TWO - SERVER MARKOVIAN QUEUES : HETEROGENEOUS Vs HOMOGENEOUS

BY

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SYNOPSIS

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In chapter I, notations, definitions and some preliminary results which are used in this dissertations are given. The earlier work done by many researchers in connection witth heterogeneous and homogeneous servers is also pointed out.

In Chapter II, we analyse a bulk service Markovian queueing system with **balking** and **two-heterogeneous servers.** Each service facility takes s customers or the whole queue (whichever is less) for [illegible] service. The service times for batches for the two channels are exponentially distributed with means μ_1^{-1} and μ_2^{-1} respectively.

Our main interest in this chapter is to determine the capacity of the slower server and obtain the optimal service rates that minimize the average characteristics of the heterogeneous system. The heterogeneous system is compared with a corresponding homogeneous system and a condition showing the efficiency of the heterogeneous system is obtained. This condition involves only the traffic intensity and the service rates.

The model we consider in this chapter is a generalisation of V.P.Singh's model [14]. He has compared the heterogeneous model $M/M_i/2/\beta$ with the homogenous model $M/M/2/\beta$. We here specifically compare $M/M_{i(1,s)}/2/\beta$ model with $M/M_{(1,s)}/2/\beta$ to obtain a corresponding condition.

The Tables representing the average queue length of both the systems are given in details.

A two server heterogeneous Markovian queueing system under general bulk service rule with a different queue discipline is discussed in Chapter III.

Arora [2] considers a bulk service queueing model with Poisson arrivals at a mean rate λ and two service facilities with capacities S_1 and S_2 $(S_2 \leq S_1)$. The first (second) facility takes $S_1(S_2)$ customers or the whole queue. The queue discipline he has considered for the above model is :

When both channels are free the service always starts at the first.

We extend this problem by assigning customers to the two servers with probabilities π_1 and π_2 respectively such that $\pi_1 + \pi_2 = 1$. Laplace transform of the transient state probabilities and the steady state probability distributions are derived for this above model.

In Chapter IV, an extension is made, in the sense that, we assume that the interarrival times have Erlang distribution

with mean for each phase of $\frac{1}{k\lambda}$ such that overall mean inter-arrival rate is $\frac{1}{\lambda}$. This system also has two heterogeneous servers whose service times are exponentially distributed with means $\frac{1}{\mu_i}$, $i = 1, 2$. The steady state queue size probabilities, the expected queue length and the expected waiting, time for this model are derived. The Little's formula is also verified on that.

CHAPTER - I

INTRODUCTION

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INTRODUCTION

A queueing system can be described by the flow of units for service, forming or joining the queue, if service is not immediately available and leaving the system after being served. By units we mean those demanding service. In practice, customers at a bank counter (or) at a reservation counter, incoming calls at a telephone exchange, vehicular traffic at a traffic intersection, machines under repair by a mechanic etc., are considered as units. The basic characteristic features are the input, the service mechanism, the queue discipline and the number of channels.

THE INPUT : describes the manner in which units arrive (either singly or in a group) and join the system. The interval between two consecutive arrivals is called the interarrival time.

1. Exponential (Markovian) distribution with density function $f(x) = \lambda e^{-\lambda x}$.

2. Erlang distribution with density function

$$f(x) = \frac{(\lambda k)^k \, x^{k-1} \, e^{-k\lambda x}}{\Gamma k}$$

are some of the interarrival time distributions we have considered in our dissertation.

SERVICE MECHANISM

The service mechanism describes the manner in which service is rendered. A unit may be served either singly or in a batch. The time required for servicing a unit (or a group, in case of batch service) is called the service time.

In the case of batch service the usual rule followed by many authors is the General Bulk service rule introduced by Neuts [12].

GENERAL BULK SERVICE RULE

If, immediately after the completion of a service, the server finds less than **a** units present, he waits until there are **a** units, where upon he takes the batch of **'a'** units for service ; if he finds **'a'** or more but atmost b, he takes them all in the batch and if he finds more than b, he takes in the batch for service b units, while others wait. The batch takes a minimum of **'a'** units and a maximum of **'b'** units. In Chapter II and III we consider the above rule with a = 1.

THE QUEUE DISCIPLINE : indicates the way in which the units form a queue and are served.

a) First come first served (FCFS)
b) Last come first served (LCFS)
c) Random ordering before service

are some of the usual queue disciplines

SYSTEM CAPACITY

The number of customers, both in the queue and service put together, is called the system capacity. It may be finite or infinite.

NUMBER OF CHANNELS

The system may have a single channel or s-parallel channels for service. In parallel channels some times all the servers may be considered to have the identical mean service rate. In this case the system is said to be the homogeneous system. In the case of heterogeneous system the mean service rates of different servers are assumed to be distinct.

Most of the multiserver waiting line problems tackled in the literature assumes the servers to be identical. However this situation is not very realistic and can prevail only when the service process is highly mechanically controlled. In the case of human servers, they cannot be expected to work at the same rate. Such situations are faced in every day life, e.g., at check out counters in a grocery stores in banks etc. So our main interest here is to discuss the heterogeneous queueing system.

EARLIER WORK IN HETEROGENEOUS QUEUEING SYSTEM

Morse [11] seems to be the first to introduce the concept of heterogeneity in service. He considers a service facility consisting of two independent branches, one of rate $2\sigma\mu$ and the other of rate $2(1 - \sigma)$, where $\sigma \varepsilon (0, \frac{1}{2})$; and obtained the steady state results for the following two cases.

1. no queue is allowed before the service facility and
2. an infinite queue is allowed before the service facility.

This problem has been generalized to n-independent branches by Gupta and Goyal [8]. These authors have obtained the system-size distribution in the case of an upper limit on the queue size.

Saaty [13] further discusses Morse's problem by designing the service rates μ_1 and μ_2 to the two branches and he replaces σ by $\mu_1 / (\mu_1 + \mu_2)$. The new feature in his discussion is that both branches act as two servers and an arrival, on finding one of the servers free, immediately enters service. He has given explicit expressions for the steady state probabilities and the mean number in the system.

Krishnamoorthy [9] further extends this problem by assigning customers to the two servers with probabilities π_1 and π_2, such that $\pi_1 + \pi_2 = 1$. He obtains both the steady state and the transient solutions and also studies the problem under a modified queue discipline by allowing a certain fixed number infront of the first server.

For an arbitrary number of servers Gumbel [7] studies the $M/M_i/S$ system in which arrivals, on finding all the servers busy, join a common waiting line and get sereved on a FIFO (first-in-first-out) basis. He compares the average number of customers in the system to the average numbeer of customers in a similar system in which the service rate for each server is replaced by the arithmetic mean of the service rates of the original system.

Ancker and Gafarian [1] have further investigated the problem posed by Gumbel. They have put an upper limit N on the queue size and thus studied the $M/M_i/C/N$ system. They have obtained various steady state results and some of its special cases. Finally they have compared a heterogeneous system with an equivalent homogeneous system.

Godini [6] has obtained the steady state probabilities for the $M/M_i/C$ system under a modified queue discipline. If a service station with a smaller rate is available, the customer may prefer to wait until a service station with a more rapid service is available, if the total time spent in the system is in this way shorter.

V.P.Singh [14] analyses a Markovian queueing system with balking and two heterogeneous servers. He has compared the two systems by replacing the service rate for each server in the homogeneous system $M/M/2/\beta$ by the average service rate of the corresponding heterogeneous system $M/M_i/2/\beta$. A condition is obtained under which the $M/M_i/2/\beta$ system is more efficient than the corresponding homogeneous system. An exact formula that gives the best allocation of the service rates to the servers and that also minimizes the operational parameters of the heterogeneous system is also derived.

HETEROGENEOUS SERVERS WITH GENERAL BULK SERVICE RULE

Arora [2] and Ghare [5] have discussed the $M/M(1, b_1)$, $M(1, b_2)/2$ and $M/M(1, b)/C$ queueing systems respectively.

M.L.Chaudhry and others [4] have considered a more general two-heterogeneous system $[M/M(a_1, b_1), M(a_2, b_2)/2]$ and obtained the steady state probability distribution of the number of customers.

NOTATION

KENDALL'S NOTATION

A very convenient notation designed by Kendall (1951) to denote queueing systems has been universally accepted and used. It consists of a three part descriptor A/B/C, where the first and the second symbols denote the interarrival and service time distributions respectively and the third denotes the number of channels or servers.

For Example

(i) $M/M_i/2/\beta$ In this case we have a two-server queue under the assumption of Poisson arrivals with rate λ and negative exponentially distributed service times with different service rates $\mu_i(i = 1, 2)$ for each of the two servers. β denotes the probability with which the customer joins the system when both servers are busy.

(ii) $E_k/M_i/2$ Here the inter-arrival times have an Erlang distribution and the service is as in (i).

(iii) $M/M_{i(1, s_i)}/2$ represents a bulk service queueing system with Poisson arrivals at a mean rate λ and two service facilities with capacities S_1 and S_2 ; the service times for batches for the two channels are exponentially distributed with mean $\frac{1}{\mu_i}$, i = 1, 2.

DEFINITIONS AND PRELIMINARY RESULTS

TRANSIENT STATE

A system is said to be in transient state if its operating characteristics are depending on time.

STEADY STATE

A system is said to be in steady state when its operating charactersitics are independent of time. That is, the number of arrivals during a certain interval becomes independent of time.

LAPLACE TRANSFORM

Let f(t) be a function of a positive real variable t. Then the Laplace transform (L.T.) of f(t) is defined by

$$F(s) = \int_0^\infty \exp(-st)\, f(t)\, dt$$ for the range of value of s for which the integral exists. We shall write F(s) = L f(t) to denote the Laplace transform of f(t).

ROUCHE'S THEOREM

If $f(z)$ and $g(z)$ are functions analytic inside and on a closed contour c and if $|g(z)| < |f(z)|$ on c then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside c.

For example the equation $\mu z^{s+1} - (\lambda\beta + \mu) z + \lambda\beta = 0$ has only one real root in $(0, 1)$ with $\frac{\lambda\beta}{s\mu} < 1$.

