SOME INTERESTING RESULTS ON FUZZY CONVERGENCE

By

GEETHA.G.R

A dissertation submitted to the
Avinashilingam Institute for Home Science
and Higher Education for Women
(Deemed University) Coimbatore - 641 043.

In partial fulfilment of the requirements
for the Degree of MASTER OF SCIENCE
in Mathematics

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Certified as Bonafide Research Work

K. Sivakamasundari
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Head of the Department
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CONTENTS

CONTENTS

PAGE NO.

SECTIONS

SECTIONS

INTRODUCTION

INTRODUCTION

The main objective of this thesis is to study the concept of fuzzy convergence in terms of filters. Fuzzy convergence was obtained by M.A.DE PRADA VICENTE [5] in 1981. In 1985 he has obtained characterisations of open sets, closed sets, adherent points and limit points interms of fuzzy filter convergence. In 1988 the concept of t-prefilter was introduced by PRADA and M.SARALEGUI ARANGUREN [8], using t-prefilters defined by them and the characterisations of t-compactness. In 1993 these authors have studied fuzzy convergence of prefilters through convergence of some family of filters on the collection of its level topologies. In this thesis the following three papers are discussed in detail.

i) "Fuzzy Filters" by M.A.DE PRADA VICENTE and M.SARALEGUI ARANGUREN [8].

ii) "t-Prefilter Theory" by M.A.DE PRADA VICENTE and M.MACHO STADLER [21].

iii) "Fuzzy Convergence Versus Convergence" by M.MACHO STADLER and M.A.DE PRADA VICENTE [23].

Section I of Chpater I is devoted to the definition of fuzzy filters and fundamental properties of filters. Here we define the concept of convergence and adherence of prefilters. The open sets and closed sets are characterised interms of convergence of filters. The continuous map is characterised interms of convergence of filters. Regarding the product spaces, the following theorem is proved.

"Let $(X, \delta) = \prod_{j \in J} (X_j, \delta_j)$ be a fuzzy product space, $\mathcal{F}$, a prefilter on X and p, a fuzzy point in X, then

i) $\mathcal{F} \to p$ iff for each $j \in J$, $p_j(\mathcal{F}) \to p_j = \pi_j(p)$

ii) If $\mathcal{F} \infty p$, then for each $j \in J$, $p_j(\mathcal{F}) \infty p_j$.

In section 2 of Chpater I, a connection between filter, prefilter, ultrafilter and F-ultrafilters are discussed. The theorem proved in this connection is as follows.

"Let X be a set of points, U the family of all ultrafilters on X and U_F, the family of all F-ultrafilters. We define two maps :

a : $U \to U_F$ by $a(G) = \{ F \in I^X / \text{Supp } F \in G \} \ \forall \ G \in U$

b : $U_F \to U$ by $b(\mathcal{F}) = \{ \text{Supp } F / F \in \mathcal{F} \} \ \forall \ \mathcal{F} \in U_F$

a and b are well defined and they are inverses to each other".

The definitions and properties of F-nets and F-ultranets given in [8] are used here to define an associated F-net, given a prefilter and vice versa.

F-net based on F-filter

"Let $\mathcal{F}$ be a prefilter on X, P_F, the collection of all fuzzy points in X and

$$\begin{aligned}\Lambda_{\mathcal{F}} &= \{ (p, F) \,/\, p \in F,\ p \in P_F(x),\ F \in \mathcal{F} \} \\ &\sqsubseteq P_F(x) \times \mathcal{F}\end{aligned}$$

directed by the relation $(p_1, F_1) \lesssim (p_2, F_2)$ iff $F_2 \lesssim F_1$. The map $\psi_{\mathcal{F}} : \Lambda_{\mathcal{F}} \rightarrow P_F(X)$ defined by $\psi_{\mathcal{F}}(p, F) = p$ is an F-net in X. It is called the F-net based on $\mathcal{F}$".

F-filter generated by F-net

Let $\psi : D \rightarrow P_F(X)$ be an F-net in X. The F-filter generated by the collection B of all the F-subsets in which the F-net ψ is residually contained is called the F-filter generated by ψ.

The relation between F-net convergence and F-filter convergence are obtained in the following theorems.

1) "Let (X, δ) be an fuzzy topological space, F_ψ, an F-filter on X and p, a fuzzy point. Then,

i) F converges to p iff ψ_F converges to p.

ii) ψ converges to p iff F_ψ converges to p.

iii) F has p as a cluster point iff ψ_F has p as cluster point.

iv) ψ has p as a cluster point iff F has p as a cluster point".

2) Let (X, δ) be an fuzzy topological space, F (respectively ψ) an F-filter (respectively F-net) on X. Then,

i) The F-net based on an F-ultrafilter is an F-ultranet.

ii) The F-filter based on an F-ultranet is an F-ultrafilter.

iii) If ψ is a trivial F-ultranet, F_ψ is a trivial F-ultrafilter.

iv) If F is a trivial F-ultrafilter, ψ_F is a trivial F-ultranet".

In Chapter II, t-prefilter and convergence of t-prefilter are studied. Given a constant t, a t-prefilter is defined as follows.

"$P^t(X)$ will denote the set of all prefilters on X, which exclude the constant function C_t. A member of $P^t(X)$ is called t-prefilter on X". With every t-prefilter on X, we can associate a filter on X which can be defined as follows.

"Let $P^t(X)$ be the set of all prefilters and $F(X)$ be the set of all filters on X. A map $i_t : P^t(X) \to F(X)$ such that for each t-prefilter F, $i_t(F) = \{ \mu^{-1}(t,1] / \mu \in F \}$".

Likewise, associated to each filter $\mathcal{F}$ on X, we define a t-prefilter with every $t \in [0, 1]$ as follows. A map $w_t : F(X) \to P^t(X)$ such that for each filter $\mathcal{F}$,

$$w_t(\mathbf{F}) = \{ \mu \in I^X / \mu^{-1}(t, 1] \in \mathcal{F} \}$$

The relations between i_t and w_t are stated in the following theorems :

"Let $\mathcal{F}$, $\mathbf{F}$ be a t-prefilter, respectively filter on X. Then,

(i) For each $s < t$, we have $i_t(\mathcal{F}) \sqsupset i_s(\mathcal{F})$ and $w_t(\mathbf{F}) \sqsubset w_s(\mathbf{F})$.

(ii) $i_t \circ w_t(\mathbf{F}) = \mathbf{F}$ and if $s < t$, $i_s \circ w_t(\mathbf{F}) = \mathbf{F}$.

(iii) $\mathcal{F} \sqsubset w_t \circ i_t(\mathcal{F})$ and if $s < t$, $\mathcal{F} \sqsubset w_s \circ i_t(\mathcal{F})$.

(iv) i_t, w_t are respectively an isotone surjection and an isotone injection".

In Section 2 of this Chapter, t-convergence of t-prefilter is defined. Regarding t-convergence of t-prefilter the following theorem is obtained. "Let $\mathcal{F}$, $\mathcal{G}$ be t-prefilters on X. Then

i) If $\mathcal{F} \sqsubset \mathcal{G}$ and $\mathcal{F}^t \to p(x, t)$, then $\mathcal{G}^t \to p(x, t)$.

ii) If $\mathcal{F} \xrightarrow{t} p(x, t)$, then $\mathcal{F}^t \infty\, p(x, t)$.

iii) If $\mathcal{F}$ is prime, then $\mathcal{F} \xrightarrow{t} p(x, t)$ iff $\mathcal{F}^t \infty\, p(x, t)$.

iv) $\mathcal{F}^t \infty\, p(x, t)$ iff there exist a t-prefilter $\mathcal{G}$ finer than $\mathcal{F}$ and $\mathcal{G} \xrightarrow{t} p(x, t)$. Consequently, if $\mathcal{F} \sqsubset \mathcal{G}$, then $\mathcal{G}^t \infty\, p(x, t)$ implies $\mathcal{F}^t \infty\, p(x, t)$".

The relation between t-convergence and convergence is obtained in the following theorem. "Let (X, δ) be a topologically generated fuzzy topological space (i.e., $\delta = w_t(\tau)$ for some $t \in [0,1)$), Then,

i) $\mathrm{Lim}(w_t(F)) = t \cdot \chi_{\mathrm{Lim}}(F)$.

ii) $\mathrm{Adh}(w_t(F)) = t \cdot \chi_{\mathrm{Adh}}(F)$."

In Chapter III, a new approach to fuzzy convergence of prefilters through the convergence of some family of filters on the collection of its level topologies is discussed. For this purpose with every filter, the characteristic set Δ_F is defined as follows :

$$\Delta_F = \{ t \in [0, 1) / F \in P^t(X) \}$$

In the collection of prefilters $P(X)$, an equivalence relation is defined as follows. "If $F_1, F_2 \in P(X)$, we say F_1 and F_2 are strong equivalent prefilters if

i) $\Delta_{F_1} = \Delta_{F_2} = \Delta$

ii) For each $t \in \Delta$, we have $i_t(F_1) = i_t(F_2)$".

With each prefilter F, a prefilter $F^{\#}$ is associated.

