

Chapter 6

Passivity Analysis of Markovian Jumping Neural Networks of Neutral Type with Time Delays in the Leakage Term and Mode-Dependent Delays

6.1 Introduction

During the past decades, various kinds of recurrent neural networks have been proposed including bidirectional associative memory neural networks, cellular neural networks, Cohen - Grossberg neural networks and Hopfield neural networks, etc. In the past decades, considerable attention has been devoted to the study of neural networks because they can be used to solve certain problems related to signal processing, static image treatment, image processing, pattern recognition, optimization and associative memory design. In practice, the applications of neural networks heavily depend on their dynamic behaviors. Hence, it is important to analyze the dynamic behaviors. Time delay is often encountered in real systems, such as networked control systems, communication systems, chemical processes, etc. Without exception, time delay should be taken into account in the electronic implementations of artificial neural networks, see Faydasicok and Arik (2012) and Wu and Zeng (2012). It is well known that time-delay is usually a cause of instability and oscillations of recurrent neural networks. Therefore, the problem of stability of recurrent neural networks with time-delay is of importance in both theory and practical applications. Since the Lyapunov functional approach can present simple and delay-independent results for the considered time-delay systems, while the Lyapunov Krasovskii functional (LKF) has been widely utilized because its analytical procedure can fully

depends on the information of delays. With the help of the linear matrix inequality (LMI) approach, a number of research works has been devoted to analysis and synthesis of neural networks with various type of delays, such as stability analysis, passivity analysis, and state estimation; and significant progress has been made in the literatures of Balasubramaniam et al.(2011), Huang et al.(2010) and Zhang et al. (2010 b) and references therein.

In many engineering problems, the theory of dissipative systems which postulates the energy dissipated inside a dynamic system is less than the energy supplied from external source often links the stability problems. Passivity is part of a broader and a general theory of dissipativeness. The main idea of passivity theory is that the passive properties of a system can keep the system internally stable. The passivity theory intimately related to the circuit analysis is a useful and significant tool to analyze the stability of nonlinear systems, signal processing, chaos control, and so on. So it has been widely employed in various fields as in Ahn (2012),Ma et al. (2011) and Yu and Li (2007). In Haddad et al.(2005), the authors proposed passivity-based neural adaptive output feedback control framework for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems. In the last decade, great attention has been paid to the passivity analysis of RNNs with time delays either in delay-independent as in Mahmoud (2011) or delay-dependent Mala and Sudamani Ramaswamy(2013), Wu et al. (2012 b), Xu et al. (2009) and Zhang et al. (2010 c) and references therein.

On the other hand, the past few decades have witnessed a significant progress on Markov jump systems due to the fact that as a special class of hybrid systems, Markov jump systems have great ability to model the dynamic systems whose structure is subject to random abrupt variation mainly due to, for example, changing in subsystem interconnections, random component failures or repairs, and sudden environmental changes. The applications of the Markovian jump systems can be found in economic systems, modeling production system, network control systems, manufacturing systems, communication systems and so on. In recent years, there have been lots of research results on the stability analysis for neural networks with Markovian jump parameters, see for references Balasubramaniam et al.(2009), Ma et al. (2011), Tian et al. (2012) and Zhu and Cao (2010 a, 2010 b). In Huang et al.(2012), the authors studied global exponential estimates of delayed stochastic neural networks with Markovian switching by constructing stochastic Lyapunov functional with as many as possible of the positive definite matrices are dependent on the system mode and a triple-integral term. In Zhu et al.(2013), the authors

discussed adaptive synchronization for stochastic neural networks of neutral-type with mixed time-delays. Additionally, the problem of state estimation of recurrent neural networks with Markovian jumping parameters and mixed delays based on mode-dependent approach was investigated in Huang et al. (2013).

It is quite common in engineering systems that the time-delays occur not only in the system states (or outputs) but also in the derivatives of system states Niu et al. (2004). Examples of such kind of neutral delay systems include chemical reactors, transmission lines, partial element equivalent circuits in VLSI systems, and Lotka Volterra systems, see Chen et al. (2006). However, some of the researchers discussed about the robust passive filtering for neutral-type neural networks with time-varying discrete and unbounded distributed delays, see Lin et al. (2013). In Rakkiyappan and Balasubramaniam (2008), the authors investigated the global asymptotic stability of neutral-type neural networks with unbounded distributed delays by utilizing the Lyapunov-Krasovskii functional and the linear matrix inequality approach. Park et al. (2008) studied the global asymptotic stability of delayed neural networks of neutral type. In Mahmoud and Ismail (2010), the authors studied the on robust exponential stability criteria for neutral type delayed neural networks. The stability analysis of neutral-type impulsive neural networks with mixed time-varying delays and Markovian jump has been studied by Zhang et al. (2010).

Motivated by the above discussion, the main purpose of this chapter is to study the global asymptotic stability of Markovian jump neural networks of neutral type with time varying delays. By construction of a new Lyapunov-Krasovskii functional involving mode-dependent Lyapunov matrices, some sufficient conditions are derived in terms of LMIs. Finally, numerical examples and its simulations are given to demonstrate the usefulness and effectiveness of the presented results.

Notations: Throughout this chapter, \mathbb{R}^n and $\mathbb{R}^{n \times n}$ denote the n -dimensional Euclidean space and the set of all $n \times n$ real matrices respectively. I denotes the identity matrix with compatible dimensions. $diag(\dots)$ denotes a block diagonal matrix. The superscript T denotes the transposition and the notation $X \geq Y$ (similarly, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (similarly, positive definite). Let $(\Omega, \mathfrak{F}, \mathcal{P})$ be a complete probability space with a natural filtration $\{\mathfrak{F}_t\}_{t \geq 0}$ and $\mathbf{E}[\cdot]$ stand for the correspondent expectation operator with respect to the given probability measure \mathcal{P} . Also, let $d > 0$ and $\mathcal{C}([-d, 0]; \mathbb{R}^n)$ denote the family of continuously differentiable function ϕ from $[-d, 0]$ to \mathbb{R}^n with the uniform norm $\|\phi\| = \max\{\max_{-\tau \leq \theta \leq 0} |\phi(\theta)|, \max_{-d \leq \theta \leq 0} |\phi'(\theta)|\}$. Denote by $\mathcal{C}_{\mathfrak{F}_0}^2([-d, 0]; \mathbb{R}^n)$ the family of bounded \mathfrak{F}_0 -measurable, $\mathcal{C}([-d, 0]; \mathbb{R}^n)$ -valued stochas-

tic variables $\xi = \{\xi(\theta) : -d \leq \theta \leq 0\}$ such that $\int_{-d}^0 \mathbf{E}|\xi(\theta)|^2 ds < \infty$. The notation $*$ always denotes the symmetric block in one symmetric matrix.

6.2 Problem Description and Preliminaries

Let $\{r(t), t \geq 0\}$ is a right-continuous Markov chain on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ taking values in a finite state space $\mathcal{S} = \{1, 2, \dots, N\}$ with generator $\Gamma = (\pi_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta t) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$, $\pi_{ij} \geq 0$ is the transition rate from i to j , if $i \neq j$ while $\pi_{ii} = -\sum_{j=1, j \neq i}^N \pi_{ij}$.