LITTLE'S FORMULA

If λ = the arrival rate

W = the expected waiting time in the system

L = the number of customers in the system

then $L = \lambda W$

CHAPTER - II

$M/M_{i(1,s)}/2/\beta$ - $M/M_{(1,s)}/2/\beta$ A COMPARISON

CHAPTER - II

$M/M_{i(1,s)}/2/\beta$ - $M/M_{(1,s)}/2/\beta$

A COMPARISON

We consider "A two-heterogeneous-server-Markovian queueing system" such that each server will serve customers in batches of size atleast one and atmost s with service rates $1/\mu_1$ and $1/\mu_2$ $(\mu_1 \geq \mu_2)$ respectively. We denote this model by $M/M_{i(1,s)}/2/\beta$ where β is the probability that the arriving customer finding both servers busy joins the system. A corresponding homogeneous model is also discussed.

Steady-state results are obtained for both the models and conditions under which the heterogeneous system is 'better' than the corresponding homogeneous system is derived. The two systems are compared by replacing the service rate for each server in the homogeneous system $M/M/2/\beta$ by the average service rate of the corresponding heterogeneous system $M/M_{i(1,s)}/2/\beta$.

Statement of the Problem

$M/M_{i(1,s)}/2/\beta$: This is a bulk service queueing system with Poisson arrivals at a mean rate λ and two service facilities each with capacity s. When both channels are free the service starts at the first. The service times for batches for the two channels are exponentially distributed with means $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$ respectively. When both channels are busy the customer either join the queue with probability $\beta (0 \leq \beta \leq 1)$ or balk with probability $1 - \beta$.

Notations and Equations

$P_{0\,0}(t)$ = Prob { the system empty at time t }

$P_{1\,0}(t)$ = Prob { the first channel busy, second empty (and no unit in queue) at time t }

$P_{0\,1}(t)$ = Prob { the second channel busy, first empty (and no unit in queue) at time t }

$P_{2\,n}(t)$ = Prob { both channels busy and $n (\geq 0)$ units waiting in queue at time t }

For n = 1, $P_1(t) = P_{0\,1}(t) + P_{1\,0}(t)$

The difference - differential equations for the system are :

$$1. \quad \frac{d\, P_{0\,0}(t)}{dt} = -\lambda\, P_{0\,0}(t) + \mu_1\, P_{1\,0}(t) + \mu_2\, P_{0\,1}(t)$$

$$2. \quad \frac{d\, P_{1\,0}(t)}{dt} = -(\lambda + \mu_1)\, P_{1\,0}(t) + \lambda\, P_{0\,0}(t) + \mu_2\, P_{2\,0}(t)$$

$$3. \quad \frac{d\, P_{0\,1}(t)}{dt} = -(\lambda + \mu_2)\, P_{0\,1}(t) + \mu_1\, P_{2\,0}(t)$$

$$4. \quad \frac{d\, P_{2\,0}(t)}{dt} = -(\lambda\beta + \mu)\, P_{2\,0}(t) + \lambda[P_{1\,0}(t) + P_{0\,1}(t)] + \mu \sum_{n=1}^{s} P_{2\,n}(t)$$

$$5. \quad \frac{d\, P_{2\,n}(t)}{dt} = -(\lambda\beta + \mu)\, P_{2\,n}(t) + \lambda\beta\, P_{2\,n-1}(t) + \mu\, P_{2\,n+s}(t) \quad \forall\, n \geq 1$$

Where $\mu = \mu_1 + \mu_2$

Assume the steady state distribution exists and let

$$\operatorname*{lt}_{t \to \infty} P_{i\,n}(t) = P_{i\,n}$$

Then the steady-state equations are :

$$6. \quad 0 = -\lambda P_{0\,0} + \mu_1 P_{1\,0} + \mu_2 P_{0\,1}$$

$$7. \quad 0 = -(\lambda + \mu_1)\, P_{1\,0} + \lambda P_{0\,0} + \mu_2 P_{2\,0}$$

$$8. \quad 9 = -(\lambda + \mu_2)\, P_{0\,1} + \mu_1 P_{2\,0}$$

$$9. \quad 0 = -(\lambda\beta + \mu)\, P_{2\,0} + \lambda[P_{1\,0} + P_{0\,1}] + \mu \sum_{n=1}^{s} P_{2\,n}$$

$$10. \quad 0 = -(\lambda\beta + \mu)\, P_{2\,n} + \lambda\beta\, P_{2\,n-1} + \mu\, P_{2\,n+s} \quad n \geq 1$$

The characteristic equation from (10) is

$$\mu z^{s+1} - (\lambda\beta + \mu) z + \lambda\beta = 0$$

Let us assume that $\frac{\lambda\beta}{s\mu} < 1$

Using Rouche's theorem, the above equation has only one zero inside $|z| = 1$. It is real. Let r be the root in $(0, 1)$. Then

10a. $$P_{2\,n} = A\, r^n \qquad \text{with } 0 < r < 1$$

$$= P_{2\,0}\, r^n \qquad (n \geq 0)$$

Solving equations (6), (7), (8) and simplifying we get :

$$P_{2\,0} = \frac{\rho\, \lambda(\lambda + \mu_2)}{\mu_1 \mu_2\, (1 + 2\rho)}\, P_{0\,0}$$

$$\text{Where } \rho = \frac{\lambda}{\mu_1 + \mu_2}$$

$$P_{1\,0} = \frac{\lambda\,(1 + \rho)}{\mu_1 (1 + 2\rho)}\, P_{0\,0}$$

$$P_{0\,1} = \frac{\rho\, \lambda}{\mu_2\, (1 + 2\rho)}\, P_{0\,0}$$

Then $$P_1 = P_{1\,0} + P_{0\,1}$$

$$= \frac{\lambda}{(1 + 2\rho)}\, \frac{(\mu_1 \rho + \mu_2 + \mu_2 \rho)}{\mu_1 \mu_2}\, P_{0\,0}$$

$$= \frac{\lambda}{(1 + 2\rho)}\, \frac{(\lambda + \mu_2)}{\mu_1 \mu_2}\, P_{0\,0}$$

10b. and $P_{2\,0} = \rho\, P_1$

The probabilities $P_{1\,0}$, $P_{0\,1}$, $P_{2\,n}$ are given in terms of $P_{0\,0}$. Thus $P_{0\,0}$ can be calculated using the normalising condition,

$$P_{0\,0} + P_{1\,0} + P_{0\,1} + \sum_{n=0}^{\infty} P_{2\,n} = 1$$

$$P_{0\,0}\left[1 + \frac{\lambda(1+\rho)}{\mu_1(1+2\rho)} + \frac{\rho\lambda}{\mu_2(1+2\rho)} + \frac{\rho\lambda(\lambda+\mu_2)}{\mu_1\mu_2(1+2\rho)}\sum_{n=0}^{\infty} r^n\right] = 1$$

$$P_{0\,0} = \left[1 + \frac{\lambda(\lambda+\mu_2)}{\mu_1\mu_2(1+2\rho)}\,\frac{(1-r+\rho)}{(1-r)}\right]^{-1}$$

10c. $$= \frac{1}{1 + C^*}$$

where

$$C^* = \frac{\lambda(\lambda+\mu_2)}{\mu_1\mu_2(1+2\rho)}\,\frac{(1-r+\rho)}{(1-r)}$$

Next we shall calculate the following resultings for the above model :

(i) Expected queue length E(Q)

(ii) Expected waiting time E(W)

(iii) Probability that both the servers are busy $\sum_{n=0}^{\infty} P_{2\,n}$

(iv) Probability that atleast one server is idle

$$P_{0\,0} + P_{1\,0} + P_{0\,1} = 1 - \sum_{n=0}^{\infty} P_{2\,n}$$

(i) Expected Queue Length

$$E(Q) = \sum_{n=1}^{\infty} n \, P_{2\,n}$$

$$= \sum_{n=1}^{\infty} n \, P_{2\,0} \, r^{n}$$

$$= P_{2\,0} \frac{r}{(1-r)^2}$$

10d. $$= \frac{\rho r}{(1-r)^2} P_1$$

(ii) Expected Waiting Time

$E(W) = \int_0^{\infty} t \; w(t) \; dt$ where $w(t)$ is the waiting time of an arriving customer in the queue.

The arriving customer has to wait only when he finds both the servers are busy and the queue contains $n = ks + q$ members with $k \geq 0$; $0 \leq q \leq s-1$. This occurs with probability $P_{2\;ks+q}$. Hence

$$W(t) \; dt = \sum_{q=0}^{s-1} \sum_{k=0}^{\infty} P_{2\;ks+q} \, \mu \, e^{-\mu t} \frac{(\mu t)^k}{\underline{|k}} \, dt$$

$$= \sum_{q=0}^{s-1} \sum_{k=0}^{\infty} \mu \, e^{-\mu t} \frac{(\mu t)^k}{\underline{|k}} \, \rho \, P_1 \, r^{ks+q} \, dt$$

$$= \rho P_1 \frac{1 - r^s}{1 - r} \mu e^{-\mu t} e^{-\mu t(r)^s} dt$$

$$= \rho P_1 \frac{1 - r^s}{1 - r} e^{-\mu t(1-r^s)} \mu dt$$

$$E(W) = \int_0^\infty t W(t) dt$$

$$= \rho P_1 \frac{1 - r^s}{1 - r} \mu \int_0^\infty e^{-\mu t(1-r^s)} dt$$

$$= \rho P_1 \frac{1}{\mu(1 - r)(1 - r^s)}$$

Since r satisfies the equation, $\mu r^{s+1} - (\lambda\beta + \mu)r + \lambda\beta = 0$ we get

$$E(W) = \frac{\rho P_1 r}{(1 - r)^2 \lambda\beta} = \frac{E(Q)}{\lambda\beta} \quad \text{(Littles formula)}$$

(iii) Probability that both the servers are busy $\sum_{n=0}^{\infty} P_{2n}$

$$= \sum_{n=0}^{\infty} P_{20} r^n$$

$$= \rho P_1 \frac{1}{1 - r}$$

Next we shall consider a corresponding two-server homogeneous queueing system, $M/M_{(1,s)}/2/\beta$.

Let $P_{2\ n}(t)$ denote the probability that both servers are busy and n customers are in the queue for all $n \geq 0$ at time t.

Let $P_1(t)$ denote the probability that only one server is busy and no unit in queue at time t.

Let $P_{0\ 0}(t)$ denote the probability that both servers are free and no unit in queue at time t.