Here $$F^{\#} = \bigcap_{t \in \Delta_F} w_t \circ i_t(F)$$

and it is the finest element of this class.

The important problem studied here is as follows. Suppose that Δ is an interval in $[0, 1]$ of type $[0, t]$ for $t \in [0, 1)$ or $[0, t)$ for $t \in (0, 1)$ and $\{ F_t \ / \ t \in \Delta \}$ is a family of filters on X. Verifying the inclusions $F_s \subset F_t$ if $s \lesssim t$, then the question is "Is there any prefilter $\mathcal{F}$ on X such that $\Delta = \Delta_{\mathcal{F}}$ for each $t \in \Delta$ we have $i_t(\mathcal{F}) = F_t$?". This problem is solved in the following theorem.

"The prefilter $\mathcal{F} = \bigcap_{t \in \Delta} w_t(F_t)$ verifies the property $\Delta = \Delta_{\mathcal{F}}$ and $i_t(\mathcal{F}) = F_t$ for $t \in \Delta$ ". With this connection the author is able to connect the fuzzy convergence of prefilter to convergence of filters defined on X.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

The problem of studying topological spaces starting with the concept of convergence has attracted many great topologists. To mention a few we have E.H.MOORE [24], H.L.SMITH [24], R.ARENS [2], G.BIRKHOFF [3], M.M.DAY [4], M.K.FORT (Jr.) [9], J.L.KELLEY [12], J.NOVAK [26], [27], J.W.TUKEY [31], etc. The idea of defining a topological space interms of convergence classes of nets and filters was considered by G.BIRKHOFF [3] and developed by J.W.TUKEY [31] and J.L.KELLEY [12].

Many authors characterized sequentially regular spaces, pseudotopological, pretopological spaces interms of convergence spaces. S.SO [30] has obtained characterisations for the one point compactification to be pseudotopological, pretopological convergence spaces.

The first important article on fuzzy convergence was published by R.LOWEN in 1979. The fuzzy convergence in terms of fuzzy filters and fuzzy nets was studied by M.A.DE PRADA VICENTE [5] in the year 1981.

In 1985 he has obtained characterisations of open sets, closed sets, adherent points and continuity interms of fuzzy filter convergence. In 1988 the concept of t-prefilter was introduced by PRADA [21] and M.MACHO STADLER [21]. Using t-prefilters they have defined t-convergence and the characterisation of t-compactness. In 1993 these authors have studied fuzzy convergence of prefilters through convergence of some family of filters in the collection of its level topologies.

CHAPTER I

CHAPTER - I

FUZZY FILTERS

INTRODUCTION

In this Chapter we shall discuss the paper entitled **"FUZZY FILTERS"** by M.A.DE PRADA VICENTE and M.SARALEGUI ARANGUREN [8]. The main concept introduced in this article is that of fuzzy filters.

In Section 1, we define the concept of convergence and adherence of prefilters. The open sets and closed sets are characterised interms of convergence of filters. The continuous map is characterised interms of convergence of filters and the theorem is proved to characterise the convergence of product space.

In Section 2, a connection between filters and prefilters is obtained. A similar relation is also proved for ultrafilters and F-ultrafilters. The definition and properties of F-nets and F-ultranets given in [7] are used here to define an associated F-net, given a prefilter and vice versa.

The last two theorems bring the relation between F-net convergence and F-filter convergence.

SECTION : 1

DEFINITION AND PROPERTIES OF FUZZY FILTERS

Definition 1.1.1

A fuzzy point p in X is a fuzzy set with membership function

$$p(x) = \begin{cases} t_0, & \text{if } x = x_0 \\ 0, & \text{Otherwise,} \end{cases}$$

where $0 < t_0 < 1$

p is said to have support x_0 and value t_0.

p is denoted by $p(x_0, t_0)$ or (x_0, t_0).

Definition 1.1.2

Let p be a fuzzy point in X and A an F-set (fuzzy set) in X ; p is said to be in the F-set A (i.e. $p \in A$) if $p(x_0) < A(x_0)$, x_0 being the support of p.

Definition 1.1.3

Let p be an F-point in X and (X, δ) an fuzzy topological space. $N \in I^X$ is said to be an F-nhood (F-neighbourhood) of p if there is some open F-set μ in X such that $p \in \mu$ and $\mu \leq N$.

Remark

The collection of all neighbourhoods of an F-point p in a topology δ is noted by N_p^{δ} or simply by N_p.

Proposition 1.1.1.

Let (X, δ) be an fuzzy topological space. An F-set μ of X is open iff for all $p \in \mu$, then $\mu \in N_p$.

Proof

Necessity

Assumption : An F-set μ of X is open.

Claim : For all $p \in \mu$, $\mu \in N_p$

μ is an fuzzy set.

Choose $N = \mu$.

i.e., $p \in \mu = N$.

$N \leq N = \mu$.

$\Longrightarrow \mu \in N_p$.

Sufficiency

Claim : Every point of μ is an interior point.

i.e., $\exists$ a neighbourhood of p contained in μ.

$\mu \in N_p \Longrightarrow \mu$ is a neighbourhood of p.

$\mu \subseteq \mu \Longrightarrow$ p is an interior point of μ.

$\Longrightarrow \mu$ is open.

Definition 1.1.4

Let (X, δ) be an fuzzy topological space and p, an fuzzy point. The collection B_p of subsets of N_p is a local basis at p if, for all N in N_p, there is some $B \in B_p$ with $p \in B$ and $B \leq N$.

Definition 1.1.5

Let A be an fuzzy set and (X, δ) an fuzzy topological space. An fuzzy point is said to be in the adherence of A if for each $N \in N_{p'}$, $N \not\subset A^c$, where $p' = (x, 1-t)$.

Proposition 1.1.2

$$\overline{A} = \cup \{ p/p \text{ is in the adherence of } A \}$$

Proof

Claim (1)

For any $p \in A$, p is an adherent point of A.

i.e. $A \subseteq \overline{A}$

i.e. To prove for each $N \in N_{p'}$ $N \not\subset A^c$

$$p \in A \Longrightarrow p(x) < A(x)$$

$$t_0 < A(x)$$

$$-t_0 > -A(x)$$

$$1 - t_0 > A^c(x)$$

Let $N \in N_{p'}$

$p' \varepsilon N \Longrightarrow p'(x) < N(x)$

$1 - t_0 < N(x)$

$N(x) > 1 - t_0 > A^c(x)$

$N(x) \nless A^c(x)$

$N \not\sqsubset A^c$

$\therefore \quad A \sqsubseteq \bar{A}$

Claim (2)

F is closed and p is an adherent point of F.

$\Longrightarrow$ $p \varepsilon F$.

Given $\forall N \varepsilon \mathcal{N}_p'$, $N \not\sqsubset F^c$ (1)

F is closed $\Longrightarrow$ F^c is open.

(1) $\Longrightarrow$ F^c cannot contian a neighbourhood of p'.

i.e. $p' \not\varepsilon F^c$

$1 - t_0 \nless F^c(x)$

$1 - t_0 \underset{\sim}{>} F^c(x)$

$F^c(x) \underset{\sim}{<} 1 - t_0$

$F(x) > t_0 \Longrightarrow p \varepsilon F$

By Claim (1) and (2),

$A \sqsubseteq \bar{A}$ and if F is closed and p is an adherent point of F, then $p \varepsilon F$

$\Longrightarrow F = \bar{A}$

N is a neighbourhood of p.

Then $\exists$ a neighbourhood N' $\ni N \underset{\sim}{<} N'$ where N' is a neighbourhood of p. $\mu \varepsilon \delta \ni p \varepsilon \mu$, $\mu \underset{\sim}{<} N \underset{\sim}{<} N'$.

Proposition 1.1.3

An F-point p is said to be in the closure of A (i.e. $p \in \overline{A}$) if there is some F-point q in the adherence of A with $p \in q$.

Proof

Claim (1) : If p and q have same support x and $p \in q$,

(i.e.) $p(x) < q(x)$, then $N_q \subseteq N_p$.

Let $N \in N_q$.

$\Longrightarrow$ $\exists$ an F-set μ in X $\ni q \in \mu$ and $\mu \leq N$.

Now $p \in q \in \mu$ and $\mu \leq N$

$\therefore$ $p \in \mu$ and $\mu \leq N \Longrightarrow N \in N_p$.

$\Longrightarrow$ $N_q \subseteq N_p$.

Claim (2) : $p \in q \Longrightarrow q' \in p'$

Let the values of p and q be t_1 and t_2 respectively.

$$\begin{aligned} p \in q &\Longrightarrow p(x) < q(x) \\ &\Longrightarrow t_1 < t_2 \\ &\Longrightarrow -t_1 > -t_2 \\ &\Longrightarrow 1 - t_1 > 1 - t_2 \\ &\Longrightarrow q' \in p' \end{aligned}$$

By Claim (1) and (2)

$p \in q \Longrightarrow q' \in p'$ and $N_p' \subseteq N_q'$

Necessity

Let $p \varepsilon \overline{A}$.

Let q be a fuzzy point $\ni$ p and q have same support x_0 and $p \varepsilon q$. i.e., $t_1 < t_2$.

