class, which consists of raising the cumulative distribution function to a positive power parameter. Many other classes can be cited such as the Marshall-Olkin-G family by Marshall and Olkin (1997)[41], beta generalized-G family by Eugene, Lee and Famoye (2002)[34], the gamma – generated family by Zografos and Balakrishnan (2009)[56], Kumaraswamy G family by Cordeiro and de Castro (2011)[22], exponentiated generalized-G family by Cordeiro, Ortega and da Cunha (2013b)[26], a new method for generating families of continuous distributions by Alzaatreh, Lee and Famoye (2013)[16], exponentiated T-X family of distributions by Alzaghal , Famoye and Lee (2013)[17], the Lomax generator of distributions by Cordeiro, Ortega, Popovie and Pescim (2014)[27], beta odd log-logistic by Cordeiro, Alizadeh, Tahir, Mansoor, Bourguignon and Hamedani (2015)[25], Kumaraswamy odd log-logistic by Alizadeh, Emadi, Doostparast, Cordeiro, Ortega and Pescim (2015b)[13], beta odd log-logistic by Cordeiro, Alizadeh, Tahir, Mansoor, Bourguignon and Hamedani (2015)[25], Kumaraswamy Marshall-Olkin by Alizadeh, Cordeiro, Mansoor, Zubair and Hamedani

(2015c)[11], transmuted exponentiated generalized-G family by Yousof, Afify, Alizadeh, Butt, Hamedani and Ali (2015) [54], generalized transmuted-G by Nofal, Afify, Yousof and Cordeiro (2015)[45], generalized transmuted family by Alizadeh, Merovci and Hamedani (2015d)[14], another generalized transmuted family by Merovci, Alizadeh and Hamedani (2015)[42], Kumaraswamy transmuted-G by Afify, Cordeiro, Yousof, Alzaatreh and Nofal (2016b)[3], beta-transmuted-H by Afify, Yousof and Nadarajah (2016c)[7], the Zografos-Balakrishnan odd log-logistic family by Cordeiro, Alizadeh, Ortega and Serrano (2016c)[24] and the type I half-logistic family by Cordeiro, Alizadeh and Diniz Marinho (2016b)[23], Burr X-G by Yousof, Afify, Hamedani and Aryal (2016)[53], odd-Burr generalized family by Alizadeh, Cordeiro, Nascimento, Lima and Ortega (2016a)[12], the complementary generalized transmuted poisson family by Alizadeh, Yousof, Afify, Cordeiro and Mansoor (2016b)[15], among others.

For an arbitrary baseline cdf G(x), Shaw and Buckey (2007)[49] defined the transmuted-G (TG) family with cumulative distribution function and probability density function given by

$$F(x) = H(x;\psi)[1+\lambda-\lambda H(x;\psi)] \qquad (5.1) \quad \text{and}$$

$$f(x) = h(x;\psi)[1+\lambda-\lambda H(x;\psi)] \qquad (5.2)$$

Respectively, where $|\lambda| \leq 1$ is a shape parameter and $x \in \mathbb{R}$. The TG density is a mixture of the baseline density and exponentiated-G (exp-G) density with power parameter two. For $\lambda = 0$, equation (1) gives the baseline distribution. Let $h(x;\psi)$ and $H(x;\psi)$ denote the density and cumulative functions of the baseline model with power parameter ψ and consider the Weibull cumulative distribution function

$F(x) = 1 - e^{-x^{\alpha}} for\ x > 0$ with positive parameter α. Based on this density, Bourguignon ,Silva and Cordeiro (2014)[21] replaced the argument x by $H(x;\psi)/\overline{H}(x;\psi)$, where $\overline{H}(x;\psi) = 1 - H(x;\psi)$ and defined the cumulative distribution function of their Weibull-G class by

$$H(x;\alpha) = \int_0^{\left\{\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right\}} \alpha t^{\alpha-1} \exp(-t^{\alpha})\, dt = 1 - exp\left\{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}\right\}$$

(5.3)

respectively, where $\psi = (\psi_k) = (\psi_1, \psi_2, \dots)$ is a parameter vector. Based on the TG family and Weibull-G (WG) family, a new generator is constructed by inserting (5.3) into (5.1). We have

$$F(x) = \left\{1 - e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}\right\}\left[1 + \lambda e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}\right], \qquad (5.4)$$

where $G(x;\psi)$ is the baseline cumulative distribution function, $\alpha > 0\ and\ |\lambda| \leq 1$ are two additional shape parameters. The TW- $G(.;\alpha,\lambda,\psi)$ is a wider class of continuous distributions. It includes the TG family of distributions.

5.1 The new family

The corresponding probability density function is

$$f(x) = \alpha g(x;\psi)\frac{G(x;\psi)^{\alpha-1}}{\bar{G}(x;\psi)^{\alpha+1}} e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}$$

$$\left\{1 - \lambda + 2\lambda e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}\right\}, x > 0. \qquad (5.5)$$

The hazard rate function for the new family can be expressed as

$$\tau(x) = \frac{\alpha g(x;\psi)\frac{G(x;\psi)^{\alpha-1}}{\bar{G}(x;\psi)^{\alpha+1}} e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}\left\{1-\lambda+2\lambda e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}\right\}}{1-\left\{1-e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}\right\}\left[1+\lambda e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}}\right]}. \qquad (5.6)$$

