Some Interesting Results From Fuzzy Topological Spaces

By

H. Tamil Selvi

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY), COIMBATORE - 641 043 IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

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Contents

CONTENTS

Introduction

INTRODUCTION

The aim of this thesis is to discuss in detail the following four papers.

1. THE FUZZY TOPOLOGICAL COMPLEMENTATION THEOREM [5]
2. STRONGLY COMPACT FUZZY TOPOLOGICAL SPACES [17]
3. SEMI-OPEN SETS, SEMI-CONTINUITY AND SEMI-OPEN MAPPINGS IN FUZZY BITOPOLOGICAL SPACES [18]
4. ON FUZZY PAIRWISE α-CONTINUITY AND FUZZY PAIRWISE PRE-CONTINUITY [19].

In his article published in 1965, **A.K.STEINER** [23] has obtained a topological complementation theorem. The fuzzy analogue of this result is obtained by **S.DANG, A.BEHERA AND S.NANDA** [5]. This paper is discussed in detail in the first chapter.

Given a nonempty set X, the lattice of all fuzzy topologies on X is denoted by $\Sigma(X)$. A fuzzy topology is said to be principal if each fuzzy point x_r has a smallest fuzzy neighbourhood. Let $\pi(X)$ denote the collection of all principal fuzzy topologies. The main theorem proved here is as follows.

"Every τ in $\Sigma(X)$ has a complement τ' in $\pi(X)$".

The second chapter deals with strongly compact fuzzy topological spaces. Strong compactness is defined in terms of pre-open and semi-open sets. A fuzzy topological space is said to be strongly compact if every pre-open cover has a finite subcover.

With every fuzzy topology τ, the author associates a topology $\tau_\emptyset$. Denoting by $FP_rO(x)$, the collection of all fuzzy pre-open sets, the fuzzy topology $\tau_\emptyset$ is generated by $FP_rO(x)$. The author generalizes important properties of compact spaces to strong fuzzy compact spaces. In fact, he has obtained parallel results to the following theorems on compactness.

1. Continuous image of a compact space is compact.
2. Every closed subset of a compact space is compact.
3. A topological space X is compact iff every collection of closed subsets of X with the finite intersection property has a nonempty intersection.

The third chapter is devoted to the study of fuzzy bitopological spaces. Here the concepts of fuzzy semi-open sets, fuzzy α-open sets, fuzzy pre-open sets have been generalized to fuzzy bitopological spaces. Using these concepts the author introduces and studies fuzzy pairwise semi-continuous mappings, fuzzy pairwise semi-open mappings, fuzzy pairwise α-continuous and fuzzy pairwise pre-continuous mappings etc. The author has obtained interesting characterizations of these mappings.

Review of Literature

REVIEW OF LITERATURE

The lattice of all topologies is studied by many topologists. In 1958, **J.HARTMANIS** [6] has raised the question whether the lattice of all topologies in a given set is complemented. He proved that this result is true in the case of a finite set. In 1965, **A.K.STEINER** [23] has proved that the lattice is always complemented. The same result is proved in a different way by **A.C.M VAN ROOIJ** [25] in 1966. In 1968, **P.S.SCHNARE** [20] proved that every proper topology on a finite set with $n \geq 2$ elements has atleast (n-1) complements in the lattice of all topologies on X. In 1969, the same author improved this result and proved the following theorem [21].

"Every proper topology on an infinite set X has atleast $|x|$ complements (resp., principal complements) and atmost $2^{2^{|x|}}$ complements (resp., $2^{|X|}$ principal complements)". Moreover, these bounds are the best possible.

In 1966, **A.K.STEINER** [24] proved that every topology has a complement which is a principal topology. In his proof of this theorem, **A.C.M.VAN ROOIJ** uses Zorn's Lemma and two applications of transfinite induction. In 1972, **P.S.SCHNARE** [22] improved this proof by using a simple trick by suitably adjoining a new point to X. This proof is generalized in fuzzy situation by **S.DANG,** Fuzzy bitopological spaces were first introduced and studied by **A.KANDIL** [7] in 1989. In 1981, **K.K.AZAD** [2] published a paper on fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity. The concept of fuzzy semi-continuous, fuzzy semi-open,fuzzy semi-closed mappings are defined exactly as in the case of topological spaces. **S.SAMPATH KUMAR** [18], [19] has generalized the results on bitopological spaces and the fuzzy topological spaces and developed a study of fuzzy bitopological spaces. In this thesis two papers of **S.SAMPATH KUMAR** dealing with semi-open sets, semi-continuity and semi-open mappings, α-continuity and pre-continuity in fuzzy bitopological spaces are discussed in detail in the third chapter.

Chapter I

CHAPTER – I

THE FUZZY TOPOLOGICAL COMPLEMENTATION THEOREM

DEFINITION: 1.1 [14].

Let X be a nonempty set. A function "A" from X to [0,1] is called a fuzzy set in X.

If A takes only the values 0,1, A is called a crisp set in X. The crisp set which always takes the value 1 on X is denoted by 1_X and the crisp set which always takes the value 0 on X is denoted by 0_X.

The set $\{x \in X | A(x) > 0\}$ is called the support of A.

DEFINITION: 1.2

Let J be an indexed set and let $\mathcal{A} = \{A_\alpha | \alpha \in J\}$ be a family of fuzzy sets in X. Then the union $\cup\mathcal{A}$ and the intersection $\cap\mathcal{A}$ are defined by

$$(\cup\mathcal{A})(x) = \sup\{A_\alpha(x) | \alpha \in J\}, \ x \in X, \text{ and}$$
$$(\cap\mathcal{A})(x) = \inf\{A_\alpha(x) | \alpha \in J\}, \ x \in X.$$

DEFINITION: 1.3

Let A be a fuzzy set. The complement of A, denoted by A' is defined by $A'(x) = 1 - A(x)$ for all x in X.

DEFINITION: 1.4

Let $f: X \to Y$ be a mapping. If A is a fuzzy set of a nonempty set X, then f(A) is a fuzzy set of Y defined by

$$f(A)(y)=\begin{cases} \sup\limits_{x\in f^{-1}(y)} A(x) & \text{if } f^{-1}(y)\neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \quad (y\in Y)$$

If B is a fuzzy set of Y, then $f^{-1}(B)$ is a fuzzy set of X defined by $f^{-1}(B)(x)=B(f(x))$, for each $x\in X$.

DEFINITION: 1.5 [4].

Let X be a nonempty set and I^X be the set of all fuzzy subsets of X. A family $\tau\subset I^X$ is called a fuzzy topology of X if

(i) $0_X, I_X\in\tau$.

(ii) $A,B\in\tau \Rightarrow A\cap B\in\tau$. (iii) $\{A_j : j\in J\}\subset\tau \Rightarrow \bigcup\limits_{j\in J} A_j\in\tau$.

The pair (X,τ) is called a fuzzy topological space (briefly, fts). A fuzzy set U is called fuzzy open if $U\in\tau$. The fuzzy complement $U'=1_X-U$ is called a fuzzy closed set.

DEFINITION: 1.6

Let A be a fuzzy set of a fuzzy topological space (X,τ). The closure of A (Cl(A)) and the interior of A (Int(A)) are defined respectively as

$$Cl(A) = \inf\{B \mid B\geq A,\ B'\in\tau\}$$

$$\text{and } Int(A) = \sup\{B \mid B\leq A,\ B\in\tau\}.$$

RESULT:

For a fuzzy set A of a fuzzy space X

(i) 1-Int(A) = Cl(1-A), and

(ii) 1-Cl(A) = Int(1-A).

DEFINITION: 1.7

Let (X,τ) be a fuzzy topological space. A subfamily $\mathcal{C}$ of τ is called a base for τ if each member of τ is a union of some members of $\mathcal{C}$.

A subfamily $\mathcal{S}$ of τ is called a subbase for τ if the collection of all finite intersection of members of $\mathcal{S}$ forms a base for τ.

DEFINITION: 1.8

Let (X,τ) be a fuzzy topological space. A fuzzy point x_r is a fuzzy set such that $x_r(y) = \begin{cases} r & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$ $(0<r\leq 1)$

x and r are respectively called the support and the value of the fuzzy point x_r.

NOTE:

1) If r = 1, the fuzzy point reduces to a crisp point, say x.

2) A fuzzy point x_r belongs to a fuzzy set A in X if $r\leq A(x)$.

DEFINITION: 1.9

A fuzzy set A in a fuzzy topological space (X,τ) is called a fuzzy neighbourhood of a fuzzy point x_r if there exists a fuzzy open set U such that $x_r \in U \subset A$.

NOTE:

A fuzzy open set U is a fuzzy neighbourhood of all of its fuzzy points.

NOTATION:

$\mathcal{N}_{\tau}(x)$ denotes the neighbourhood system of x in the fuzzy topology τ.

DEFINITION: 1.10

Let X be a nonempty set and let $F \subset I^X$. Define a relation $\leq$ on F by $A_1 \leq A_2$ if $A_1 \subseteq A_2$ (fuzzy inclusion). Then $(F, \leq)$ is called a fuzzy partially ordered set (f-poset) if it satisfies

1) $A \leq A$ for all $A \in F$ (Reflexive)
2) $A \leq B$ and $B \leq A$ implies $A = B$ (Antisymmetric)
3) $A \leq B$ and $B \leq C$ implies $A \leq C$ (Transitivity).

DEFINITION:1.11

An fuzzy partially ordered set $(F, \leq)$ is called a fuzzy lattice if any pair of elements A, B in F has an infimum and a supremum in F, written as $A \wedge B$ and $A \vee B$ respectively.

DEFINITION: 1.12

A fuzzy lattice F is said to be complete if every nonempty subset of F has an infimum and supremum. A subset $E \subset F$ is called a sublattice if $(E, \leq)$ itself forms a lattice.

DEFINITION: 1.13

Let $(F,\leq)$ be a fuzzy partially ordered set. Let $A\subset F$ and $A \neq \emptyset$, then $(A,\leq|A)$ is a chain in $(F,\leq)$ iff $\leq|A$ is a total ordering on A.

DEFINITION: 1.14

A fuzzy topological space (X,τ) is said to be principal if every fuzzy point x_r has a smallest fuzzy neighbourhood $N_\tau(x_r)$.

NOTATIONS:

1) Smallest fuzzy neighbourhood of a crisp point x is denoted by $N_\tau(x)$.

2) Set of all principal fuzzy topologies in a nonempty set X is denoted by $\pi(X)$.

3) Set of all fuzzy topologies in a nonempty set X is denoted by $\Sigma(X)$.

DEFINITION: 1.15

We define a relation $\leq$ on $\Sigma(X)$ by saying $\tau_1 \leq \tau_2$ iff τ_1 is weaker than τ_2, where $\tau_1,\tau_2\in\Sigma(X)$.

THEOREM: 1.16

$(\Sigma(X),\leq)$ is a complete fuzzy lattice with the greatest and the least element.