Let us consider the following Mode-dependent Markov jump neutral type neural networks with mixed time-delays:

$$\left\{ \begin{array}{l} \dot{x}(t) = -A(r(t))x(t - \sigma(r(t))) + B(r(t))f(x(t)) + C(r(t))f(x(t - \tau(t, r(t)))) \\ \quad + D(r(t))\dot{x}(t - h(t, r(t))) + E(r(t)) \int_{t-d(t, r(t))}^t f(x(s))ds + u(t) \\ y(t) = f(x(t)) \end{array} \right. \quad (6.2.1)$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector associated with the n neurons. The diagonal matrix $A(r(t)) = \text{diag}(A_1(r(t)), A_2(r(t)), \dots, A_n(r(t)))$ has positive entries $A_i(r(t)) > 0$ ($i = 1, 2, \dots, n$). $B(r(t))$, $C(r(t))$, $D(r(t))$, $E(r(t))$ are known constant matrices of appropriate dimensions.

$$f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \dots, f_n(x_n(t))]^T$$

is the neuron activation function. $u(t)$ denotes a constant input. $\tau(t, r(t))$, $h(t, r(t))$, $d(t, r(t))$ are mode dependent discrete, neutral and distributed delays, respectively

and $\sigma(r(t))$ mode dependent leakage delay.

Throughout this chapter, the following assumption is necessary.

Assumption 1. For any $j = 1, 2, \dots, n$ $f_j(0) = 0$ and there exist constants \widehat{l}_i^- and \widehat{l}_i^+ such that

$$\widehat{l}_j^- \leq \frac{f_j(\gamma_1) - f_j(\gamma_2)}{\gamma_1 - \gamma_2} \leq \widehat{l}_j^+, \quad (6.2.2)$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$, and $\gamma_1 \neq \gamma_2$.

For the sake of convenience, denote $A(r(t) = i) = A_i, B(r(t) = i) = B_i, C(r(t) = i) = C_i, D(r(t) = i) = D_i, E(r(t) = i) = E_i$, respectively.

The system (6.2.1) can be rewritten as

$$\left\{ \begin{array}{l} \dot{x}(t) = -A_i x(t - \sigma_i) + B_i f(x(t)) + C_i f(x(t - \tau_i(t))) + D_i \dot{x}(t - h_i(t)) \\ \quad + E_i \int_{t-d_i(t)}^t f(x(s)) ds + u(t) \\ y(t) = f(x(t)) \end{array} \right. \quad (6.2.3)$$

and the parameters associated with time delays are assumed to satisfy followings:

$$0 \leq \tau_i(t) \leq \tau_i, \dot{\tau}_i(t) \leq \tau_{\mu_i}, 0 \leq h_i(t) \leq h_i, \dot{h}_i(t) \leq h_{\mu_i}, 0 \leq d_i(t) \leq d_i, \dot{d}_i(t) \leq d_{\mu_i}, \sigma_i > 0 \quad (6.2.4)$$

where $\tau_i, h_i, d_i, \tau_{\mu_i}, h_{\mu_i}$ and d_{μ_i} are some real constants and $\tau = \max_{i \in S} \{\tau_i\}$, $h = \max_{i \in S} \{h_i\}$, $d = \max_{i \in S} \{d_i\}$, $\sigma = \max_{i \in S} \{\sigma_i\}$.

Throughout this chapter, we need the following Lemmas and definition.

Lemma 6.2.1. *Gu et al. (2003) For any matrix $M \geq 0$, any scalars a and b with $a \leq b$, and a vector function $x(t) : [a, b] \rightarrow \mathcal{R}^n$ such that the integrals concerned are well-defined, then the following inequality holds:*

$$(b - a) \left[\int_a^b x(s)^T M x(s) ds \right] \geq \left[\int_a^b x(s) ds \right]^T M \left[\int_a^b x(s) ds \right].$$

Lemma 6.2.2. *For any real vectors $x, y \in \mathcal{R}^n$ and positive definite matrix $\mathcal{M} = \mathcal{M}^T$ it follows that:*

$$\pm 2x^T y \leq x^T \mathcal{M} x + y^T \mathcal{M}^{-1} y.$$

Lemma 6.2.3. (Schur complement) *Li and Huang (2009)*; Given constant matrix Ω_1, Ω_2 and Ω_3 with appropriate dimensions, where $\Omega_1^T = \Omega_1$ and $\Omega_2^T = \Omega_2 > 0$, then

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0.$$

Definition 6.2.1. *Fu et al. (2008)*; The system (6.2.3) is said to be passive, if there exists a scalar $\nu \geq 0$ such that for all $t_p \geq 0$ and for all the solutions of (6.2.1), the following inequality

$$2 \int_0^{t_p} E\{y(s)^T u(s)\} ds \geq -\gamma \int_0^{t_p} E\{u(s)^T u(s)\} ds \quad (6.2.5)$$

holds under zero initial conditions.

Now, we establish the following passivity condition for the system (6.2.3).

6.3 Main Results

For presentation convenience, in the following, we denote

$$\widehat{L}_1 = \text{diag}\{\widehat{l}_1^- \widehat{l}_1^+, \widehat{l}_2^- \widehat{l}_2^+, \dots, \widehat{l}_m^- \widehat{l}_m^+\}, \quad \widehat{L}_2 = \text{diag}\left\{\frac{\widehat{l}_1^- + \widehat{l}_1^+}{2}, \frac{\widehat{l}_2^- + \widehat{l}_2^+}{2}, \dots, \frac{\widehat{l}_m^- + \widehat{l}_m^+}{2}\right\}.$$

Theorem 6.3.1. *For given scalars $\tau_i > 0, h_i > 0, d_i > 0, \tau_{\mu_i} > 0, h_{\mu_i} > 0, d_{\mu_i} > 0$ and $\sigma_i > 0$, the system (6.2.3) is passive if there exist symmetric positive definite*

$$\text{matrices } P_i > 0, Q_i = \begin{bmatrix} Q_{1i} & Q_{2i} \\ Q_{2i}^T & Q_{3i} \end{bmatrix} > 0, W_i > 0, R_i > 0, S_i > 0, V_i > 0, U_i > 0,$$

$$X_i > 0, Y_i > 0, T_i > 0, Z_i > 0, L_i > 0, K_i > 0, Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} > 0, W > 0,$$

$R > 0, S > 0, V > 0, U > 0, X > 0, Y > 0, T > 0, Z > 0, L > 0, M > 0$ and the diagonal matrices $H_{1i} > 0, H_{2i} > 0, H_{3i} > 0, H_{4i} > 0, H_{5i} > 0, H_{6i} > 0$ and any matrices N_i, J_1, J_2 with appropriate dimensions such that the following LMIs are satisfied for $i = 1, \dots, N$:

$$\begin{bmatrix} X_i & N_i \\ * & X_i \end{bmatrix} \geq 0, \quad (6.3.1)$$

$$\sum_{j=1}^N \pi_{ij} V_j - V \leq 0, \quad (6.3.2)$$

$$\sum_{j=1, j \neq i}^N \pi_{ij} \mathcal{G}_j - \mathcal{G} \leq 0, \quad (6.3.3)$$

$$\Phi = \begin{bmatrix} \Omega & \Gamma^T \\ * & -\frac{1}{\pi_{ij}} K_j \end{bmatrix} < 0, \quad (6.3.4)$$

where \mathcal{G}_j in (6.3.3) respectively represents $Q_i, W_i, R_i, S_i, U_i, X_i, Y_i, T_i, Z_i, L_i$ and correspondingly \mathcal{G} represents $Q, W, R, S, U, X, Y, T, Z, L$ (e.g., when \mathcal{G}_j is Q_j ,