For simulation of this family, if $U \sim u(0,1), when\ \lambda = 0$, then

$X_U = G^{-1}\left\{\frac{[-\log(1-U)]^{1/\alpha}}{1+[-\log(1-U)]^{1/\alpha}}\right\}$ and for $\lambda \neq 0$, we have

$$X_U = G^{-1}\left\{\frac{\left\{-log\left[\frac{\lambda-1+\sqrt{(1+\lambda)^2-4\lambda U}}{2\lambda}\right]\right\}^{1/\alpha}}{1+\left\{-log\left[\frac{\lambda-1+\sqrt{(1+\lambda)^2-4\lambda U}}{2\lambda}\right]\right\}^{1/\alpha}}\right\}$$

has cumulative distribution function (4). Below is a simple motivation for the development of TW-G family of distributions. Suppose ”T_1 and T_2” are two independent random v ariables from cumulative distribution function in (5.3). Define

$$X = \begin{cases} T_{1:2} \;\; with\, probability \; \dfrac{1}{2}(\lambda+1); \\ T_{2:2} \;\; with\, probability \; \dfrac{1}{2}(1-\lambda); \end{cases}$$

where $T_{1:2} = \min\{T_1, T_2\}\, and\, T_{2:2} = \max\{T_1, T_2\}$. Then the cumulative distribution function of X is given by (5.4). The TW-G family of distributions appears to be more flexible and could be used for modeling various types of data. For illustration propose we provide probability density function and hazard rate function of some special models of this family in Figures 5.1, 5.2. It can be seen that the hazard rate can take increasing, decreasing, upside down and bathtub shapes. Therefore, this family of distributions could be used to model diverse nature of data sets. Furthermore, the basic motivations for using the TW-G family in practice are the following:

i. To make the kurtosis more flexible compared to the baseline model;
ii. To produce a skewness for symmetrical distributiuons;
iii. To construct heavy-tailed distributions for modeling real data;
iv. To generate distributions with symmetric, left-skewed, right-skewed or reversed-J shape;
v. To define special models with all types of the hazard rate function;
vi. To provide consistently better fits than other generated models under the same underlying distribution;

The cumulative distribution function of the TW-G family in (5.4) can be expressed as

$$F(x) = 1 + (\lambda - 1)e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}} - \lambda e^{-2\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}} \qquad (5.7)$$

and after some algebra, we get

$$F(x) = 1 + (\lambda - 1)\sum_{i=0}^{\infty} \frac{(-1)^i\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha i}}{i!} - \lambda \sum_{i=0}^{\infty} \frac{2^i(-1)^i\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha i}}{i!}$$

$$= 1 + (\lambda - 1)\sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\frac{G(x;\psi)^{\alpha i}}{\bar{G}(x;\psi)^{\alpha i}} - \lambda \sum_{i=0}^{\infty} \frac{2^i(-1)^i}{i!}\frac{G(x;\psi)^{\alpha i}}{\bar{G}(x;\psi)^{\alpha i}}$$

$$= 1 + (\lambda - 1)\sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{-\alpha i}{j} G(x;\psi)^{\alpha i+j}$$

$$-\lambda \sum_{i,j=0}^{\infty} \frac{2^{i}(-1)^{i+j}}{i!} \binom{-\alpha i}{j} G(x;\psi)^{\alpha i+j}$$

$$= 1 + \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{-\alpha i}{j} (\lambda - 1 - \lambda \, x \, 2^{i}) G(x;\psi)^{\alpha i+j}$$

$$F(x) = 1 + \sum_{i,j=0}^{\infty} w_{i,j}\, G(x;\psi)^{\alpha i+j} = 1 + \sum_{i,j=0}^{\infty} w_{i,j}\, \Pi_{\alpha i+j(x)} , \tag{5.8}$$

Where $w_{i,j} = \frac{(-1)^{i+j}}{i!} \binom{-\alpha i}{j} (\lambda - 1 - \lambda \, x \, 2^{i})$ and

$\Pi_{\delta}(x) = G(x;\psi)^{\delta}$ is the cumulative density function of the exp-G distribution with power parameter δ. The corresponding TW-G density function is obtained by differentiating

$$f(x) = \sum_{i,j=0}^{\infty} w_{i,j}\, \Pi_{\alpha i+j(x)} , \tag{5.9}$$

Where $\pi_{\delta}(x) = \delta g(x;\psi) G(x;\psi)^{\delta-1}$ is the probability density function of the exp-G distribution with power parameter δ.

5.2 Mathematical Properties

5.2.1 Asymptotics

Let $c = inf\{x | G(x;\psi)^{\alpha}\}$, then the asymptotics of cdf, pdf and hrf as $x \to c$ are given by

$F(x) \sim (1 + \lambda) G(x;\psi)^{\alpha}$ as $x \to c$,

$f(x) \sim \alpha(1 + \lambda)\, g(x;\psi) G(x;\psi)^{\alpha-1}$ as $x \to c$,

$h(x) \sim \alpha(1 + \lambda)\, g(x;\psi) G(x;\psi)^{\alpha-1}$ as $x \to c$.

the asymptotics of cdf, pdf and hrf as $x \to \infty$ are given by

$1 - F(x) \sim e^{-\bar{G}(x;\psi)^{-\alpha}}$ as $x \to \infty$,

$f(x) \sim \alpha\, g(x;\psi) \bar{G}(x;\psi)^{-\alpha-1} e^{-\bar{G}(x;\psi)^{-\alpha}}$ as $x \to \infty$,

$h(x) \sim \alpha\, g(x;\psi)\bar{G}(x;\psi)^{-\alpha-1}$ as $x \to \infty$.

5.2.2 Probability weighted moments

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWMs method can generally be used

for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The (s, r)th PWMs of X following the TW-G family , say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\{X^s F(X)^r\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x)dx.$$

Using equations (5) and (6), we can write

$$f(x)\, F(x)^r = \sum_{i,j=0}^{\infty} m_{i,j}\pi_{\alpha(i+1)+j}(x)$$

where $m_{i,j} = \sum_{k,h=0}^{\infty} \frac{(-1)^{k+h+i+j}(h+1)^i\left[(1+h)\binom{r+k}{h}-2\lambda\binom{r+k+1}{h}\right]}{i!\alpha^{-1}\lambda^{-k}(1+\lambda)^{k-r}[\alpha(i+1)+j]}\binom{r}{k}\binom{-[\alpha(i+1)+1]}{j}$

Then, the (s, r)th PWMs of X can be expressed as

$$\rho_{s,r} = \sum_{i,j=0}^{\infty} m_{i,j} E\left(Y^s_{\alpha(i+1)+j}\right)$$

5.2.3 Residual life and reversed residual life functions

The nth moment of the residual life, say $m_n(t)=E[(X-t)^n \mid X>t]$, n=1,2,…, uniquely determined F(x). The nth moment of the residual life of X is given by $m_n(t)= \frac{1}{R(t)}\int_t^{\infty}(x-t)^n\, dF(x)$.