PROOF:

Claim 1: $(\Sigma(X), \leq)$ is a fuzzy partially ordered set.

i) Reflexivity: Obvious.

ii) Antisymmetric:

If $\tau_1 \leq \tau_2$ and $\tau_2 \leq \tau_1$ for $\tau_1, \tau_2 \in \Sigma(X)$, then τ_1 is weaker than τ_2 and τ_2 is weaker than τ_1. Therefore, $\tau_1 = \tau_2$.

iii) Transitivity:

If $\tau_1 \leq \tau_2$ and $\tau_2 \leq \tau_3$ for $\tau_1, \tau_2, \tau_3 \in \Sigma(X)$, then τ_1 is weaker than τ_2 and τ_2 is weaker than τ_3. This implies that τ_1 is weaker than $\tau_3 \Rightarrow \tau_1 \leq \tau_3$.

Therefore, $(\Sigma(X), \leq)$ is a fuzzy partially ordered set.

Claim 2: $(\Sigma(X), \leq)$ is a fuzzy lattice.

Let $\tau_1, \tau_2 \in \Sigma(X)$.

Define $\tau_1 \wedge \tau_2 = \{U \mid U$ is both τ_1 - open and τ_2 - open in $X\}$ and $\tau_1 \vee \tau_2 = \{\bigcap_k \tau_k : \tau_1 \leq \tau_k, \tau_2 \leq \tau_k\}$.

To prove that $\tau_1 \wedge \tau_2$ is a fuzzy topology on X.

i) Since 0_X, 1_X belongs to both τ_1 and τ_2,

$$0_X, 1_X \in \tau_1 \wedge \tau_2.$$

ii) Let $U, V \in \tau_1 \wedge \tau_2$.

Since U and V are both τ_1-open and τ_2-open in X, $U \cap V$ is also both τ_1-open and τ_2-open in X.

iii) Let $\{U_j : j \in J\} \subset \tau_1 \wedge \tau_2$. Since $\{U_j : j \in J\}$ is a sequence of both τ_1 -open and τ_2- open sets in X. $\bigcup_{j \in J} U_j$ is also both τ_1 - open and τ_2 - open set in X.

$\Rightarrow \bigcup_{j\in J} U_j \in \tau_1 \wedge \tau_2$.

Therefore, $\tau_1 \wedge \tau_2 \in \Sigma(X)$.

Next to prove that $\tau_1 \vee \tau_2$ is a fuzzy topology on X.

i) Since 0_X and $1_X \in \tau_k$ for all k,

$$0_X, 1_X \in \tau_1 \vee \tau_2.$$

ii) Let $U_1, U_2 \in \tau_1 \vee \tau_2$.

Then $U_1, U_2 \in \tau_k$ for all k.

So $U_1 \cap U_2 \in \tau_k$ for all k.

$\Rightarrow U_1 \cap U_2 \in \bigcap_k \tau_k$.

$\Rightarrow U_1 \cap U_2 \in \tau_1 \vee \tau_2$.

iii) Let $\{U_j : j\in J\}$ be a family of fuzzy open sets in $\tau_1 \vee \tau_2$

Then $\{U_j : j\in J\} \subset \tau_k$ for all k and since each τ_k being a fuzzy topology, we have $\bigcup_{j\in J} U_j \in \tau_k$ for all k.

$\Rightarrow \bigcup_{j\in J} U_j \in \bigcap_k \tau_k$.

$\Rightarrow \bigcup_{j\in J} U_j \in \tau_1 \vee \tau_2$.

Therefore, $\tau_1 \vee \tau_2 \in \Sigma(X)$.

Claim 3: $\Sigma(X)$ is complete.

For that we have to prove that every nonempty subset of $\Sigma(X)$ has an infimum and supremum.

Let F be a nonempty subset of $\Sigma(X)$. We define infimum F and supremum F as,

$$\inf F = \{\bigcap_j \tau_j : \tau_j \in F\} \text{ and}$$

$$\sup F = \{\bigcap_k \tau_k : \tau_k \in \Sigma(X) \text{ and } \tau \leq \tau_k \text{ for all } \tau \in F\}.$$

Then inf F and sup F are in $\Sigma(X)$. So $\Sigma(X)$ is a complete fuzzy lattice.

Furthermore, $\{0_X, 1_X\}$ = ID (the indiscrete fuzzy topology on X) and I^X=D (the discrete fuzzy topology on X) are respectively the least and the greatest elements of $\Sigma(X)$.

REMARK:

$\pi(X)$ is a fuzzy sublattice of $\Sigma(X)$.

DEFINITION: 1.17

Let $\tau \in \Sigma(X)$. Then a fuzzy topology τ' is called the complement of τ if

$\tau \vee \tau'$ = D (the discrete fuzzy topology on X) and

$\tau \wedge \tau'$ = ID (the indiscrete fuzzy topology on X).

NOTE:

When X = {p} and $\tau = (0_{\{p\}}, 1_{\{p\}})$ we have $\tau = \tau'$. Here both are indiscrete fuzzy topologies.

THEOREM: 1.18

THE FUZZY COMPLEMENTATION THEOREM

"Every $\tau \in \Sigma(X)$ has a complement $\tau' \in \pi(X)$"

We expand the set X by adding a point (crisp) p as $Y = X \cup \{p\}$. Define a fuzzy topology η on Y as follows.

$$\eta = \tau \cup \{U \cup 1_{\{p\}} \mid U \in \tau\}$$

where τ is the fuzzy topology on X.

The method of proof of this theorem is to construct a fuzzy topology on Y, whose restriction to X is the required complement. The construction of fuzzy topology on Y uses Zorn's lemma.

Proof of the theorem:

By our definition of a fuzzy topology η on Y, every object on η is either a fuzzy open set in (X,τ) or is of the form $U \cup 1_{\{p\}}$, where U is fuzzy open in (X,τ).

Let $\mathcal{A} = \{(A,s):$

(i) $p \in A \subset Y$ (ii) $s \in \pi(A)$

(iii) s is a complement for $\eta|A$, that is $s = (\eta|A)'\}$.

If $A = \{p\}$, then $\eta|A = \{0_{\{p\}}, 1_{\{p\}}\}$ and its complement is itself. Therefore $(\{p\},\{0_{\{p\}},1_{\{p\}}\}) \in \mathcal{A} \Rightarrow \mathcal{A} \neq \emptyset$.

Define a relation $\leq$ in $\mathcal{A}$ by saying $(A_1,s_1) \leq (A_2,s_2) \Leftrightarrow$

(1) $A_1 \subset A_2$

(2) $N_{s_1}(x) = N_{s_2}(x)$ for $x \in A_1 - \{p\}$

(3) $N_{s_1}(p) \subset N_{s_2}(p) \subset N_{s_1}(p) \cup (A_2 - A_1)$.

To prove $(\mathcal{A}, \leq)$ is a fuzzy partially ordered set.

(i) Reflexive:

Since (1) $A_1 \subset A_1$

(2) $N_{s_1}(x) = N_{s_1}(x)$ for $x \in A_1 - \{p\}$

and (3) $N_{s_1}(p) \subset N_{s_1}(p) \subset N_{s_1}(p) \cup (A_1 - A_1)$

we have $(A_1, s_1) \leq (A_1, s_1)$ for every $(A_1, s_1) \in \mathcal{A}$.

(ii) Antisymmetric:

Let $(A_1, s_1) \leq (A_2, s_2)$ and $(A_2, s_2) \leq (A_1, s_1)$.

Then (1) $A_1 \subset A_2, A_2 \subset A_1 \Rightarrow A_1 = A_2$.

(2) $N_{s_1}(x) = N_{s_2}(x)$ for $x \in A_1 - \{p\}$,

$N_{s_2}(x) = N_{s_1}(x)$ for $x \in A_2 - \{p\}$.

$\Rightarrow N_{s_1}(x) = N_{s_2}(x)$ for $x \in A_1 \cup A_2 - \{p\}$.

(3) $N_{s_1}(p) \subset N_{s_2}(p) \subset N_{s_1}(p) \cup (A_2 - A_1)$,

$N_{s_2}(p) \subset N_{s_1}(p) \subset N_{s_2}(p) \cup (A_1 - A_2)$.

$\Rightarrow N_{s_1}(p) \subset N_{s_2}(p) \subset N_{s_1}(p) \cup \emptyset$ (Since $A_1 = A_2$).

$\Rightarrow N_{s_1}(p) = N_{s_2}(p)$.

Therefore, $(A_1, s_1) = (A_2, s_2)$.

(iii) Transitivity:

Let $(A_1, s_1), (A_2, s_2), (A_3, s_3) \in \mathcal{A}$.

such that $(A_1, s_1) \leq (A_2, s_2)$ and $(A_2, s_2) \leq (A_3, s_3)$.

Then (1) $A_1 \subset A_2, A_2 \subset A_3 \Rightarrow A_1 \subset A_3$.

(2) $N_{s_1}(x) = N_{s_2}(x)$ for $x \in A_1 - \{p\}$,

$N_{s_2}(x) = N_{s_3}(x)$ for $x \in A_2 - \{p\}$.

$\Rightarrow N_{s_2}(x) = N_{s_3}(x)$ for $x \in A_1 - \{p\}$(since $A_1 \subset A_2$).

$\Rightarrow N_{s_1}(x) = N_{s_3}(x)$ for $x \in A_1 - \{p\}$.

(3) $N_{s_1}(p) \subset N_{s_2}(p) \subset N_{s_1}(p) \cup (A_2 - A_1)$,

$N_{s_2}(p) \subset N_{s_3}(p) \subset N_{s_2}(p) \cup (A_3 - A_2)$.

$\Rightarrow N_{s_1}(p) \subset N_{s_2}(p) \subset N_{s_3}(p) \subset N_{s_2}(p) \cup (A_3 - A_2)$.

$\subset N_{s_1}(p) \cup (A_2 - A_1) \cup (A_3 - A_2)$.

$= N_{s_1}(p) \cup (A_3 - A_1)$.

$\Rightarrow N_{s_1}(p) \subset N_{s_3}(p) \subset N_{s_1}(p) \cup (A_3 - A_1)$.

Therefore, $(A_1, s_1) \leq (A_3, s_3)$.

If $\mathcal{B} = \{(A_j, s_j) : j \in J\} \subset \mathcal{A}$ is totally ordered, then define (A, s) as follows,

(a) $A = \bigcup_{j \in J} Aj$.

(b) $N_s(x) = N_{s_j}(x)$ if $x \in Aj - \{p\}$.

(c) $N_s(p) = \bigcup_{j \in J} N_{s_j}(p)$.

Then $(A, s) \in \mathcal{A}$. For,

(i) Since $p \in A_j \subset Y$ for every j

we have $p \in A_j \in \bigcup_{j \in J} A_j \subset Y$.

$\Rightarrow p \in A \subset Y$.

(ii) For any $x \in A$, $x \in A_j$ for some $j \in J$.

$\Rightarrow$ x has a smallest neighbourhood.

$N_{s_j}(x) = N_s(x)$ if $x \in A_j - \{x\}$ and $N_{s_j}(p) \subset N_s(p)$ if $x = p$.

Therefore, $s \in \pi(A)$.

(iii) Since $s_j = (\eta|A_j)'$ for $j \in J$

we have $S = (\eta|A)'$.

Hence $(A,S) \in \mathcal{A}$.

Here (A,S) is an upper bound for $\mathcal{B}$. For,

1) $A_j \subset A$, for each $j \in J$

2) $N_{s_j}(x) = N_s(x)$ if $x \in A_j - \{P\}$ by (b)

3) $N_{s_j}(p) \subset N_s(p)$ by (c)

$$= N_{s_j}(p)$$

$$= N_{s_j}(p) \underset{k \neq j}{U} N_{s_k}(p)$$

$$= N_{s_j}(p) U (A-Aj)$$

Thus $(A_j, s_j) \leq (A,S)$ for each $j \in J$.