Claim - q is an adherent point of A.

$p \varepsilon \overline{A}$

$\Longrightarrow$ p is an adherent point of A (from prop. 1.1.2)

$\Longrightarrow$ $\forall N \varepsilon \mathcal{N}_p', N \underline{\sqsubset} A^c$

$\Longrightarrow$ $N \varepsilon \mathcal{N}_p' \underline{\sqsubset} \mathcal{N}_q', N \not\sqsubset A^c$

$\Longrightarrow$ $\forall N \varepsilon \mathcal{N}_q', N \underline{\sqsubset} A^c$

$\Longrightarrow$ q is an adherent point of A.

Sufficiency

On retracing the steps, we get $p \varepsilon \overline{A}$

Definition 1.1.6

Let (X, δ) and (Y, γ) be fuzzy topological spaces and f, a map from X into Y. f is said to be F-continuous if for each F point p in X and each neighbourhood N of f(p), there is some neighbourhood M of p with $f(M) \lesssim N$.

Theorem 1.1.1

Let (X, δ), (Y, γ) be fuzzy topological spaces and f, a map from X into Y. Then f is F-continuous iff for each $\mu \varepsilon \gamma$, $f^{-1}(\mu) \varepsilon \delta$.

Proof

Necessity

Assume that f is F-continuous.

$\Longrightarrow$ $\forall$ $p \in X$ and $\forall$ neighbourhood N of $f(p)$, there is some neighbourhood M of p with $f(M) \lesssim N$.

Claim - $f^{-1}(\mu)$ is open in X.

Let any arbitrary $p \in f^{-1}(\mu)$.

$\Longrightarrow$ $f(p) \in \mu$

Let $f(p) = q$.

$\because$ $\mu \in N_q$, μ is a neighbourhood of q, By definition of continuous function, there is a neighbourhood M of p with $f(M) \lesssim \mu$.

i.e. $M \lesssim f^{-1}(\mu)$

$\therefore$ $f^{-1}(\mu)$ is a neighbourhood of p.

$p \in f^{-1}(\mu)$ and $f^{-1}(\mu) \in N_p$

$\Longrightarrow$ $f^{-1}(\mu)$ is an fuzzy open set

$\Longrightarrow$ $f^{-1}(\mu) \in \delta$

Sufficiency

Let $p \in X$. Let $f(p) = q$.

Let N be a neighbourhood of q.

Let μ be an open F-set in Y $\ni$

$q \in \mu$, $\mu \lesssim N$.

Then $f^{-1}(\mu)$ is F-open in X.

$\therefore$ All $p \in f^{-1}(\mu)$, $f^{-1}(\mu) \in N_p$.

$f(f^{-1}(\mu)) = \mu \leq N$.

$\therefore$ f is F continuous.

Definition 1.1.7

A prefilter F on X is a nonempty collection of subsets of I^X with the properties.

(i) If $F_1, F_2 \in F$ then $F_1 \wedge F_2 \in F$

(ii) If $F \in F$ and $F' \geq F$, then $F' \in F$

(iii) $0 \notin F$

Definition 1.1.8

A collection B of subsets of I^X is a base for some prefilter iff $B \neq \phi$ and

(i) If $B_1, B_2 \in B$ then $B_3 \leq B_1 \wedge B_2$ for some $B_3 \in B$.

(iii) $0 \notin B$.

Definition 1.1.9

The collection $F = \{ F \in I^X / \exists\, B \in B \ni F \geq B \}$ is a prefilter. F is said to be generated by B and is denoted $\langle B \rangle$.

Criterion

A collection B of subsets of F is a base for F iff for each $F \in F$ there is some $B \in B$ such that $B \leq F$.

Definition 1.1.10

If F_1 and F_2 are prefilters on X, F_1 is said to be finer than F_2 (equivalently F_2 is coarser than F_1) iff $F_1 \supset F_2$.

Definition 1.1.11

A prefilter F is said to converge to the fuzzy point p iff $N_p \subset F$.

Definition 1.1.12

F has p as a cluster point (written $F \infty p$) iff $\forall N \in N_p$, then $N \wedge F \neq 0$, $\forall F \in F$.

Definition 1.1.13 (in terms of bases of prefilters)

i) A base for a prefilter converges to a fuzzy point p ($\mathbf{B} \rightarrow p$) iff each $N \in N_p$ contains some $B \in \mathbf{B}$.

ii) A base for a prefilter has p as a cluter point ($\mathbf{B} \infty p$) iff each $N \in N_p$ meets some $B \in \mathbf{B}$.

Note

These definitions are still valid if we use neighbourhood bases at p, B_p, instead of neighbourhood systems at p, N_p.

Result

If $F \rightarrow p$, then $F \infty p$.

Proof

$\mathcal{F} \to p \Longrightarrow \mathcal{N}_p \sqsubset \mathcal{F}$

i.e. $N \in \mathcal{N}_p$

i.e. To prove : $\mathcal{F} \infty p$

i.e. To prove : $\forall N \in \mathcal{N}_p,\ N \wedge F \neq 0,\ \forall_F \in \mathcal{F}$

i.e. $N \in \mathcal{N}_p \sqsubset \mathcal{F} \Longrightarrow N = F$ for some $F \in \mathcal{F}$

$\therefore$ $N \wedge F = F \wedge F \neq 0$ for that F.

$\therefore$ $\mathcal{F} \infty p$.

Proposition 1.1.4

Let (X, δ) be a fuzzy topological space and $\mathcal{F}$ a prefilter on X. Then $\mathcal{F} \infty p$ iff there is a prefilter $\mathcal{G}$ such that $\mathcal{G} \sqsupset \mathcal{F}$ and $\mathcal{G} \to p$.

Proof

Let $\mathbf{B} = \{ N \wedge F \ / \ N \in \mathcal{N}_p$ and $F \in \mathcal{F} \}$

Let $\mathcal{G}$ be a prefilter generated by $\mathbf{B}$.

i.e. $\mathcal{G} = \{ G \in I^X \ / \ G \geq B,\ B \in \mathbf{B} \}$

Claim (1) : $\mathcal{G}$ is finer than $\mathcal{F}$ (i.e.) $\mathcal{F} \sqsubset \mathcal{G}$

Let $F \in \mathcal{F}$

We know that $F \geq N \wedge F$

$\therefore$ By definition of $\mathcal{G}$, $F \in \mathcal{G}$

$\therefore$ $\mathcal{F} \sqsubset \mathcal{G}$

Claim (2) : $\mathcal{N}_p \sqsubseteq \mathcal{G}$ (i.e. $\mathcal{G} \to p$)

$\mathcal{G} = \{ G \in I^X \,/\, \exists\, B \in \mathbf{B} \ni G \geq B \}$

$\mathbf{B} = \{ N \wedge F \,/\, N \in \mathcal{N}_p, F \in \mathcal{F} \}$

$N \geq N \wedge F \Longrightarrow N \in \mathcal{G}$

$\therefore \quad \mathcal{N}_p \sqsubseteq \mathcal{G}$

$\therefore \quad \mathcal{G} \to p$

$\therefore \quad \mathcal{G} \propto p$

Converse

Assume that there is a prefilter $\mathcal{G} \ni$ $\mathcal{G} \sqsupseteq \mathcal{F}$ and $\mathcal{G} \to p$.

To Prove : $\mathcal{F} \propto p$

To Prove : $\forall\, N \in \mathcal{M}_p, N \wedge F \neq 0, \forall\, F \in \mathcal{F}$

$\mathcal{G} \to p \Longrightarrow \mathcal{G} \propto p$

$\Longrightarrow \forall\, N \in \mathcal{N}_p, N \wedge \mathcal{G} \neq 0, \forall\, G \in \mathcal{G}$

$\therefore \quad \forall\, N \in \mathcal{N}_p, N \wedge F \neq 0, \forall\, F \in \mathcal{F} \sqsubset \mathcal{G}$

$\therefore \quad \mathcal{F} \propto p$

Proposition 1.1.5

Let (X, δ) be a fuzzy topological space, A, a fuzzy set of X, and p, a fuzzy point. Then $A \in \delta$ iff for each prefilter $\mathcal{F}$ on X converging to $p \in A$, we have $A \in \mathcal{F}$.

Proof

Necessity

Assume that $A \in \delta$

i.e. A is an open fuzzy set.

Let $\mathcal{F}$ on $X \to p$ in A.

$\therefore$ By definition, $\mathcal{N}_p \sqsubseteq \mathcal{F}$ (1)

Any open F-set containing p is a neighbourhood of p.

$\therefore$ $A \in \mathcal{N}_p$ (2)

$\Longrightarrow$ $A \in \mathcal{F}$ (from (1))

Sufficiency

Consider arbitrary $p \in A$.

Claim : $\mathcal{N}_p$ is a prefilter and it converges to p.

i) $N_1, N_2 \in \mathcal{N}_p$, $N_1 \wedge N_2$ is a neighbourhood of p.

$\Longrightarrow$ $N_1 \wedge N_2 \in \mathcal{N}_p$

ii) If $N \in \mathcal{N}_p$ and $N' \geq N$, then N' is a neighbourhood of p $\Longrightarrow$ $N' \in \mathcal{N}_p$.

iii) 0 is not a neighbourhood

$\therefore$ $0 \notin \mathcal{N}_p$

Hence $\mathcal{N}_p$ is a prefilter.

