Chapter II

CHAPTER II

FUZZY PARAMETERIZED FUZZY SOFT SETS

Definition: 2.1

Let U be an initial universe, P(U) be the power set of U, E be the set of all parameters and X be a fuzzy set over E with the membership function $\mu_X : E \to [0, 1]$. Then, **Fuzzy Parameterized Soft Set** (*fps*-set) F_X over U is a set defined by a function f_X representing a mapping

$$f_X : E \to P(U) \text{ such that } f_X(x) = \phi \text{ if } \mu_X(x) = 0$$

Here, f_X is called approximate function of the *fps*-set F_X, and the value $f_X(x)$ is a set called x- element of the *fps*-set for all $x \in E$. Thus, an *fps*-set F_X over U can be represented by the set of ordered pairs

$$F_X = \{ \ (\mu_X(x)/x, \ f_X(x)) : x \in E, \ f_X(x) \in P(U), \ \mu_X(x) \in [0,1] \ \}$$

The set of all *fps*-sets over U will be denoted by FPS(U).

Example: 2.2

Let U = $\{u_1, u_2, u_3, u_4, u_5\}$ be a universal set and E = $\{x_1, x_2, x_3, x_4\}$ be a set of parameters.

If X $= \{0.2/x_2, \ 0.5/x_3, \ 1/x_4\}$, $f_X(x_2) = \{u_2, u_4\}$, $f_X(x_3) = \phi$ and $f_X(x_4) = U$, then the *fps*-set F_X is written by

$$F_X = \{ (\ 0.2/x_2, \ \{u_2, u_4\} \ ), (\ 1/x_4, \ U \ ) \}$$

Notation: 2.3

We use $\Gamma_X, \Gamma_Y, \Gamma_Z$,.....etc. for fuzzy parameterized fuzzy soft sets (*fpfs- sets*) and $\gamma_X, \gamma_Y, \gamma_Z$,.............etc. for their fuzzy approximate functions, respectively.

Definition: 2.4

Let U be an initial universe, E be the set of parameters and X be a fuzzy set over E with the membership function $\mu_X : E \rightarrow [0, 1]$ and $\gamma_X(x)$ be a fuzzy set over U for all $x \in E$. Then, an **Fuzzy Parameterized Fuzzy Soft Set** (*fpfs - set*) Γ_X over U is a set defined by a function $\gamma_X(x)$ representing a mapping

$$\gamma_X : E \rightarrow F(U) \text{ such that } \gamma_X(x) = \phi \text{ if } \mu_X(x) = 0$$

Here, γ_X is called fuzzy approximate function of the *fpfs - set* Γ_X, and the value $\gamma_X(x)$ is a fuzzy set called x - element of the *fpfs - set* for all $x \in E$. Thus, an *fpfs - set* Γ_X over U can be represented by set of ordered pairs

$$\Gamma_X = \{ (\mu_X(x)/x ,\ \gamma_X(x)) : x \in E,\ \gamma_X(x) \in F(U),\ \mu_X(x) \in [0, 1] \}$$

The set of all *fpfs - set*s over U will be denoted by FPFS(U).

Example: 2.5

Assume that U = $\{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and E = $\{x_1, x_2, x_3, x_4\}$ is a set of parameters.

If X = $\{0.2/x_2, 0.5/x_3, 1/x_4\}$ and $\gamma_X(x_2) = \{0.5/u_1, 0.3/u_5\}$, $\gamma_X(x_3) = \phi$ and $\gamma_X(x_4) = U$, then the *fpfs - set* Γ_X is written by

$$\Gamma_X = \{ (0.2/x_2 ,\ \{0.5/u_1, 0.3/u_5\}) , (1/x_4 , U) \}$$

Definition: 2.6

Let $\Gamma_X \in FPFS(U)$. If $\gamma_X(x) = \phi$ for all $x \in E$, then Γ_X is called X **– empty *fpfs* - set**, denoted by Γ_{Φ_X}.

If $X = \phi$, then the X – empty *fpfs* - set (Γ_{Φ_X}) is called **empty *fpfs* - set**, denoted by Γ_Φ.

Definition: 2.7

Let $\Gamma_X \in FPFS(U)$. If $\mu_X(x)=1$ and $\gamma_X(x)=U$ for all $x \in X$, then Γ_X is called X – **universal *fpfs* - set**, denoted by $\Gamma_{\tilde{X}}$.