Hence by Zorn's lemma $\mathcal{A}$ has a maximal element, say, (M,m).

To prove $M = Y$.

If $M \neq Y$, let $q \in Y-M$. Take $M^* = M \, U\{q\}$.

Define a fuzzy topology m^* on M^* (By extending m to m^* on M^*) as follows:

$$N_{m^*}(p) = \begin{bmatrix} N_m(p) & \text{if } 1_m \text{ is not fuzzy open in } M^* \\ N_m(p) \; U \; 1_{\{q\}} & \text{if } 1_m \text{ is fuzzy open in } M^* \end{bmatrix}$$

$$N_{m^*}(q) = \begin{bmatrix} 1_{\{q\}} & \text{if } 1_{\{q\}} \text{ is not fuzzy open in } M^* \\ N_{m(q)} U 1_{\{q\}} & \text{if } 1_{\{q\}} \text{ is fuzzy open in } M^* \end{bmatrix}$$

Then we have,

1) $P \in M^* \subset Y$, since $p \in M$

2) for each $x \in M^*$, if $x \neq q$, x has a smallest fuzzy neighbourhood of the form$N_m(x)$ or $N_m(x) U 1_{\{q\}}$ and if $x = q$ then x has a

smallest neighbourhood of the form $1_{\{q\}}$ or $N_m(q) \cup 1_{\{q\}}$.

Hence $m^* \in \pi(M^*)$.

3) Since $m = (\eta|M)'$ and since the complement of $\{0_{\{q\}}, 1_{\{q\}}\}$ is itself, we obtain $m^* = (\eta|M)'$

Therefore $(M^*, m^*) \in \mathcal{A}$.

Also since

1) $M \subset M^*$.

2) $N_m(x) = N_{m^*}(x)$ for $x \in M - \{q\}$.

3) $N_m(q) \subset N_m(q) \cup 1_{\{q\}}$.

$= N_{m^*}(q)$.

$= N_{m^*}(q) \cup (M^* - M)$.

We obtain $(M,m) \leq (M^*,m^*)$. Which is a contradiction to the maximality of (M.m). Therefore we have $M = Y$.

This implies that $m \in \pi(M) = \pi(Y)$ and $m = (\eta|M)' = (\eta|Y)'$. Thus m is a fuzzy principal complement for the topology η of Y.

So $m \vee \eta = D$ (the discrete fuzzy topology on Y), and

$m \wedge \eta = ID$ (the indiscrete fuzzy topology on Y).

Since m is a fuzzy principal complement for η and since both 1_X and $1_{\{q\}}$ are fuzzy η-open, $m|x$ is a fuzzy principal complement of τ, that is,

$m|X \vee \tau = D$ (the discrete fuzzy topology on X).

$m|X \wedge \tau = ID$ (the indiscrete fuzzy topology on X).

Hence $\tau' = m|x$ which is a fuzzy principal topology on X.

Hence the proof.

Chapter II

CHAPTER – II

STRONGLY COMPACT FUZZY TOPOLOGICAL SPACES

DEFINITION: 2.1

A fuzzy subset A of a fuzzy topological space (x,τ) is said to be pre-open if $A \subset Int(ClA)$.

DIFINITION: 2.2

A fuzzy subset A of a fuzzy topological space (X,τ) is said to be semi-open if $A \subset Cl(Int(A))$.

NOTATION:

The set of all fuzzy pre-open subsets of X is denoted by $FP_rO(X)$, and the set of all fuzzy semi-open sets in X is denoted by FSO(X).

DEFINITION: 2.3

A fuzzy topological space (X,τ) is said to be strongly compact if every pre-open cover of X has a finite subcover.

DEFINITION: 2.4

Let (X,τ) and (Y,η) be fuzzy topological spaces. A mapping $f:X \to Y$ is called fuzzy continuous if $f^{-1}(U) \in \tau$ for each $U \in \eta$.

DIFINITION: 2.5

Let (X,τ) and (Y,η) be fuzzy topological spaces. A mapping $f:X \to Y$ is called pre-continuous if the inverse image of each fuzzy open set in Y is pre-open in X.

DIFINITION: 2.6

Let (X,τ) and (Y,η) be fuzzy topological spaces. A mapping $f:X\to Y$ is called M-pre-continuous if the inverse image of every fuzzy pre-open set in Y is pre-open in X.

DEFINITION: 2.7

Let (X,τ) and (Y,η) be fuzzy topological spaces. A mapping $f:X\to Y$ is called M-preclosed if the image of each fuzzy pre-closed set in X is pre-closed in Y.

DEFINITION: 2.8

Let (X,τ) and (Y,η) be fuzzy topological spaces and let $\tau_\emptyset$ be a fuzzy topology on X which has $FP_rO(X)$ as a subbase. A mapping $f:X\to Y$ is called $\emptyset$-continuous if $f:(X,\tau_\emptyset)\to(Y,\eta)$ is continuous. f is said to be $\emptyset'$ - continuous if $f:(X,\tau_\emptyset)\to(Y,\eta_\emptyset)$ is continuous.

THEOREM: 2.9

A fuzzy topological space X is strongly compact if and only if every family of fuzzy pre-closed subsets of X with finite intersection property has nonempty intersection.

THEOREM: 2.10

Let (X,τ) and (Y,η) be fuzzy topological spaces and let $\tau_\emptyset$ be a fuzzy topology on X which has $FP_rO(X)$ as a subbase. If $f:(X,\tau)\to(Y,\eta)$ is pre-continuous then f is $\emptyset$-continuous.

PROOF:

Assume that $f: (X,\tau)\to(Y,\eta)$ is pre-continuous.

Then for each fuzzy open set U in Y,

$f^{-1}(U)$ is fuzzy pre-open set in X,

$f^{-1}(U)$ is fuzzy pre-open set in X,

$\Rightarrow f^{-1}(U) \in FP_rO(X)$.

But $\tau_\varnothing$ has $FP_rO(X)$ as a subbase.

Therefore, $f^{-1}(U) \in \tau_\varnothing$

$\Rightarrow$ f is $\varnothing$-continuous.

THEOREM: 2.11

Let (X,τ) and (Y,η) be fuzzy topological spaces. Let $\tau_\varnothing$ and $\eta_\varnothing$ be respectively the fuzzy topologies on X and Y which has $FP_rO(X)$ and $FP_rO(Y)$ as subbases. If $f:(X,\tau)\to(Y,\eta)$ is M-pre-continuous, then f is $\varnothing'$-continuous.

PROOF:

Assume that $f:(X,\tau)\to(Y,\eta)$ is M-pre-continuous.

Let $U \in \eta_\varnothing$.

Then $U = \bigcup_j \left[\bigcap_{j=1}^{n} \eta_{jn_j}\right]$ where $\eta_{jn_j} \in FP_rO(Y,\eta)$.

Consider $f^{-1}(U) = f^{-1}\left[\bigcup_j \bigcap_{j=1}^{n} \eta_{jn_j}\right]$.

$$= \bigcup_j \left[f^{-1}\left(\bigcap_{j=1}^{n} \eta_{jn_j}\right)\right]$$

$$= \bigcup_j \left[\bigcap_{j=1}^{n} \left(f^{-1}(\eta_{jn_j})\right)\right].$$

But $f^{-1}(\eta_{jn_j}) \in FP_rO(X,\tau)$

Hence $f^{-1}(U) \in \tau_\varnothing$

$\therefore$ f is $\varnothing'$ continuous.

THEOREM: 2.12

Let (X,τ) be a fuzzy topological space and let $\tau_{\emptyset}$ be a fuzzy topology on X which has $FP_rO(X)$ as a subbase. Then (X,τ) is strongly compact if any only if $(X,\tau_{\emptyset})$ is compact.

NOTE:

The converse is a consequence of Alexander's subbase theorem for fuzzy topological spaces, which can be obtained in a manner similar to the corresponding result for the non-fuzzy case.

THEOREM: 2.13

Let (X,τ) be a fuzzy topological space which is strongly compact. Then each $\tau_{\emptyset}$-fuzzy closed set in X is strongly compact.

PROOF:

Let V be any $\tau_{\emptyset}$-fuzzy closed set in X. And let $\{U_{\alpha_j}\}_{\alpha_j \in J}$ be a $\tau_{\emptyset}$- open cover of V.

Since X-V is $\tau_{\emptyset}$-open, $\{U\alpha_j : \alpha_j \in J\}$ **U**$(X-V)$ is a $\tau_{\emptyset}$-open cover of X. Since $(X,\tau_{\emptyset})$ is compact, there exists a finite subset $J_o \subset J$ such that $X=\mathbf{U}\{U_{\alpha_j} : \alpha_j \in Jo\}\mathbf{U}(X-V)$

$\Rightarrow V \subset \mathbf{U}\{U_{\alpha_j} : \alpha_j \in Jo)$

That is, there exists a finite subcollection which covers V.

Hence V is strongly compact relative to X.

THEOREM: 2.14

Let the fuzzy topological space (X,τ) be strongly compact. Then every family of $\tau_{\emptyset}$-fuzzy closed subsets of X with finite intersection property has nonempty intersection.

PROOF:

Let X be strongly compact and let $V=\{B_{\alpha_j}:\alpha_j\in J\}$ be any family of $\tau_{\emptyset}$-fuzzy closed subsets of X with finite intersection property. Suppose

$\cap\{B_{\alpha_j}:\alpha_j\in J\}=\emptyset$

Then $\{X-B_{\alpha_j}:\alpha_j\in J\}$ is a $\tau_{\emptyset}$-open cover of X

Hence it has a finite subcover

$\{X-B_{\alpha_{j_k}}:k=1,2,\ldots\ldots..n)$ for X.

$\Rightarrow \cap\{B_{\alpha_{j_k}}: K=1,2.\ldots.n\} = \emptyset.$

Which contradicts the assumption that V has finite intersection property. Hence

$\cap\{B_{\alpha_j} \mid \alpha_j\in J\}\neq\emptyset.$

THEOREM: 2.15

Let (X,τ) and (Y,η) be fuzzy topological spaces and let $f:X\to Y$ be $\emptyset'$-continuous. If a fuzzy subset G of X is strongly compact relative to X, then f(G) is strongly compact relative to Y.

PROOF:

Let $\{U_{\alpha_j} : \alpha_j\in J\}$ be a cover of f(G) by $\eta_{\emptyset}$-open fuzzy sets in Y. Since f is ϕ'-continuous, $\{f^{-1}(U_{\alpha_j}) : \alpha_j \in J\}$ is a cover of G by τ_ϕ-fuzzy open sets in X.

Since G is strongly compact relative to X, by theorem 2.12, G is τ_ϕ-compact. So there exists a finite subset $J_o \subset J$ such that $G \subset \cup\{f^{-1}(U_{\alpha_j} : \alpha_j \in Jo\}$.

$\Rightarrow f(G) \subset \{U_{\alpha_j} : \alpha_j \in J_o\}$.

$\Rightarrow f(G)$ is τ_ϕ-compact relative to Y.

Thus f(G) is strongly compact relative to X.