Also $\mathcal{N}_p \leq \mathcal{N}_p = \mathcal{F}$

$\therefore$ $\mathcal{N}_p \to p$

By assumption, $A \in \mathcal{F} = \mathcal{N}_p$

$A \varepsilon \mathcal{N}_p, \quad \forall p \varepsilon A$

$\therefore$ By proposition 1.1.1, A is an open F-set.

$\Longrightarrow$ $A \varepsilon \delta$

Proposition 1.1.6

Let (X, δ) be a fuzzy topological space, A, a fuzzy set of X and p, a fuzzy point. Then p is adherent to A iff there is some prefilter F on X $\ni$ $F \to p'$.

Proof

Assume that p is adherent to A.

i.e. $\forall N \varepsilon \mathcal{N}_p, N \not\sqsubset A^c$

Let $F = \mathcal{N}_p'$

We have proved F is a prefilter converging to p'.

i.e. $F \to p'$ and $A^c \not\sqsubset \mathcal{N}_p'$

$\because$ If $A^c \sqsubset \mathcal{N}_p'$, then $A^c \not\sqsubset A^c$ which is a contradiction.

Converse

Let F be a prefilter with $F \to p'$ and $A^c \notin F$

$F \to p' \Longrightarrow \mathcal{N}_p' \sqsubseteq F$

Suppose for some $N \varepsilon \mathcal{N}_p'$, $N \sqsubseteq A^c \notin F$. Then $N \notin F$ contradicting $\mathcal{N}_p' \sqsubseteq F$.

$\therefore$ $\forall N \varepsilon \mathcal{N}_p', N \not\sqsubset A^c$

$\therefore$ p is adherent point of A.

Proposition 1.1.7

Let (X, δ) be a fuzzy topological space, A, a fuzzy set and p, a fuzzy point. Then $A \in \delta^C$ iff whenever F is a prefilter on X such that $A^C \notin F$ and $F \rightarrow p'$, then $p \sqsubseteq A$.

Proof

Assume that $A \in \delta^C$.

Let F be a prefilter on $X \ni A^C \notin F$ and $F \rightarrow p'$.

Then by proposition 1.1.6, p is adherent to A.

$\overline{A} = \cup \{ p/p \text{ is adherent to } A \}$

i.e. $p \in \overline{A}$

$\Longrightarrow A = \overline{A}$ ($\because A \in \delta^C$)

$\therefore p \in A$

Converse

Each fuzzy point p is adherent to A and $p \sqsubset A$.

$\therefore \overline{A} \sqsubset A$ and $A = \overline{A} \Longrightarrow A \in \delta^C$.

Lemma

Let f be a map from X into Y and F, a prefilter on X. Then $\{ f(F) / F \in F \}$ is a base for a prefilter on Y.

Proof

Let F be a prefilter on X.

$G = \{ f(F) / F \in F \}$

Claim

$\mathcal{G}$ is a base for a prefilter on Y. $\mathcal{G}$ is non empty, $\because$ $\mathcal{F}$ is nonempty as it is a prefilter.

i) If $G_1, G_2 \in \mathcal{G}$, then

$G_1 = f(F_1), G_2 = f(F_2)$

$G_1 \wedge G_2 = f(F_1) \wedge f(F_2) = f(F_1 \wedge F_2)$

$f(F_1 \wedge F_2) = G_3 \lesssim G_1 \wedge G_2$

ii) $0 \notin \mathcal{F}$

$\therefore$ $f(0) = \overline{0} \notin \mathcal{G}$

$\therefore$ $\mathcal{G}$ is a base for a prefilter on Y.

Proposition 1.1.8

Let (X, δ) and (Y, δ') be fuzzy topological spaces and f, a map from X into Y. Then f is fuzzy continuous iff whenever $\mathcal{F}$ converges to p, q being a fuzzy point, $f(\mathcal{F})$ converges to f(p).

Proof

Assume that $f : X \to Y$ is fuzzy continuous.

Let $\mathcal{F} \to p$.

Then $\mathcal{N}_p \sqsubseteq \mathcal{F}$

Let $N \in \mathcal{N}_{f(p)}$

From the definition of F-continuous,

$$\exists\ M \in \mathcal{N}_p \ni f(M) \lesssim N \in \mathcal{N}_{f(p)} \tag{1}$$

$$\because\ M \in \mathcal{N}_p \sqsubseteq \mathcal{F} \Longrightarrow f(M) \sqsubseteq f(\mathcal{F}) \tag{2}$$

To Prove : $f(\mathcal{F}) \to f(p)$

Claim : $\mathcal{N}_{f(p)} \sqsubseteq f(\mathcal{F})$

From (1) and (2), whenever $f(M) \in f(\mathcal{F})$, any superset N also $\in f(\mathcal{F})$.

$\therefore \quad N \in f(\mathcal{F})$

$\therefore \quad \mathcal{N}_{f(p)} \sqsubseteq f(\mathcal{F})$

Converse

Assume that whenever $\mathcal{F} \to p$, then $f(\mathcal{F}) \to f(p)$.

To Prove : f is continuous.

Claim : if $N \in \mathcal{N}_{f(p)}$, $\mathcal{F}\ M \in \mathcal{N}_p \ni f(M) \leq N$.

Let $\mathcal{F} = N_p$ (by above claim)

($\mathcal{N}_p$ is a filter)

Let $\mathcal{N}_p \to p$, then $f(\mathcal{N}_p) \to f(p)$

i.e. $\mathcal{N}_p \sqsubseteq \mathcal{F}$

$\mathcal{N}_{f(p)} \sqsubseteq f(\mathcal{F}) = f(\mathcal{N}_p)$

Let $N \in \mathcal{N}_{f(p)} \Longrightarrow N = f(M)$

$\therefore \quad M \in \mathcal{N}_p$

$\therefore \quad f(M) = N \leq N$

$\therefore \quad f(M) \leq N$

Hence the claim.

Theorem 1.1.2

Let $(X, \delta) = \underset{j \in J}{\pi} (X_j, \delta_j)$ be a fuzzy product space, $\mathcal{F}$, a fuzzy point in X. Then,

i) $\mathcal{F} \rightarrow p$ iff for each $j \in J$, $p_j(\mathcal{F}) \rightarrow p_j = \pi_j(p)$.

ii) If $\mathcal{F} \infty p$, then for each $j \in J$, $p_j(F) \infty p_j$.

Proof

Note (1)

Any $p \in X$ is of the form $(p_j)_{j \in J}$

With $p_j \in X_j \ \forall\ j$ and $p_j = \pi_j(p)$ where $\pi_j : X \rightarrow X_j$ is the projection on X_j.

Note (2)

Every projection map is F-continuous.

i.e., π_j is F-continuous.

Sufficiency follows from Proposition 2.3 and the F-continuity of the projection maps.

Necessity

$\beta = \inf \{ V_j \pi_j \ / \ V_j \in \delta_j, \ \forall\ j \in k, \ k \subset I$ and k is a finite set $\}$

$$\begin{tikzcd} X \arrow[r, "\pi_j"] \arrow[rr, bend right, "V_j \circ \pi_j"'] & X_j \arrow[r, "V_j"] & I \end{tikzcd}$$

$\therefore \quad V_j \pi_j \in \delta$

Claim : β is a base for δ

Definition : Product Topology

$\mathcal{S} = \cup \{ \pi_\beta^{-1}(U_\beta) \}$ for product topology.

$B = \cap \{ \pi_\beta^{-1} (U_\beta) \}$ is base for product topology.

$V_j \; \pi_j = \pi_j^{-1}(V_j)$

$V_j \circ \pi_j : X \to I, \; x \in X, \; x = (x_1, x_2, \ldots .)$

$V_j \circ \pi_j(x) = V_j(x_j)$

We know $p_j(\mathcal{F}) \to p_j$

To Prove : $\mathcal{F} \to p$

Claim : $\mathcal{N}_p \sqsubseteq \mathcal{F}$

$\mu \in \beta$ and $p \in \mu$. $\mathcal{F} \; F \in \mathcal{F} \ni F \sqsubset \mu$

$\therefore$ $\mathcal{F}$ is a filter, $\mu \in \mathcal{F}$. i.e., $\mathcal{N}_p \sqsubseteq \mathcal{F}$

$p = ((x_j)_{j \in J}, t)$ be a fuzzy point in

$X = \pi X_i$ and $p \in \mu$

$$p((x_j)_{j \in J}) < \mu((x_j)_{j \in J})$$

$$= \inf \{ V_j(\pi_j(x_j)) j \in J \}$$

$$= \inf \{ V_j(x_j) \}$$

$$t < \inf_{j \in k} (V_j(x_j))$$

But $\pi_j(p_1, p_2, \ldots, p_j, \ldots) = p_j(x_j) = t$

$\therefore$ $\pi_j(p) (x_j) = t < \inf\limits_{j \in k} (V_j(x_j)) < V_j \, x_j \; \forall j$

$\pi_j(p) (x_j) < V_j x_j, \; \forall j$

$p_j(x_j) \leq V_j(x_j)$

$\Longrightarrow \quad p_j \in v_j, \ \forall \ j \in k$

$\therefore \quad V_j$ is a nhd of p_j.

i.e. $\quad V_j \in \mathcal{N}_{p_j} \sqsubseteq p_j(\mathcal{F})$

$\therefore \quad \exists \ \ F_j \in \mathcal{F} \ni V_j = P_j(F)$

i.e. $\quad p_j(F) \lesssim v_j$

Choose $\quad F = \inf_{j \in k} \{ F_j \} \in \mathcal{F}$

Then $F \sqsubset \mu \Longrightarrow \mu \in \mathcal{F}$

$\therefore \quad \Longrightarrow \quad \mathcal{N}_p \sqsubset \mathcal{F} \Longrightarrow \quad \mathcal{F} \to p$

ii) Let $\mathcal{F} \infty p$

Proposition (1.1.4) $\Longrightarrow$

There is a prefilter $\mathcal{G} \ni \mathcal{G} \sqsupset \mathcal{F}$ and $\mathcal{G} \to p$.