\mathcal{G} is Q) and

$$\begin{aligned}
\Omega &= (\vartheta_{i,j})_{15 \times 15}, \\
\vartheta_{1,1} &= -P_i A_i - A_i^T P_i + \pi_{ii} P_i + \sum_{j \neq i} \pi_{ij} P_j + \sum_{j \neq i} \pi_{ij} K_j + Q_{1i} + \tau Q_1 + W_i + \tau W + R_i + \sigma R \\
&\quad + \tau_i U_i + \frac{\tau^2}{2} U - \frac{1}{\tau_i} X_i - \frac{1}{h_i} Y_i - 2T_i - 2Z_i - 2L_i - \widehat{L}_1 H_{1i} - \widehat{L}_1 H_{3i} - \widehat{L}_1 H_{4i} \\
&\quad - \widehat{L}_1 H_{5i} - \widehat{L}_1 H_{6i} - 2\sigma^2 M, \quad \vartheta_{1,2} = -\frac{1}{\tau_i} N_i^T + \frac{1}{\tau_i} X_i, \quad \vartheta_{1,3} = \frac{1}{\tau_i} N_i^T + \widehat{L}_2 H_{3i}, \\
\vartheta_{1,4} &= \frac{2}{\tau_i} T_i, \quad \vartheta_{1,5} = P_i B_i + Q_{2i} + Q_2 + \widehat{L}_2 H_{1i} + J_1 B_i, \quad \vartheta_{1,6} = P_i C_i + J_1 C_i, \\
\vartheta_{1,7} &= -J_1 A_i - A_i^T J_1^T + \widehat{L}_2 H_{5i}, \quad \vartheta_{1,8} = A_i^T P_i A_i - \pi_{ii} P_i A_i - \frac{2}{\sigma_i} L_i, \quad \vartheta_{1,9} = 2\sigma M, \\
\vartheta_{1,10} &= P_i, \quad \vartheta_{1,11} = P_i D_i + J_1 D_i, \quad \vartheta_{1,12} = \frac{1}{h_i} Y_i + \widehat{L}_2 H_{4i}, \quad \vartheta_{1,13} = \frac{2}{h_i} Z_i, \\
\vartheta_{1,14} &= J_1 E_i, \quad \vartheta_{1,15} = -J_1 + \widehat{L}_2 H_{6i}, \quad \vartheta_{2,2} = -(1 - \tau_{\mu_i}) Q_{1i} - \widehat{L}_1 H_{2i} - \frac{2}{\tau_i} X_i + \frac{1}{\tau_i} N_i^T, \\
\vartheta_{2,3} &= \frac{1}{\tau_i} X_i - \frac{1}{\tau_i} N_i^T, \quad \vartheta_{2,6} = -(1 - \tau_{\mu_i}) Q_{1i} + \widehat{L}_2 H_{2i}, \quad \vartheta_{3,3} = -W_i - \frac{1}{\tau_i} X_i - H_{3i}, \\
\vartheta_{4,4} &= -\frac{1}{\tau_i} U_i - \frac{2}{\tau_i^2} T_i, \quad \vartheta_{5,5} = Q_{3i} + Q_3 + d_i V_i + \frac{d^2}{2} V - H_{1i}, \quad \vartheta_{5,8} = -B_i^T P_i A_i, \\
\vartheta_{5,15} &= B_i^T J_2^T, \quad \vartheta_{6,6} = -(1 - \tau_{\mu_i}) Q_{3i} - H_{2i}, \quad \vartheta_{6,8} = -C_i^T P_i A_i, \quad \vartheta_{6,15} = C_i^T J_2^T, \\
\vartheta_{7,7} &= -R_i - H_{5i}, \quad \vartheta_{7,15} = -A_i^T J_2^T, \quad \vartheta_{8,8} = \pi_{ii} A_i^T P_i A_i - \frac{2}{\sigma_i^2} L_i, \quad \vartheta_{8,10} = -A_i^T P_i, \\
\vartheta_{8,11} &= -A_i^T P_i D_i, \quad \vartheta_{8,14} = -A_i^T P_i E_i, \quad \vartheta_{9,9} = \sum_{j \neq i} \pi_{ij} A_j^T P_j A_j - 2M, \quad \vartheta_{10,10} = -\gamma I, \\
\vartheta_{10,15} &= J_2^T, \quad \vartheta_{11,11} = -S_i(1 - h_{\mu_i}), \quad \vartheta_{11,15} = D_i^T J_2^T, \quad \vartheta_{12,12} = -\frac{1}{h_i} Y_i - H_{4i}, \\
\vartheta_{13,13} &= -\frac{2}{h_i^2} Z_i, \quad \vartheta_{14,14} = -\frac{(1 - d_{\mu_i})}{d_i} V_i, \quad \vartheta_{14,15} = E_i^T J_2^T, \quad \vartheta_{15,15} = S_i + hS + \tau_i X_i + \frac{\tau^2}{2} X \\
&\quad + h_i Y_i + \frac{h^2}{2} Y + \frac{\tau_i^2}{2} T_i + \frac{\tau^3}{6} T + \frac{h_i^2}{2} Z_i + \frac{h^3}{6} Z + \frac{\sigma_i^2}{2} L_i + \frac{\sigma^3}{6} L + \frac{\sigma^4}{2} Z - (J_2 + J_2^T) - H_{6i}, \\
\Gamma^T &= [\underbrace{0 \dots 0}_{8 \text{ elements}} \quad \sum_{j \neq i} \pi_{ij} (A_j^T P_j)^T \quad \underbrace{0 \dots 0}_{15 \text{ elements}}]^T
\end{aligned}$$

and the remaining coefficients are all zero.

Proof. Consider the following Lyapunov-Krasovskii functional candidate:

$$V(x_t, i, t) = \sum_{\kappa=1}^{13} V_{\kappa}(x_t, i, t), \tag{6.3.5}$$

where

$$\begin{aligned}
V_1(x_t, i, t) &= \left[x(t) - A_i \int_{t-\sigma_i}^t x(s) ds \right]^T P_i \left[x(t) - A_i \int_{t-\sigma_i}^t x(s) ds \right], \\
V_2(x_t, i, t) &= \int_{t-\tau_i}^t \zeta^T(s) Q_i \zeta(s) ds + \int_{-\tau}^0 \int_{t+\theta}^t \zeta^T(s) Q \zeta(s) ds d\theta, \\
V_3(x_t, i, t) &= \int_{t-\tau_i}^t x^T(s) W_i x(s) ds + \int_{-\tau}^0 \int_{t+\theta}^t x^T(s) W x(s) ds d\theta, \\
V_4(x_t, i, t) &= \int_{t-\sigma_i}^t x^T(s) R_i x(s) ds + \int_{-\sigma}^0 \int_{t+\theta}^t x^T(s) R x(s) ds d\theta, \\
V_5(x_t, i, t) &= \int_{t-h_i}^t x^T(s) S_i x(s) ds + \int_{-h}^0 \int_{t+\theta}^t x^T(s) S x(s) ds d\theta, \\
V_6(x_t, i, t) &= \int_{-d_i(t)}^0 \int_{t+\theta}^t f^T(x(s)) V_i f(x(s)) ds d\theta + \int_{-d}^0 \int_{\theta}^0 \int_{t+\beta}^t f^T(x(s)) V f(x(s)) ds d\beta d\theta, \\
V_7(x_t, i, t) &= \int_{-\tau_i}^0 \int_{t+\theta}^t x^T(s) U_i x(s) ds d\theta + \int_{-\tau}^0 \int_{\theta}^0 \int_{t+\beta}^t x^T(s) U x(s) ds d\beta d\theta, \\
V_8(x_t, i, t) &= \int_{-\tau_i}^0 \int_{t+\theta}^t \dot{x}^T(s) X_i \dot{x}(s) ds d\theta + \int_{-\tau}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) X \dot{x}(s) ds d\beta d\theta, \\
V_9(x_t, i, t) &= \int_{-h_i}^0 \int_{t+\theta}^t \dot{x}^T(s) Y_i \dot{x}(s) ds d\theta + \int_{-h}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) Y \dot{x}(s) ds d\beta d\theta, \\
V_{10}(x_t, i, t) &= \int_{-\tau_i}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) T_i \dot{x}(s) ds d\beta d\theta + \int_{-\tau}^0 \int_{\theta}^0 \int_{\beta}^0 \int_{t+\alpha}^t \dot{x}^T(s) T \dot{x}(s) ds d\alpha d\beta d\theta, \\
V_{11}(x_t, i, t) &= \int_{-h_i}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) Z_i \dot{x}(s) ds d\beta d\theta + \int_{-h}^0 \int_{\theta}^0 \int_{\beta}^0 \int_{t+\alpha}^t \dot{x}^T(s) Z \dot{x}(s) ds d\alpha d\beta d\theta, \\
V_{12}(x_t, i, t) &= \int_{-\sigma_i}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) L_i \dot{x}(s) ds d\beta d\theta + \int_{-\sigma}^0 \int_{\theta}^0 \int_{\beta}^0 \int_{t+\alpha}^t \dot{x}^T(s) L \dot{x}(s) ds d\alpha d\beta d\theta, \\
V_{13}(x_t, i, t) &= \sigma^2 \int_{-\sigma}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) M \dot{x}(s) ds d\beta d\theta
\end{aligned}$$