The steady-state equations for the homogeneous system $M/M_{(1,s)}/2/\beta$ are given by

$$(11) \qquad 0 = -\lambda P_{0\ 0} + \mu P_1$$

$$(12) \qquad 0 = -(\lambda + \mu) P_1 + \lambda P_{0\ 0} + 2\mu P_{2\ 0}$$

$$(13) \qquad 0 = -(\lambda\beta + 2\mu) P_{2\ 0} + 2\mu \sum_{n=1}^{s} P_{2\ n} + \lambda P_1$$

$$(14) \qquad 0 = -(\lambda\beta + 2\mu) P_{2\ n} + \lambda\beta P_{2\ n-1} + 2\mu P_{2\ n+s} \quad (n \geq 1)$$

from (14)

$P_{2\ n} = P_{2\ 0}\, r^n$ where r is the root which lies in (0, 1) of the equation

$$2\mu z^{s+1} - (\lambda\beta + 2\mu) z + \lambda\beta = 0$$

Solving equations (11) and (12) we get

$$P_1 \quad = \quad 2\,\rho\,P_{0\,0} \qquad\qquad \text{Where } \rho = \frac{\lambda}{2\mu} < 1$$

$$P_{2\,0} = \quad P\,\rho^2\,P_{0\,0}$$

$$= \quad \rho\,P_1$$

$P_{0\,0}$ is derived using the normalising condition,

$$P_{0\,0} + P_1 + \sum_{n=0}^{\infty} P_{2\,n} = 1 \qquad \text{as}$$

$$P_{0\,0} \quad = \quad [1 + 2\rho + \frac{2\rho^2}{(1 - r)}]^{-1}$$

For the homogeneous system,

10e. $E(Q) \quad = \quad \dfrac{\rho r}{(1 - r)^2} \; P_1$

$$E(W) \quad = \quad \frac{E(Q)}{\lambda\beta}$$

COMPARISON BETWEEN $M/M_{(1,s)}/2/\beta$ AND $M/M_{i(1,s)}/2/\beta$

We compare the two systems by replacing the service rate μ of the homogeneous system by the average service rate $\frac{\mu_1 + \mu_2}{2}$ of the corresponding heterogeneous system. This substitution leaves ρ unaffected for both systems and simplifies some of the computations. The difference between the two systems lies in the probabilities $P_{0\ 0}$ and P_1 Otherwise the probabilities $P_{2\ n}$ in both have the geometric distribution for $n \geq 0$.

We say that a heterogeneous system is 'better' than the corresponding homogeneous system if ${}^{(i)}P_{0\ 0} > P_{0\ 0}$, ${}^{(i)}P_1 < P_1$ and ${}^{(i)}P_{2\ n} < P_{2\ n}$ for $n \geq 0$. The superscript (i) on the top left side of the symbols corresponds to the heterogeneous system.

The following theorem gives a condition in terms of ρ and the service parameters guaranteeing that a two-server heterogeneous system is better than the corresponding homogeneous system

THEOREM : 1

Given ρ and λ, a necessary and sufficient condition for a two-server heterogeneous system to be better than the corresponding homogeneous system is that $\rho < \frac{\mu_2}{\mu_1 - \mu_2}$ then $\delta E(.) = E(.) - {}^{(i)}E(.) > 0$.

Proof

Since for both systems $P_{2\ n} = \rho r^n P_1 \ n \geq 1$, it suffices to compare the probabilities P_1 and $P_{0\ 0}$. We have

$$P_1 = 2\rho P_{0\ 0}$$

$$= \frac{2\rho}{1 + 2\rho + \frac{2\rho^2}{1 - r}}$$

$$^{(i)}P_1 = \frac{\lambda}{1 + 2\rho} \frac{\lambda + \mu_2}{\mu_1 \mu_2} \frac{1}{1 + \frac{\lambda(\lambda + \mu_2)(1 - r\rho\rho)}{\mu_1 \mu_2 (1 + 2\ell)(1 - r)}}$$

$$= [\frac{\mu_1 \mu_2 (1 + 2\rho)}{\lambda(\lambda + \mu_2)} + \frac{1 - r + \rho}{1 - r}]^{-1}$$

Now

$$^{(i)}P_1 < P_1$$

$$\Longleftrightarrow P_1^{-1} < {}^{(i)}P_1^{-1}$$

$$\Longleftrightarrow 1 + 2\rho + \frac{2\rho^2}{1 - r} < 2\rho [\frac{\mu_1 \mu_2 (1 + 2\rho)}{\lambda(\lambda + \mu_2)} + \frac{\rho + 1 - r}{1 - r}]$$

$$\Longleftrightarrow \frac{1}{2\rho} < \frac{\mu_1 \mu_2 (1 + 2\rho)}{\lambda(\lambda + \mu_2)}$$

Substituting for $\rho = \frac{\lambda}{\mu_1 + \mu_2}$

$$\Longleftrightarrow \quad \frac{\mu_1 + \mu_2}{2\lambda} < \frac{\mu_1 \mu_2 \left(1 + \frac{2\lambda}{\mu_1 + \mu_2}\right)}{\lambda(\lambda + \mu_2)}$$

$$\Longleftrightarrow \quad \frac{\lambda(\mu_1 - \mu_2)^2}{\mu_1 + \mu_2} < 2(\mu_1 - \mu_2)$$

$$14 \quad \Longleftrightarrow \quad \rho < \frac{\mu_2}{\mu_1 - \mu_2}$$

We can prove that the above condition also guarantees that ${}^{(i)}P_0 > P_0$

ie $\quad P_{0\,0}^{-1} > {}^{(i)}P_{0\,0}^{-1}$

$$\Longleftrightarrow \quad 1 + 2\rho + \frac{2\rho^2}{1 - r} > 1 + \frac{\lambda(\lambda + \mu_2)(\rho + 1 - r)}{\mu_1 \mu_2 (1 + 2\rho)(1 - r)}$$

Simplifying

$$-\lambda(\mu_1 - \mu_2)^2 + \mu_2(\mu_1^2 - \mu_2^2) > 0$$

$$\Longleftrightarrow \quad -\frac{\lambda}{\mu_1 + \mu_2} > -\frac{\mu_2}{\mu_1 - \mu_2}$$

$$15 \quad \Longleftrightarrow \quad \rho < \frac{\mu_2}{\mu_1 - \mu_2}$$

Next we show that $\delta E(.) > 0$. From equations (10d) and (10e)

$$\delta E(Q) = E(Q) - {}^{(i)}E(Q)$$

$$= \frac{\rho r}{(1 - r)^2} [P_1 - {}^{(i)}P_1]$$

$$\delta E(W) = E(W) - {}^{(i)}E(W)$$

$$= \frac{\rho r}{(1 - r)^2 \ \lambda\beta} [P_1 - {}^{(i)}P_1]$$

Since for given λ and ρ the quantities infront of the final paranthesis in the above two expressions are positive and also $P_1 - {}^{(i)}P_1 > 0$ from (14) and (15) we have

$$\delta E(.) > 0$$

REMA RK

Suppose we change the queue discipline in this two-server heterogeneous system to the following : "An arrival finding both servers free chooses the first server with probability π_1 and second server with probability π_2 such that $\pi_1 + \pi_2 = 1$; then the condition given in theorem 1 takes the form

$$\rho < \frac{\mu_2 \pi_1 - \mu_1 \pi_2}{\mu_1 - \mu_2}$$

The steady-state results and transient solutions with the above discipline for a more general case are discussed in the next chapter.

Now the best allocation of the service rates μ_1 and μ_2 to the two servers is determined in such a way that the average charactereistics of the heterogeneous system are minimized and thereis an improvement over the corresponding homogeneous system. In this context the following lemma and theorem are proved.

Lemma 1

For a fixed λ and ρ, $f(\mu_2) = \frac{\lambda + \mu_2}{\mu_2 (\lambda - \rho \mu_2)}$ is a strictly convex function of μ_2 and achieves a unique minimum at $\mu_2^o = \lambda [(1 + \frac{1}{\rho})^{\frac{1}{2}} - 1]$

Proof

Given $f(\mu_2) = \frac{\lambda + \mu_2}{\mu_2 (\lambda - \rho \mu_2)}$

$$f'(\mu_2) = \frac{\lambda \mu_2 - \rho \mu_2^2 - (\lambda + \mu_2)(\lambda - 2\rho \mu_2)}{(\lambda \mu_2 - \rho \mu_2^2)^2}$$

$$f'(\mu_2) = 0 \implies \mu_2^{o} = \pm \lambda[(1 + \frac{1}{\rho})^{\frac{1}{2}} - 1]$$

positive value

The / gives the minimum condition for the function $f(\mu_2)$ as $f''(\mu_2^{o}) > 0$

THEOREM : 2

For a given λ and ρ, $\mu_2^{o} = \lambda[(1 + \frac{1}{\rho})^{\frac{1}{2}} - 1]$ is the best allocation to the second server. This μ_2^{o} minimizes the average characteristics ${}^{(i)}E(Q)$ and ${}^{(i)}E(W)$ of the two-server heterogeneous system and gives the following reductions over the corresponding average characteristics of the homogeneous system :

16. $$\delta E(Q) = \rho\, r\, k^{*}$$

17. $$\delta E(W) = (\frac{\rho^2}{\lambda})\, k^{*}$$

18. where $$k^{*} = \frac{1}{(1-r)}\left[\frac{2\rho}{2\rho^2 + (1+2\rho)(1-r)} - \frac{1}{\rho + (1-r)[1 + (1+2\rho) \times ((1+\frac{1}{\rho})^{\frac{1}{2}} - 1)^2]}\right]$$

and

$$\delta E(.) = E(.) - {}^{(i)}E(.) \text{ is positive.}$$

Note that μ_2^{o} is independent of β.

PROOF

In order to show that the choice of μ_2^o stated in the theorem is the best for the second server it will be sufficient to show that ${}^{(i)}P_{2\,n}$'s are minimized and ${}^{(i)}P_{0\,0}$ is maximized by μ_2^o

From (10a) and (10b)

$$ {}^{(i)}P_{2\,n} = r^n \rho \, {}^{(i)}P_1 $$

$$ {}^{(i)}P_{0\,0} = \frac{1}{1 + C^*} \qquad \text{(from 10c)} $$

and $$ {}^{(i)}P_1 = \frac{(1 - r)}{(1 + \rho - r)} \frac{1}{(1 + \frac{1}{C^*})} $$

where

$$ C^* = \frac{\lambda(\lambda + \mu_2)}{\mu_1 \mu_2 (1 + 2\rho)} \frac{(1 - r + \rho)}{(1 - r)} $$

C^* can be written as

$$ C^* = k \, f(\mu_2) $$

with

$$ K = \frac{\lambda(1 + \rho - r)}{(1 - r)(1 + 2\rho)} > 0 $$

Lemma 1 shows that $f(\mu_2)$ is minimized by μ_2^o. Hence C^* is minimized by μ_2^o. Therefore ${}^{(i)}P_0 = \frac{1}{1 + C^*}$ is maximized by μ_2^o and

$${}^{(i)}P_1 = \frac{(1 - r)}{(1 + \rho - r)} \frac{1}{(1 + \frac{1}{C^*})}$$ is minimized by μ_2^o.