(i) $\Longrightarrow \mathcal{G} \to p \Longrightarrow \quad p_j(\mathcal{G}) \to p_j$ and $p_j(\mathcal{F}) \sqsubset p_j(\mathcal{G})$

Applying the converse of proposition (1.1.4),

$p_j(\mathcal{G}) \sqsupset p_j(\mathcal{F})$ and $p_j(\mathcal{G}) \to p_j$

$\therefore \quad p_j(\mathcal{F}) \infty p_j$

Hence the proof.

SECTION : 2

ASSOCIATION BETWEEN FILTER AND PREFILTER

Definition 1.2.1

A prefilter $\mathcal{F}$ on X is an F-ultrafilter if there is no other prefilter finer than $\mathcal{F}$.

Definition 1.2.2

Let $\mathcal{F}$ be a prefilter on X. A subset Y of X is included in $\mathcal{F}$ if every fuzzy subset with support Y is an element of $\mathcal{F}$.

Proposition 1.2.1

If X is a set and $\mathcal{F}$ a prefilter on X, the following are equivalent.

i) $\mathcal{F}$ is an F-ultrafilter.

ii) Let $A \in I^X$. If $A \notin \mathcal{F}$ then there is some $F \in \mathcal{F}$ such that $A \wedge F = 0$.

iii) Let Y be a subset of X. Then either Y or Y^C is included in $\mathcal{F}$.

(i) $\Longrightarrow$ (ii)

(i) $\mathcal{F}$ is an ultrafilter.

Let $A \notin \mathcal{F}$ for every $F \in \mathcal{F}$ say $A \wedge F \neq 0$.

Consider $B = \{ A \wedge F \,/\, F \in \mathcal{F} \}$

Consider $\mathcal{G}$ be prefilter generated by B.

Claim : $\mathcal{G}$ is strictly finer than $\mathcal{F}$.

Proof : By the definition of prefilter generated by B,

$A \supset A \wedge F$ is an element of $\mathcal{G}$ (1)

$\forall F$, $F \supset A \wedge F$ is an element of $\mathcal{G}$ (2)

$\therefore$ (2) $\Longrightarrow$ $\mathcal{F} \subset \mathcal{G}$

(1) $\Longrightarrow$ $\mathcal{G} \not\subset \mathcal{F}$. $\because$ $A \notin \mathcal{F}$, but $A \in \mathcal{G}$

$\therefore$ $\mathcal{G}$ is strictly finer than $\mathcal{F}$

$\therefore$ Contradiction to the definition of ultrafilter. Hence there is some $F \in \mathcal{F} \ni A \wedge F \neq 0$.

(ii) $\Longrightarrow$ (iii)

Let $Y \subset X$

Let A be a fuzzy set with support Y. Then

$$Y = \{ x \in X \,/\, A(X) > 0 \}$$

Let B be a fuzzy set with support Y^c. Then

$$Y^c = \{ x \in X \,/\, B(x) > 0 \}$$

Let $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$

By (ii), $\exists\ F_A, F_B \in \mathcal{F} \ni F_A \wedge A = 0$

$$F_B \wedge B = 0$$

$\forall\ x \in X,\ (F_A \wedge A)(x) = 0$

$$\inf \{ F_A(x), A(x) \} = 0$$

i.e. $F_A(x) = 0,\ \forall\ x \in Y$

$|||^{ly}$ $F_B(x) = 0,\ \forall\ x \in Y^c$

Consider $F_A \wedge F_B = 0$

But $0 \notin \mathcal{F}$

Contradiction that $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$ which is not possible.

i.e. Either $A \in \mathcal{F}$ or $B \in \mathcal{F}$

$\therefore$ Either Y is included in $\mathcal{F}$

(or) Y^c is included in $\mathcal{F}$

(iii) $\rightarrow$ (i)

Let $\mathcal{F}$ is not an ultrafilter.

$\therefore$ $\exists$ another prefilter $\mathcal{G} \ni \mathcal{F} \subset \mathcal{G}$

$\therefore$ $\exists\ G \in \mathcal{G} \ni G \notin \mathcal{F}$

Let Y be the support of G.

i.e. $Y = \{ x \in X \, / \, G(x) > 0 \}$ (1)

Let B be any fuzzy set with support Y^c.

$$Y^c = \{ x \in X \, / \, B(x) > 0 \} \quad (2)$$

Since by definition 1.2.2 of inclusion and from (1) Y is not included in $\mathcal{F}$ and from (iii) and from (2), Y^c is included in $\mathcal{F}$ and again by definition, $B \in \mathcal{F} \sqsubset \mathcal{G}$

Now $B \in \mathcal{G}$ and $G \in \mathcal{G}$

$\therefore \quad B \wedge G \in \mathcal{G}$

But from (1) and (2), $B \wedge G = 0 \notin \mathcal{G}$

which is a contradiction.

i.e. $\mathcal{F}$ is an ultra-filter.

Definition 1.2.3

A prefilter $\mathcal{F}$ is free iff $\cap \{ F/F \in \mathcal{F} \} = \varphi$.

Definition 1.2.4

A prefilter $\mathcal{F}$ is fixed iff $\cap \{ F/F \in \mathcal{F} \} \neq \varphi$

Note

If a prefilter is not free, then it is fixed.

Proposition 1.2.2

Every fuzzy ultra-filter $\mathcal{F}$ is free.

Proof

Let $\mathcal{F}$ be a fuzzy ultra-filter.

i.e. If $\mathcal{G} \supset \mathcal{F}$ then, $\mathcal{G} = \mathcal{F}$

To prove : $\mathcal{F}$ is free.

i.e., **Claim** : $\cap \{F/F \in \mathcal{F}\} = 0$.

Let take $\cap \{F/F \in \mathcal{F}\} \neq \phi$

$\Longrightarrow \exists\ p \in \cap \{F/F \in \mathcal{F}\}$

$\Longrightarrow p \in F, \forall F \in \mathcal{F}$

Consider a fuzzy point $q = (x, s)$

where $p = (x, t)\ \forall s \in (0, 1]$

$Y = \text{Supp } p = \{x\}$

$\because \text{Supp } q = \{x\} = Y$.

By (iii) of proposition 1.2.1,

$\therefore p \in \mathcal{F}$, Y is included in $\mathcal{F}$.

Hence $q \in \mathcal{F}$

$\because q = F$ for some $F \in \mathcal{F}$, $p \in q$

i.e., $t < s$

But sometimes $s < t$ also $\because s \in (0, 1]$ whiich is a contradiction.

Hence the proof.

Definition 1.2.5

An F-ultrafilter $\mathcal{F}$ is strong free iff

$\cap \{\text{supp } F/F \in \mathcal{F}\} = \phi$

Clearly if $\cap \{\text{Supp } F/F \in \mathcal{F}\} \neq \varphi$ where $\mathcal{F}$ is an ultrafilter, then this intersection contains only one point.

Example

$\mathcal{F}_x = \{ F \in I^X / F(x) > 0 \}$ is an s-fixed ultrafilter.

Proof

i) $\mathcal{F}_x$ is a prefilter.

(a) $F_1, F_2 \in \mathcal{F}_x \Longrightarrow F_1 \wedge F_2 \in \mathcal{F}$

i.e. $F_1(x) > 0, \quad F_2(x) > 0$

$\therefore \quad (F_1 \wedge F_2)(x) > 0$

$\therefore \quad F_1 \wedge F_2 \in \mathcal{F}$

(b) $F \in \mathcal{F}$

Let $F' \geq F$

Then, $F'(x) \geq F(x) > 0$

$\Longrightarrow F'(x) > 0 \Longrightarrow F' \in \mathcal{F}$

(c) $\because F(x) > 0, \quad 0 \notin \mathcal{F}$

ii) $\mathcal{F}_X$ is an ultrafilter.

Let $A \notin \mathcal{F}_x$, if for $\forall F \in \mathcal{F}_x$

$A \wedge F \neq 0$

$\inf \{ A(x), F(x) \} \neq 0$

i.e., $A(x) > 0 \Longrightarrow A \in \mathcal{F}_x$

Which is a contradiction.