THEOREM: 2.16

Let A and B be fuzzy subsets of a fuzzy topologial space (X,τ) such that A is strongly compact relative to X and B is τ_ϕ-closed in X. Then $A \cap B$ is strongly compact relative to X.

PROOF:

Let $\{U_{\alpha_j} : \alpha_j \in J\}$ be a cover of $A \cap B$ by τ_ϕ-fuzzy open subsets of X.Since X-B is a τ_ϕ-fuzzy open set, $\{U_{\alpha_j} : \alpha_j \in J\} \cup (X-B)$ is a cover of A.

A is strongly compact relative to X and thus A is τ_ϕ-compact relative to X. Hence there exists a finite subset $J_o \subseteq J$ such that $A \subset (\cup\{U_{\alpha_j} : \alpha_j \in Jo\}) \cup (X-B)$

Therefore,

$A \cap B \subset \cup\{U_{\alpha_j} : \alpha_j \in Jo\}$

This is the required subcover for $A \cap B$.Hence $A \cap B$ is τ_ϕ-compact.

Thus $A \cap B$ is strongly compact relative to X.

Chapter III

CHAPTER – III

FUZZY BITOPOLOGICAL SPACES

SECTION: 1

SEMI-OPEN SETS, SEMI-CONTINUITY AND SEMI-OPEN MAPPINGS IN FUZZY BITOPOLOGICAL SPACES.

DEFINITION: 3.1.1

A system (X,τ_1,τ_2) consisting of a set X with two fuzzy topologies τ_1 and τ_2 on X is called a fuzzy bitopological space (briefly, fbts).

NOTATION: In this chapter, λ, μ, ν, etc denote fuzzy sets. τ_i-Int(λ) and τ_j-Cl(λ) mean the interior and closure of a fuzzy set λ with respect to the fuzzy topologies τ_i and τ_j in a fuzzy bitopological space. Throughout this chapter, the indices i and j take values in {1,2} and $i \neq j$. Also i=j gives the known results in fuzzy topological spaces.

DIFINITION: 3.1.2

Let λ be a fuzzy set of a fbts X. λ is called

(i) a (τ_i,τ_j)-fuzzy semi-open (briefly, (τ_i,τ_j)-fso) set of X if there exists a $\nu \in \tau_i$ such that $\nu \leq \lambda \leq \tau_j\text{-Cl}(\nu)$, and

(ii) a (τ_i,τ_j)- fuzzy semi-closed (briefly, (τ_i,τ_j)-fsc) set of X if there exists a $\nu' \in \tau_i$ such that $\tau_j\text{-Int}(\nu) \leq \lambda \leq \nu$.

NOTE:

The set of all (τ_i,τ_j)-fso(resp.(τ_i,τ_j)- fsc) sets of a fbts X will be denoted by (τ_i,τ_j)-**FSO**(X) (resp. (τ_i,τ_j)-FSC(X)).

THEOREM: 3.1.3

Let λ be a fuzzy set of a fbts X. Then the following are equivalent:

(i) λ is a (τ_i,τ_j)-fsc set,

(ii) λ' is a (τ_i,τ_j)-fso set,

(iii) $\tau_j\text{-Int}(\tau_i\text{-Cl}(\lambda)) \leq \lambda$,

(iv) $\tau_j\text{-Cl}(\tau_i\text{-Int}(\lambda)) \geq \lambda'$.

PROOF:

To prove (i)=>(ii).

Assume that λ is a (τ_i,τ_j)-fsc set.

Then there exists a $\nu' \in \tau_i$ such that $\tau_j\text{-Int}(\nu) \leq \lambda \leq \nu$.

=> there exists a $\nu' \in \tau_i$ such that

$1-\nu \leq 1-\lambda \leq 1-(\tau_j\text{-Int}(\nu))$.

=> there exists a $\nu' \in \tau_i$ such that $\nu' \leq \lambda' \leq \tau_j\text{-Cl}(\nu')$.

=> λ' is a (τ_i,τ_j)-fso set.

To prove (ii)=>(i)

Assume that λ' is a (τ_i,τ_j)-fso set.

Then there exists a $\nu \in \tau_i$, such that $\nu \leq \lambda' \leq \tau_j\text{-Cl}(\nu)$.

=> there exists a $(\nu')' \in \tau_i$ such that

$1-(\tau_j\text{-Cl}(\nu)) \leq 1-\lambda' \leq 1-\nu$.

=> there exists a $(\nu')' \in \tau_i$ such that

$\tau_j\text{-Int}(1-\nu) \leq \lambda \leq \nu'$.

=> there exists a $(\nu')' \in \tau_i$ such that $\tau_j\text{-Int}(\nu') \leq \lambda \leq \nu'$.

$\Rightarrow \lambda$ is a (τ_i, τ_j)-fsc set.

To prove (i) $\Rightarrow$(iii).

Assume that λ is a (τ_i, τ_j)-fsc set.

Then there exists a $\nu' \in \tau_i$ such that $\tau_j\text{-Int}(\nu) \leq \lambda \leq \nu$.

$\Rightarrow \tau_j\text{-Int}(\nu) \leq \lambda \leq \tau_i\text{-Cl}(\lambda) \leq \nu$.

$\Rightarrow \tau_j\text{-Int}(\tau_i\text{-Cl}(\lambda)) \leq \tau_j\text{-Int}(\nu)$.

$\Rightarrow \tau_j\text{-Int}(\tau_i\text{-Cl}(\lambda)) \leq \lambda$.

To prove (iii)$\Rightarrow$(i).

Assume that $\tau_j\text{-Int}(\tau_i\text{-Cl}(\lambda)) \leq \lambda$.

If $\nu = \tau_i\text{-Cl}(\lambda)$ then $\nu' \in \tau_i$.

Hence there exists a $\nu' \in \tau_i$ such that $\tau_j\text{-Int}(\nu) \leq \lambda \leq \nu$.

$\Rightarrow \lambda$ is a (τ_i, τ_j)-fsc set.

To prove (ii)$\Rightarrow$(iv).

Assume that λ' is a (τ_i, τ_j)-fso set.

Then there exists a $\nu \in \tau_i$ such that $\nu \leq \lambda' \leq \tau_j\text{-Cl}(\nu)$.

Since $\nu \leq \tau_i\text{-Int}(\lambda)$ we have $\lambda' \leq \tau_j\text{-Cl}(\tau_i\text{-Int}(\lambda))$.

THEOREM: 3.1.4

(i) Any union of (τ_i, τ_j)-fso sets is a (τ_i, τ_j)-fso set, and

(ii) Any intersection of (τ_i, τ_j)-fsc sets is a (τ_i, τ_j)-fsc set.

PROOF:

(i) Let X be a fbts and let $\{\lambda_\alpha : \alpha \in A\}$ be a collection of (τ_i, τ_j)-fso sets of X.

Each λ_α is a (τ_i, τ_j)-fso sets of X.

$\Rightarrow$ there exists a $V_\alpha \in \tau_i$ such that $V_\alpha \leq \lambda_\alpha \leq \tau_j\text{-Cl}(V_\alpha)$ for each α.

Then $\cup V_\alpha \leq \cup \lambda_\alpha \leq \cup \tau_j\text{-Cl}(V_\alpha) \leq \tau_j\text{-Cl}(\cup V_\alpha)$.

$\Rightarrow U\lambda_\alpha$ is a (τ_i,τ_j)-fso set of X.

(ii) If $\{\lambda_\alpha : \alpha \in \mathcal{A}\}$ is a collection of (τ_i,τ_j)-fso sets of X.

Then $\{\lambda_\alpha' : \alpha \in \mathcal{A}\}$ is a collection of (τ_i,τ_j)-fsc sets of X.

$\Rightarrow U\lambda_\alpha$ is a (τ_i,τ_j)-fso sets of X.

$\Rightarrow (U\lambda_\alpha)'$ is a (τ_i,τ_j)-fsc sets of X.

$\Rightarrow \cap \lambda_\alpha'$ is a (τ_i,τ_j)-fsc set of X.

REMARK:

1. Every τ_i-fuzzy open, briefly, τ_i-fo(resp. τ_i-fuzzy closed, briefly, τ_i-fc)set is a (τ_i,τ_j)-fso (resp. (τ_i,τ_j)-fsc)set.
2. Every (τ_i,τ_j)-fso (resp. (τ_i,τ_j)-fsc) set need not be a τ_i-fo(resp. τ_i-fc)set.
3. The intersection (resp. Union) of any two (τ_i,τ_j)-fso(resp. (τ_i,τ_j)-fsc) sets need not be a (τ_i,τ_j)-fso(resp. (τ_i,τ_j)-fsc) set.
4. The intersection (resp. Union) of a (τ_i,τ_j)-fso (resp. (τ_i,τ_j)-fsc)set with a (τ_i,τ_j)-fso(resp. (τ_i,τ_j)-fsc) set.

The following counter example proves the above statements.

EXAMPLE: 3.1.5

Let $X = \{a,b,c\}$ and let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 be fuzzy sets of X defined by,

$\lambda_1(a) = 0, \quad \lambda_1(b) = 0.3, \quad \lambda_1(c) = 0.2,$

$\lambda_2(a) = 0.2, \lambda_2(b) = 0.4, \quad \lambda_2(c) = 0.3,$

$\lambda_3(a) = 0.3, \lambda_3(b) = 0.5, \quad \lambda_3(c) = 0.3,$

$\lambda_4(a) = 0, \quad \lambda_4(b) = 0.2, \quad \lambda_4(c) = 0.4,$

$\lambda_5(a) = 0.3, \lambda_5(b) = 0.4, \quad \lambda_5(c) = 0.4,$

$\lambda_6(a) = 0.2, \lambda_6(b) = 0.3, \quad \lambda_6(c) = 0.6.$

Let $\tau_1=\{0,\lambda_1,\lambda_5,1\}$and $\tau_2\{0,\lambda_2,\lambda_3,1\}$ be fuzzy topologies on X. Then λ_2,λ_3 and λ_6 are (τ_1,τ_2)-fso sets which are not τ_1-fo sets and by theorem 3.1.3,λ_2',λ_3' and λ_6' are (τ_1,τ_2)-fsc sets which are not τ_1-fc sets in (X,τ_1,τ_2).

Further, λ_5 is a τ_1-fo set and λ_3 is a (τ_1,τ_2)-fso set but $\lambda_3 \cap \lambda_5$ need not be a (τ_1,τ_2)-fso set. Also, λ_2 and λ_6 are (τ_1,τ_2) - fso sets but $\lambda_2 \cap \lambda_6$ is not a (τ_1,τ_2)-fso set and $\lambda_2' \cup \lambda_6'$ is not a (τ_1,τ_2)-fsc set in (X,τ_1,τ_2).

Moreover, λ_5 is a (τ_2,τ_1)-fso set but not a τ_2 - fo set. Further,$\lambda_3 \cap \lambda_5$ need not be a (τ_2,τ_1)-fso set and $\lambda_3' \cup \lambda_5'$ is not a (τ_2,τ_1)-fsc set.

THEOREM:3.1.6

Let λ and μ be two fuzzy sets of a fbts X.

(i) If λ is (τ_i,τ_j)-fso and $\tau_i\text{-Int}(\lambda) \leq \mu \leq \tau_j\text{-Cl}(\lambda)$, then μ is (τ_i,τ_j)-fso.