Next we shall consider $\delta E(Q)$ and $\delta E(W)$ at μ_2^o

$$\delta E(Q) = E(Q) - {}^{(i)}E(Q)$$

$$= \frac{\rho r}{(1 - r)^2} (P_1 - {}^{(i)}P_1) \quad \text{from (10d) and (10e)}$$

Substituting for P_1 and ${}^{(i)}P_1$ and simplifying we get

$$\delta E(Q) = \frac{\rho r}{(1 - r)^2} \left[\frac{2\rho}{(1 + 2\rho + \frac{2\rho^2}{1 - r})} - \frac{(1 - r)}{(1 + \rho - r) + (1 + 2\rho)(1 - r)((1 + \frac{1}{\rho})^{\frac{1}{2}} - 1)^2} \right]$$

$$= \rho r k^* \quad \text{(from 18)}$$

Similarly,

$$\delta E(W) = \frac{\rho r}{(1 - r)^2 \lambda\beta} (P_1 - {}^{(i)}P_1)$$

$$= \frac{\rho r}{\lambda\beta} k^* \quad \text{(from 18)}$$

REMARK

With this allocation of $\mu_2^{\,0}$ we can show that $\delta\, E(.)$ is positive. For this it is enough if we prove that the condition of theorem 1 is satisfied by $\mu_1^{\,0}$ and $\mu_2^{\,0}$. Recall that

$$\mu_2^{\,0} = \lambda\,[(1+\frac{1}{\rho})^{\frac{1}{2}} - 1] \text{ and } \mu_1^{\,0} = (\frac{\lambda}{\rho} - \mu_2^{\,0})$$

We now show that $\rho < \dfrac{\mu_2^{\,0}}{\mu_1^{\,0} - \mu_2^{\,0}}$

Assume on the contrary that $\dfrac{\mu_2^{\,0}}{\mu_1^{\,0} - \mu_2^{\,0}} \leq \rho$

then

$$\frac{\mu_2^{\,0}}{\frac{\lambda}{\rho} - \mu_2^{\,0} - \mu_2^{\,0}} \leq \rho$$

$$\Longrightarrow \quad \lambda\,[(1+\frac{1}{\rho})^{\frac{1}{2}} - 1] \leq \frac{\lambda}{(1+2\rho)}$$

$$\Longrightarrow \quad [(1+\rho)^{\frac{1}{2}} - (\rho)^{\frac{1}{2}}]^2 \leq 0 \text{ which is impossible.}$$

Hence $\mu_1^{\,0}$ and $\mu_2^{\,0}$ satisfy the required condition.

We next proceed to establish that $\mu_2^{\,0} = \lambda\,[(1+\frac{1}{\rho})^{\frac{1}{2}} - 1]$ also reduces the variance of queue size, waiting time distributions etc. Our next theorem demonstrates this fact and gives reductions in the variance of these quentities of the heterogeneous system, over the corresponding homogeneous system. We shall denote the variance of these quantities by var(.). We shall also denote by δ var(.) the differencee in the variance of the two-server homogeneous and heterogeneous system.

THEOREM : 3

For fixed β, λ and ρ, $\mu_2^{o} = \lambda\,[(1+\frac{1}{\rho})^{\frac{1}{2}} - 1]$ minimizes the variances of the queue length and the waiting-time distributions respectively. Then,

We have

$$19 \qquad \delta\, \mathrm{Var}(Q) = \rho\, r\, k^{*}\, [\frac{2r}{1-r} + 1 + \frac{\rho r}{(1-r)^2}\, [(1-r)^2 k^{*} - 2\, P_1]]$$

$$20 \qquad \delta\, \mathrm{Var}(W) = k^{*}\, [\frac{2\,\rho\, r^2}{(1-r)\,(\lambda\,\beta\,)^2} - \frac{\rho^2\, r^2}{(1-r)^2\,(\lambda\,\beta\,)^2} \; x \; [2P_1 - k^{*}\,(1-r)^2\,]]$$

Where

$$21 \qquad k^{*} = \frac{1}{1-r}\, [\frac{2\rho}{2\rho^2 + (1+2\rho)\,(1-r)} - \frac{1}{\rho + (1-r)\,[1 + (1+2\rho\,)\,((1+\frac{1}{\rho})^{\frac{1}{2}} - 1)^2\,]}]$$

$$P_1 = \frac{2\rho(1-r)}{2\rho^2 + (1-r)(1+2\rho)}$$

and

$$\delta\, \mathrm{Var}(.) = \mathrm{Var}(.) - {}^{(i)}\mathrm{Var}(.) > 0$$

Proof

$${}^{(i)}\mathrm{Var}(Q) = \sum_{n=1}^{\infty} n^2\, {}^{(i)}P_n - ({}^{(i)}E(Q))^2$$

$$= \rho\, {}^{(i)}P_1 \sum_{n=1}^{\infty} n(n-1)r^n + {}^{(i)}E(Q) - ({}^{(i)}E(Q))^2$$

$$= \rho\, {}^{(i)}P_1\, r^2 \frac{d^2}{dr^2}\left(\sum_{n=0}^{\infty} r^n\right)$$

$$+ \frac{\rho r}{(1-r)^2}\, {}^{(i)}P_1 [1 - \frac{\rho r}{(1-r)^2}\, {}^{(i)}P_1]$$

$$= \frac{\rho r}{(1-r)^2} [\frac{2r}{1-r} + 1 - \frac{\rho r}{(1-r)^2}\, {}^{(i)}P_1]\, {}^{(i)}P_1$$

Similarly

$$\mathrm{Var}(Q) = \frac{\rho r}{(1-r)^2} [\frac{2r}{1-r} + 1 - \frac{\rho r}{(1-r)^2} P_1] P_1$$

22 $$\delta\, \mathrm{Var}(Q) = \mathrm{Var}(Q) - {}^{(i)}\mathrm{Var}(Q)$$

$$= \frac{\rho r}{(1 - r)^2} [(\frac{2r}{1 - r} + 1) (P_1 - {}^{(i)}P_1)$$

$$- \frac{\rho r}{(1 - r)^2} (P_1^2 - {}^{(i)}P_1^2)]$$

$$= \frac{\rho r}{(1 - r)^2} [(\frac{2r}{1 - r} + 1) (P_1 - {}^{(i)}P_1)$$

$$- \frac{\rho r}{(1 - r)^2} (P_1 - {}^{(i)}P_1) (P_1 + {}^{(i)}P_1)]$$

$P_1 - {}^{(i)}P_1$ at μ_2^o is

$$P_1 - {}^{(i)}P_1 = k^*(1 - r)^2$$

$P_1 + {}^{(i)}P_1$ at μ_2^o is

$$P_1 + {}^{(i)}P_1 = - k^*(1 - r)^2 + 2 P_1$$

Substituting these in (22) we get

$$\delta \text{ Var}(Q) = \rho r k^* [\frac{2r}{1-r} + 1 + \frac{\rho r}{(1-r)^2} [(1-r)^2 k^* - 2 P_1]]$$

To show that $\delta \text{ Var}(Q) > 0$

Assume $\delta \text{ Var}(Q) \leq 0$ then it follows that

$$\rho r k^* [\frac{2r}{1 - r} + 1 + \frac{\rho r}{(1 - r)^2} [(1 - r)^2 k^* - 2 P_1]] \leq 0$$

$$\Longrightarrow \quad \frac{\rho r}{(1-r)^2}(1-r)^2 k^* \leq -\frac{2r}{1-r} - 1 + \frac{2\rho r}{(1-r)^2} P_1$$

$$\Longrightarrow \quad \rho r k^* \leq -\frac{2r}{1-r} - 1 + 2 P_1 \frac{\rho r}{(1-r)^2}$$

Substituting $P_1 = \dfrac{2\rho(1-r)}{2\rho^2 + (1-r)(1+2\rho)}$

$$\rho r k^* \leq -\frac{2r}{1-r} - 1 + \frac{2\rho r}{(1-r)^2} \frac{2\rho(1-r)}{2\rho^2 + (1-r)(1+2\rho)}$$

$$\leq -\frac{2r}{1-r} - 1 + \frac{2r}{(1-r)} \frac{2\rho^2}{(2\rho^2 + (1-r)(1+2\rho))} < 0$$

But since k^* is positive, δ Var(Q) is also positive

We shall calculate δ Var(W)

$${}^{(i)}\mathrm{Var}(W) = {}^{(i)}E(W^2) - ({}^{(i)}E(W))^2$$

$${}^{(i)}E(W^2) = \int_0^\infty t^2 W(t)\, dt = \frac{2\rho r^2}{(1-r)^3 (\lambda\beta)^2} {}^{(i)}P_1$$

Therefore

$${}^{(i)}\mathrm{Var}(W) = \frac{2\rho r^2\, {}^{(i)}P_1}{(1-r)^3 (\lambda\beta)^2} - \left(\frac{\rho r\, {}^{(i)}P_1}{(1-r)^2 \lambda\beta}\right)^2$$

$$= \frac{\rho r}{(1-r)^2 (\lambda\beta)^2} \left[\frac{2r}{1-r} - \frac{\rho r}{(1-r)^2} {}^{(i)}P_1\right] {}^{(i)}P_1$$

Similarly

$$\mathrm{Var}(W) = \frac{\rho r}{(1 - r)^2 (\lambda \beta)^2} \left[\frac{2 r}{(1 - r)} - \frac{\rho r}{(1 - r)^2} P_1\right] P_1$$

$$\delta\ \mathrm{Var}(W) = \mathrm{Var}(W) - {}^{(i)}\mathrm{Var}(W)$$

$$= \frac{\rho r}{(1 - r)^2 (\lambda \beta)^2} \left[\frac{2 r}{(1 - r)} (P_1 - {}^{(i)}P_1) - \frac{\rho r}{(1 - r)^2} (P_1^2 - {}^{(i)}P_1^2)\right]$$

$$= k^* \left[\frac{2 \rho r^2}{(1 - r) (\lambda \beta)^2} - \frac{\rho^2 r^2}{(1 - r)^2 (\lambda \beta)^2} \times [2 P_1 - k^*(1 - r)^2]\right]$$

APPENDIX

In the APPENDIX we present the table that gives the average number of customers waiting in the queue in both the homogeneous and heterogeneous system. In the first three tables we calculate the average queue length for different ρ such that $\frac{\rho}{s} = \frac{\lambda}{s\mu} < 1$. when s = 1, 2 and 12. The percentage reductions in the mean of the indicated quantities are also given. In Table 4 we give the average queue length for fixed ρ and different s. From the first table it is noted that the expressions coincide with the corresponding expressions of V.P.Singh [14] when s = 1.