Therefore,

$$m_n(t) = \frac{1}{R(t)}\sum_{i,j=0}^{\infty} w_{i,j}^{\star}\int_t^{\infty} x^r \pi_{\alpha i+j}(x)\, dx,$$

where $w_{i,j}^{\star} = w_{i,j}\sum_{r=0}^{n}\binom{n}{r}(-t)^{n-r}$. Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $m_1(t) = E[(X-t) \mid X>t]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting n=1in the last equation. The nth moment of the reversed residual life ,

$M_n(t)$ = E[(t -X)^n | X $\leq$ t] for t >0 and n = 1,2,… uniquely determines F(x). We obtain $M_n(t) = \frac{1}{F(t)}\int_0^t (t-x)^n$ dF(x).

Therefore,

$$M_n(t) = \frac{1}{F(t)}\sum_{i,j=0}^{\infty} w_{i,j}^{**} \int_0^t x^r \pi_k(x)\, dx,$$

where $w_{i,j}^{**} = w_{i,j} \sum_{r=0}^{n}\binom{n}{r}(t)^{n-r}$. The mean inactivity time (MIT) or mean waiting time (MWT) , also called the mean reversed residual life function, is given by $M_1(t)$ = E[(t - X) | X$\leq$ t], and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in (0, t). By setting n=1 in the above equation, the MIT of the TW-G family of distribution can be obtained.

5.2.4 Stress-strength model

Stress-strength model is most widely approach used for reliability estimation. This model is used in many applications in physics and engineering such as strength failure and system collapse. In stress-strength modeling , R = Pr($X_2 < X_1$) is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1. The system fails if and only if the applied stress greater than its strength and the component will function satisfactorily whenever $X_1 < X_2$. R can be considered as measure of system performance and naturally arise in electrical and electronic systems. The reliability of the system is the probability that the system is strong enough to overcome the stress imposed on it . Let X_1 and X_2 be two independent random variables with TW-G$(\lambda_1, \alpha_1, \psi)$ and TW-G$(\lambda_1, \alpha_1, \psi)$ distributions .

The reliability is defined by

$$R = \int_o^{\infty} f_1(x; \lambda_1, \alpha_1, \psi)\, F_2(x; \lambda_1, \alpha_1, \psi)\, dx.$$

Then, one can write

$$R = \sum_{i,j=0}^{\infty} \alpha_{i,j} \int_0^{\infty} \pi_{\alpha i+j}(x)dx$$

$$+\sum_{i,j,h,k=0}^{\infty} b_{i,j,k,h} \int_0^{\infty} \pi_{\alpha_1 i+j+\alpha_2 h+k}(x)dx,$$

Where $\alpha_{i,j} = \frac{(-1)^{i+j}}{i!}\binom{-\alpha_1{}^i}{j}\left(\lambda_1 - 1 - \lambda_2 \times 2^i\right),$

and $\quad b_{i,j,k,h} = \frac{(-1)^{k+h+i+j}(\lambda_1 - 1 - \lambda_2 \times 2^i)\binom{-\alpha_1 i}{j}\binom{-\alpha_2 h}{k}}{i!(\alpha_1 i + j)[\alpha_1 i + j + \alpha_2 h + k](\lambda_2 - 1 - \lambda_2 \times 2^h)^{-1}}$

Thus , the reliability, R can be expressed as $R = \sum_{i,j=0}^{\infty} \alpha_{i,j} + \sum_{i,j,h,k=0}^{\infty} b_{i,j,k,h},$

5.2.5 Moments, incomplete moments and generating functions

The rth ordinary moment of X is given by

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx. \text{ Then,}$$

$$\mu'_r = \sum_{i,j=0}^{\infty} w_{i,j}\, E(Y^r{}_{\alpha i+j}) \qquad (5.10)$$

Henceforth , Y_δ denotes the exp-G distributions with power parameters(δ). Setting r=1 in (5.10), we have the mean of X. The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The nth central moment of X, say M_n, follows as

$M_n = E(X-\mu)^n = \sum_{h=0}^{n}(-1)^h\binom{n}{h}(\mu'_1)^n\mu'_{n-h}$.The cumulants ($k_n$) of X follow recursively from

$k_n = \mu'_n - \sum_{r=0}^{n-1}\binom{n-1}{r-1}k_r\,\mu'_{n-r}$ where $k_1 = \mu'_1$, $k_2 = \mu'_2 - \mu'^2_1$,

$k_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu'^3_1$, etc. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The main application of the first incomplete moment refer to the main derivations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The rth incomplete moment, say φ_r(t) can be expressed from (5.10)

$$\varphi_r(t) = \int_{-\infty}^{t} x^r f(x)\,dx = \sum_{i,j=0}^{\infty} w_{i,j}\int_{-\infty}^{t} x^r \pi_{\alpha i+j}(x)\,dx \qquad (5.11)$$

The mean derivations about the mean [δ_1= E (|X - μ'_1 |)] and about the mean

[δ_2= E (|X - M |)] of X are given by $\delta_1 = 2\mu'_1 F(\mu'_1) - 2\varphi_1(\mu'_1)$ and

$\delta_2 = \mu'_1 - 2\varphi_1(M)$, respectively, where μ'_1= E(X) , M= Median(X) =Q(0.5) is the median , $F(\mu'_1)$ is easily calculated from (4) and $\varphi_1(t)$ is the first incomplete moment given by (5.11) with r=1. A general equation for $\varphi_1(t)$ can be derived from (5.11) as $\varphi_1(t) = \sum_{i,j=0}^{\infty} w_{i,j}\, I_{\alpha i+j}(x)$, where $I_\delta(x) = \int_{-\infty}^{t} x\pi_\delta(x)\, dx$ is the first complete moment of the exp-G distribution. The moment generating function

$M_X(t) = E(e^{tX})$ of X can be derived from the equation (10) as

$$M_X(t) = \sum_{i,j=0}^{\infty} w_{i,j}\, M_{\alpha i+j}(t),$$

Where $M_\delta(t)$ is the mgf of Y_δ. Hence $M_X(t)$ can be determined from the exp-G generating function.