where $\zeta^T(t) = [x^T(t), f^T(x(t))]^T$.

From (6.2.1), it can be seen that

$$\mathcal{L}V(x_t, i, t) = \sum_{\kappa=1}^{13} \mathcal{L}V_{\kappa}(x_t, i, t), \quad (6.3.6)$$

where

$$\begin{aligned}
\mathcal{L}V_1(x_t, i, t) &= 2 \left[x(t) - A_i \int_{t-\sigma_i}^t x(s) ds \right]^T P_i \frac{d}{dt} \left[x(t) - A_i \int_{t-\sigma_i}^t x(s) ds \right] \\
&\quad + \sum_{j=1}^N \pi_{ij} \left[x(t) - A_j \int_{t-\sigma_j}^t x(s) ds \right]^T P_j \left[x(t) - A_j \int_{t-\sigma_j}^t x(s) ds \right],
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \left[x(t) - A_i \int_{t-\sigma_i}^t x(s) ds \right]^T P_i [-A_i x(t) + B_i f(x(t)) + C_i f(x(t - \tau_i(t)))] \\
&+ D_i \dot{x}(t - h_i(t)) + E_i \int_{t-d_i(t)}^t f(x(s)) ds + u(t) \\
&+ \pi_{ii} \left[x(t) - A_i \int_{t-\sigma_i}^t x(s) ds \right]^T P_i \left[x(t) - A_i \int_{t-\sigma_i}^t x(s) ds \right] \\
&+ \sum_{j \neq i} \pi_{ij} \left[x^T(t) P_j x(t) + \int_{t-\sigma_j}^t x^T(s) ds A_j P_j K_j^{-1} P_j A_j \int_{t-\sigma_j}^t x(s) ds \right. \\
&\left. + \int_{t-\sigma}^t x^T(s) ds A_j^T P_j A_j \int_{t-\sigma}^t x(s) ds \right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_2(x_t, i, t) &\leq \zeta^T(t) Q_i \zeta(t) - \zeta^T(t - \tau_i(t)) Q_i \zeta(t - \tau_i(t)) (1 - \tau_{\mu_i}) + \sum_{j=1}^N \pi_{ij} \int_{t-\tau_j(t)}^t \zeta^T(s) Q_j \zeta(s) ds \\
&+ \tau \zeta^T(t) Q \zeta(t) - \int_{t-\tau}^t \zeta^T(s) Q \zeta(s) ds,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_3(x_t, i, t) &= x^T(t) W_i x(t) - x^T(t - \tau_i) W_i x(t - \tau_i) + \sum_{j=1}^N \pi_{ij} \int_{t-\tau_j}^t x^T(s) W_j x(s) ds \\
&+ \tau x^T(t) W x(t) - \int_{t-\tau}^t x^T(s) W x(s) ds,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_4(x_t, i, t) &= x^T(t) R_i x(t) - x^T(t - \sigma_i) R_i x(t - \sigma_i) + \sum_{j=1}^N \pi_{ij} \int_{t-\sigma_j}^t x^T(s) R_j x(s) ds \\
&+ \sigma x^T(t) R x(t) - \int_{t-\sigma}^t x^T(s) R x(s) ds,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_5(x_t, i, t) &\leq \dot{x}^T(t) S_i \dot{x}(t) - \dot{x}^T(t - h_i(t)) S_i \dot{x}(t - h_i(t)) (1 - h_{\mu_i}) + \sum_{j=1}^N \pi_{ij} \int_{t-h_j(t)}^t \dot{x}^T(s) S_j \dot{x}(s) ds \\
&+ h \dot{x}^T(t) S \dot{x}(t) - \int_{t-h}^t \dot{x}^T(s) S \dot{x}(s) ds,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_6(x_t, i, t) &= d_i(t) f^T(x(t)) V_i f(x(t)) - (1 - d_{\mu_i}) \int_{t-d_i(t)}^t f^T(x(s)) V_i f(x(s)) ds \\
&+ \sum_{j=1}^N \pi_{ij} \int_{-d_j(t)}^0 \int_{t+\theta}^t f^T(x(s)) V_j f(x(s)) ds d\theta \\
&+ \frac{d^2}{2} f^T(x(t)) V f(x(t)) - \int_{-d}^0 \int_{t+\theta}^t f^T(x(s)) V f(x(s)) ds d\theta,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_7(x_t, i, t) &= \tau_i x^T(t) U_i x(t) - \int_{t-\tau_i}^t x^T(s) U_i x(s) ds + \sum_{j=1}^N \pi_{ij} \int_{-\tau_j}^0 \int_{t+\theta}^t x^T(s) U_j x(s) ds d\theta \\
&\quad + \frac{\tau_i^2}{2} x^T(t) U x(t) - \int_{-\tau}^0 \int_{t+\theta}^t x^T(s) U x(s) ds d\theta, \\
\mathcal{L}V_8(x_t, i, t) &= \tau_i \dot{x}^T(t) X_i \dot{x}(t) - \int_{t-\tau_i}^t \dot{x}^T(s) X_i \dot{x}(s) ds + \sum_{j=1}^N \pi_{ij} \int_{-\tau_j}^0 \int_{t+\theta}^t \dot{x}^T(s) X_j \dot{x}(s) ds d\theta \\
&\quad + \frac{\tau_i^2}{2} \dot{x}^T(t) X \dot{x}(t) - \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}^T(s) X \dot{x}(s) ds d\theta, \\
\mathcal{L}V_9(x_t, i, t) &= h_i \dot{x}^T(t) Y_i \dot{x}(t) - \int_{t-h_i}^t \dot{x}^T(s) Y_i \dot{x}(s) ds + \sum_{j=1}^N \pi_{ij} \int_{-h_j}^0 \int_{t+\theta}^t \dot{x}^T(s) Y_j \dot{x}(s) ds d\theta \\
&\quad + \frac{h_i^2}{2} \dot{x}^T(t) Y \dot{x}(t) - \int_{-h}^0 \int_{t+\theta}^t \dot{x}^T(s) Y \dot{x}(s) ds d\theta, \\
\mathcal{L}V_{10}(x_t, i, t) &= \frac{\tau_i^2}{2} \dot{x}^T(t) T_i \dot{x}(t) - \int_{-\tau_i}^0 \int_{t+\theta}^t \dot{x}^T(s) T_i \dot{x}(s) ds d\theta + \frac{\tau_i^3}{6} \dot{x}^T(t) T \dot{x}(t) \\
&\quad + \sum_{j=1}^N \pi_{ij} \int_{-\tau_j}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) T_j \dot{x}(s) ds d\beta d\theta - \int_{-\tau}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) T \dot{x}(s) ds d\beta d\theta, \\
\mathcal{L}V_{11}(x_t, i, t) &= \frac{h_i^2}{2} \dot{x}^T(t) Z_i \dot{x}(t) - \int_{-h_i}^0 \int_{t+\theta}^t \dot{x}^T(s) Z_i \dot{x}(s) ds d\theta + \frac{h_i^3}{6} \dot{x}^T(t) Z \dot{x}(t) \\
&\quad + \sum_{j=1}^N \pi_{ij} \int_{-h_j}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) Z_j \dot{x}(s) ds d\beta d\theta - \int_{-h}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) Z \dot{x}(s) ds d\beta d\theta, \\
\mathcal{L}V_{12}(x_t, i, t) &= \frac{\sigma_i^2}{2} \dot{x}^T(t) L_i \dot{x}(t) - \int_{-\sigma_i}^0 \int_{t+\theta}^t \dot{x}^T(s) L_i \dot{x}(s) ds d\theta + \frac{\sigma_i^3}{6} \dot{x}^T(t) L \dot{x}(t) \\
&\quad + \sum_{j=1}^N \pi_{ij} \int_{-\sigma_j}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) L_j \dot{x}(s) ds d\beta d\theta - \int_{-\sigma}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) L \dot{x}(s) ds d\beta d\theta, \\
\mathcal{L}V_{13}(x_t, i, t) &= \frac{\sigma_i^4}{2} \dot{x}^T(t) M \dot{x}(t) - \sigma^2 \int_{-\sigma}^0 \int_{t+\theta}^t \dot{x}^T(s) M \dot{x}(s) ds d\theta.
\end{aligned}$$