If $X = E$, then the X – universal *fpfs* -set(Γ_{Φ_X}) is called **universal *fpfs* - set**, denoted by $\Gamma_{\tilde{E}}$.

Example: 2.8

Assume that U = $\{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and E = $\{x_1, x_2, x_3, x_4\}$ is a set of parameters.

If X = $\{0.2/x_2, 0.5/x_3, 1/x_4\}$ and $\gamma_X(x_2)=\{0.5/u_1, 0.3/u_5\}$, $\gamma_X(x_3)=\phi$ and $\gamma_X(x_4)=U$, then the *fpfs - set* Γ_X is written by

$$\Gamma_X = \{\, (0.2/x_2 \,,\, \{0.5/u_1, 0.3/u_5\})\,,\, (1/x_4 \,, U)\,\}$$

If Y = $\{1/x_1, 0.7/x_4\}$ and $\gamma_X(x_1)=\phi, \gamma_X(x_4)=\phi$ then the *fpfs - set* Γ_Y is a Y - empty *fpfs - set, (i.e.)* $\Gamma_Y = \Gamma_{\Phi_Y}$.

If Z = $\{1/x_1, 1/x_2\}$, $\gamma_X(x_1)=U$ and $\gamma_X(x_2)=U$, then the *fpfs - set* Γ_Z is a Z - universal *fpfs - set, (i.e.)*$\Gamma_Z = \Gamma_{\tilde{Z}}$

If $X=\phi$, then the *fpfs - set* Γ_X is an empty *fpfs - set, (i.e.)*$\Gamma_X = \Gamma_\Phi$.

If $X = E$ and $\gamma_X(x_i)=U$, for all $x_i \in E$(i = 1,2,3,4), then the *fpfs - set* Γ_X is a universal *fpfs - set, (i.e.)*$\Gamma_X = \Gamma_{\tilde{E}}$

Definition: 2.9

Let $\Gamma_X, \Gamma_Y \in FPFS(U)$. Then, Γ_X is an ***fpfs* - subset** of Γ_Y, denoted by $\Gamma_X \tilde{\subseteq} \Gamma_Y$, if $\mu_X(x) \le \mu_Y(x)$ and $\gamma_X(x) \subseteq \gamma_Y(x)$ for all $x \in E$.

Theorem: 2.10

Let $\Gamma_X, \Gamma_Y \in FPFS(U)$. Then,

1) $\Gamma_X \tilde{\subseteq} \Gamma_{\tilde{E}}$

2) $\Gamma_{\Phi_X} \tilde{\subseteq} \Gamma_X$

3) $\Gamma_{\Phi} \tilde{\subseteq} \Gamma_X$

4) $\Gamma_X \tilde{\subseteq} \Gamma_X$

5) $\Gamma_X \tilde{\subseteq} \Gamma_Y$ and $\Gamma_Y \tilde{\subseteq} \Gamma_Z \Rightarrow \Gamma_X \tilde{\subseteq} \Gamma_Z$

They can be proved easily by using the fuzzy approximate and membership functions of the *fpfs* - set.

Definition: 2.11

Let $\Gamma_X, \Gamma_Y \in FPFS(U)$. Then, Γ_X and Γ_Y are ***fpfs*-equal**, written as $\Gamma_X = \Gamma_Y$, if and only if $\mu_X(x) = \mu_Y(x)$ and $\gamma_X(x) = \gamma_Y(x)$ for all $x \in E$.

Proposition: 2.12

Let $\Gamma_X, \Gamma_Y, \Gamma_Z \in FPFS(U)$. Then,

1) $(\Gamma_X = \Gamma_Y$ and $\Gamma_Y = \Gamma_Z) \Leftrightarrow \Gamma_X = \Gamma_Z$

2) $(\Gamma_X \tilde{\subseteq} \Gamma_Y$ and $\Gamma_Y \tilde{\subseteq} \Gamma_X) \Leftrightarrow \Gamma_X = \Gamma_Y$

The proofs are trivial.

Definition: 2.13

Let $\Gamma_X \in FPFS(U)$. Then, the **Complement** of Γ_X, denoted by $\Gamma_X^{\tilde{C}}$, is defined by $\mu_{X^C}(x) = 1 - \mu_X(x)$ and $\gamma_{X^{\tilde{C}}} = \gamma_X^C(x)$ for all $x \in E$, where $\gamma_X^C(x)$ is the complement of the set $\gamma_X(x)$, that is, $\gamma_X^C(x) = U / \gamma_X(x)$ for every $x \in E$.