5.2.6 Order statistics

Let $X_1, X_2,\ldots, X_n$ be a random sample from the TW-G family of distributions and let $X_{1:n},\ldots, X_{n:n}$ be the corresponding order statistics. The pdf of the ith order statistic, $X_{1:n}$,can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \qquad (5.12)$$

where B(·,·) is the beta function. Substituting (5.4) and (5.5) in equation (5.12) and using a power series expansion, we arrive at

$$f(x)\, F(x)^{j+i-1} = \sum_{m,w=0}^{\infty} t_{m,w} \pi_{\alpha(m+1)+w}(x), \text{where}$$

$$t_{m,w} = \sum_{k,h=0}^{\infty} \frac{(-1)^{k+h+m+w}(h+1)^m [(1+\lambda)\binom{i+j+k-1}{h} - 2\lambda\binom{j+i+k}{h}]}{i!\alpha^{-1}\lambda^{-k}(1+\lambda)^{k-(j+i-1)}[\alpha(m+1)+w]}$$

$$\binom{j+i-1}{k}\binom{-[\alpha(m+1)+1}{w}).$$

The pdf of $X_{1:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-1} \frac{(-1)^j \binom{n-i}{j}}{B(i,n-i,+1)} \sum_{m,w=0}^{\infty} t_{m,w} \pi_{\alpha(m+1)+w}(x).$$

Then the density function of the TW-G order statistics is a mixture of exp-G densities. Based on the last equation, we note that the properties of $X_{i:n}$ can be expressed as

$$E(X^q{}_{i:n}) = \sum_{m,w=0}^{\infty} \sum_{j=0}^{n-1} \frac{(-1)^j \binom{n-i}{j} t_{m,w}}{B(i,n-i+1)} E(Y^q{}_{\alpha(m+1)+w}). \qquad (5.13)$$

5.3 Estimation

For parameter, the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used for constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood. Let $x_1, \ldots, x_n$ be a random sample from the TW-G distribution with parameters λ, α and ψ. Let $\Theta = (\lambda, \alpha, \psi)^T$ be the p x 1 parameter vector. For determining the MLE of Θ, we have the log-likelihood function

$$l = l(\Theta) = nlog\alpha + \sum_{i=1}^{n} \log g(x_i;\psi) + (\alpha - 1)\sum_{i=1}^{n} \log G(x_i;\psi)$$

$$-(\alpha + 1)\sum_{i=1}^{n} \log \bar{G}(x_i;\psi) - \sum_{i=1}^{n} S_i$$

$$+\sum_{i=1}^{n} log\{1 - \lambda + 2\lambda e^{-S_i}\}$$

Where $S_i = \left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{\alpha}$. The components of the score vector,

$$U(\Theta) = \frac{\partial l}{\partial \Theta} = \left(\frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \psi}\right)^T, \text{ are}$$

$$U_\lambda = \sum_{i=1}^{n} \frac{2e^{-S_i} - 1}{1 - \lambda + 2\lambda e^{-S_i}},$$

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n} \log G(x_i;\psi)$$

$$-\sum_{i=1}^{n} \log \bar{G}(x_i;\psi) - \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} \frac{2\lambda p_i e^{-S_i}}{1 - \lambda + 2\lambda e^{-S_i}}, \text{and}$$

$$U_\psi = \sum_{i=1}^{n} \frac{g'(x_i;\psi)}{g(x_i;\psi)} + (\alpha - 1)\sum_{i=1}^{n} \frac{G'(x_i;\psi)}{G(x_i;\psi)}$$

$$+(\alpha + 1)\sum_{i=1}^{n} \frac{G'(x_i;\psi)}{\bar{G}(x_i;\psi)} - \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} \frac{2\lambda q_i e^{-S_i}}{1 - \lambda + 2\lambda e^{-S_i}},$$

Where

$$g'(x_i;\psi) = \frac{\partial g(x_i;\psi)}{\partial \psi}, p_i = \left[\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right]^{\alpha} log\left[\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right]$$

$$q_i = \alpha \left[\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right]^{\alpha - 1} \frac{G'(x_i;\psi)}{[\bar{G}(x_i;\psi)]^2} \text{ and } G'(x_i;\psi) = \frac{\partial G(x_i;\psi)}{\partial \psi}.$$

Setting the nonlinear system of equations $U_\lambda = U_\alpha = 0 \; and \; U_\psi = 0$ and solving them simultaneously yields the MLE $\widehat{\Theta} = \left(\hat{\lambda}, \hat{\alpha}, \hat{\psi}\right)^T$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-newton algorithm to numerically maximize l. For interval estimation of the parameters, we obtain the pxp observed information matrix

$J(\Theta) = \left\{\frac{\partial^2 l}{\partial r \, \partial s}\right\}\{for \; r, s = \lambda, \alpha, \psi\}$, whose elements can be computed numerically. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\Theta}$ can be approximated by a multivariate normal $N_p\left(0, J(\widehat{\Theta})^{-1}\right)$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method.