$\therefore \exists$ atleast one $F \in \mathcal{F}_x \ni$

$A \wedge F = 0$ (proposition 1.2.1)

iii) $\mathcal{F}_x$ is s-fixed.

To prove : $\cap \{ \text{Supp } F / F \in \mathcal{F}_x \} \neq \varphi$

Claim : $x \in \cap \{ \text{Supp } F / F \in \mathcal{F}_x \}$

$\because F(x) > 0, \forall F \in \mathcal{F}$

$x \in \text{supp } F, \forall F \in \mathcal{F}$

$\therefore x \in \cap \{ \text{Supp } F / F \in \mathcal{F} \}$

Hence $\cap \{ \text{Supp } F / F \in \mathcal{F} \} \neq 0$

$\therefore \mathcal{F}_x$ is s-fixed ultrafilters.

Proposition 1.2.3

If X and Y are set of point, $f : X \to Y$ a map from X into Y, and $\mathcal{F}$ an F-ultrafilter on X, then $f(\mathcal{F})$ is an F-ultrafilter on Y.

Proof

$Y_0 \subset Y$

Let $X_0 = f^{-1}(Y_0)$

$\because \mathcal{F}$ is an F-ultrafilter, either X_0 or X_0^c is included in $\mathcal{F}$.

If $f^{-1}(Y_0) = X_0 = \phi$, X_0 is not included in $\mathcal{F}$, in that case X_0^c is included.

Claim : Y_0 is included in $f(\mathcal{F})$.

Let A be a F-set with Supp Y_0.

Supp A $= Y_0$

Let B $= f^{-1}(A)$

Supp B $=$ Supp $f^{-1}(A) = f^{-1}(\text{Supp } A)$

$= f^{-1}(Y_0) = X_0$

$\therefore$ Supp B $= X_0$

$\therefore$ $B \in F$ ($\because$ X_0 is included in F, every fuzzy set with Supp X_0 is an element of F).

$\therefore$ $f(B) \in f(F)$

But $f(B) \leq A$

$A \in f(F)$ ($\because$ $f(F)$ is prefilter)

$\therefore$ A is arbitrary fuzzy set with support Y_0, every fuzzy set with supp Y_0 is in $f(F)$.

$\therefore$ Y_0 is included in $f(F)$.

Hence $f(F)$ is an ultrafilter.

Definition 1.2.6

An ultrafilter on a set X is a maximal filter on X. i.e. a filter a on a set X is an ultrafilter on X if given any filter a' finer than a, then $a = a'$.

Theorem 1.2.1

Let X be a set of points, U, the family of all ultrafilters on X and U_r, the family of all F-ultrafilters. We define two maps.

$a : U \to U_r$ by : $a(G) = \{ F \in I^X / \text{Supp } F \in G \}$, $\forall\, G \in U$

and

$b : U_F \to U$ by : $b(F) = \{ \text{supp } F / F \in F \}$, $\forall\, F \in U_F$

a and b are well defined and they are inverse to each other.

Proof

a(G) is an F-ultrafilter.

i) It is a prefilter.

$F_1, F_2 \in a(\mathbf{G})$

$\Longrightarrow$ $\text{Supp } F_1 \in \mathbf{G}$

$\text{Supp } F_2 \in \mathbf{G}$

$\text{Supp } (F_1 \wedge F_2) \subseteq (\text{Supp } F_1) \cap (\text{Supp } F_2)$

Proof

Let $x \in \text{Supp } (F_1 \wedge F_2)$

Then $(F_1 \wedge F_2)(x) > 0$

$\inf \{ F_1(x), F_2(x) \} > 0$

$F_1(x) > 0$

$F_2(x) > 0$

$\therefore$ $x \in (\text{Supp } F_1) \cap (\text{Supp } F_2)$

$\therefore$ $\text{Supp } (F_1 \wedge F_2) \in \mathbf{G}$ ($\because$ $\mathbf{G}$ is an ultrafilter)

(ii) To prove

If $F \in a(\mathbf{G})$ and $F' > F$, then $F' \in a(\mathbf{G})$.

$F \in a(\mathbf{G}) \Longrightarrow \text{Supp } F \in \mathbf{G}$ (1)

$\because$ $F' > F$, $\text{Supp } F' \supset \text{Supp } F$ (2)

Proof

Let $x \in \text{Supp } F$

$\Longrightarrow$ $F(x) > 0$

$\because$ $F < F'$, $F(x) < F'(X)$

$\therefore$ $F'(x) > 0$

$\Longrightarrow$ $x \in \text{Supp } F'$

$\therefore$ Supp $F' \supset$ Supp F

$\Longrightarrow$ Supp $F' \in G$ ($\because$ G is an ultrafilter)

$\therefore$ $F' \in a(G)$

Hence $a(G)$ is a prefilter.

To prove : $a(G)$ is an F-ultrafilter.

Let Y be a subset of X.

If Y is not included in $a(G)$,

$\exists$ a fuzzy set $A \ni$ Supp $A = Y$.

i.e., Supp $A \notin G$

$\because$ G is an ultrafilter on X.

$\therefore$ X - Supp $A \in G$.

Then $F \in a(G) \ni$ Supp F = X - Supp A.

To Prove : $a(G)$ is an ultrafilter.

Let $A \notin a(G)$

$\Longrightarrow$ Supp $A \notin G$

X - Supp $A \in G$

Let $F \in I^X$ is such that

Supp F = X - Supp A

$Q = X - P$

$\forall\, x \in P$, $(A \wedge F)(x) = \inf (A(x), F(x))$

$= 0$ ($\because$ $F(x) = 0$)

$\forall\, x \in Q$, $(A \wedge F)(x) = \inf (A(x), F(x))$

$= 0$ ($\because$ $A(x) = 0$)

$\therefore\ A \wedge F = 0$

Hence $a(\mathbf{G})$ is an ultrafilter.

$x \notin \mathrm{Supp}\ A \iff x \in \mathrm{Supp}\ F$

$|||^{ly}$ $b(\mathcal{F})$ is an ultrafilter.

Let $\mathbf{G} \in \mathbf{U}$

$a(\mathbf{G}) \in \mathbf{U}_F$

$$\begin{aligned} \text{Consider } b(a(\mathbf{G})) &= \{ \mathrm{Supp}\ F \ / \ F \in a(\mathbf{G}) \} \\ &= \{ \mathrm{Supp}\ F \ / \ F \in a(\mathbf{G}) \Longrightarrow \mathrm{Supp}\ F \in \mathbf{G} \} \\ &\sqsubseteq \mathbf{G} \end{aligned}$$

$\therefore\ (b \circ a)\ (\mathbf{G}) \sqsubseteq \mathbf{G}$

To prove $\mathbf{G} \sqsubseteq (b \circ a)\ (\mathbf{G})$

$$\begin{aligned} \text{Define}\quad \mathcal{F} &= \{ F \in I^X \ / \ \mathrm{Supp}\ F = G,\ G \in \mathbf{G} \} \\ &= a(\mathbf{G}) \end{aligned}$$

$b(\mathcal{F}) = b \circ a(\mathbf{G})$

Now, $\mathbf{G} \sqsubseteq b(\mathcal{F}) = (b \circ a)\ (\mathbf{G})$

$\therefore\ \mathbf{G} \sqsubseteq (b \circ a)\ (\mathbf{G})$

$|||^{ly}$ $(a \circ b)\ (\mathcal{F}) = \mathcal{F}\ \ \forall\ \mathcal{F} \in \mathbf{U}_F$

$\therefore$ a and b are inverse maps.

SECTION : 3

RELATION BETWEEN F-NETS AND F-FILTERS

Definition 1.3.1

Let $\mathcal{F}$ be a prefilter or X, P_F, the collection of all fuzzy points in X, and

$\Lambda_{\mathcal{F}} = \{ (p, F) / p \in F, p \in \mathcal{P}_F(X), F \in \mathcal{F} \} \subset \mathcal{P}_F(X) \times \mathcal{F}$ directed by the relation $(p_1, F_1) \leq (p_2, F_2)$ iff $F_2 \leq F_1$. The map $\psi_{\mathcal{F}} : \Lambda_{\mathcal{F}} \to \mathcal{P}_F(X)$ defined by $\psi_{\mathcal{F}}(p, F) = p$ is an F-net in X. It is called the F-net based on $\mathcal{F}$.

Definition 1.3.2.

Let $\psi : D \to \mathcal{P}_F(X)$ be an F-net in X. The F-filter generated by the collection B of all the F-subsets in which the F-net ψ is residually contained is called the F-filter generated by ψ. It is denoted by $\mathcal{F}_\psi$.

Theorem 1.3.1

Let (X, δ) be an fuzzy topological space, $\mathcal{F}_\psi$ an F-filter on X and p, a fuzzy point. Then

i) $\mathcal{F}$ converges to p iff $\psi_{\mathcal{F}}$ converges to p.

ii) ψ converges to p iff $\mathcal{F}_\psi$ converges to p.

iii) $\mathcal{F}$ has p as a cluster point iff $\psi_{\mathcal{F}}$ has p as a cluster point.

iv) ψ has p as a cluster point iff $\mathcal{F}_\psi$ has p as a cluster point.