(ii) If λ is (τ_i,τ_j)-fsc and $\tau_j\text{-Int}(\lambda) \leq \mu \leq \tau_i\text{-Cl}(\lambda)$, then

μ is (τ_i,τ_j)-fsc.

PROOF:

(i) Let λ be a (τ_i,τ_j)-fso set, and $\tau_i\text{-Int}(\lambda) \le \mu \le \tau_j\text{-Cl}(\lambda)$.

=> there exists a $\nu \in \tau_i$ such that $\nu \le \lambda \le \tau_j\text{-Cl}(\nu)$.

=> there exists a $\nu \in \tau_i$ such that $\nu \le \tau_i\text{-Int}(\lambda) \le \lambda \le \tau_j\text{-Cl}(\lambda) \le \tau_j\text{-Cl}(\nu)$.

=> there exists a $\nu \in \tau_i$ such that $\nu \le \tau_i\text{-Int}(\lambda) \le \mu \le \tau_j\text{-Cl}(\lambda) \le \tau_j\text{-Cl}(\nu)$.

=> there exists a $\nu \in \tau_i$ such that $\nu \le \mu \le \tau_j\text{-Cl}(\nu)$.

=> μ is a (τ_i,τ_j)-fso set.

(ii) Let λ be a (τ_i,τ_j) - fsc, and $\tau_j\text{-Int}(\lambda) \le \mu \le \tau_i\text{-Cl}(\lambda)$.

=> there exists a $\nu' \in \tau_i$ such that $\tau_j\text{-Int}(\nu) \le \lambda \le \nu$.

=> there exists a $\nu' \in \tau_i$ such that $\tau_j\text{-Int}(\nu) \le \tau_j\text{-Int}(\lambda) \le \lambda \le \nu$.

=> there exists a $\nu' \in \tau_i$ such that $\tau_j\text{-Int}(\nu) \le \tau_j\text{-Int}(\lambda) \le \lambda \le \tau_i\text{-Cl}(\lambda) \le \nu$.

=> there exists a $\nu' \in \tau_i$ such that $\tau_j\text{-Int}(\nu) \le \mu \le \tau_i\text{-Cl}(\lambda) \le \nu$.

=> there exists a $\nu' \in \tau_i$ such that $\tau_j\text{-Int}(\nu) \le \mu \le \nu$.

=> μ is a (τ_i,τ_j)-fsc set.

THEOREM: 3.1.7

Let λ be a fuzzy set of a fbts X. λ is a (τ_i,τ_j)-fso set if and only if for every fuzzy point $x_\alpha \in \lambda$ there exists a (τ_i,τ_j)-fso set ν_{x_α} such that $x_\alpha \in \nu_{x_\alpha} \leq \lambda$.

PROOF:

Assume that λ is a (τ_i,τ_j)-fso set.

Then there exists a $\nu \in \tau_i$ such that $\nu \leq \lambda \leq \tau_j\text{-Cl}(\nu)$.

Take $\lambda = \nu_{x_\alpha}$ for every $x_\alpha \in \lambda$.

$\Rightarrow \nu_{x_\alpha}$ is a (τ_i,τ_j)-fso set such that $x_\alpha \in \nu_{x_\alpha} = \lambda$.

Conversely, assume that for every fuzzy point $x_\alpha \in \lambda$, there exists a (τ_i,τ_j)-fso set ν_x such that $x_\alpha \in \nu_{x_\alpha} \leq \lambda$.

Then $\lambda = \{\cup x_\alpha : x_\alpha \in \lambda\} \leq \cup\{\nu_{x_\alpha} : x_\alpha \in \lambda\} \leq \lambda$.

$\Rightarrow \lambda = \cup\{\nu_{x_\alpha} : x_\alpha \in \lambda\}$.

$\Rightarrow \lambda$ is the union of (τ_i,τ_j)-fso sets of X.

$\Rightarrow \lambda$ is a (τ_i,τ_j)-fso set of X.

DEFINITION:3.1.8

Let λ be a fuzzy set of a fbts X.

(i) The (τ_i,τ_j)-semi-interior of λ, $(\tau_i,\tau_j)\text{-sInt}(\lambda)$, is defined by $(\tau_i,\tau_j)\text{-sInt}(\lambda)=\text{Sup}\{\nu : \nu \leq \lambda, \nu \text{ is } (\tau_i,\tau_j)\text{-fso}\}$, and

(ii) The (τ_i,τ_j)-semi-closure of λ, $(\tau_i,\tau_j)\text{-sCl}(\lambda)$, is defined by $(\tau_i,\tau_j)\text{-sCl}(\lambda)=\text{Inf}\{\nu : \nu \geq \lambda, \nu \text{ is } (\tau_i,\tau_j)\text{-fsc}\}$.

We have obtained the following results by using the above definitions directly.

RESULT (1):

Let λ be a fuzzy set of a fbts (X,τ_i,τ_j). Then $\tau_i\text{-Int}(\lambda) \leq (\tau_i,\tau_j)\text{-sInt}(\lambda) \leq \lambda \leq (\tau_i,\tau_j)\text{-sCl}(\lambda) \leq \tau_i\text{-Cl}(\lambda)$.

RESULT (2):

Let λ and μ be two fuzzy sets of a fbts X.

Then(i) If $\lambda \leq \mu$, $(\tau_i,\tau_j)\text{-sInt}(\lambda) \leq (\tau_i,\tau_j)\text{-sInt}(\mu)$,

(ii) If $\lambda \leq \mu$, $(\tau_i,\tau_j)\text{-sCl}(\lambda) \leq (\tau_i,\tau_j)\text{-sCl}(\mu)$,

(iii) $(\tau_i,\tau_j)\text{-sInt}(\lambda)$ is a (τ_i,τ_j)-fso set,

(iv) $(\tau_i,\tau_j)\text{-sCl}(\lambda)$ is a (τ_i,τ_j)-fsc set.

RESULT (3):

Let λ be a fuzzy set of a fbts X. Then

(i) λ is (τ_i,τ_j)-fso if and only if $\lambda = (\tau_i,\tau_j)\text{-sInt}(\lambda)$, and

(ii) λ is (τ_i,τ_j)-fsc if and only if $\lambda = (\tau_i,\tau_j)\text{-sCl}(\lambda)$.

Next, the author introduces the concepts of fuzzy pairwise semi-continuous, fuzzy pairwise semi-open (semi-closed) briefly, fpsc, fps open (fps closed), mappings by using (τ_i,τ_j)-fso and (τ_i,τ_j)-fsc sets and studies some of their basic properties. Several characterisations of these mappings are also obatined.

DEFINITION: 3.1.9

Let $f:X \to Y$ be a mapping from a fbts X to another fbts Y. f is called

(i) a fpsc mapping, if $f^{-1}(\lambda)$ is a (τ_i,τ_j)-fso set of X for set of X for each η_i-fo set λ of Y.

(ii) a fps open (fps closed) mapping, if $f(\lambda)$ is a (η_i,η_j)-fso(resp. (η_i,η_j)-fsc) set of Y for each τ_i-fo (resp. τ_i-fc) set λ of X.

DEFINITION: 3.1.10

Let f:X->Y be a mapping from a fbts X to another fbts Y. f is called a fuzzy pairwise continuous(resp. fuzzy pairwise open and fuzzy pairwise closed), briefly, fpc (fp open and fp closed) mapping if and only if the induced mappings $f:(X,\tau_k)\to(Y,\eta_k)$ $(k=1,2)$ are fuzzy continuous (resp. fuzzy open and fuzzy closed).

REMARK:

Every fpc(resp. fp open and fp closed) mapping is fpsc(resp. fps open and fps closed). The following examples show that the converse need not be true.

EXAMPLE: 3.1.11

Let X=Y=I and let λ_1, λ_2 and λ_3 be fuzzy sets of I defined as follows:

$$\lambda_1(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ 2x-1, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$\lambda_2(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{4}, \\ -4x+2, & \frac{1}{4} \le x \le \frac{1}{2}, \\ 0, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$\lambda_3(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{4}, \\ 1/3(4x-1), & \frac{1}{4} \le x \le 1. \end{cases}$$

Consider the fuzzy topologies

$\tau_1=\{0,\lambda_1, \lambda_2, \lambda_1 \cup \lambda_2,1\}$, $\tau_2=\{0, \lambda_2,1\}$ $\eta_1=\{0,\lambda_3,1\}$

and $\eta_2=\{0,\lambda_1',1\}$ and the mapping $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ defined by $f(x)=x$. Then f is a fpsc mapping but f is not fpc.

EXAMPLE: 3.1.12

Let X, λ_1, λ_2, λ_3, λ_4, λ_5 and λ_6 be described as in example 3.1.5. Consider the fuzzy topologies
$\tau_1=\{0,\lambda_2,\lambda_6,\lambda_2 \cup \lambda_6,\lambda_2 \cap \lambda_6,1\}$,
$\tau_2=\{0,\lambda_3,\lambda_5,\lambda_3 \cup \lambda_5,\lambda_3 \cap \lambda_5,1\}$, $\eta_1=\{0,\lambda_1,\lambda_3,1\}$ and $\eta_2=\{0,\lambda_1,\lambda_2,1\}$ and the mapping $f:(X,\tau_1,\tau_2)\to(X,\eta_1,\eta_2)$ defined by $f(x)=x$ for each $x \in X$. Then f is a fps open mapping but not fp open mapping. The characteristic properties of fpsc, fps open, fps closed mappings in fuzzy bitopological spaces are discussed in the following theorems.

THEOREM: 3.1.13

A mapping $f:X\to Y$ is fpsc iff for any fuzzy point x_α of X and any η_i-fo set μ in Y with $f(x_\alpha)\in\mu$, there exists a $\lambda\in(\tau_i,\tau_j)$-FSO(x) such that $x_\alpha\in\lambda$ and $f(\lambda)\le\mu$.

THEOREM: 3.1.14

A mapping $f:X\to Y$ is fpsc iff for every fuzzy set is X, $f((\tau_i,\tau_j)\text{-sCl}(\lambda))\le\eta_i\text{-Cl}(f(\lambda))$.

PROOF:

Assume that f is fpsc.

Let λ be any fuzzy set in X.

Then $f^{-1}(\eta_i\text{-Cl}(f(\lambda)))$ is a (τ_i,τ_j)-fsc set in X.

Further more, $\lambda\le f^{-1}(f(\lambda))\le f^{-1}(\eta_i\text{-Cl}(f(\lambda)))$.

Therefore, $(\tau_i,\tau_j)\text{-sCl}(\lambda) \leq f^{-1}(\eta_i\text{-Cl}(f(\lambda)))$.

$\Rightarrow f((\tau_i,\tau_j)\text{-sCl}(\lambda)) \leq f(f^{-1}(\eta_i\text{-Cl}(f(\lambda))))$.

$\Rightarrow f((\tau_i,\tau_j)\text{-sCl}(\lambda)) \leq \eta_i\text{-Cl}(f(\lambda))$.

Conversely, assume that for every fuzzy set λ in X, $f((\tau_i,\tau_j)\text{-sCl}(\lambda)) \leq \eta_i\text{-Cl}(f(\lambda))$. Let μ be any η_i-fc set in Y. Then $f((\tau_i,\tau_j)\text{-sCl}(f^{-1}(\mu))) \leq \eta_i\text{-Cl}(f(f^{-1}(\mu)))$.