TABLE 1

AVERAGE NUMBER OF CUSTOMERS WAITING IN THE QUEUE when s=1

ρ	r	E(Q)	$^{(i)}E(Q)$	100 δ E(Q)/E(Q)
0.10	0.100000	0.002020	0.001635	19.059406
0.15	0.150000	0.006905	0.005938	14.004345
0.20	0.200000	0.016667	0.014911	10.535789
0.25	0.250000	0.033333	0.030650	8.049081
0.30	0.300000	0.059341	0.055649	6.221668
0.35	0.350000	0.097721	0.092985	4.846451
0.40	0.400000	0.152381	0.146600	3.793780
0.45	0.450000	0.228527	0.221722	2.977763
0.50	0.500000	0.333333	0.325542	2.337302
0.55	0.550000	0.477061	0.468332	1.829745
0.60	0.600000	0.675000	0.665386	1.424296
0.65	0.650000	0.951082	0.940640	1.097907
0.70	0.700000	1.345098	1.333885	0.833619
0.75	0.750000	1.928571	1.916644	0.618437
0.80	0.80000	2.844444	2.831858	0.442476
0.85	0.850000	4.426126	4.412934	0.298048
0.90	0.900000	7.673684	7.659937	0.179145

TABLE 2

AVERAGE NUMBER OF CUSTOMERS WAITING IN THE QUEUE when s = 2

ρ	r	E(Q)	$^{(i)}E(Q)$	$100\ \delta\ E(Q)/E(Q)$
0.20	0.170820	0.013282	0.011880	10.555639
0.30	0.2416	0.041151	0.038565	5.947459
0.40	0.306225	0.090034	0.086515	4.070864
0.50	0.366020	0.163278	0.159179	2.510442
0.60	0.421950	0.263877	0.259524	1.649632
0.70	0.474670	0.395170	0.390806	1.104334
0.80	0.524695	0.561655	0.557442	0.750104

TABLE 3

When s = 12

ρ	r	E(Q)	$^{(i)}E(Q)$	$100\ \delta\ E(Q)/E(Q)$
0.6	0.3750	0.2062073	0.2027121	1.6949933
1.2	0.5456	0.7814825	0.7796718	0.2317006
2.4	0.7093	2.128453	2.1281014	0.0165143
3.6	0.7933	3.602369	3.6022692	0.0027815
4.8	0.8477	5.377954	5.3775592	0.0007865
6.0	0.8874	7.724864	7.724343	0.0002679

TABLE 4

ρ	$s = 2$				$s = 4$				$s = 12$			
	r	$E(Q)$	${}^{(i)}E(Q)$	$100\,\delta\, E(Q)/E(Q)$	r	$E(Q)$	${}^{(i)}E(Q)$	$100\,\delta\, E(Q)/E(Q)$	r	$E(Q)$	${}^{(i)}E(Q)$	$100\,\delta\, E(Q)/E(Q)$
0.6	0.421950	0.2638771	0.2595239	1.64971	0.3799489	0.211694	0.2081157	1.69031	0.3750019	0.2062073	0.2027121	1.695
1.2	0.7041594	1.7640822	1.761051	0.171828	0.5737046	0.895245	0.893256	0.22173	0.5456	0.7814825	0.7796718	0.2317

CHAPTER - III

$M/M_{i(1,s_i)}/2 \quad i = 1, 2$ SYSTEM

CHAPTER - III

$M/M_{i(1,s_i)}/2$ $i = 1, 2$ SYSTEM

In this chapter, we consider a bulk service queueing system with Poisson arrivals at a mean rate λ and two service facilities with capacities S_1 and S_2 with $S_2 \leq S_1$. The first facility takes S_1 customers or the whole queue whichever is less and the second takes S_2 or the whole queue whichever is less. The service times for the batches for the two channels are exponentially distributed with mean $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$. The queue discipline namely when both channels are free, the arriving customer joins the first service facility (whose maximum capacity is S_1) with probability π_1 and the other service facility with probability π_2 such that $\pi_1 + \pi_2 = 1$, is followed. When $\pi_1 = 1$, the model under discussion coincides with Arora's [2] model.

The steady-state probabilities and the Laplace transform of the transient state probability distribution of queue length for the above model are derived.

Let

$P_{0\,0}(t)$ = Prob { the system empty at time t }

$P_{1\,0}(t)$ = Prob { the first channel busy, second empty (and no unit in queue) at time t }

$P_{0\,1}(t)$ = Prob { the second channel busy, first empty (and no unit in queue) at time t }

$P_{2\,n}(t)$ = Prob { both channels busy and n($\geq$ 0) units waiting in queue at time t }

$$P_1 = P_{1\,0} + P_{0\,1}$$

The above probabilities satisfy the following difference-differential equations :

$$\frac{d\,P_{0\,0}(t)}{dt} = -\lambda P_{0\,0}(t) + \mu_1 P_{1\,0}(t) + \mu_2 P_{0\,1}(t) \tag{1}$$

$$\frac{d\,P_{1\,0}(t)}{dt} = -(\lambda + \mu_1) P_{1\,0}(t) + \lambda \pi_1 P_{0\,0}(t) + \mu_2 P_{2\,0}(t) \tag{2}$$

$$\frac{d\,P_{0\,1}(t)}{dt} = -(\lambda + \mu_2) P_{0\,1}(t) + \lambda \pi_2 P_{0\,0}(t) + \mu_1 P_{2\,0}(t) \tag{3}$$

$$\frac{d\,P_{2\,0}(t)}{dt} = -(\lambda + \mu_1 + \mu_2) P_{2\,0}(t) + \lambda [P_{1\,0}(t) + P_{0\,1}(t)] + \mu_1 \sum_{m=1}^{S_1} P_{2\,m}(t) + \mu_2 \sum_{k=1}^{S_2} P_{2\,k}(t) \tag{4}$$

$$\frac{d\,P_{2\,n}(t)}{dt} = -(\lambda + \mu_1 + \mu_2) P_{2\,n}(t) + \lambda P_{2\,n-1}(t) + \mu_1 P_{2\,n+s_1}(t) + \mu_2 P_{2\,n+s_2}(t) \qquad (n \geq 1) \tag{5}$$

TRANSIENT SOLUTION

First we discuss the Laplace transform of the transient solution for the above model.

Define $F(x, t) = \sum_{n=0}^{\infty} P_{2\,n}(t)\, x^n$

From equations (5) and (4) we see

6 $$\frac{\partial F(x, t)}{\partial t} = -[\lambda + \mu_1 + \mu_2 - \lambda x - \mu_1 x^{-s_1} - \mu_2 x^{-s_2}]F(x,t)$$
$$+ \mu_1 \sum_{m=0}^{s_1-1} (1 - x^{m-s_1})\, P_{2\,m}(t)$$
$$+ \mu_2 \sum_{k=0}^{s_2-1} (1 - x^{k-s_2})\, P_{2\,k}(t)$$
$$- (\mu_1 + \mu_2)\, P_{2\,0}(t) + \lambda [P_{1\,0}(t) + P_{0\,1}(t)]$$

Let $p_{0\,0}(s)$, $p_{1\,0}(s)$, $p_{0\,1}(s)$, $p_{2\,0}(s)$ and $f(x, s)$ denote the Laplace transform of $P_{0\,0}(t)$, $P_{1\,0}(t)$, $P_{0\,1}(t)$, $P_{2\,0}(t)$ and $F(x, t)$ respectively. Then from (1), (2), (3) and (6),

7 $$-(s + \lambda)\, p_{0\,0}(s) + 1 + \mu_1\, p_{1\,0}(s) + \mu_2\, p_{0\,1}(s) = 0$$

8 $$-(s + \lambda + \mu_1)\, p_{1\,0}(s) + \lambda\, \pi_1\, p_{0\,0}(s) + \mu_2\, p_{2\,0}(s) = 0$$

9 $$-(s + \lambda + \mu_2)\, p_{0\,1}(s) + \lambda\, \pi_2\, p_{0\,0}(s) + \mu_1\, p_{2\,0}(s) = 0$$

$$\tag{10} f(x,\ s) = \frac{x^{s_1}\left[\mu_1 \sum_{m=0}^{s_1-1} (1 - x^{m-s_1})\, p_{2\ m}(s) + \mu_2 \sum_{k=0}^{s_2-1} (1 - x^{k-s_2})\, p_{2\ k}(s) - (\mu_1 + \mu_2)\, p_{2\ 0}(s) + \lambda [p_{1\ 0}(s) + p_{0\ 1}(s)]\right]}{\left[- \lambda x^{s_1+1} + (s + \lambda + \mu_1 + \mu_2)\, x^{s_1} - \mu_2\, x^{s_1-s_2} - \mu_1\right]}$$

We assume that the system starts with empty state, so that $P_{0\ 0}(0) = 1$ and

$$P_{1\ 0}(0) = P_{0\ 1}(0) = P_{2\ n}(0) = 0 \qquad (n \geq 0)$$

The zeros of the denominator of (10) can be obtained from the solution of the equation

$$\lambda\, x^{s_1+1} - (s + \lambda + \mu_1 + \mu_2)\, x^{s_1} + \mu_2\, x^{s_1-s_2} + \mu_1 = 0$$

Applying Rouche's theorem we find that above equation has s_1 roots inside and one root outside $|x| = 1$. Let the roots inside $|x| = 1$ be denoted by $x_i(s)$, $i = 1, 2, \ldots, s_1$ and one outside by $x_0(s)$.