Proof

((i) $\Longrightarrow$)

Let $N \in \mathcal{N}_p \subset \mathcal{F}$

Then $(p, N) \in \Lambda_{\mathcal{F}}$ and $\psi(q, M) = q \in M \subset N$, $\forall (q, M) \geq (p, N)$. Thus the F-net is residually in every neighbourhood of p.

($\Longleftarrow$)

Let $N \in \mathcal{N}_p$

There is some $(q, F) \in \Lambda_{\mathcal{F}}$ such that if $(p', F') \gtrsim (q, F)$, then $\psi_{\mathcal{F}}(p', F') = p' \in N$. Hence $F' \sqsubset N$ and $N \in \mathcal{F}$.

((ii) $\Longrightarrow$) Let $N \in \mathcal{N}_p$

There is some $d_0 \in D$ such that $p_d \in N$, $\forall\, d \gtrsim d_0$.

$$p_d = (x_d, t_d)$$

Consider the fuzzy set,

$$A = \{\, q_d \equiv (x_d, \frac{t_d + N(x_d)}{2}),\ d \gtrsim d_0 \,\}$$

$A \in \mathcal{F}_\psi$, since $p_d(x_d) = t_d < q_d(x_d) \lesssim A(x_d)$, $\forall\, d \gtrsim d_0$ and $A \sqsubset N$. Since $\mathrm{Supp}\, A = \{\, x_d / d \gtrsim d_0 \,\}$ and

$$A(x_d) = \mathrm{Sup}_{d \gtrsim d_0} \{\, q_d(x_d) \,\} \lesssim N(x_d).\ \text{Hence } N \in \mathcal{F}.$$

($\Longleftarrow$) Let $N \in \mathcal{N}_p \sqsubset \mathcal{F}_\psi$. There is some F-subset B such that $p_d \in B$, $\forall\, d \gtrsim d_B$ and $B \sqsubset N$. Then $p_d \in N$, $\forall\, d \gtrsim d_B$.

((iii) $\Longrightarrow$)

Let N be a neighbourhood of p, and (p_0, F_0) an element of $\Delta_{\mathcal{F}}$. By hypothesis, there is some $q \in N \wedge F_0$. Hence, there is $(q, F_0) \gtrsim (p_0, F_0)$ such that $\psi_{\mathcal{F}}(q, F_0) = q \in N$. So $\psi_{\mathcal{F}} \infty\, p$.

($\Longleftarrow$) Let N be an neighbourhood of p ; F, an element of $\mathcal{F}$, and $p \in F$. Then $(p, F) \in \Lambda_{\mathcal{F}}$ and there is, by hypothesis, some $(p_0, F_0) \geq (p, F)$ such that $\psi_F(p_0, F_0) = p_0 \in N$. Then $p_0 \in F \wedge N$ and $F \infty p$.

((iv) $\Longrightarrow$)

Let N be an neighbourhood of p and $A \in I^X$ such that there is some $d_0 \in D$ with $\psi(d) \in A$, $\forall\, d \geq d_0$. By hypothesis, there is some $d' \geq d_0$ such that $\psi(d') \in N$.

Since $\psi(d')(x_{d'}) < A(x_{d'}) \wedge N(x_{d'})$ and $N \wedge A \neq \phi$ for each fuzzy subset which contains a tail of the net, $F_\psi \infty p$.

($\Longleftarrow$) Let $N \in \mathcal{N}_p$ and $d \in D$. Consider the F-subset :

$$A = \{ q_{d'} = (x_{d'}, \frac{(1 + t_{d'})}{2}) \;/\; d' \geq d \}, \text{ where}$$

$$(x_{d'}, t_{d'}) = \psi(d')$$

Clearly $\psi(d')(x_{d'}) = t_{d'} < \dfrac{1 + t_{d'}}{2} \leq A(x_{d'})$ $\forall\, d' \geq d$ and then $A \in B$. Besides, $A \wedge N \neq \phi$, as, by hypothesis,

$$\text{Supp } A = \{ x_{d'} / d' \geq d_0 \}$$

Hence the theorem.

Definition 1.3.3

An F-ultranet ψ in X is a trivial F-ultranet if $\exists\, d_0 \in D$ such that $x_d = x$, $\forall\, d \geq d_0$ with $\psi_d = (x_d, t_d)$.

Proposition 1.3.1

Let $\{p_d\}$ be an F-net. Then $\{p_d\}$ is a trivial F-ultranet if $\exists$ $d_0 \in D$ and $\{x_d\}_{d \gtrsim d_0}$ is a trivial ultranet and $t_d \to 0$.

Theorem 1.3.2

Let (X, δ) be an fuzzy topological spaces, F, an F-filter, ψ, an F-net on X. Then

i) The F-net based on an F-ultrafilter is an F-ultranet.

ii) The F-filter based on an F-ultranet is an F-ultrafilter.

iii) If ψ is a trivial F-ultranet, F_ψ is a trivial F-ultrafilter.

iv) If F is a trivial F-ultrafilter, ψ_F is a trivial F-ultranet.

Proof

(i) Let F be an F-ultrafilter and $Y \subset X$. Suppose that Y is included in F. Then for each $A \in I^X$ with Supp $A = Y$, $A \in F$. Let $p \in A$ $(A \neq \phi)$. Infact, $(p, A) \in \Lambda_F$ and for each $(q, F) \in \Lambda_F$ with $(q, F) \gtrsim (p, A)$, $\psi_F(q, F) = q \in F \lesssim A$. Then, ψ_F is residually in A, A being a fuzzy set with support A.

(ii) It follows from Definition 1.3.2.

(iii) If $\psi \equiv \{p_d\}$ is a trivial F-ultranet, there is some $d_0 \in D$ with $x_d = x$ whenever $d \gtrsim d_0$. Besides, for each $s > 0$, there is some $d_1 \in D$ with $t_d < s$, whenever $d \gtrsim d_1$.

We will prove that $\mathcal{F}_{\psi} = \{ F \in I^X / F(x) > 0 \}$. If $F \in \mathcal{F}_{\psi}$, there is some $B \in I^X$ and some $d_2 \in D$ such that $p_d \in B \;\forall\; d \gtrsim d_2$ and $B \subset F$. Choose $d^* \in D$ such that $d^* \gtrsim d_0$ and $d^* \gtrsim d_2$. Then $p_d \in B, \;\forall\; d \gtrsim d^*$ and $x_d = x$. Then, $p_d(x_d = x) < B(x) \lesssim F(x)$ if $d \gtrsim d^*$.

Consequently, $F(x) > 0$

If F is a fuzzy set and $x \in \text{Supp } F$, there is some $s > 0$ such that $s < F(x)$, and, by hypothesis, $\exists\; d_1 \in D$ such that $t_d < s < F(x), \;\forall\; d \gtrsim d_1$. This is to say that $F \in \mathcal{F}_{\psi}$.

(iv) If $\mathcal{F}$ is a trivial F-ultrafilter,

$\mathcal{F} = \{ F \in I^X / F(x) > 0 \}$. Then, any F-point with support x is an element of the F-filter $\mathcal{F}$. Let $p \equiv (x, t_p)$, $q \equiv (x, t_q)$ be two F-points with support x and such that $p \in q$. Then $(p, q) \in \Lambda_{\mathcal{F}}$ and for each $(p', F) \in \Lambda_{\mathcal{F}}$ with $(p', F) \gtrsim (p, q)$, $\text{Supp } F = x$ and hence $\text{Supp } \psi_{\mathcal{F}}(p', F) = x$. Therefore, the net $\{ \text{Supp } \psi_{\mathcal{F}}(p', F) \}_{(p',F) \gtrsim (p,q)}$ is a trivial ultranet.

Besides, for each $s > 0$, let $p_s \equiv (x, \frac{s}{4})$ and $q_s \equiv (x, \frac{s}{2})$ be two F-points with $p_s \in q_s$.

Then $(p_s, q_s) \in \Lambda_{\mathcal{F}}$, and for every $(p, F) \in \Lambda_{\mathcal{F}}$ With $(p, F) \gtrsim (p_s, q_s)$, $F \equiv (x, t)$ with $t \lesssim \frac{s}{2}$ and $p \equiv (x,r)$ With $r < t$. Hence $(p, F)(x) = r < s$.

Hence the proposition.

CHAPTER II

CHAPTER - II

t - PREFILTER THEORY

INTRODUCTION

In this Chapter, definitions of the two concepts, t-prefilter and the convergence of t-prefilter are discussed in detail as in the paper "t-Prefilter Theory" by M.A.DE PRADA VICENTE and M.MACHO STADLER [21].

In Section 1, with every t-prefilter on X, we associate a filter on X. In a similar way associated to each filter on X, we define a t-prefilter with every t in [0, 1]. The relation between these associations are proved in a theorem. (Proposition 2.1.1).

In Section 2 of this Chapter, t-convergence of t-prefilter is defined. The relation between t-convergence and t-prefilter are stated in the theorem 2.2.1. The relation between t-convergence and convergence are obtained in the theorem 2.2.2.