$\Rightarrow f((\tau_i,\tau_j)\text{-sCl}(f^{-1}(\mu))) \leq \eta_i\text{-Cl}(\mu)$.

$\Rightarrow (\tau_i,\tau_j)\text{-sCl}(f^{-1}(\mu)) \leq f^{-1}(\eta_i\text{-Cl}(\mu))=f^{-1}(\mu)$.

Hence $f^{-1}(\mu)$ is (τ_i,τ_j)-fsc set in X.

Therefore, f is fpsc.

The following characterizations of fpsc mappings are easily verified for an ordered pair $i \neq j$ and $i,j = 1,2$.

THEOREM: 3.1.15

A mapping $f:X\to Y$ is fpsc iff for every fuzzy set μ in Y, $(\tau_i,\tau_j)\text{-sCl}(f^{-1}(\mu)) \leq f^{-1}(\eta_i\text{-Cl}(\mu))$.

THEOREM: 3.1.16

A mapping $f:X\to Y$ is fpsc iff for each fuzzy set μ in Y, $f^{-1}(\eta_i\text{-Int}(\mu)) \leq (\tau_i,\tau_j)\text{-sInt}(f^{-1}(\mu))$.

THEOREM: 3.1.17

Let $f:X\to Y$ be a one-one and onto mapping. f is fpsc iff for every fuzzy set λ of X,

$\eta_i\text{-Int}(f(\lambda)) \leq f((\tau_i,\tau_j)\text{-sInt}(\lambda))$.

THEOREM:3.1.18

For a mapping $f:X\to Y$ the following statements are equivalent:

(i) f is fps closed,

(ii) $f(\tau_i\text{-Cl}(\lambda)) \geq \eta_j\text{-Int}(\eta_i\text{-Cl}(f(\lambda)))$, for any fuzzy set λ

of X,

(iii) (η_i,η_j)-sCl$(f(\lambda)) \le f(\tau_i$-Cl$(\lambda))$, for any fuzzy set of X.

PROOF:

To prove (i) =>(ii)

Assume that f is fps closed.

Let λ be any fuzzy set of X.

Then $f(\tau_i$-Cl$(\lambda))$ is a (η_i,η_j)-fsc set of Y.

=> $f(\tau_i$-Cl$(\lambda)) \ge \eta_j$-Int$(\eta_i$-Cl$(f(\tau_i$-Cl$(\lambda))))$.(by theorem 3.1.3)

=> $f(\tau_i$-Cl$(\lambda)) \ge \eta_j$-Int$(\eta_i$-Cl$(f(\lambda)))$.

To prove (ii) =>(i)

Assume that statement (ii) is true.

Let λ be any τ_i-fc set of X.

Then η_j-Int$(\eta_i$-Cl$(f(\lambda))) \le f(\tau_i$-Cl$(\lambda)) = f(\lambda)$

=> $f(\lambda)$ is (η_i,η_j)-fsc set of Y.(by theorem 3.1.3)

=> f is a fps closed mapping.

To prove (i)=>(iii)

Assume that f is fps closed.

Let λ be any fuzzy set of X.

Then $f(\tau_i$-Cl$(\lambda))$ is a (η_i,η_j)-fsc set of Y.

Since $f(\lambda) \le f(\tau_i$-Cl$(\lambda))$, we have

(η_i,η_j)-sCl$(f(\lambda)) \le f(\tau_i$-Cl$(\lambda))$.

To prove (iii) =>(i)

Assume that statement (iii) is true.

Let λ be any τ_i-fc set of X. Then

$f(\lambda) \le (\eta_i,\eta_j)$ - sCl$(f(\lambda)) \le f(\tau_i$-Cl$(\lambda)) = f(\lambda)$.

=> $f(\lambda) = (\eta_i,\eta_j)\text{-sCl}(f(\lambda))$.

=> $f(\lambda)$ is a (η_i,η_j)-fsc set of Y.

=> f is fps closed mapping.

COROLLARY:

For a mapping $f:X \rightarrow Y$ the following statements are equivalent.

(i) f is fps open,

(ii) $f(\tau_i\text{-Int}(\lambda)) \leq (\eta_i,\eta_j)\text{-sInt}(f(\lambda))$ for each fuzzy set λ of X,

(iii) $\tau_i\text{-Int}(f^{-1}(\mu)) \leq f^{-1}((\eta_i,\eta_j)\text{-sInt}(\mu))$ for each fuzzy set μ of Y.

THEOREM: 3.1.19

For a mapping $f:X \rightarrow Y$ the following statements are equivalent:

(i) f is fpsc,

(ii) for every fuzzy point x_α in X and for every $\mu \in \mathcal{N}_{\eta_i}(f(x_\alpha))$, there exists a $\lambda \in (\tau_i,\tau_j)$-FSO (X) such that $x_\alpha \in \lambda$ and $\lambda \leq f^{-1}(\mu)$,

(iii) for every x_α in X and for every $\mu \in \mathcal{N}_{\eta_i}(f(x_\alpha))$, there exists a $\lambda \in (\tau_i,\tau_j)$-FSO (X) such that $x_\alpha \in \lambda$ and $f(\lambda) \leq \mu$,

(iv) for every η_i-fo set μ of Y,
$f^{-1}(\mu) \leq \tau_j\text{-Cl}(\tau_i\text{-Int}(f^{-1}(\mu)))$,

(v) for every η_i-fc set γ of Y, $f^{-1}(\gamma) \in (\tau_i,\tau_j)$-FSC(X),

(vi) for every η_i-fc set γ of Y,
$f^{-1}(\gamma) \geq \tau_j\text{-Int}(\tau_i\text{-Cl}(f^{-1}(\gamma)))$.

To prove (i)==>(ii)

Assume that f is fpsc.

Let x_α be any fuzzy point in X and let $\mu \in \mathcal{N}_{\eta_i}(f(x_\alpha))$.

Then there exists a $\gamma \in \eta_i$ such that $f(x_\alpha) \in \gamma \leq \mu$.

=> $f^{-1}(\gamma) \in (\tau_i,\tau_j)$-FSO(X)

=> $x_\alpha \in f^{-1}(\gamma) = \lambda \leq f^{-1}(\mu)$.

To prove(ii)=>(iii)

Assume that statement (ii) is true.

Let $x_\alpha \in X$ and let $\mu \in \mathcal{N}_{\eta_i}(f(x_\alpha))$. Then there exists a $\lambda \in (\tau_i,\tau_j)$-FSO(X) such that $x_\alpha \in \lambda$ and $\lambda \leq f^{-1}(\mu)$.

=> $x_\alpha \in \lambda$ and $f(\lambda) \leq f(f^{-1}(\mu)) \leq \mu$.

To prove(iii)=>(i)

Assume that statement (iii) is true.

Let μ be any η_i-fo set of Y.

Consider $f^{-1}(\mu)$ and let $x_\alpha \in f^{-1}(\mu)$.

Then $f(x_\alpha) \in f(f^{-1}(\mu) \leq \mu$.

Since $\mu \in \mathcal{N}_{\eta_i}(f(x_\alpha))$, there exists a $\lambda \in (\tau_i,\tau_j)$-FSO(X) such that $x_\alpha \in \lambda$ and $f(\lambda) \leq \mu$.

=> $x_\alpha \in \lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(\mu)$.

=> $f^{-1}(\mu)$ is a (τ_i,τ_j)-fsc set (by theorem 3.1.7).

To prove(i)=>(iv)

Assume that f is fpsc.

Let μ be any η_i-fo set of Y.

Then $f^{-1}(\mu)$ is a (τ_i,τ_j)-fso set of X.

$\Rightarrow f^{-1}(\mu) \leq \tau_j\text{-}Cl(\tau_i\text{-}Int(f^{-1}(\mu)))$ (by theorem 3.1.3)

To prove (iv)=>(i)

Assume that statement (iv) is true.

Let μ be any η_i-fo set of Y.

Then $f^{-1}(\mu) \leq \tau_j\text{-}Cl(\tau_i\text{-}Int(f^{-1}(\mu)))$.

$\Rightarrow f^{-1}(\mu)$ is a (τ_i,τ_j)-fso set of X. (by theorem 3.1.3)

To prove (i) =>(v)

Assume that f is fpsc.

Let ν be any η_i-fc set of Y.

Then ν' is η_i-fo set of Y.

$\Rightarrow f^{-1}(\nu')$ is a (τ_i,τ_j)-fso set of X and

$f^{-1}(\nu')=(f^{-1}(\nu))'$.

$\Rightarrow (f^{-1}(\nu))'$ is a (τ_i,τ_j)-fso set of X.

$\Rightarrow f^{-1}(\nu)$ is a (τ_i,τ_j)-fsc set of X.

The converse is obvious.

To prove (v)=>(vi)

Assume that statement (v) is true.

Let ν be any η_i-fc set of Y.

Then $f^{-1}(\nu)$ is a (τ_i,τ_j)-fsc set of X.

$\Rightarrow f^{-1}(\nu) \geq \tau_j\text{-}Int(\tau_i\text{-}Cl(f^{-1}(\nu)))$.

The converse can be easily proved.

SECTION: 2

ON FUZZY PAIRWISE α-CONTINUITY AND FUZZY PAIRWISE PRE-CONTINUITY

DEFINITION: 3.2.1

Let λ be a fuzzy set of a fbts (X,τ_1,τ_2). λ is called

i) a (τ_i,τ_j) - fuzzy α- open (briefly, (τ_i,τ_j) - fαo) set of X if $\lambda \leq (\tau_i - Int(\tau_j\text{-}Cl(\tau_i\text{-}Int\,\lambda))$.

ii) a (τ_i,τ_j) - fuzzy α- closed (briefly, (τ_i,τ_j)-fαc) set of X if $\tau_i\text{-}Cl(\tau_j\text{-}Int(\tau_i\text{-}Cl\,\lambda)) \leq \lambda$.

iii) a (τ_i,τ_j)-fuzzy pre-open set of X if $\lambda \leq \tau_i\text{-}Int(\tau_j\text{-}Cl\,\lambda)$.

iv) a (τ_i,τ_j)- fuzzy pre-closed set of X if $\tau_i\text{-}Cl(\tau_j\text{-}Int\,\lambda) \leq \lambda$.

THEOREM: 3.2.2

Let λ be a fuzzy set in a fbts(x,τ_1,τ_2).

i) λ is (τ_i,τ_j)-fαc if and only if λ' is (τ_i,τ_j) - fαo.

ii) λ is (τ_i,τ_j)-fuzzy pre-closed if and only if λ' is (τ_i,τ_j)-fuzzy pre-open.

PROOF:

i) λ is (τ_i,τ_j)-fαc.

<=> $\tau_i\text{-}Cl(\tau_j\text{-}Int(\tau_j\text{-}Cl\,\lambda)) \leq \lambda$.

<=> $1-\lambda \leq 1-(\tau_i\text{-}Cl(\tau_j\text{-}Int(\tau_i\text{-}Cl\,\lambda)))$.

<=> $\lambda' \leq \tau_i\text{-}Int(\tau_j\text{-}Cl(\tau_i\text{-}Int\,\lambda'))$.

<=> λ' is (τ_i,τ_j)-fαo .

ii) λ is (τ_i,τ_j)-fuzzy pre-closed.