5.4 Special TW-W models

5.4.1 The TW-W distribution

Consider the probability density function $g(x) = ba^b x^{b-1} e^{-(ax)^b}$ and cumulative distribution function $G(x) = 1 - e^{-(ax)^b}$ of the W distribution with scale a>0 and shape b>0 parameters. Inserting these functions in

$$f(x) = \alpha g(x;\psi) \frac{G(x;\psi)^{\alpha-1}}{\bar{G}(x;\psi)^{\alpha+1}} e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^\alpha} \left\{1 - \lambda + 2\lambda e^{-\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^\alpha}\right\}, x > 0$$

the probability density function of the TW-W model(for x>0) is given by

$$f(x) = \alpha a b a^b x^{b-1} e^{-(ax)^b} \left[1 - e^{-(ax)^b}\right]^{\alpha-1}$$

$$e^{-\left[e^{-(ax)^b} - 1\right]^\alpha} \left\{1 - \lambda + 2\lambda e^{-\left[e^{-(ax)^b} - 1\right]^\alpha}\right\} \qquad (5.14)$$

A random variable having probability density function (5.14) is denoted by

$X \sim TW - W(\alpha, \lambda, a, b)$. For b=1, we have the TW-exponential distribution. The

TW-W density and hrf plots for selected parameter values are displayed in Figure 5.1.

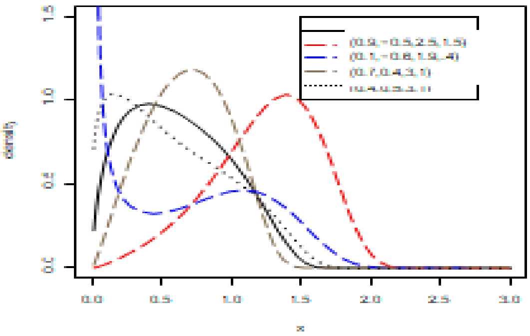


Figure 5.1: pdf of transmuted weibull distribution

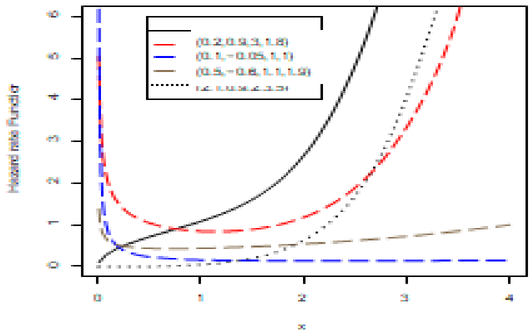


Figure5.1: hrf of transmuted weibull distribution

5.4.2 The TW-L distribution

Consider the probability density function $g(x) = \frac{a^2}{1+a}(1+x)e^{-ax}$ (for x>0) of the Li distribution with positive shape parameter a. The pdf of the TW-L model is given by

$$f(x) \quad = \frac{\alpha a^2(1+x)e^{\alpha ax}\left[1-\left(1+\frac{ax}{1+a}\right)e^{-ax}\right]^{\alpha-1}}{(1+a)\left(1+\frac{ax}{1+a}\right)^{\alpha+1}}$$

$$\text{x } e^{-\left[\frac{(1+a)e^{ax}}{1+a+ax}-1\right]^{\alpha}}\left\{1+\lambda-2\lambda e^{-\left[\frac{(1+a)e^{ax}}{1+a+ax}-1\right]^{\alpha}}\right\}.$$

The TW-L density and hrf plots for some parameter values are displayed in Figure 5.2.

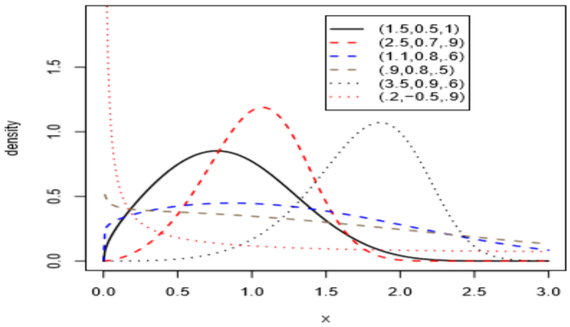


Figure 5.2: pdf of Transmuted Weibull-Lindey distribution

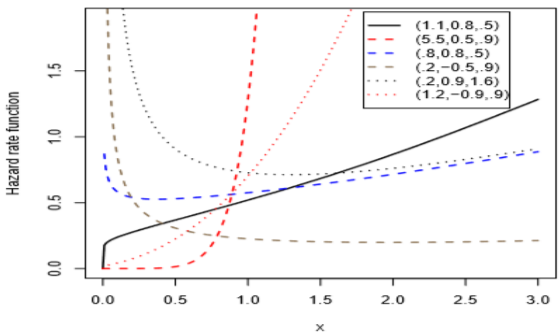


Figure 5.2: hrf of Transmuted Weibull-Lindey distribution

5.5 Simulation Study

The performance of the maximum likelihood estimators presented in transmuted Weibull-Weibull distribution with respect to sample size n is investigated. The evaluation is based on a simulation study:

Generate 5000 samplesof size n from TW-W distribution. The inversion method was used to generate samples.Calculate the maximum likelihood estimates for the five thousand samples, say $(\hat{\alpha}_i, \hat{\lambda}_i, \hat{a}_i, \hat{b}_i)$ for i=1,2,…5000.Calculate the biases and mean squared errors given by $Bias(n) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{h}_i - h)$

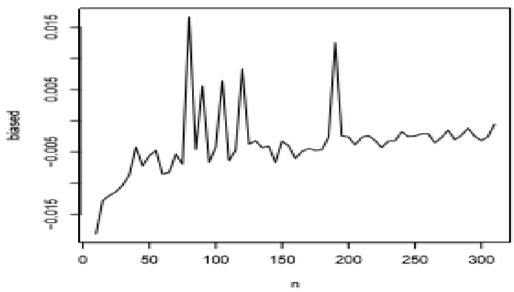
and $MSE(n) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{h}_i - h)^2$ for $h = \alpha, \lambda, a, b$.