Theorem: 2.14

Let $\Gamma_X \in FPFS(U)$. Then,

1) $(\Gamma_X^{\tilde{C}})^{\tilde{C}} = \Gamma_X$

2) $\Gamma_\Phi^{\tilde{C}} = \Gamma_{\tilde{E}}$

By using the fuzzy approximate and membership functions of the *fpfs* - sets, the proofs can be straight forward.

Definition: 2.15

Let $\Gamma_X, \Gamma_Y \in FPFS(U)$. Then, **Union** of Γ_X and Γ_Y, denoted by $\Gamma_X \tilde{\cup} \Gamma_Y$, is defined by $\mu_{X\tilde{\cup}Y}(x) = \max\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{X\tilde{\cup}Y}(x) = \gamma_X(x) \cup \gamma_Y(x)$ for all $x \in E$

Theorem: 2.16

Let $\Gamma_X, \Gamma_Y, \Gamma_Z \in FPFS(U)$. Then,

1) $\Gamma_X \tilde{\cup} \Gamma_X = \Gamma_X$

2) $\Gamma_{\Phi_X} \tilde{\cup} \Gamma_X = \Gamma_X$

3) $\Gamma_X \tilde{\cup} \Gamma_\Phi = \Gamma_X$

4) $\Gamma_X \tilde{\cup} \Gamma_{\tilde{E}} = \Gamma_{\tilde{E}}$

5) $\Gamma_X \tilde{\cup} \Gamma_Y = \Gamma_Y \tilde{\cup} \Gamma_X$

6) $(\Gamma_X \tilde{\cup} \Gamma_Y) \tilde{\cup} \Gamma_Z = \Gamma_X \tilde{\cup} (\Gamma_Y \tilde{\cup} \Gamma_Z)$

The proofs can be easily obtained from Definition 2.15

Definition: 2.17

Let $\Gamma_X, \Gamma_Y \in FPFS(U)$. Then, **Intersection** of Γ_X and Γ_Y, denoted by $\Gamma_X \tilde{\cap} \Gamma_Y$, is defined by $\mu_{X\tilde{\cap}Y}(x) = \min\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{X\tilde{\cap}Y}(x) = \gamma_X(x) \cap \gamma_Y(x)$ for all $x \in E$.

Theorem: 2.18

Let $\Gamma_X, \Gamma_Y, \Gamma_Z \in FPFS(U)$. Then,

1) $\Gamma_X \tilde{\cap} \Gamma_X = \Gamma_X$
2) $\Gamma_{\Phi_X} \tilde{\cap} \Gamma_X = \Gamma_X$
3) $\Gamma_X \tilde{\cap} \Gamma_\Phi = \Gamma_\Phi$
4) $\Gamma_X \tilde{\cap} \Gamma_{\tilde{E}} = \Gamma_X$
5) $\Gamma_X \tilde{\cap} \Gamma_Y = \Gamma_Y \tilde{\cap} \Gamma_X$
6) $(\Gamma_X \tilde{\cap} \Gamma_Y) \tilde{\cap} \Gamma_Z = \Gamma_X \tilde{\cap} (\Gamma_Y \tilde{\cap} \Gamma_Z)$

The proofs can be easily obtained from Definition 2.17

Remark: 2.19

Let $\Gamma_X \in FPFS(U)$. If $\Gamma_X \neq \Gamma_{\tilde{E}}$ or $\Gamma_X \neq \Gamma_\Phi$, then $\Gamma_X \tilde{\cup} \Gamma_X^{\tilde{c}} \neq \Gamma_{\tilde{E}}$ and $\Gamma_X \tilde{\cap} \Gamma_X^{\tilde{c}} \neq \Gamma_\Phi$.