<=> $\tau_i\text{-}Cl(\tau_j\text{-}Int\,\lambda) \leq \lambda$.

<=> $1-\lambda \leq 1-[\tau_i\text{-}Cl(\tau_j - Int\,\lambda)]$.

<=> $\lambda' \leq \tau_i\text{-}Int(\tau_j\text{-}Cl\,\lambda')$.

<=> λ' is (τ_i, τ_j) - fuzzy pre-open.

REMARK:

The following implications can be obtained directly from the definitions.

$$
\begin{array}{ccc}
 & & (\tau_i,\tau_j) - fso \; ((\tau_i,\tau_j) - fsc). \\
 & & \uparrow \\
\tau_i\text{-}fo(\tau_i\text{-}fc). & \longrightarrow & (\tau_i,\tau_j) - fao((\tau_i,\tau_j)\text{-}fac). \\
 & & \downarrow \\
 & & (\tau_i,\tau_j)\text{-fuzzy pre-open.} \\
 & & ((\tau_i,\tau_j)\text{-fuzzy pre-closed}).
\end{array}
$$

From the following examples we can see that the converse of the implications are not true.

EXAMPLE: 3.2.3.

Let μ_1, μ_2 and μ_3 be fuzzy sets on I defined as follows:

$$
\mu_1(x) = \begin{cases} 0, & 0 \le x \le 1/2, \\ 2x-1, & 1/2 \le x \le 1, \end{cases}
$$

$$
\mu_2(x) = \begin{cases} 1, & 0 \le x \le 1/4, \\ -4x+2, & 1/4 \le x \le 1/2, \\ 0, & 1/2 \le x \le 1, \end{cases}
$$

$$
\mu_3(x) = \begin{cases} 0, & 0 \le x \le 1/4, \\ 1/3(4x-1), & 1/4 \le x \le 1. \end{cases}
$$

Consider fuzzy topologies $\tau_1 = \{0, \mu_3, 1\}$ and $\tau_2 = \{0, \mu_1, 1\}$. The fuzzy set μ_2'(resp. μ_2) is (τ_1, τ_2) - fao(resp.(τ_1, τ_2) - fac) which is not τ_1-fo (resp.τ_1-fc) in (I, τ_1, τ_2).

The fuzzy set μ_3(resp.μ_3') is (τ_2,τ_1) - fao(resp. (τ_2,τ_1)-fac) which is not τ_2-fo (resp. τ_2-fc) in (I,τ_1,τ_2).

Consider fuzzy topologies $\tau_3=\{0,\mu_2,1\}$ and $\tau_2=\{0,\mu_1,1\}$. The fuzzy set μ_3' (resp. μ_3) is (τ_3,τ_2) - fso(resp.(τ_3,τ_2) - fsc) which is not (τ_3,τ_2)-fao(resp. (τ_3,τ_2)-fac) and the fuzzy set μ_2' (resp.μ_2) is(τ_2,τ_3) - fso(resp.(τ_2,τ_3) - fsc) which is not (τ_2,τ_3) - fao(resp.(τ_2,τ_3)-fac) in (I,τ_3,τ_2).

Consider fuzzy topologies $\tau_1 = \{ 0,\mu_3,1\}$ and $\tau_3 = \{0,\mu_2,1\}$. The fuzzy set μ_1(resp.μ_1') is both (τ_1,τ_3) - fuzzy pre-open (resp.(τ_1,τ_3)-fuzzy pre-closed) and (τ_3,τ_1)-fuzzy pre-open (resp.(τ_3,τ_1)-fuzzy pre-closed) which is neither (τ_1,τ_3)-fao (resp.(τ_1,τ_3) - fac) nor (τ_3, τ_1) - fao (resp.(τ_3,τ_1)-fac).

THEOREM: 3.2.4

i) Any union of (τ_i,τ_j)-fao(resp.(τ_i,τ_j)-fuzzy pre-open) sets is a (τ_i,τ_j) - fao (resp.(τ_i,τ_j)-fuzzy pre-open) set, and

ii) any intersection of (τ_i,τ_j)-fac (resp.(τ_i,τ_j)- fuzzy pre-closed) sets is a (τ_i,τ_j)-fac (resp.(τ_i,τ_j)-fuzzy pre-closed) set .

PROOF:

i) Let $\{\lambda_k\}$ be a collection of (τ_i,τ_j)-fao sets in a fbts(X,τ_1,τ_2).

Then $\lambda_k \leq \tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_i\text{-Int}\ \lambda_k))$ for each k.

$\Rightarrow \cup\lambda_k \leq \cup\ [\tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_i\text{-Int}\ \lambda_k))]$.

$\leq \tau_i\text{-Int}[\tau_j\text{-Cl}(\tau_i\text{-Int}(\cup\lambda_k))]$.

$\Rightarrow \cup\lambda_k$ is a (τ_i,τ_j)-fao set.

ii) Let $\{\lambda_k\}$ be a collection of (τ_i,τ_j)-fac sets in a fbts(x,τ_i,τ_j).

Then τ_i-Cl$(\tau_j$-Int$(\tau_i$-Cl$\lambda_k)) \leq \lambda_k$ for each k.

$\Rightarrow \tau_i$-Cl$\{\tau_j$-Int$(\tau_i$-Cl$(\cap\lambda_k))\} \leq \cap\tau_i$-Cl$\{\tau_j$-Int$(\tau_i$-Cl$\lambda_k)\} \leq \cap\lambda_k$.

$\Rightarrow \cap\lambda_k$ is a (τ_i,τ_j)-fac set.

Similar proof follows for fuzzy pre-open set and fuzzy pre-closed set. The following theorem gives a characterization of (τ_i,τ_j) - fao sets.

THEOREM: 3.2.5

A fuzzy set λ in a fbts (X,τ_1,τ_2) is a (τ_i,τ_j)-fao set if and only if it is (τ_i,τ_j)-fso and (τ_i,τ_j)-fuzzy pre-open.

PROOF:

It is obvious that a (τ_i,τ_j)-fao set is both (τ_i,τ_j)-fso set and (τ_i,τ_j)-fuzzy pre-open set.

Conversely, let λ be both(τ_i,τ_j)-fso set and (τ_i,τ_j)-fuzzy pre-open set. Then there exists a τ_i-Int$\lambda \in \tau_i$ such that τ_i-Int$\lambda \leq \lambda \leq \tau_j$-Cl$(\tau_i$-Int$(\lambda))$.

$\Rightarrow \tau_j$-Cl$(\lambda) \leq \tau_j$-Cl$(\tau_i$Int$(\lambda))$.

$\Rightarrow \tau_i$-Int$(\tau_j$-Cl$(\lambda)) \leq \tau_i$-Int$(\tau_j$-Cl$(\tau_i$-Int$(\lambda)))$.

But $\lambda \leq \tau_i$-Int$(\tau_j$-Cl $(\lambda))$.

Therefore $\lambda \leq \tau_i$-Int$(\tau_j$-Cl$(\tau_i$-Int$(\lambda)))$.

$\Rightarrow \lambda$ is a (τ_i,τ_j)-fao set of X.

THEOREM: 3.2.6:

If μ is a fuzzy set in a fbts(X,τ_1,τ_2) and λ is a (τ_i,τ_j)-fso set such that $\lambda \leq \mu \leq \tau_i$-Int$(\tau_j$-Cl$(\lambda))$, then μ is a (τ_i,τ_j)-fao set.

PROOF:

Let λ be a $(\tau_i \tau_j)$-fso set such that

$\lambda \leq \mu \leq \tau_i\text{-Int}(\tau_j\text{-Cl}(\lambda))$.

Then $\lambda \leq \tau_j\text{-Cl}(\tau_i\text{-Int}(\lambda))$.

$\Rightarrow \mu \leq \tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_j\text{-Cl}(\tau_i\text{-Int}(\lambda))))$.

$\Rightarrow \mu \leq \tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_i\text{-Int}(\lambda)))$.

$\leq \tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_i\text{-Int}(\mu)))$.

$\Rightarrow \mu$ is a (τ_i, τ_j)-fαo set.

DEFINITION: 3.2.7

Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \eta_1, \eta_2)$ be a mapping from a fbts X to another fbts Y. f is called

i) a fuzzy pairwise α-continuous (fpαc, in short) mapping if $f^{-1}(\lambda)$ is a (τ_i, τ_j)-fαo set in X for each $\lambda \in \eta_i$.

ii) a fuzzy pairwise pre-continuous (fppc, in short) mapping if $f^{-1}(\lambda)$ is a (τ_i, τ_j)-fuzzy pre-open set in X for each $\lambda \in \eta_i$.

REMARK:

The following implications can be obtained directly from the definitions

fpsc.
↑
fpc. ——> fpαc.
↓
fppc.

From the following examples we can see that the converse of the implications are not true.

EXAMPLE: 3.2.8:

Let μ_1, μ_2 and μ_3 be fuzzy sets of I as described in Example 3.2.3. Consider fuzzy topologies $\tau_1 = \{0, \mu_1, \mu_2', 1\}$, $\tau_2 = \{0, \mu_3, 1\}$, $\eta_1 = \{0, \mu_1 \cup \mu_2, 1\}$ and $\eta_2 = \{0, \mu_2', 1\}$ and the identity mapping $f:(I, \tau_1, \tau_2) \to (I, \eta_1, \eta_2)$.

Then f is a fpac mapping but not a fpc mapping.

Consider fuzzy topologies $\tau_3 = \{0, \mu_2, 1\}$, $\tau_4 = \{0, \mu, 1\}$, $\eta_3 = \{0, \mu_3', 1\}$ and $\eta_2 = \{0, \mu_2', 1\}$ and the identity mapping $f:(I, \tau_3, \tau_4) \to (I, \eta_3, \eta_2)$. Then f is a fpsc mapping which is not fppc and thus also not fpac mapping.

Consider fuzzy topologies $\tau_2 = \{0, \mu_3, 1\}$, $\tau_3 = \{0, \mu_2, 1\}$, $\eta_3 = \{0, \mu_3', 1\}$ and $\eta_2 = \{0, \mu_2', 1\}$. Consider the identity mapping $f:(I, \tau_2, \tau_3) \to (I, \eta_3, \eta_2)$. Then f is a fppc mapping which is not fpsc and thus also not fpac mapping.

The following theorem provides several characterizations of fpac mappings.

THEOREM: 3.2.9

Let $f:(X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2)$ be a mapping. The following statements are equivalent:

(i) f is fpac.

(ii) For each fuzzy point x_β of X and each η_i-fo set μ of Y containing $f(x_\beta)$, there exists a (τ_i, τ_j)-fao set λ of X containing x_β such that $f(\lambda) \le \mu$.

(iii) The inverse image of each η_i-fc set of Y is a (τ_i, τ_j)-fac set in X.

(iv) τ_i-Cl(τ_j-Int(τ_i-Cl($f^{-1}(\mu)$))) $\leq$ f^{-1}(η_i-Cl(μ)) for each fuzzy set μ of Y.

(v) f(τ_i-Cl(τ_j-Int(τ_i-Cl(λ)))) $\leq$ η_i-Cl(f(λ)) for each fuzzy set λ of X.

PROOF:

To prove (i)=>(ii).

Assume that f is fpac.

Let μ be a η_i-fo set of Y containing $f(x_\beta)$ where $x_\beta \in X$.