Here by using upper bounds of discrete, neutral, distributed time-varying delays

and leakage delays, Lemma 6.2.2 and $\pi_{ii} < 0$, the following relationship is obtained

$$\begin{aligned}
& \sum_{j \neq i} \left(-2\pi_{ij} x^T(t) P_j A_j \int_{t-\sigma_j}^t x(s) ds \right) \\
& \leq \sum_{j \neq i} \pi_{ij} \left(x^T(t) K_j x(t) + \int_{t-\sigma_j}^t x^T(s) ds A_j^T P_j K_j^{-1} P_j A_j \int_{t-\sigma_j}^t x(s) ds \right) \\
& \leq \sum_{j \neq i} \pi_{ij} \left(x^T(t) K_j x(t) + \int_{t-\sigma}^t x^T(s) ds A_j^T P_j K_j^{-1} P_j A_j \int_{t-\sigma}^t x(s) ds \right), \quad (6.3.7)
\end{aligned}$$

$$\begin{aligned}
& \sum_{j \neq i} \pi_{ij} \left(\int_{t-\sigma_j}^t x^T(s) ds \right) A_j^T P_j A_j \left(\int_{t-\sigma_j}^t x(s) ds \right) \\
& \leq \sum_{j \neq i} \pi_{ij} \left(\int_{t-\sigma}^t x^T(s) ds \right) A_j^T P_j A_j \left(\int_{t-\sigma}^t x(s) ds \right), \quad (6.3.8)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{j=1}^N \pi_{ij} \int_{t-\tau_j(t)}^t \zeta^T(s) Q_j \zeta(s) ds & \leq \sum_{j=1, j \neq i}^N \pi_{ij} \int_{t-\tau_j(t)}^t \zeta^T(s) Q_j \zeta(s) ds \\
& \leq \sum_{j=1, j \neq i}^N \pi_{ij} \int_{t-\tau}^t \zeta^T(s) Q_j \zeta(s) ds \\
& \leq \int_{t-\tau}^t \zeta^T(s) Q \zeta(s) ds,
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^N \pi_{ij} \int_{t-\tau_j}^t x^T(s) W_j x(s) ds & \leq \int_{t-\tau}^t x^T(s) W x(s) ds, \\
\sum_{j=1}^N \pi_{ij} \int_{t-\sigma_j}^t x^T(s) R_j x(s) ds & \leq \int_{t-\sigma}^t x^T(s) R x(s) ds, \\
\sum_{j=1}^N \pi_{ij} \int_{t-h_j(t)}^t \dot{x}^T(s) S_j \dot{x}(s) ds & \leq \int_{t-h}^t \dot{x}^T(s) S \dot{x}(s) ds, \\
\sum_{j=1}^N \pi_{ij} \int_{-d_j(t)}^0 \int_{t+\theta}^t f^T(x(s)) V_j f(x(s)) ds d\theta & \leq - \int_{-d}^0 \int_{t+\theta}^t f^T(x(s)) V f(x(s)) ds d\theta,
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^N \pi_{ij} \int_{-\tau_j}^0 \int_{t+\theta}^t x^T(s) U_j x(s) ds d\theta &\leq \int_{-\tau}^0 \int_{t+\theta}^t x^T(s) U x(s) ds d\theta, \\
\sum_{j=1}^N \pi_{ij} \int_{-\tau_j}^0 \int_{t+\theta}^t \dot{x}^T(s) X_j \dot{x}(s) ds d\theta &\leq \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}^T(s) X \dot{x}(s) ds d\theta, \\
\sum_{j=1}^N \pi_{ij} \int_{-h_j}^0 \int_{t+\theta}^t \dot{x}^T(s) Y_j \dot{x}(s) ds d\theta &\leq \int_{-h}^0 \int_{t+\theta}^t \dot{x}^T(s) Y \dot{x}(s) ds d\theta, \\
\sum_{j=1}^N \pi_{ij} \int_{-\tau_j}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) T_j \dot{x}(s) ds d\beta d\theta &\leq \int_{-\tau}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) T \dot{x}(s) ds d\beta d\theta, \\
\sum_{j=1}^N \pi_{ij} \int_{-h_j}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) Z_j \dot{x}(s) ds d\beta d\theta &\leq \int_{-h}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) Z \dot{x}(s) ds d\beta d\theta, \\
\sum_{j=1}^N \pi_{ij} \int_{-\sigma_j}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) L_j \dot{x}(s) ds d\beta d\theta &\leq \int_{-\sigma}^0 \int_{\theta}^0 \int_{t+\beta}^t \dot{x}^T(s) L \dot{x}(s) ds d\beta d\theta.
\end{aligned}$$