We note that the numerator of equation (10) has s_1 roots. Since $f(x, s)$ exists for $|x| < 1$, the zeros of numerator and denominator must cancel leaving

$$f(x, s) = \frac{A(s)}{x_0(s) - x}$$

$$= \frac{A(s)}{x_0(s)\,[1 - \frac{x}{x_0(s)}]}$$

$$\sum_{n=0}^{\infty} p_{2\;n}(s)\, x^n = \frac{A(s)}{x_0(s)} \sum_{n=0}^{\infty} \left(\frac{x}{x_0(s)}\right)^n$$

$$p_{2\;n}(s) = \frac{A(s)}{x_0(s)^{n+1}} \qquad n = 0, 1, 2, \ldots$$

$$f(1, s) = \frac{A(s)}{x_0(s) - 1}$$

$$\frac{-(\mu_1 + \mu_2)\, p_{2\;0}(s) + \lambda(p_{1\;0}(s) + p_{0\;1}(s))}{s} = \frac{A(s)}{x_0(s) - 1}$$

ie.,

$$A(s) = \frac{(x_0(s) - 1)}{s}\,[-(\mu_1 + \mu_2)\, p_{2\;0}(s) + \lambda(p_{1\;0}(s) + p_{0\;1}(s))]$$

The expressions $p_{1\;0}(s)$, $p_{2\;0}(s)$ and $p_{0\;1}(s)$ can be obtained from (7), (8) and (9).

$$P_{1\ 0}(s) = \frac{[[(s+\lambda)(s+\lambda+\mu_2) - \lambda(\mu_2\pi_2 - \mu_1\pi_1)] P_{0\ 0}(s) - (s+\lambda+\mu_2)]}{\mu_1(2s+2\lambda+\mu_1+\mu_2)}$$

$$P_{0\ 1}(s) = \frac{[[(s+\lambda)(s+\lambda+\mu_1) - \lambda(\mu_1\pi_1 - \mu_2\pi_2)] P_{0\ 0}(s) - (s+\lambda+\mu_1)]}{\mu_2(2s+2\lambda+\mu_1+\mu_2)}$$

$P_{0\ 0}(s)$ can be calculated from the normalising condition.

Next we shall calculate the steady state probabilities for the above model. Assuming the steady state probabilities $P_{i\ n}(t) = \underset{t\to\infty}{\text{lt}} P_{1\ n}(t)$ exist the steady state equations become

12 $$0 = -\lambda P_{0\ 0} + \mu_1 P_{1\ 0} + \mu_2 P_{0\ 1}$$

13 $$0 = -(\lambda+\mu_1) P_{1\ 0} + \lambda\pi_1 P_{0\ 0} + \mu_2 P_{2\ 0}$$

14 $$0 = -(\lambda+\mu_2) P_{0\ 1} + \lambda\pi_2 P_{0\ 0} + \mu_1 P_{2\ 0}$$

15 $$0 = -(\lambda+\mu_1+\mu_2) P_{2\ 0} + \lambda[P_{1\ 0} + P_{0\ 1}] + \mu_1 \sum_{m=1}^{S_1} P_{2\ m} + \mu_2 \sum_{k=1}^{S_2} P_{2\ k}$$

16 $$0 = -(\lambda + \mu_1 + \mu_2) P_{2\,n} + \lambda P_{2\,n-1} + \mu_1 P_{2\,n+s_1} + \mu_2 P_{2\,n+s_2} \qquad (n \geq 1)$$

Let

$$E(P_{2\,n}) = P_{2\,n+1} \text{ be the shifting operator.}$$

Then from (16),

$$[\mu_2 E^{s_2+1} + \mu_1 E^{s_1+1} - (\lambda + \mu_1 + \mu_2) E + \lambda] P_{2\,n} = 0$$

The characteristic equation is,

$$\mu_1 z^{s_1+1} + \mu_2 z^{s_2+1} - (\lambda + \mu_1 + \mu_2) z + \lambda = 0$$

By the usual arguments

17 $$P_{2\,n} = A\, \alpha^n \quad (n \geq 0) \text{ such that } |\alpha| < 1$$

$$\therefore \quad P_{2\,n} = P_{2\,0}\, \alpha^n$$

Solving equations (12), (13) and (14) we have

$$P_{0\,1} = \frac{\lambda(\lambda + \overline{\mu_1 + \mu_2}\, \pi_2)}{\mu_2 (2\lambda + \mu_1 + \mu_2)} P_{0\,0}$$

$$P_{1\,0} = \frac{\lambda(\lambda + \overline{\mu_1 + \mu_2}\, \pi_1)}{\mu_1 (2\lambda + \mu_1 + \mu_2)} P_{0\,0}$$

$$P_{2\,0} = \frac{\lambda^2 (\lambda + \mu_2 \pi_1 + \mu_1 \pi_2)}{\mu_1 \mu_2 (2\lambda + \mu_1 + \mu_2)} P_{0\,0}$$

$P_{0\,0}$ can be derived using the normalising condition that

$$P_{0\,0} + P_{1\,0} + P_{0\,1} + \sum_{n=0}^{\infty} P_{2\,n} = 1$$

ie.,

$$P_{0\,0} = 1 + \frac{\lambda(\lambda + \mu_1 \pi_2 + \mu_2 \pi_1)(1 - \alpha + \rho)}{\mu_1 \mu_2 (1 + 2\rho)(1 - \alpha)}$$

where $\rho = \dfrac{\lambda}{\mu_1 + \mu_2}$

From the relation between steady-state and transient probabilities we have

$$P_{2\,n} = \lim_{t \to \infty} P_{2\,n}(t) = \lim_{s \to 0} s\, f_n(s)$$

Hence we find that the steady-state probabilities obtained from this relation coincide with the previous solution.

$$\lim_{s \to 0} s\, p_{1\,0}(s) = \lim_{s \to 0} \left[\frac{s[(s + \lambda)(s + \lambda + \mu_2) - \lambda(\mu_2 \pi_2 - \mu_1 \pi_1)]\, P_{0\,0}(s) - s(s + \lambda + \mu_2)]}{\mu_1 (2s + 2\lambda + \mu_1 + \mu_2)} \right]$$

$$= \frac{\lambda(\lambda + \mu_2) + \lambda(\mu_1 \pi_1 - \mu_2 \pi_2)\, P_{0\,0}}{\mu_1 (2\lambda + \mu_1 + \mu_2)}$$

$$= \frac{\lambda(\lambda + \overline{\mu_1 + \mu_2}\, \pi_1)}{\mu_1 (2\lambda + \mu_1 + \mu_2)} P_{0\,0} = P_{1\,0}$$

Similarly we can prove

$$\lim_{s \to 0} s\, p_{0\,1}(s) = P_{0\,1}$$

$$\lim_{s \to 0} s\, p_{2\,0}(s) = P_{2\,0}$$

Next we derive the following measures for the above model.

a. The expected queue length E(Q)

$$E(Q) = \sum_{n=1}^{\infty} n\, P_{2\,n} + (P_{1\,0} + P_{0\,1})$$

$$= \sum_{n=1}^{\infty} n\, \alpha^{n}\, P_{2\,0} + (P_{1\,0} + P_{0\,1})$$

$$= P_{2\,0} \frac{\alpha}{(1 - \alpha)^{2}} + P_{1\,0} + P_{0\,1}$$

Substituting and simplifying we get,

$$= \frac{\alpha}{(1 - \alpha)^{2}} \; \frac{\lambda^{2} (\lambda + \mu_2 \pi_1 + \mu_1 \pi_2)}{\mu_1 \mu_2 (2\lambda + \mu_1 + \mu_2)} P_{0\,0}$$

$$+ \frac{\lambda (\lambda + \overline{\mu_1 + \mu_2}\, \pi_1)}{\mu_1 (2\lambda + \mu_1 + \mu_2)} P_{0\,0}$$

$$+ \frac{\lambda (\lambda + \overline{\mu_1 + \mu_2}\, \pi_2)}{\mu_2 (2\lambda + \mu_1 + \mu_2)} P_{0\,0}$$

Simplifying we get,

$$E(Q) = \frac{\lambda(\lambda + \mu_1 \pi_2 + \mu_2 \pi_1)}{\mu_1 \mu_2 (2\lambda + \mu_1 + \mu_2)} \left[\frac{\lambda \alpha}{(1 - \alpha)^2} + \mu_1 + \mu_2\right] P_{0\,0}$$

b. Probability that both the servers are busy $\sum_{n=0}^{\infty} P_{2\,n}$

$$= \sum_{n=0}^{\infty} P_{2\,0}\, \alpha^n$$

$$= \frac{1}{(1 - \alpha)} \; \frac{\lambda^2 (\lambda + \mu_2 \pi_1 + \mu_1 \pi_2)}{\mu_1 \mu_2 (2\lambda + \mu_1 + \mu_2)} P_{0\,0}$$

c. Probability that atleast one server is busy $\sum_{n=0}^{\infty} P_{2\,n} + P_{1\,0} + P_{0\,1}$

$$= \sum_{n=0}^{\infty} P_{2\,0}\, \alpha^n + P_{1\,0} + P_{0\,1}$$

$$= \left[\frac{1}{(1 - \alpha)} \; \frac{\lambda^2 (\lambda + \mu_2 \pi_1 + \mu_1 \pi_2)}{\mu_1 \mu_2 (2\lambda + \mu_1 + \mu_2)}\right.$$

$$\left. + \frac{\lambda(\lambda + \overline{\mu_1 + \mu_2}\, \pi_1)}{\mu_1 (2\lambda + \mu_1 + \mu_2)} + \frac{\lambda(\lambda + \overline{\mu_1 + \mu_2 \pi_2})}{\mu_2 (2\lambda + \mu_1 + \mu_2)}\right] P_{0\,0}$$

$$= \frac{\lambda(\lambda + \mu_1 \pi_2 + \mu_2 \pi_1)}{\mu_1 \mu_2 (2\lambda + \mu_1 + \mu_2)} \left(\frac{\lambda}{1 - \alpha} + \mu_1 + \mu_2\right)$$

CHAPTER - IV

$E_k/M_i/2$ HETEROGENEOUS SERVERS

CHAPTER - IV

$E_k/M_i/2$ HETEROGENEOUS SERVERS

This chapter analyses the model with two heterogeneous servers in which the inter-arrival times have an Erlang distribution with a mean for each phase of $\frac{1}{k\lambda}$ such that the over all mean inter arrival time is $\frac{1}{\lambda}$. The service times for server i are independently distributed random variables with an exponential distribution having mean $\frac{1}{\mu_i}$ $(i = 1, 2)$.

An arriving customer finding both servers free, chooses for his service, the faster one. If he finds both servers busy, then he waits in the queue and is served according to the first in first out rule. This model is denoted by $E_k/M_i/2$.

Steady state queue size probabilities, the mean queue length and the expected waiting time of an arriving customers are derived. Little's formula is also verified.