Then $f^{-1}(\mu)$ is a (τ_i,τ_j)-fao set of X containing x_β.

Let $\lambda = f^{-1}(\mu)$.

Then λ is a (τ_i,τ_j)-fao set of X containing x_β such that $f(\lambda) \leq \mu$.

To prove(ii) => (i).

Assume that statement (ii) is true.

Let μ be any η_i-fo set of Y.

Then $f^{-1}(\mu)$ is a fuzzy set of X.

Let $x_\beta \in f^{-1}(\mu)$. Then $f(x_\beta) \in \mu$.

Hence by our assumption, there exists a (τ_i,τ_j)-fao set λ_{x_β} of X such that $x_\beta \in \lambda_{x_\beta}$ and $f(\lambda_{x_\beta}) \leq \mu$.

Thus $x_\beta \in \lambda_{x_\beta} \leq f^{-1}(\mu)$. Therefore, we have

$\cup\{x_\beta : x_\beta \in f^{-1}(\mu)\} \leq \cup\{\lambda_{x_\beta} : x_\beta \in f^{-1}(\mu)\} \leq f^{-1}(\mu)$.

Hence $f^{-1}(\mu) = \cup\{\lambda_{x_\beta} : x_\beta \in f^{-1}(\mu)\}$ which is a (τ_i,τ_j)-fac set of X.

To prove (i) => (iii).

Assume that f is fpac.

Let μ be a η_i-fc set of Y. Then $1-\mu$ is a η_i-fo set of Y.

$\Rightarrow f^{-1}(1-\mu)$ is (τ_i,τ_j)-fao set of X.

$\Rightarrow 1-f^{-1}(\mu)$ is (τ_i,τ_j)-fao set of X.

$\Rightarrow f^{-1}(\mu)$ is (τ_i,τ_j)-fac set of X.

To prove (iii) => (iv).

Assume that the inverse image of each η_i-fc set of Y is a (τ_i,τ_j)-fac set in X.

Let μ be any fuzzy set of Y.

Then $f^{-1}(\eta_i\text{-Cl}(\mu))$ is a (τ_i,τ_j)-fac set of X.

$$\Rightarrow f^{-1}(\eta_i\text{-Cl}(\mu)) \geq \tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}(f^{-1}(\eta_i\text{-Cl}(\mu))))).$$

$$\geq \tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}(f^{-1}(\mu)))).$$

To prove (iv)=>(v).

Assume that $\tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}(f^{-1}(\mu)))) \leq f^{-1}(\eta_i\text{-Cl}(\mu))$ for each fuzzy set μ of Y.

We get the result (v) by replacing $\mu = f(\lambda)$ in our assumption where λ is a fuzzy set of X.

To prove(v)=>(i)

Assume that $f(\tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}(\lambda)))) \leq \eta_i\text{-Cl}(f(\lambda))$ for each fuzzy set λ of X.

Let μ be a η_i-fo set of Y.

Then $1-\mu$ is a η_i-fc set of Y.

$$\Rightarrow f(\tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}(f^{-1}(1-\mu))))) \leq \eta_i\text{-Cl}(f(f^{-1}(1-\mu))).$$

$$= \eta_i\text{-Cl}(1-\mu).$$

$$= 1-\mu.$$

Therefore, $\tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}(f^{-1}(1-\mu)))) \leq f^{-1}(1-\mu)$.

$\Rightarrow f^{-1}(1-\mu)$ is a (τ_i,τ_j)-fac set of X.

$\Rightarrow 1-f^{-1}(\mu)$ is a (τ_i,τ_j)-fac set of X.

$\Rightarrow f^{-1}(\mu)$ is a (τ_i, τ_j)-fao set of X.

$\Rightarrow$ f is fpac mapping.

THEOREM: 3.2.10

A mapping $f:(X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ is fpac if and only if it is fpsc and fppc.

PROOF:

It is obvious that a fpac mapping is both fpsc mapping and fppc mapping.

Conversely, let f be both fpsc mapping and fppc mapping and let λ be a η_i-fo set of Y.

Then $f^{-1}(\lambda)$ is both (τ_i,τ_j)-fso and (τ_i,τ_j)-fuzzy pre-open set of X.

$\Rightarrow f^{-1}(\lambda)$ is (τ_i,τ_j)-fao set of X.

$\Rightarrow$ f is fpac mapping.

DEFINITION: 3.2.11

Let $f:(X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ be a mapping from a fbts X to another fbts Y. f is called

(i) a fuzzy pairwise α-open (resp.fuzzy pairwise α-closed), briefly, fpα open (fpα closed), mapping, if $f(\lambda)$ is a (η_i,η_j)-fαo (resp.(η_i,η_j)-fαc) set of Y for each τ_i-fo (resp. τ_i-fc) set of X.

ii) a fuzzy pairwise pre-open (resp. fuzzy pairwise pre-closed) briefly, fpp open (fpp closed) mapping, if $f(\lambda)$ is a (η_i,η_j)-fuzzy pre-open (resp. (η_i,η_j)-fuzzy pre-closed set of Y for each τ_i-fo(resp.τ_i-fc) set λ of X.

REMARK:

The following implications can be obtained directly from the definitions

fpp open (resp. fpp closed)
↑
fp open (resp. fp closed) ⟶ fpα open (resp. fpα closed)
↓
fps open (resp. fps closed)

From the following example we see that the converse of the implications are not true.

EXAMPLE: 3.2.12

Let X = {a,b,c} and Y = {x,y,z}. The fuzzy sets λ_1, λ_2 of X and the fuzzy sets μ_1, μ_2 of Y are defined as follows:

$\lambda_1(a)=0.3$, $\lambda_1(b)=0.4$, $\lambda_1(c)=0.5$,

$\lambda_2(a)=0.3$, $\lambda_2(b)=0.5$, $\lambda_2(c)=0.6$,

$\mu_1(x)=0.2$, $\mu_1(y)=0.3$, $\mu_1(z)=0.4$,

$\mu_2(x)=0$, $\mu_2(y)=0.2$, $\mu_2(z)=0.3$.

Consider fuzzy topologies $\tau_1\{0,\lambda_1,1\}$, $\tau_2=\{0,\lambda_2,1\}$, $\eta_1=\{0,\mu_2,1\}$, $\eta_2=\{0,\mu_1,\mu_2,1\}$ and the mapping $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ defined by f(a)=x, f(b)=y and f(c)=z. Then f is a fps open mapping but not fpα open mapping.

Consider fuzzy topologies $\tau_3=\{0,\lambda_2', 1\}$, $\tau_2=\{0,\lambda_2, 1\}$, $\eta_3=\{0,\mu_1,1\}$, $\eta_1=\{0,\mu_2,1\}$ and the mapping $f=(Y,\eta_3,\eta_1)\to(X,\tau_3,\tau_2)$ defined by f(x)=a, f(y)=b and f(z)=c. Then f is a fpp open mapping but not fpα open mapping.

Consider fuzzy topologies $\tau_2=\{0,\lambda_2, 1\}$, $\tau_1=\{0,\lambda_1, 1\}$, $\eta_4=\{0,\mu_2',1\}$, $\eta_5=\{0,\mu_1',1\}$ and the mapping $f=(Y,\eta_4,\eta_5)\to(X,\tau_2,\tau_1)$ defined by $f(x)=a$, $f(y)=b$ and $f(z)=c$. Then f is a fpα open mapping but not a fp open mapping.

THEOREM: 3.2.13

Let $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ be a fpα open (resp. fpα closed) mapping. If μ is a fuzzy set of Y and λ is a τ_i-fc (resp. τ_i-fo) set of X containing $f^{-1}(\mu)$, then there exists a (η_i,η_j)-fαc (resp. (η_i,η_j)-fαo) set ν of Y containing μ such that $f^{-1}(\nu)\le\lambda$.

PROOF:

Assume that $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ is a fpα open mapping. Let μ be a fuzzy set of Y and λ be a τ_i-fc set of X containing $f^{-1}(\mu)$. Let $\nu=1-f(1-\lambda)$. Then $f^{-1}(\mu)\le\lambda$.

$$\Rightarrow f(\lambda)\ge\mu.$$

$$\Rightarrow 1-f(\lambda)\le 1-\mu.$$

$$\Rightarrow f(1-\lambda)\le 1-\mu.$$

Since $1-\lambda$ is a τ_i-fo set of X, we have $f(1-\lambda)$ is a (η_i,η_j)-fαo set of Y.

$\Rightarrow 1-f(1-\lambda)=\nu$ is a (η_i,η_j)-fαc set of Y.

And $f^{-1}(\nu) = f^{-1}(1-f(1-\lambda))$.

$$= 1-f^{-1}(f(1-\lambda)).$$

$$\le 1-1-\lambda.$$

$$= \lambda.$$

Hence there exists a (η_i,η_j)-fαc set of Y containing μ such that $f^{-1}(\nu)\le\lambda$.

The proof for the class of fpα closed mapping is similar.

THEOREM: 3.2.14

If $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ is a fpα open mapping, then $f^{-1}(\eta_i Cl(\eta_j\text{-}Int(\eta_i\text{-}Cl(\mu))))\le\tau_i\text{-}Cl(f^{-1}(\mu))$ for each fuzzy set μ of Y.

PROOF:

Assume that $f:(X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ is a fpα open mapping and μ is a fuzzy set of Y. Since $\tau_i\text{-}Clf^{-1}(\mu)$ is a fuzzy closed set of X containing $f^{-1}(\mu)$ for each fuzzy set μ of Y, there exists a (η_i, η_j)-fαc set λ of Y containing μ such that $f^{-1}(\lambda) \leq \tau_i\text{-}Cl(f^{-1}(\mu))$.

Since $\mu \leq \lambda$,

$$f^{-1}(\eta_i\text{-}Cl(\eta_j\text{-}Int(\eta_i\text{-}Cl(\mu)))) \leq f^{-1}(\eta_i\text{-}Cl(\eta_j\text{-}Int(\eta_i\text{-}Cl(\lambda)))).$$
$$\leq f^{-1}(\lambda).$$
$$\leq \tau_i\text{-}Cl(f^{-1}(\mu)).$$

for each fuzzy set μ of Y.

The following theorem gives a characterization of fpα open mappings.

THEOREM: 3.2.15

A mapping $f:(X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ is fpα open (fpα closed) if and only if it is fps open (fps closed) and fpp open (fpp closed).

PROOF:

It is obvious that a fpα open mapping is both fps open mapping and fpp open mapping.

Conversely, let f be both fps open mapping, and fpp open mapping and let λ be a τ_i-fo set of X.

Then $f(\lambda)$ is both (η_i,η_j)-fso set and (η_i,η_j)-fuzzy pre-open set of Y.

=> $f(\lambda)$ is (η_i,η_j)-fαo set of Y.

=> f is fpα open mapping.

The proof for the class of fpα closed mapping is similar.

Summary and Conclusion

SUMMARY AND CONCLUSION

The concept of fuzzy bitopological spaces was first introduced by K.K.Azad in 1981. Since then, many results on topological spaces are generalized to fuzzy bitopological spaces. In this thesis, we have discussed in detail the following three important concepts.

1. Fuzzy complementation
2. Fuzzy strongly compactness
3. Semi-open sets, semi-continuity, α-open sets, α-continuity in fuzzy bitopological spaces.

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