As from Lemma 6.2.1, we have

$$\begin{aligned}
-(1 - d_{\mu_i}) \int_{t-d_i(t)}^t f^T(x(s)) V_i f(x(s)) ds &\leq -\frac{(1 - d_{\mu_i})}{d_i} \int_{t-d_i(t)}^t f^T(x(s)) ds V_i \int_{t-d_i(t)}^t f(x(s)) ds, \\
-\int_{t-\tau_i}^t x^T(s) U_i x(s) ds &\leq -\frac{1}{\tau_i} \int_{t-\tau_i}^t x^T(s) ds U_i \int_{t-\tau_i}^t x(s) ds, \\
-\int_{t-h_i}^t \dot{x}^T(s) Y_i \dot{x}(s) ds &\leq -\frac{1}{h_i} \int_{t-h_i}^t \dot{x}^T(s) ds Y_i \int_{t-h_i}^t \dot{x}(s) ds, \\
-\int_{-\tau_i}^0 \int_{t+\theta}^t \dot{x}^T(s) T_i \dot{x}(s) ds d\theta &\leq -\frac{2}{\tau_i^2} \int_{-\tau_i}^0 \int_{t+\theta}^t \dot{x}^T(s) ds d\theta T_i \int_{-\tau_i}^0 \int_{t+\theta}^t \dot{x}(s) ds d\theta, \\
-\int_{-h_i}^0 \int_{t+\theta}^t \dot{x}^T(s) Z_i \dot{x}(s) ds d\theta &\leq -\frac{2}{h_i^2} \int_{-h_i}^0 \int_{t+\theta}^t \dot{x}^T(s) ds d\theta Z_i \int_{-h_i}^0 \int_{t+\theta}^t \dot{x}(s) ds d\theta, \\
-\int_{-\sigma_i}^0 \int_{t+\theta}^t \dot{x}^T(s) Z_i \dot{x}(s) ds d\theta &\leq -\frac{2}{\sigma_i^2} \int_{-\sigma_i}^0 \int_{t+\theta}^t \dot{x}^T(s) ds d\theta Z_i \int_{-\sigma_i}^0 \int_{t+\theta}^t \dot{x}(s) ds d\theta, \\
\int_{-\sigma}^0 \int_{t+\theta}^t \dot{x}^T(s) M \dot{x}(s) ds d\theta &\leq -\frac{2}{\sigma^2} \int_{-\sigma}^0 \int_{t+\theta}^t \dot{x}^T(s) ds d\theta Z_i \int_{-\sigma}^0 \int_{t+\theta}^t \dot{x}(s) ds d\theta.
\end{aligned}$$

Note that (6.3.1) and using the reciprocally convex combination technique in Park

et al. (2011), we obtain

$$\begin{aligned}
-\int_{t-\tau_i}^t \dot{x}^T(s)X_i\dot{x}(s)ds &\leq -\int_{t-\tau_i}^{t-\tau_i(t)} \dot{x}^T(s)X_i\dot{x}(s)ds - \int_{t-\tau_i(t)}^t \dot{x}^T(s)X_i\dot{x}(s)ds \\
&\leq -\frac{1}{\tau_i}\varpi^T(t) \begin{bmatrix} R_i & X_i \\ * & R_i \end{bmatrix} \varpi(t),
\end{aligned}$$

where $\varpi(t) = [x^T(t - \tau_i(t)) - x^T(t - \tau_i), x^T(t) - x^T(t - \tau_i(t))]$. For positive diagonal matrices $H_{1i}, H_{2i}, H_{3i}, H_{4i}, H_{5i}, H_{6i}$, we can get from Assumption 1 that

$$\begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} \widehat{L}_1 H_{1i} & -\widehat{L}_2 H_{1i} \\ * & H_{1i} \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \leq 0, \quad (6.3.9)$$

$$\begin{bmatrix} x(t - \tau_i(t)) \\ f(x(t - \tau_i(t))) \end{bmatrix}^T \begin{bmatrix} \widehat{L}_1 H_{2i} & -\widehat{L}_2 H_{2i} \\ * & H_{2i} \end{bmatrix} \begin{bmatrix} x(t - \tau_i(t)) \\ f(x(t - \tau_i(t))) \end{bmatrix} \leq 0, \quad (6.3.10)$$

$$\begin{bmatrix} x(t) \\ x(t - \tau_i) \end{bmatrix}^T \begin{bmatrix} \widehat{L}_1 H_{3i} & -\widehat{L}_2 H_{3i} \\ * & H_{3i} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau_i) \end{bmatrix} \leq 0, \quad (6.3.11)$$

$$\begin{bmatrix} x(t) \\ x(t - h_i) \end{bmatrix}^T \begin{bmatrix} \widehat{L}_1 H_{4i} & -\widehat{L}_2 H_{4i} \\ * & H_{4i} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - h_i) \end{bmatrix} \leq 0, \quad (6.3.12)$$

$$\begin{bmatrix} x(t) \\ x(t - \sigma_i) \end{bmatrix}^T \begin{bmatrix} \widehat{L}_1 H_{5i} & -\widehat{L}_2 H_{5i} \\ * & H_{5i} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \sigma_i) \end{bmatrix} \leq 0, \quad (6.3.13)$$

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} \widehat{L}_1 H_{6i} & -\widehat{L}_2 H_{6i} \\ * & H_{6i} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \leq 0. \quad (6.3.14)$$

Hence, for any matrices J_1, J_2 with appropriate dimensions, we get

$$\begin{aligned} 0 = & 2[x^T(t)J_1 + \dot{x}^T(t)J_2][-A_i x(t - \sigma_i) + B_i f(x(t)) + C_i f(x(t - \tau_i(t))) \\ & + D_i \dot{x}(t - h_i(t)) + E_i \int_{t-d_i(t)}^t f(x(s))ds + u(t) - \dot{x}(t)]. \end{aligned} \quad (6.3.15)$$

Using (6.3.6) and adding (6.3.9)-(6.3.15), we have

$$\mathcal{L}V(x_t, i, t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) \leq \mathcal{L}\left\{ \eta^T(t)\Phi\eta(t) \right\} < 0, \quad (6.3.16)$$

where

$$\begin{aligned} \eta^T(t) = & \left[x^T(t) \quad x^T(t - \tau_i(t)) \quad x^T(t - \tau_i) \quad \int_{t-\tau_i}^t x^T(s)ds \quad f^T(x(t)) \quad f^T(x(t - \tau_i(t))) \right. \\ & x^T(t - \sigma_i) \quad \int_{t-\sigma_i}^t x^T(s)ds \quad \int_{t-\sigma}^t x^T(s)ds \quad u^T(t) \quad \dot{x}^T(t - h_i(t)) \quad x^T(t - h_i) \\ & \left. \int_{t-h_i}^t x^T(s)ds \quad \int_{t-d_i(t)}^t f^T(x(s))ds \quad \dot{x}^T(t) \right]. \end{aligned}$$

Hence we can obtain from (6.3.6) that,

$$\mathcal{L}V(x_t, i, t) - 2y(t)^T u(t) - \gamma u(t)^T u(t) \leq 0.$$

Now, to show the passivity of the delayed neural networks in (6.2.3), we set

$$J(t_p) = \mathbb{E} \left\{ \int_0^{t_p} [-\gamma u(t)^T u(t) - 2y(t)^T u(t)] dt \right\} \quad (6.3.17)$$

where $t_p \geq 0$.

Using Dynkin's formula, we have

$$\mathbb{E} \left[\int_0^{t_p} \mathcal{L}V(x_t, i, t) dt \right] = \mathbb{E} [V(x_{t_p}, i, t_p)] - \mathbb{E} [V(x_0, r(0), 0)].$$

Now, we can deduce that

$$\begin{aligned} J(t_p) &= \mathbb{E} \left\{ \int_0^{t_p} [-\gamma u(t)^T u(t) - 2y(t)^T u(t) + \mathcal{L}V(x_t, i, t)] dt \right\} - \mathbb{E} \left[\int_0^{t_p} \mathcal{L}V(x_t, i, t) dt \right] \\ &= \mathbb{E} \left\{ \int_0^{t_p} [-\gamma u(t)^T u(t) - 2y(t)^T u(t) + \mathcal{L}V(x_t, i, t)] dt \right\} \\ &\quad - \mathbb{E} [V(x_{t_p}, i, t_p)] + \mathbb{E} [V(x_0, r(0), 0)]. \end{aligned} \tag{6.3.18}$$