We repeat these steps for n = 10, 15, 20,..... , 320 with α = 0.1, $\lambda = -0.6$, a = 1.9.and b = 0.4 (a special case of Figure(5.1), so computing $Bias_{\alpha}(n)$, $Bias_{\lambda}(n)$, $Bias_{a}(n)$, $Bias_{b}(n)$ and $MSE_{\alpha}(n)$, $MSE_{\lambda}(n)$, $MSE_{a}(n)$, $MSE_{b}(n)$ for n = 10, 15, 20, ..., 320. Figures 5.3 and 5.4 shows how the biases and mean squared errors change with respect to n.

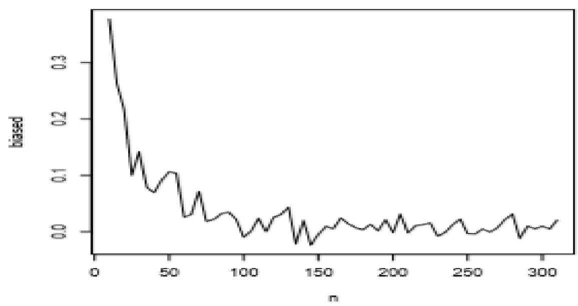
The following observations can be made:

1. the biases for λ, a and b are generally positive,
2. the biases for α have both sign,
3. the biases for each parameter generally approach zero ,n→ ∞
4. the biases appear smallest for α,
5. the mean squared errors for each parameter generally decrease to 0, as $n \to \infty$
6. the mean squared errors appear smallest for all parameters for n large enough ($n \geq 200$).

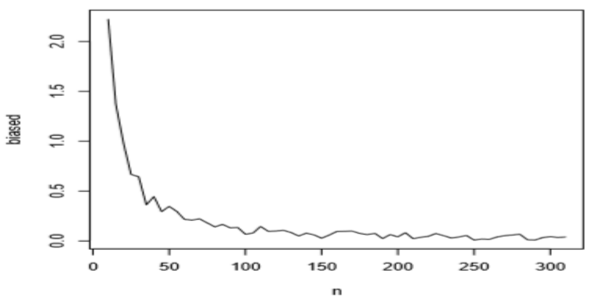
Figure 5.3:



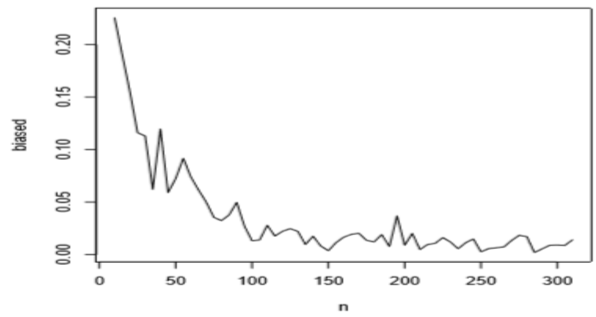
$Bias_{\alpha}(n)$ for n=10,15,20,….,320



$Bias_{\lambda}(n)$ for n=10,15,20,….,320

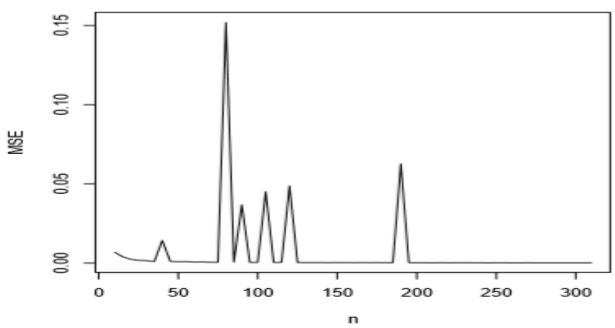


$Bias_a(n)$ for n=10,15,20,….,320

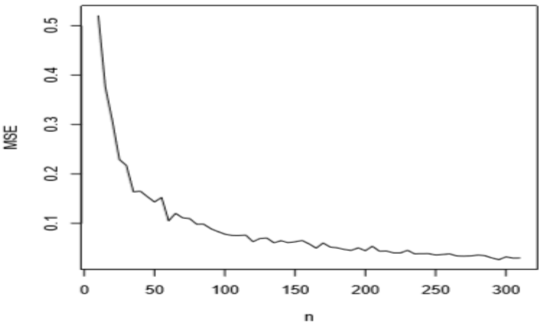


$Bias_b(n)$ for n=10,15,20,….,320

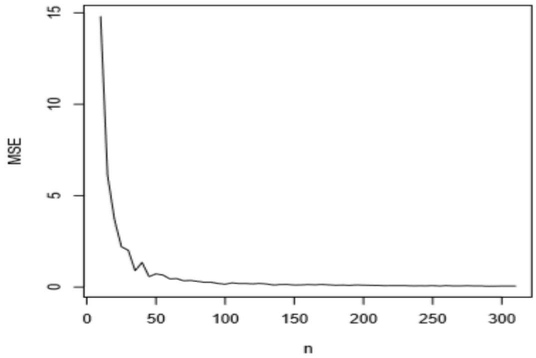
Figure 5.4:



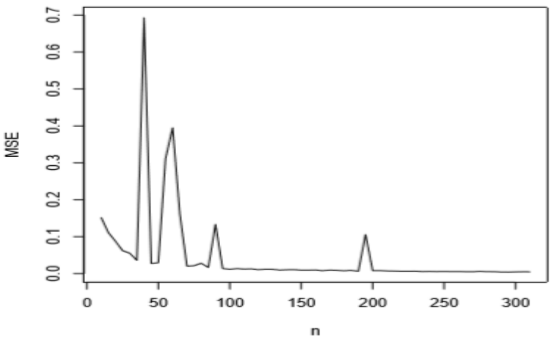
$MSE_\alpha(n)$for n=10,15,20,…,320.