Theorem: 2.20

Let $\Gamma_X, \Gamma_Y \in FPFS(U)$. Then De-Morgan's laws are valid

1) $(\Gamma_X \tilde{\cup} \Gamma_Y)^{\tilde{c}} = \Gamma_X^{\tilde{c}} \tilde{\cap} \Gamma_Y^{\tilde{c}}$
2) $(\Gamma_X \tilde{\cap} \Gamma_Y)^{\tilde{c}} = \Gamma_X^{\tilde{c}} \tilde{\cup} \Gamma_Y^{\tilde{c}}$

Proof: For all $x \in E$,

1) $\mu_{(X \tilde{\cup} Y)^c}(x) = 1 - \mu_{X \tilde{\cup} Y}(x)$

$$= 1 - \max\{\mu_X(x), \mu_Y(x)\}$$

$$= \min\{1 - \mu_X(x), 1 - \mu_Y(x)\}$$

$$= \min\{\mu_{X^c}(x), \mu_{Y^c}(x)\}$$

$$= \mu_{X^c \tilde{\cap} Y^c}(x)$$

and $\gamma_{(X\cup Y)^C}(x) = \gamma^C_{X\cap Y}(x)$

$$= (\gamma_X(x) \cup \gamma_Y(x))^C$$

$$= (\gamma_X(x))^C \cap (\gamma_Y(x))^C$$

$$= \gamma^C_X(x) \cap \gamma^C_Y(x)$$

$$= \gamma_{X^{\tilde{c}}}(x) \cap \gamma_{Y^{\tilde{c}}}(x)$$

$$= \gamma_{X^{\tilde{c}} \tilde{\cap} Y^{\tilde{c}}}(x)$$

Likewise, the proof of (2) can be made similarly.

Theorem: 2.21

Let $\Gamma_X, \Gamma_Y, \Gamma_Z \in FPFS(U)$. Then,

1) $\Gamma_X \tilde{\cup} (\Gamma_Y \tilde{\cap} \Gamma_Z) = (\Gamma_X \tilde{\cup} \Gamma_Y) \tilde{\cap} (\Gamma_X \tilde{\cup} \Gamma_Z)$
2) $\Gamma_X \tilde{\cap} (\Gamma_Y \tilde{\cup} \Gamma_Z) = (\Gamma_X \tilde{\cap} \Gamma_Y) \tilde{\cup} (\Gamma_X \tilde{\cap} \Gamma_Z)$

Proof: For all $x \in E$,

(i) $\mu_{X \tilde{\cup} (Y \tilde{\cap} Z)}(x)$

$$= \max\{\mu_X(x),\ \mu_{Y\tilde{\cap} Z}(x)\}$$

$$= \max\{\ \mu_X(x),\ \min\{\mu_Y(x), \mu_Z(x)\}\ \}$$

$$= \min\{\ \max\{\mu_X(x), \mu_Y(x)\}\ ,\ \max\{\mu_X(x), \mu_Z(x)\}\ \}$$

$$= \min\ \{\ \mu_{X\tilde{\cup} Y}(x), \mu_{X\tilde{\cup} Z}(x)\ \}$$

$$= \mu_{(X\tilde{\cup} Y)\tilde{\cap}(X\tilde{\cup} Z)}(x)$$

and $\gamma_{X \tilde{\cup} (Y \tilde{\cap} Z)}(x)$

$$= \gamma_X(x) \cup \gamma_{Y\tilde{\cap} Z}(x)$$

$$= \gamma_X(x) \cup (\gamma_Y(x) \cap \gamma_Z(x))$$

$$= (\gamma_X(x) \cup \gamma_Y(x)) \cap (\gamma_X(x) \cup \gamma_Z(x))$$

$= \gamma_{X \tilde{\cup} Y}(x) \cap \gamma_{X \tilde{\cup} Z}(x)$

$= \gamma_{(X \tilde{\cup} Y) \tilde{\cap} (X \tilde{\cup} Z)}(x)$

Likewise, the proof of (2) can be made similarly.

Definition: 2.22

Let $\Gamma_X \in FPFS(U)$. Then ***fpfs*-aggregation operator**, denoted by $FPFS_{agg}$, is defined by

$$FPFS_{agg} : F(E) \times FPFS(U) \rightarrow F(U),$$

$$FPFS_{agg}(X, \Gamma_X) = \Gamma_X^*$$

where $\Gamma_X^* = \{\mu_{\Gamma_X^*}(u) / u \ : \ u \in U\}$ which is a fuzzy set over U. The value Γ_X^* is called aggregate fuzzy set of the Γ_X. Here, the membership degree $\mu_{\Gamma_X^*}(u)$ of u is defined as follows

$$\mu_{\Gamma_X^*}(u) = \frac{1}{|E|} \sum_{x \in E} \mu_X(x) \, \mu_{\gamma_X(x)}(u)$$

where $|E|$ is the cardinality of E.