Applying Schur complement to (6.3.5), we have

$$\Phi < 0. \tag{6.3.19}$$

Thus, if (6.3.19) holds, then since $\mathbb{E}[V(x_{t_p}, i, t_p)] \geq 0$ and $V(x_0, r(0), 0) = 0$ holds under zero initial condition, from (6.3.19) it follows that

$$J(t_p) \leq 0$$

for any $t_p \geq 0$, which implies that (6.2.5) is satisfied and therefor the delayed neural networks (6.2.3) is locally passive. Next we shall prove that $\mathbb{E}[\|x(t)\|^2] \rightarrow 0$ as $t \rightarrow \infty$. Taking expectation on both sides of (6.3.16) and integrating from 0 to t we have,

$$\int_0^t \mathbb{E}[\mathcal{L}V(x_s, r(s), s)] ds - 2 \int_0^t \mathbb{E}[y^T(s)u(s)] ds - \gamma \int_0^t \mathbb{E}[u^T(s)u(s)] ds \leq \int_0^t \mathbb{E}[\eta^T(s)\Phi\eta(s)] ds$$

By using Dynkin's formula, we have

$$\begin{aligned} &\mathbb{E}[\mathcal{L}V(x_t, i, t)] - \mathbb{E}[\mathcal{L}V(x_0, r(0), 0)] - 2 \int_0^t \mathbb{E}[y^T(s)u(s)] ds - \gamma \int_0^t \mathbb{E}[u^T(s)u(s)] ds \\ &\leq \int_0^t \mathbb{E}[\eta^T(s)\Phi\eta(s)] ds \end{aligned}$$

Hence

$$\begin{aligned}
\mathbf{E}[\mathcal{L}V(x_t, i, t)] - \int_0^t \mathbf{E}[\eta^T(s)\Phi\eta(s)]ds &\leq \mathbf{E}[\mathcal{L}V(x_0, r(0), 0)] + 2 \int_0^t \mathbf{E}[y^T(s)u(s)]ds \\
&\quad + \gamma \int_0^t \mathbf{E}[u^T(s)u(s)]ds \\
&< \infty, \quad t \geq 0.
\end{aligned} \tag{6.3.20}$$

Using Jensen's inequality and (6.3.6), we have

$$\begin{aligned}
\mathbf{E} \left\| A_i \int_{t-\sigma_i}^t x(s)ds \right\|^2 &= \mathbf{E} \left[A_i \int_{t-\sigma_i}^t x(s)ds \right]^T \left[A_i \int_{t-\sigma_i}^t x(s)ds \right] \\
&\leq \lambda_{\max}(A_i^2) \mathbf{E} \left[\int_{t-\sigma_i}^t x(s)ds \right]^T \left[\int_{t-\sigma_i}^t x(s)ds \right] \\
&\leq \frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_i)} \left[\int_{t-\sigma_i}^t \mathbf{E}x(s)ds \right]^T R_i \left[\int_{t-\sigma_i}^t \mathbf{E}x(s)ds \right] \\
&\leq \sigma_i \frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_i)} \left\{ \int_{t-\sigma_i}^t \mathbf{E}x^T(s)R_ix(s)ds \right\} \\
&\leq \sigma_i \frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_i)} \mathbf{E}V_4(x_t, i, t) \\
&\leq \sigma \frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_i)} \mathbf{E}V(x_t, i, t) \\
&\leq \sigma \frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_i)} \mathbf{E}V(x_0, r(0), 0), \quad t \geq 0.
\end{aligned} \tag{6.3.21}$$

Similarly, it follows from the definition of $V_1(x_t, i, t)$ that

$$\begin{aligned}
\mathbf{E} \left\| x(t) - A_i \int_{t-\sigma_i}^t x(s)ds \right\|^2 &= \mathbf{E} \left[A_i \int_{t-\sigma_i}^t x(s)ds \right]^T \left[A_i \int_{t-\sigma_i}^t x(s)ds \right] \\
&\leq \frac{\mathbf{E}V_1(x_t, i, t)}{\lambda_{\min}(P_i)} \\
&\leq \frac{\mathbf{E}V(x_t, i, t)}{\lambda_{\min}(P_i)} \\
&\leq \frac{\mathbf{E}V(x_0, r(0), 0)}{\lambda_{\min}(P_i)}, \quad t \geq 0.
\end{aligned}$$

Hence, it can be obtained that

$$\begin{aligned}
\mathbf{E} \|x(t)\|^2 &= \mathbf{E} \left\| x(t) - A_i \int_{t-\sigma_i}^t x(s) ds + A_i \int_{t-\sigma_i}^t x(s) ds \right\|^2 \\
&\leq 2\mathbf{E} \left\| A_i \int_{t-\sigma_i}^t x(s) ds \right\|^2 + 2\mathbf{E} \left\| x(t) - A_i \int_{t-\sigma_i}^t x(s) ds \right\|^2 \\
&\leq 2\sigma \frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_i)} \mathbf{E}V(x_0, r(0), 0) + 2 \frac{\mathbf{E}V(x_0, r(0), 0)}{\lambda_{\min}(P_i)} < \infty, \quad t \geq 0,
\end{aligned} \tag{6.3.22}$$

$$\begin{aligned}
&\mathbf{E}V(x_0, r(0), 0) \\
&\leq \left\{ 2\lambda_{\max}(P_i)(1 + \sigma_i^2 \max_{i \in S} A_i) + \tau \max_{i \in S} \{\lambda_{\max}(Q_i)\} + \tau^2 \lambda_{\max}(Q) + \tau \max_{i \in S} \{\lambda_{\max}(W_i)\} \right. \\
&+ \tau^2 \lambda_{\max}(W) + \sigma \max_{i \in S} \{\lambda_{\max}(R_i)\} + \sigma^2 \lambda_{\max}(R) + h \max_{i \in S} \{\lambda_{\max}(S_i)\} + h^2 \lambda_{\max}(S) \\
&+ d^2 \max_{i \in S} \{\lambda_{\max}(V_i)\} + d^3 \lambda_{\max}(V) + \tau^2 \max_{i \in S} \{\lambda_{\max}(U_i)\} + \tau^3 \lambda_{\max}(U) + \tau^2 \max_{i \in S} \{\lambda_{\max}(X_i)\} \\
&+ \tau^3 \lambda_{\max}(X) + h^2 \max_{i \in S} \{\lambda_{\max}(Y_i)\} + h^3 \lambda_{\max}(Y) + \tau^3 \max_{i \in S} \{\lambda_{\max}(T_i)\} + \tau^4 \lambda_{\max}(T) \\
&\left. + h^3 \max_{i \in S} \{\lambda_{\max}(Z_i)\} + h^4 \lambda_{\max}(Z) + \sigma^3 \max_{i \in S} \{\lambda_{\max}(L_i)\} + \sigma^4 \lambda_{\max}(L) + \sigma^5 \lambda_{\max}(M) \right\} < \infty,
\end{aligned} \tag{6.3.23}$$

From (6.3.22) and (6.3.23), it can be deduced that the trivial solution of system (6.2.3) is locally passive. Then the solutions $x(t) = x(t, 0, \phi)$ of system (3) is bounded on $[0, \infty)$. considering (6.2.3), we know that $\dot{x}(t)$ is bounded on $[0, \infty)$, which leads to the uniform continuity of the solution $x(t)$ on $[0, \infty)$. From (6.3.20), we note that the following inequality holds:

$$\begin{aligned}
\lambda_{\min}(\Phi) \int_0^t \mathbf{E}[x^T(s)x(s)] ds &\leq \mathbf{E}[\mathcal{L}V(x_t, i, t)] - \int_0^t \mathbf{E}[\eta^T(s)\Phi\eta(s)] ds \\
&\leq \mathbf{E}[\mathcal{L}V(x_0, r(0), 0)] + 2 \int_0^t \mathbf{E}[y^T(s)u(s)] ds \\
&+ \gamma \int_0^t \mathbf{E}[u^T(s)u(s)] ds \\
&< \infty, \quad t \geq 0.
\end{aligned}$$