$MSE_{\lambda}(n)$for n=10,15,20,…,320.



$MSE_{a}(n)$for n=10,15,20,…,320.



$MSE_{b}(n)$for n=10,15,20,…,320.

5.6 Applications

The two applications based on the two real data sets to demonstrate the flexibility of the TW-W and TW-L distributions has been presented. We compare TW-W with Kw-Weibull (Kw-W) (Cordeiro, Ortega and Nadarajah 2010)[29], Beta-Weibull (BW) (Lee, 2007)[39], Beta-Exponentiated Weibull (BEW) (Cordeiro, Gomes, da-siva and Ortega 2013)[28], Kw-Exponentiated Weibull (Kw-Ew) (Cordeiro, Saboor, Nauman Khan and Pascoa, 2016)[31]. Also,we compare TW-L with Exponentiated power Lindley(EPL) ,extended Lindley (EXL) (Bakouch, Al-Zahrani and Al-Shomrani, Marchi and Louzada 2012)[20] and generalized Lindley (GL) (Zakerzadeh and Dolati, 2009)[55] distributions.

The first data is given by Murthy (2004) on failure times for a particular model aircraft windshield. The data set consists of 84 observations and was also analyzed by Ramos, Marinho, da Silva and Cordeiro (2013)[46].The second data set is the fracture toughness of Alumina (Al2O3) (in the units MPa ml/2), Nadarajah and Kotz (2008)[44]. These data are 5.5, 5, 4.9, 6.4, 5.1, 5.2 , 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7,5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.The MLEs of parameters, maximized log-likelihood function, Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Consistent Akaike information criterion (CAIC) statistics are determined by fitting mentioned distributions using the two data sets. In general, the smaller values of these statistics show the better fit to the data sets. The MLEs are computed using the “optim” function in R statistical program. The estimated parameters based on MLE procedure reports in Tables 5.1 and 5.2, whereas the values of goodness-of-fit statistics are given in Tables 5.3 and 5.4.In the applications, the information about the hazard shape can help in selecting a particular model. The TTT plot is obtained by plotting

$$G(r/n) = \left[\left(\sum_{i=1}^{r} y_{(i)}\right) + (n-r)y_{(r)}\right] / \sum_{i=1}^{r} y_{(i)} \ ,$$

Where r=1,…,n and $y_{(i)}$ (i=1,…,n) are the order statistics of the sample, against r/n. If the shape is a straight diagonal the hazard is constant. It is convex shape for decreasing hazards and concave shape for increasing hazards.The TTT plot for both data sets presented in Figure 5.6. These figures indicates that first and second data set has increasing hazard rate functions. In both real data sets, the results show that the TW-W and TW-L distribution yields a better fit than other generalizations of

Weibull and Lindley distributions. Figure 5.5 shows the fitted pdf on histogram of both data sets.

Parameters estimates and standard deviation in parenthesis for first data set are presented in Table 5.1

Table 5.1:

data set

Model	Estimates	−Log Likelihood
TW-W(α,λ,a,b)	$0.10(9e^{-3})$, 1.00(0.12), 21.85(1.57), 4.21(0.04)	102.602
Kw-W(a,b,β,c)	17.47(0.33), 594.78(69.14), 0.29(0.01), 0.60(0.02)	107.756
BW(a,b,β,c)	1.28(0.10), 39.78(4.08), $0.07(3e^{-3})$, 2.34(0.07)	107.705
BEW(a,b,α,c,λ)	11.86(0.77), 3.86(0.24), $0.02(1e^{-3})$, 1.22(0.66), $0.16(6e^{-3})$	106.523
Kw-EW(a,b,α,c,λ)	0.24(0.01), $0.01(2e^{-3})$, 2.44(0.18), 3.17(0.19), 4.89(0.56)	107.752
BMW$(a,b,\alpha,\lambda,\gamma)$	4.84(1.50), 0.11(0.01), 1.03(0.02), 0.51(0.01), 0.65(0.02)	106.632

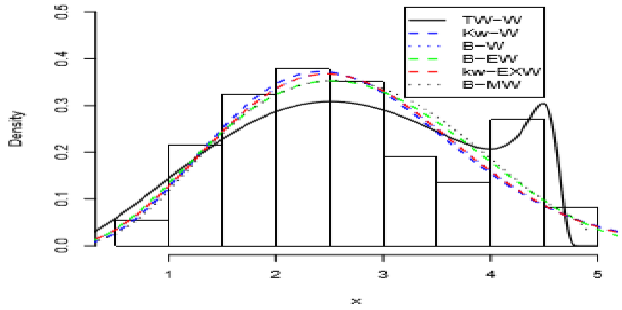
Parameters estimates and standard deviation in parenthesis for second data set are presented in Table 5.2.

Table 5.2:

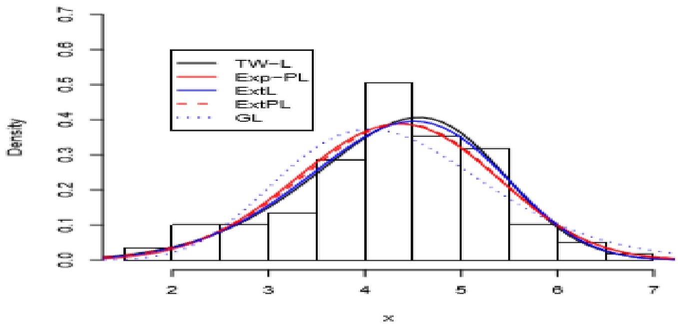
	Model	Estimates	−Log Likelihood
data set	TW-L(α,λ,θ)	3.00(0.21), 0.78(0.11), $0.27(4e^{-3})$	168.250
	EPL(θ,α,β)	$0.01(6e^{-4})$, 3.43(0.04), 0.88(0.08)	169.948
	EXL(θ,α,β)	$0.21(3e^{-3})$, -0.01(0.08), 4.99(0.34)	168.887
	EXPL(θ,α,β)	$0.01(6e^{-4})$3.44(0.04), 0.07(0.05)	169.381
	GL(θ,α,β)	3.64(0.08), 15.05(0.36), 8.03(12.30)	177.271

Figure 5.5:

Fitted probability density functions on histogram for example 5.1 and 5.2 are shown below in following Figure as follows



Fitted pdfs on histogram: example 5.1



Fitted pdfs on histogram: example 5.2

Formal goodness of fit statistics for the first data set and second data set are given in the following Table 5.3 and 5.4 respectively.