Let $Q_{rn}(t)$, $n \geq 0$, $1 \leq r \leq k$ denote at time t, the probability that the arriving customer is in the r^{th} phase and there are n customers waiting in the queue and both the servers are busy.

$P_{r\ 1\ 0}(t)$ = Prob. that the faster server is busy and the arriving customer is in the r^{th} phase at time t. $1 \leq r \leq k$

$P_{r\ 0\ 1}(t)$ = Prob that the slower server is busy and the arriving customer is in the r^{th} phase at time t.

$P_{r\ 0\ 0}(t)$ = Prob. that both servers are idle and the arriving customer is at the r^{th} phase at time t.

With this definition the probabilities $Q_{r\ n}(t)$, $P_{r\ 1\ 0}(t)$, $P_{r\ 0\ 1}(t)$ and $P_{r\ 0\ 0}(t)$ satisfy the following difference differential equation.

For $2 \leq r \leq k,\ n \geq 0$

$$Q_{r\ n}(t + \Delta t) = Q_{r\ n}(t)\ [1 - k\lambda\,\Delta t - \overline{\mu_1 + \mu_2}\,\Delta t]$$

$$+ k\lambda\,\Delta t + Q_{r-1\ n}(t) + (\mu_1 + \mu_2)\Delta t\ Q_{r\ n+1}(t)$$

$$\lim_{\Delta t \to 0} \frac{Q_{r\ n}(t + \Delta t) - Q_{r\ n}(t)}{\Delta t} = -(k\lambda + \mu_1 + \mu_2)\ Q_{r\ n}(t) + k\lambda\, Q_{r-1\ n}(t) + (\mu_1 + \mu_2)\ Q_{r\ n+1}(t)$$

$$Q_{r\ n}'(t) = -(k\lambda + \mu_1 + \mu_2)\ Q_{r\ n}(t) + k\lambda\, Q_{r-1}\ n(t) + (\mu_1 + \mu_2)\ Q_{r\ n+1}(t)$$

$2 \leq r \leq k$

$n \geq 0$

By similar arguments

$$Q_{1\,n}'(t) = -(k\lambda + \mu_1 + \mu_2)\,Q_{1\,n}(t) + (\mu_1 + \mu_2)Q_{1\,n+1}(t) + k\lambda\;Q_{k\,n-1}(t) \qquad (n \geq 1)$$

$$Q_{1\,0}'(t) = -(k\lambda + \mu_1 + \mu_2)\,Q_{1\,0}(t) + (\mu_1 + \mu_2)\,Q_{1\,1}(t) + k\lambda\;(P_{k\,1\,0}(t) + P_{k\,0\,1}(t))$$

$$P_{r\,1\,0}'(t) = -(k\lambda + \mu_1)\,P_{r1\,0}(t) + k\lambda\;P_{r-1\,1\,0}(t) + \mu_2\;Q_{r\,0}(t) \qquad 2 \leq r \leq k$$

$$P_{1\,1\,0}'(t) = -(k\lambda + \mu_1)\,P_{1\,1\,0}(t) + \mu_2\;Q_{1\,0}(t) + \lambda\,k\;P_{k\,0\,0}(t)$$

$$P_{r\,0\,1}'(t) = -(k\lambda + \mu_2)\,P_{r\,0\,1}(t) + k\lambda\;P_{r-1\,0\,1}(t) + \mu_1\;P_{r\,0\,0} \qquad 2 \leq r \leq k$$

$$P_{1\,0\,1}'(t) = -(k\lambda + \mu_2)\,P_{1\,0\,1}(t) + \mu_1\;P_{1\,0\,0}(t)$$

$$P_{r\,0\,0}'(t) = -k\lambda\;P_{r\,0\,0}(t) + k\lambda\;P_{r-1\,0\,0}(t) + \mu_2\;P_{r\,0\,1}(t) + \mu_1\;P_{r\,1\,0}(t) \qquad 2 \leq r \leq k$$

$$P_{1\,0\,0}'(t) = -k\lambda\;P_{1\,0\,0}(t) + \mu_2\;P_{1\,0\,1}(t) + \mu_1\;P_{1\,1\,0}(t)$$

Assuming steady-state probabilities exist at t → ∞

1 $$(k\lambda + \mu_1 + \mu_2)\;Q_{r\,n} = k\lambda\;Q_{r-1\,n} + (\mu_1 + \mu_2)\;Q_{r\,n+1} \qquad 2 \leq r \leq k \quad (n \geq 0)$$

2 $$(k\lambda + \mu_1 + \mu_2)\, Q_{1\ n} = k\lambda\, Q_{k\ n-1} + (\mu_1 + \mu_2)\, Q_{1\ n+1} \qquad (n \geq 1)$$

3 $$(k\lambda + \mu_1 + \mu_2)\, Q_{1\ 0} = k\lambda\, (P_{k\ 1\ 0} + P_{k\ 0\ 1}) + (\mu_1 + \mu_2)\, Q_{1\ 1}$$

4 $$(k\lambda + \mu_1)\, P_{r\ 1\ 0} = k\lambda\, P_{r-1\ 1\ 0} + \mu_2\, Q_{r\ 0} \qquad 2 \leq r \leq k$$

5 $$(k\lambda + \mu_1)\, P_{1\ 1\ 0} = k\lambda\, P_{k\ 0\ 0} + \mu_2\, Q_{1\ 0}$$

6 $$(k\lambda + \mu_2)\, P_{r\ 0\ 1} = k\lambda\, P_{r-1\ 0\ 1} + \mu_1\, Q_{r\ 0} \qquad 2 \leq r \leq k$$

7 $$(k\lambda + \mu_2)\, P_{1\ 0\ 1} = \mu_1\, Q_{1\ 0}$$

8 $$k\lambda\, P_{r\ 0\ 0} = k\lambda\, P_{r-1\ 0\ 0} + \mu_1\, P_{r\ 1\ 0} + \mu_2\, P_{r\ 0\ 1} \qquad 2 \leq r \leq k$$

9 $$k\lambda\, P_{1\ 0\ 0} = \mu_1\, P_{1\ 1\ 0} + \mu_2\, P_{1\ 0\ 1}$$

Let $$Q_r(z) = \sum_{n=0}^{\infty} Q_{r\ n}\, z^n$$

$$\theta(x, z) = \sum_{r=1}^{k} Q_r(z)\, x^r$$

From (1) we get

10 $$(k\lambda + \mu)\, Q_r(z) = k\lambda\, Q_{r-1}(z) + \frac{\mu}{z}\,[Q_r(z) - Q_{r\ 0}]$$

Where $\mu = \mu_1 + \mu_2$

From (2) we get

11 $$(k\lambda + \mu)\, Q_r(z) = k\lambda z\, Q_k(z) + \frac{\mu}{z}\,[Q_1(z) - Q_{10}] + k\lambda\,[P_{k\,1\,0} + P_{k\,0\,1}]$$

Multiplying equation (10) by x^r and equation (11) by x and adding we get

$$\begin{aligned}(k\lambda + \mu)\,\theta(x, z) = {} & k\lambda x\,[\theta(x, z) - Q_k(z)\, x^k] + k\lambda\, zx\, Q_k(z) \\ & + \frac{\mu}{z}\,\theta(x, z) - \frac{\mu}{z}\sum_{r=1}^{k} Q_{r\,0}\, x^r \\ & + k\lambda x\,[P_{k\,1\,0} + P_{k\,0\,1}]\end{aligned}$$

$$\begin{aligned}&\left[(k\lambda + \mu) - k\lambda x - \frac{\mu}{z}\right]\theta(x, z) \\ &\quad = Q_k(z)\, k\lambda x\,(z - x^k) - \frac{\mu}{z}\sum_{r=1}^{k} Q_{r\,0}\, x^r \\ &\quad\quad + k\lambda x(P_{k\,1\,0} + P_{k\,0\,1})\end{aligned}$$

Setting $x = 1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z}$ the left hand side becomes zero.

Cancelling x we get,

$$Q_k(z) = \frac{-\dfrac{\mu}{k\lambda z}\displaystyle\sum_{r=1}^{k} Q_{r\,0}\, x^{r-1} + (P_{k\,1\,0} + P_{k\,0\,1})}{z - x^k}$$

$$= \frac{-\frac{\mu}{k\lambda z}\sum_{r=1}^{k} Q_{r\,0}\, x^{r-1} + (1 + \frac{\mu}{k\lambda})\, Q_{1\,0} - \frac{\mu}{k\lambda}\, Q_{1\,1}}{z - x^{k}}$$

$$12 \quad Q_k(z) = \frac{-\frac{\mu}{k\lambda z}\sum_{r=1}^{k} Q_{r\,0}\,(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z})^{r-1} + (1 + \frac{\mu}{k\lambda})\, Q_{1\,0} - \frac{\mu}{k\lambda}\, Q_{1\,1}}{z - (1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z})^{k}}$$

The zeros of the denominator of (12) can be obtained from the solution of equation

$$z - (1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z})^{k} = 0$$

Applying Rouche's theorem we find if $\rho = \frac{\lambda}{\mu} < 1$ the above equation has k-1 roots inside and one on and one root outside $|z| = 1$. Let z_0 be the root which lies outside the circle $|z| = 1$.

In can be written as

$$Q_k(z) = \frac{[-\frac{\mu}{k\lambda}\sum_{r=0}^{k} Q_{r\,0}(Az + B)^{r-1}\, z^{-r} + (1 + \frac{\mu}{k\lambda})\, Q_{1\,0} - \frac{\mu}{k\lambda}\, Q_1]\, z^{k}}{z^{k+1} - (Az + B)^{k}}$$

Where A, B are constants.