By Barbalats lemma in Gopalsamy (1992), it holds that $\mathbb{E}[\|x(t)\|^2] \rightarrow 0$ as $t \rightarrow \infty$ and this completes the proof of the global passivity of the system (6.2.1). Hence the proof is completed. \square

Remark 6.3.1. *Recently, the authors in Balasubramaniam et al.(2011) studied passivity analysis for neural networks of neutral type with Markovian jumping parameters and time delay in the leakage term. By constructing proper Lyapunov–Krasovskii functional, new delay-dependent passivity conditions are derived in terms of LMIs and it can be checked easily via standard numerical packages. Moreover, it is well known that the passivity behavior of neural networks is very sensitive to the time delay in the leakage term. Triple and quadruple integrals have not been taken into account to derive the passivity conditions in Balasubramaniam et al.(2011). Mode-dependent time delays have not been included in Balasubramaniam et al.(2011). Very recently, a mode-dependent approach is proposed by constructing a novel Lyapunov functional, where some terms involving triple or quadruple integrals are taken into account to study the state estimation problem in Huang et al. (2013). Motivating this reason, the author has introduced improved Lyapunov–Krasovskii functional with triple and quadruple integrals for deriving the reported stability results in this chapter. Based on this discussion, results will give less conservative results than those studied in Balasubramaniam et al.(2011)and Huang et al.(2013).*

6.4 Numerical Example

In this section, numerical example is provided to demonstrate the effectiveness and applicability of the proposed method.

Example 6.4.1. *Consider the 2-D Mode-dependent Markov jump neutral type neural networks with mixed time-delays (6.2.3) with the following parameters*

$$A_1 = \begin{bmatrix} 8.4 & 0 \\ 0 & 9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 7.8 & 0 \\ 0 & 8.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.21 & -0.19 \\ -0.24 & 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.9 & -0.9 \\ 0.5 & -0.8 \end{bmatrix},$$

$$\begin{aligned}
C_1 &= \begin{bmatrix} -0.09 & -0.2 \\ 0.2 & 0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.09 \end{bmatrix}, \\
D_2 &= \begin{bmatrix} 0.1 & 0 \\ 0.5 & -0.1 \end{bmatrix}, \\
E_1 &= \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.1 & -0.02 \\ -0.2 & 0.07 \end{bmatrix}, \quad \hat{L}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{L}_2 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}.
\end{aligned}$$

Take $f_1(s) = f_2(s) = \tanh(s)$, $\tau_1(t) = \tau_2(t) = h_1(t) = h_2(t) = d_1(t) = d_2(t) = 0.1 \cos t + 0.4$, $\sigma_1 = \sigma_2 = 0.1$, $\tau_{\mu_1} = \tau_{\mu_2} = h_{\mu_1} = h_{\mu_2} = d_{\mu_1} = d_{\mu_2} = 0.1$.

$$\Gamma = \begin{bmatrix} -7 & 7 \\ 6 & -6 \end{bmatrix}.$$

By using the MATLAB LMI toolbox, the following feasible solution is obtained for the LMIs (6.3.1)-(6.3.5):

$$\begin{aligned}
P_1 &= \begin{bmatrix} 0.0157 & 0.0008 \\ 0.0008 & 0.0143 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.0068 & 0.0004 \\ 0.0004 & 0.0060 \end{bmatrix}, \quad Q_{11} = \begin{bmatrix} 0.2401 & 0.0007 \\ 0.0007 & 0.2592 \end{bmatrix}, \\
Q_{12} &= \begin{bmatrix} 0.3770 & -0.0192 \\ -0.0192 & 0.3992 \end{bmatrix}, \quad Q_{21} = \begin{bmatrix} -0.4101 & 0.0043 \\ 0.0201 & -0.4457 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} -0.6428 & 0.0773 \\ 0.0231 & -0.6959 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
Q_{31} &= \begin{bmatrix} 1.3786 & 0.0133 \\ 0.0133 & 1.4063 \end{bmatrix}, & Q_{32} &= \begin{bmatrix} 1.9101 & -0.0284 \\ -0.0284 & 1.9903 \end{bmatrix}, & W_1 &= \begin{bmatrix} 0.1039 & 0.0088 \\ 0.0088 & 0.1018 \end{bmatrix}, \\
W_2 &= \begin{bmatrix} 0.1517 & 0.0121 \\ 0.0121 & 0.1487 \end{bmatrix}, & R_1 &= \begin{bmatrix} 0.1758 & 0.0160 \\ 0.0160 & 0.1721 \end{bmatrix}, & R_2 &= \begin{bmatrix} 0.5675 & 0.0333 \\ 0.0333 & 0.5563 \end{bmatrix}, \\
S_1 &= \begin{bmatrix} 1.0817 & -0.0075 \\ -0.0075 & 0.7204 \end{bmatrix}, & S_2 &= \begin{bmatrix} 1.2231 & 0.0093 \\ 0.0093 & 0.9256 \end{bmatrix}, & V_1 &= \begin{bmatrix} 10.7113 & 0.5692 \\ 0.5692 & 9.9973 \end{bmatrix}, \\
V_2 &= \begin{bmatrix} 10.8483 & 0.5953 \\ 0.5953 & 10.2619 \end{bmatrix}, & U_1 &= \begin{bmatrix} 0.5120 & 0.0419 \\ 0.0419 & 0.5056 \end{bmatrix}, & U_2 &= \begin{bmatrix} 0.6238 & 0.0492 \\ 0.0492 & 0.6133 \end{bmatrix}, \\
X_1 &= \begin{bmatrix} 21.0470 & -0.1153 \\ -0.1153 & 21.2070 \end{bmatrix}, & X_2 &= \begin{bmatrix} 18.5642 & -0.3139 \\ -0.3139 & 18.6620 \end{bmatrix}, & Y_1 &= \begin{bmatrix} 27.5946 & -0.0019 \\ -0.0019 & 26.6557 \end{bmatrix}, \\
Y_2 &= \begin{bmatrix} 28.4115 & 0.1096 \\ 0.1096 & 27.4131 \end{bmatrix}, & T_1 &= \begin{bmatrix} 3.0928 & 0.0943 \\ 0.0943 & 3.0505 \end{bmatrix}, & T_2 &= \begin{bmatrix} 3.8182 & 0.0431 \\ 0.0431 & 3.7941 \end{bmatrix}, \\
Z_1 &= \begin{bmatrix} 3.0867 & 0.1080 \\ 0.1080 & 3.0591 \end{bmatrix}, & Z_2 &= \begin{bmatrix} 3.8647 & 0.0504 \\ 0.0504 & 3.8446 \end{bmatrix}, & L_1 &= \begin{bmatrix} 0.0068 & 0.0001 \\ 0.0001 & 0.0068 \end{bmatrix},
\end{aligned}$$

$$L_2 = \begin{bmatrix} 0.0059 & 0.0004 \\ 0.0004 & 0.0062 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 1.3461 & -0.1410 \\ -0.1410 & 1.3280 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.0919 & 0.0135 \\ 0.0135 & 0.0907 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 4.0913 & -0.1850 \\ -0.1850 & 4.4495 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -6.6332 & 0.7686 \\ 0.3444 & -7.3948 \end{bmatrix}, \quad \gamma = 0.2272.$$

This shows that the given system (6.2.3) is globally passive in the mean square.