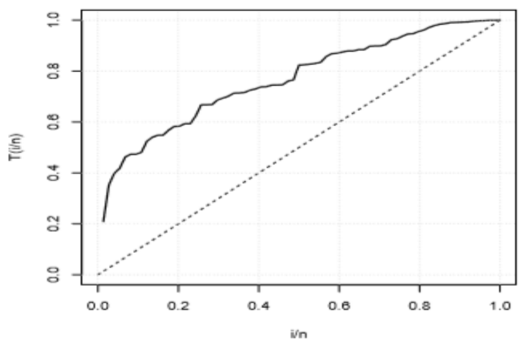
Table 5.3:

Model	Goodness of fit criteria			
	AIC	*BIC*	*HQIC*	*CAIC*
TW-W	213.20	224.32	211.46	213.55
Kw-W	223.51	234.63	221.77	223.86
BW	223.41	234.52	221.66	223.76
BEW	223.04	236.94	220.86	223.57
Kw-EW	225.50	239.40	223.32	226.03
BMW	223.26	237.15	221.08	223.79

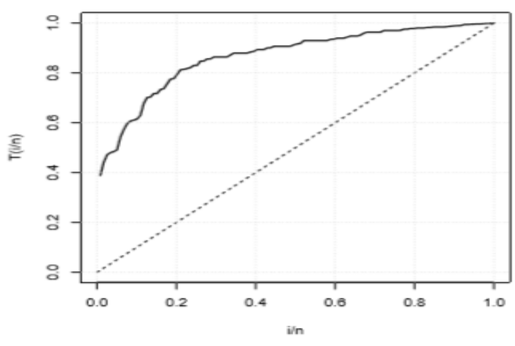
Table 5.4:

Model	Goodness of fit criteria			
	AIC	BIC	$HQIC$	$CAIC$
TW-L	342.50	350.83	342.70	341.19
EPL	345.89	354.23	346.10	344.59
EXL	343.77	352.11	343.98	342.46
EXPL	348.76	353.10	348.97	343.45
GL	364.54	368.88	364.75	359.23

Figure 5.6:



TTT-plot for the first data set



TTT-plot for the second data set

The mathematical properties of this new family including explicit expansions for the ordinary and incomplete moments, generating function, mean deviations , order statistics ,probability weighted moments were provided . Characterizations based on the ratio of two truncated moments as well as based on hazard function were presented. The model parameters were estimated by the maximum likelihood estimation method and the observed Information matrix was determined. It was shown, by means of two real data sets, that special cases of the TW-G class could give better fit than other models generated by the Well-known families.

Summary and Conclusion

In statistical inference, maximum likelihood estimation is a method of estimating the parameters of a statistical model, given observations. The method obtains the parameter estimates by finding the parameter values that maximize the likelihood function. The method of maximum likelihood is used with a wide range of statistical analyses. Reliability of a unit is the probability that the unit performs its intended function adequately for a given period of time under the stated operating conditions or environment. By a unit we mean an element, a system or a part of a system. Reliability analysis allows to study the properties of measurement scales and items that compose the scales. The reliability analysis procedure calculates a number of commonly used measures of scale reliability and also provides information about the relationships between individual items in the scale. The order statistics is the most fundamental tools in non-parametric statistics and inference. Important special cases of the order statistics are the minimum and maximum value of a sample, and the sample median and other sample quantiles.

The first chapter deals with basic concepts of Reliability, Statistical inference, Order statistics, Lifetime Distributions, Notations and Review of literature.

In the second chapter, the transmuted Rayleigh distribution which extends the Rayleigh distribution in the analysis of data with real support was proposed. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. The expansions for the mean, variance, moments and for the moment generating function was derived. The estimation of parameters was approached by the method of maximum likelihood, also the information matrix was derived. The likelihood ratio statistic was considered to compare the model with its baseline model. An application of the transmuted Rayleigh distribution to real data shows that the new distribution could be used quite effectively to provide better fits than the Rayleigh distribution.

In the third chapter, the introduction of a new class of transmuted Rayleigh distribution was discussed. The estimation of parameters of transmuted Rayleigh distribution were obtained by using new method of moments. A new distribution contains as a special case was introduced. The characterizing properties of the model were also determined.

The fourth chapter deals with a new transmuted geometric-G (TG-G) family of distributions, which extends the transmuted family by adding one extra shape parameter. The mathematical properties of the new family including explicit expansions for the ordinary and incomplete moments, generating function, mean deviations, entropies, order statistics and probability weighted moments were provided. The model parameters were estimated by the maximum likelihood estimation method and the observed information matrix was determined. It was shown, by means of two real data sets, that special cases of the TG-G class could give a better fit than other models generated by well-known families.

In the fifth chapter, a new class of distributions called the transmuted Weibull-G (TW-G) family of distributions, which extends the transmuted family by adding one extra shape parameter has been presented. The mathematical properties of this new family including explicit expansions for the ordinary and incomplete moments, generating function, mean deviations , order statistics ,probability weighted moments were provided . The model parameters were estimated by the maximum likelihood estimation method and the observed Information matrix was determined. It was shown, by means of two real data sets, that special cases of the TW-G class could give better fit than other models generated by the Well-known families.

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