The numerator is of degree k and the denominator is of degree k+1. Hence the numerator has k zeros. Since $Q_k(z)$ exists, the zeros of the numerator and denominator must cancel leaving

$$Q_k(z) = \frac{A}{z - z_0}$$

$$= \sum_{n=0}^{\infty} - \frac{A}{z_0^{n+1}} z^n$$

$$\sum_{n=0}^{\infty} Q_{k\ n} z^n = - \frac{A}{z_0} \sum_{n=0}^{\infty} \frac{z^n}{z_0^n}$$

$$\Longrightarrow Q_{k\ n} = \frac{Q_{k\ 0}}{z_0^n} \qquad (n \geq 0)$$

From equation (1)

$$(k\lambda + \mu) Q_{r\ n} = k\ \lambda\ Q_{r-1\ n} + \mu\ Q_{r\ n+1} \qquad n \geq 0,\ 2 \leq n \leq k$$

when $r = k$,

$$Q_{k-1\ n} = (1 + \frac{\mu}{k\ \lambda}) Q_{k\ n} - \frac{\mu}{k\ \lambda} Q_{k\ n+1}$$

$$= (1 + \frac{\mu}{k\ \lambda}) \frac{Q_{k\ 0}}{z_0^n} - \frac{\mu}{k\ \lambda} \frac{Q_{k\ 0}}{z_0^{n+1}}$$

$$= \frac{Q_{k\ 0}}{z_0^n} (g_{z_o} \quad)$$

Where $g_{z_o} = 1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0}$

Similarly,

$$Q_{k-2\;n} = \frac{Q_{k\;0}}{z_0^{\,n}}\, g_{z_0}^{\;2}$$

Proceeding so on we get

$$Q_{r\;n} = \frac{Q_{k\;0}}{z_0^{\,n}}\,(g_{z_0})^{k-r} \qquad n \geq 0 \qquad 1 \leq r \leq k$$

From (7)

$$P_{1\;0\;1} = \frac{\mu_1}{k\lambda + \mu_2}\, Q_{1\;0}$$

Substituting in (6) we get

$$P_{2\;0\;1} = \frac{1}{k\lambda + \mu_2}\,[\theta_2\, Q_{1\;0} + Q_{2\;0}], \qquad \theta_2 = \frac{k\lambda}{k\lambda + \mu_2}$$

$$P_{3\;0\;1} = \theta_2\, P_{2\;0\;1} + \frac{\mu_1}{k\lambda + \mu_2}\, Q_{3\;0}$$

$$= \theta_2\left[\frac{\mu_1}{k\lambda + \mu_2}\,\theta_2\, Q_{1\;0} + \frac{\mu_1}{k\lambda + \mu_2}\, Q_{2\;0}\right] + \frac{\mu_1}{k\lambda + \mu_2}\, Q_{3\;0}$$

$$= \frac{\mu_1}{k\lambda + \mu_2}\, Q_{k\;0}\,\frac{z_0}{g_{z_0}^{\;3}}\,[(\theta_2\, g_{z_0})^2 + \theta_2\, g_{z_0} + 1]$$

13 $$P_{r\,0\,1} = \frac{\mu_1}{k\lambda + \mu_2} Q_{k\,0}\, z_0 \left[\frac{g_{z_0}^{-r} - \theta_2^{\,r}}{1 - \theta_2\, g_{z_0}}\right] \qquad 1 \le r \le k$$

Substituting (5) in (4) when r = 2 we get

$$P_{2\,1\,0} = \theta_1^{\,2} P_{k\,0} + \theta_1 \frac{\mu_2}{k\lambda + \mu_1} Q_{1\,o} + \frac{\mu_2}{k\lambda + \mu_1} Q_{2\,0}$$

$$P_{3\,1\,0} = \theta_1^{\,3} P_{k\,0} + \theta_1^{\,2} \frac{\mu_2}{k\lambda + \mu_1} Q_{1\,0} + \theta_1 \frac{\mu_2}{k\lambda + \mu_1} Q_{2\,0} + \frac{\mu_2}{k\lambda + \mu_1} Q_{3\,0}$$

$$= \theta_1^{\,3} P_{k\,0} + z_0 \frac{\mu_2}{k\lambda + \mu_1} g_{z_0}^{-3} \left[\frac{1 - (\theta_1\, g_{z_0})^3}{1 - \theta_1\, g_{z_0}}\right]$$

14 $$P_{r\,1\,0} = \theta_1^{\,r} P_{k\,0\,0} + \frac{\mu_2}{k\lambda + \mu_1} z_0 \left[\frac{g_{z_0}^{-r} - \theta_1^{\,r}}{1 - \theta_1\, g_{z_0}}\right] Q_{k\,0} \qquad 1 \le r \le k$$

Taking summation over r = 2 to n in (8) we get

$$k\lambda\, P_{n\,0} = k\lambda\, P_{1\,0} + \mu_1 \sum_{s=2}^{n} P_{s\,1\,0} + \mu_2 \sum_{s=2}^{n} P_{s\,0\,1}$$

Substituting (9) in the above equation

$$P_{n\,0\,0} = \frac{\mu_1}{k\lambda} \sum_{s=1}^{n} P_{s\,1\,0} + \frac{\mu_2}{k\lambda} \sum_{s=1}^{n} P_{s\,0\,1}$$

Substituting $P_{s\,1\,0}$, $P_{s\,0\,1}$ from (13) and (14),

$$P_{n\,0\,0} = \frac{\mu_1}{k\lambda}[P_{k\,0\,0}\frac{\theta_1 - \theta_1^{n+1}}{1-\theta_1} + \frac{\mu_2 z_0}{(k\lambda+\mu_1)}\frac{Q_{k\,0}}{(1-\theta_1 g_{z_0})} \times$$

$$[\frac{g_{z_0}^{-1} - g_{z_0}^{-\overline{n+1}}}{1-g_{z_0}^{-1}} - \frac{\theta_1 - \theta_1^{n+1}}{1-\theta_1}]$$

$$+ \frac{\mu_2}{k\lambda}[\frac{\mu_1}{(k\lambda+\mu_2)}\frac{Q_{k\,0}\ z_0}{(1-\theta_2 g_{z_0})}(\frac{g_{z_0}^{-1} - g_{z_0}^{-(n+1)}}{1-g_{z_0}^{-1}}$$

$$- \frac{\theta_2 - \theta_2^{n+1}}{1-\theta_2})]$$

when n = k,

$$P_{k\,0\,0} = \frac{\mu_1}{k\lambda}[P_{k\,0\,0}\frac{\theta_1 - \theta_1^{k+1}}{1-\theta_1} + \frac{\mu_2 z_0}{(k\lambda+\mu_1)}\frac{Q_{k\,0}}{(1-\theta_1 g_{z_0})}$$

$$\times (\frac{g_{z_0}^{-1} - g_{z_0}^{-k+1}}{1-g_{z_0}^{-1}} - \frac{\theta_1 - \theta_1^{k+1}}{1-\theta_1})]$$

$$+ \frac{\mu_2}{k\lambda}[\frac{\mu_1}{k\lambda+\mu_2}\frac{Q_{k\,0}\ z_0}{(1-\theta_2 g_{z_0})}(\frac{g_{z_0}^{-1} - g_{z_0}^{-k+1}}{1-g_{z_0}^{-1}}$$

$$- \frac{\theta_2 - \theta_2^{k+1}}{1-\theta_2})]$$

$$\theta_1^{k} P_{k\,0\,0} = \frac{\mu_1 \mu_2 z_0 Q_{k\,0}}{k\lambda(k\lambda + \mu_1)(1 - \theta_1 g_{z_0})} \left[\frac{g_{z_0}^{-1} - g_{z_0}^{-k+1}}{1 - g_{z_0}^{-1}} - \frac{\theta_1 - \theta_1^{k+1}}{1 - \theta_1}\right] + \frac{\mu_1 \mu_2 z_0 Q_{k\,0}}{k\lambda(k\lambda + \mu_2)(1 - \theta_2 g_{z_0})} \times \left[\frac{g_{z_0}^{-1} - g_{z_0}^{-k+1}}{1 - g_{z_0}^{-1}} - \frac{\theta_2 - \theta_2^{k+1}}{1 - \theta_2}\right]$$

Using the recursion relations one can show that

15 $$\sum_{n=0}^{\infty} Q_{k\,n} + P_{k\,0\,1} + P_{k\,1\,0} + P_{k\,0\,o} = \frac{1}{k} \text{ for } E_k/M/1$$

model, so that

$$\sum_{n=0}^{\infty} \sum_{r=1}^{k} Q_{r\,n} + \sum_{r=1}^{k} (P_{r\,0\,1} + P_{r\,1\,0} + P_{r\,0}) = 1$$

Substituting the corresponding values in equation (15) we can calculate $Q_{k\,0}$

Expected Queue Length and Expected Waiting Time

Next we shall calculate the expected queue length and the expected waiting time of an arriving customer.

Let L_q denote the expected queue length. Then

$$Lq = \sum_{n=0}^{\infty} \sum_{r=1}^{k} n \, Q_{r\,n}$$

$$= \sum_{n=0}^{\infty} \sum_{r=1}^{k} \frac{n \, Q_{k\,0}}{z_0^{\,n}} (g_{z_0})^{k-r}$$

$$= Q_{k\,0} \sum_{r=1}^{k} \left(\frac{1}{g_{z_0}}\right)^{r} \sum_{n=0}^{\infty} \frac{n}{z_0^{\,n-1}}$$

Simplifying

$$Lq = \frac{k\,\lambda}{\mu} Q_{k\,0} \frac{z_0^{\,2}}{(z_0 - 1)^2}$$

Waiting Time

Now we consider the virtual waiting time distribution of an arriving customer.

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It is impossible at this point to consider the relationship of the random-point probabilities to those of the arrival epoch. Defining

Q_n^- = Probability of n in queue at arrival epoch and server busy we can see, using the standard procedure.

$$Q_n^- = k\, Q_{n,\,k} \; ; \quad n \geq 0$$

Similarly if one considers the idle server case

$$P_n^- = k\, P_{n,\,k} \; ; \quad 0 \leq n \leq a-1$$

Using the above relations one may derive the virtual waiting time distribution in the following way :

An arriving unit which must necessarily be in phase 'k' of the arrival channel has to wait only if he finds the system is busy with two servers, and n, ($n \geq 0$) customers are waiting in the queue. In this case he has to wait for $n + 1$ service completions which has probability density function $\mu e^{-\mu t} \frac{(\mu t)^n}{n!}$. Hence if $W_q(t)$ denote the probability density function of the waiting time of the customer in the k^{th} phase, we have

$$W_q(t) = \sum_{n=0}^{\infty} Q_{k\,n}\, \mu\, e^{-\mu t} \frac{(\mu t)^n}{n!}$$

$$= k\, Q_{k\,0}\, \mu\, e^{-\mu t \left(1 - \frac{1}{z_0}\right)}$$

Expected Waiting Time

$$E(W_q) = \int_0^{\infty} t\, W_q(t)\, dt$$

$$= \quad k \frac{Q_{k\,0}}{\mu} \quad \frac{z_0^{\,2}}{(z_0 - 1)^2}$$

ie, $\quad L_q \quad = \quad \lambda\, E(W_q)$

Thus Little's formula is justified.

BIBLIOGRAPHY

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