SECTION : 1

t-PREFILTERS ON X

Definition 2.1.1

A prefilter $\mathcal{F}$ is called a prime prefilter iff for all $\mu, \gamma \in I^X$ such that $\mu \vee \gamma \in \mathcal{F}$, we have either $\mu \in \mathcal{F}$ or $\gamma \in \mathcal{F}$.

Definition 2.1.2

Let $P^t(X)$ denote the set of all prefilters on X, which exclude the constant function c_t. A member of $p^t(X)$ will be called a t-prefilter on X.

Definition 2.1.3

Associated to each t-prefilter $\mathcal{F}$, we define a filter on X through an application i_t as follows :

$i_t \; : \; P^t(X) \rightarrow F(X)$ such that for each t-prefilter ,

$i_t(\mathcal{F}) = \{ \mu^{-1}(t, 1] \, / \, \mu \in \mathcal{F} \}$

Note

Since $0 \notin \mathcal{F}$ for each prefilter $\mathcal{F}$ on X i_0 is always defined.

Definition 2.1.4

Associated to each filter **F** on X, we define a t-prefilter for each $t \in [0, 1)$ as follows.

$$w_t \;:\; F(X) \rightarrow P^t(X)$$

such that for each filter $\mathbf{F}$,

$$w_t(\mathbf{F}) \;=\; \{\, \mu \in I^X \;/\; \mu^{-1}(t,\, 1] \in \mathbf{F} \,\}$$

Proposition 2.1.1

Let $\mathcal{F}$ be a prefilter on X. Let $\mathbf{F}$ be a filter on X. Then,

i) for each $s < t$, we have $i_t(\mathcal{F}) \sqsupset i_s(\mathcal{F})$ and $w_t(\mathbf{F}) \sqsubset w_s(\mathbf{F})$.

ii) $i_t \circ w_t(\mathbf{F}) = \mathbf{F}$ and if $s < t$, $i_s \circ w_t(\mathbf{F}) = \mathbf{F}$.

iii) $\mathcal{F} \sqsubset w_t \circ i_t(\mathcal{F})$ and if $s < t$, $\mathcal{F} \sqsubset w_s \circ i_t(\mathcal{F})$.

iv) i_t, w_t are respectively an isotone surjection and an isotone injection.

Note

If $\mathcal{F}$ is a maximal t-prefilter, then since $w_t \circ i_t(\mathcal{F})$ is also a t-prefilter, it follows from the proposition stated above that $\mathcal{F} = w_t \circ i_t(\mathcal{F})$.

Proposition 2.1.2

If $\mathcal{F}$ is either a prime t-prefilter or a maximal t-prefilter, then $i_t(\mathcal{F})$ is an ultrafilter on X. If $\mathbf{F}$ is an ultrafilter on X, then $w_t(\mathbf{F})$ is a maximal t-prefilter.

Proof

Let A be a subset of X. Since $\mathcal{F}$ is prime, we have, either $\psi_A \in \mathcal{F}$ or $\psi_{X \setminus A} \in \mathcal{F}$ and thus either

$$\psi_A^{-1}(t, 1] = A \in i_t(\mathcal{F})$$

or

$$\psi_{X \setminus A}^{-1}(t, 1] = X \setminus A \in i_t(\mathcal{F})$$

If $\mathcal{F}$ is t-maximal, we know that $\mathcal{F} = w_t \circ i_t(\mathcal{F})$. If there exists a filter $\mathbf{F}$ finer than $i_t(\mathcal{F})$, then $\mathcal{F} = w_t \circ i_t(\mathcal{F}) \subset w_t(\mathbf{F})$ and so $\mathcal{F} = w_t(\mathbf{F})$. Then we conclude that $i_f(\mathcal{F}) = \mathbf{F}$. As for the second statement, let $\mathcal{G}$ be a t-prefilter strictly less than $w_t(\mathcal{F})$. Then there exists $\mu \in \mathcal{G}$ such that $\mu \notin w_t(\mathbf{F})$ which means $\mu^{-1}(t, 1] \notin \mathbf{F}$ and $\mu^{-1}(t, 1] \in i_t(\mathbf{G})$. Then $i_t(\mathcal{F})$ is strictly finer than $\mathbf{F}$, which is impossible since $\mathbf{F}$ is an ultrafilter.

Proposition 2.1.3

In $\mathcal{F}$ is a maximal t-prefilter, then it is a prime t-prefilter.

Proof

Let $\mu, \gamma \in I^X$ such that $\mu \vee \gamma \in \mathcal{F}$;
Then $(\mu \vee \gamma)^{-1}(t, 1] \in i_t(\mathcal{F})$ which is an ultrafilter. Now, if $\mu^{-1}(t, 1] \in i_t(\mathcal{F})$, then $\mu \in \mathcal{F}$; if it is not so, then $\gamma^{-1}(t, 1] \in i_t(\mathcal{F})$ and $\gamma \in \mathcal{F}$. Then $\mathcal{F}$ is a prime t-prefilter.

Corollary

If F is a filter on X, F is an ultrafilter iff $w_t(F)$ is a prime t-prefilter, iff $w_t(F)$ is a maximal t-prefilter.

SECTION : 2

t-CONVERGENCE OF PREFILTERS

Let (X, δ) be an fuzzy topological space ;
For each $t \in [0, 1)$ we consider the topology

$$i_t(\delta) = \{ i^{-1}(t, 1] ; \mu \in \delta \}$$

If (X, τ) is a topological space, and if for all $t \in [0,1)$ we put $w_t(\tau) = \{ \mu \in I^X : \mu^{-1}(t_1)] \in \tau \}$ then $w_t(\tau)$ is a fuzzy topology on X, which will be called t-topologically generated fuzzy topology. Clearly, $w(\tau) = \bigcap_{t \in [0,1)} w_t(\tau)$ where w is Lowen's application.

Proposition 2.2.1

For each topology τ on X, fuzzy topology δ on X and $s \in [0, 1)$ we have

i) $i_t \circ w_t(\tau) = \tau$

ii) w_t, i_t are respectively an isotone injection and an isotone surjection for each $t \in [0, 1)$.

iii) $\tau \sqsubset i_s \circ w_t(\tau)$

iv) $\delta \sqsubset w_t \circ i_t(\delta)$

Note

To define fuzzy filter convergence, we need to consider the following set containing $\mathcal{B}_F(X)$:

$$\mathcal{B}_F^*(X) = \{ p = (x, t) \text{ where } t \in [0, 1), x \in X \}$$

Definition 2.2.1

Let (X, δ) be an fuzzy topological space, $F \in P(X)$ and $p(x, t) \in \mathcal{B}_F^*(X)$. We say that F t-convergences to $p(x, t)$ (denoted by $F \overset{t}{\to} p(x, t)$) if

i) F is a t-prefilter.

ii) The filter $i_t(F)$ converges to x in $(X, i_t(\delta))$.

And we say that F has $p(x, t)$ as a t-cluster point (denoted by $F \overset{t}{\infty} p(x, t)$) if

i) F is a t-prefilter.

ii) The filter $i_t(F)$ on X has x as a cluster point in $(X, i_t(\delta))$.

If F is either a fuzzy ultrafilter on X or a prefilter on X including all the constants, only i_0 can be defined ; then the convergence of F in both cases is equivalent to convergence in the support.

Note

i) If $\mathcal{F} \in p^t(X)$, $\mathcal{F} \overset{t}{\rightarrow} p(x, t) \in \mathcal{B}_F(X)$ iff $\mathcal{N}_p \sqsubset w_t \circ i_t(\mathcal{F})$. Thus if $\mathcal{F}$ is a maximal t-prefilter, then $\mathcal{F} \overset{t}{\rightarrow} p(x, t)$ iff $\mathcal{N}_p \sqsubset \mathcal{F}$.

ii) If $\mathcal{F}$ is a t-prefilter and $\mathcal{F} \overset{t}{\rightarrow} p(x, t) \in \mathcal{B}_F(X)$, then $V \wedge F \neq 0$ for each $V \in \mathcal{N}_p$ and $F \in \mathcal{F}$.

iii) A t-prefilter $\mathcal{F} \overset{t}{\infty} p(x, t) \in \mathcal{B}_F(X)$ iff there exists an ultrafilter U on X, t-compatible with both $\mathcal{N}_p$ and $\mathcal{F}$.

Theorem 2.2.1

Let $\mathcal{F}$, $\mathcal{G}$ be t-prefilters on X.

i) If $\mathcal{F} \sqsubset \mathcal{G}$ and $\mathcal{F} \overset{t}{\rightarrow} p(x, t)$, then $\mathcal{G} \overset{t}{\rightarrow} p(x, t)$.

ii) If $\mathcal{F} \overset{t}{\rightarrow} p(x, t)$, then $\mathcal{F} \overset{t}{\infty} p(x, t)$.

iii) If $\mathcal{F}$ is prime, then $\mathcal{F} \overset{t}{\rightarrow} p(x, t)$ iff $\mathcal{F} \overset{t}{\infty} p(x, t)$.

iv) $\mathcal{F} \overset{t}{\infty} p(x, t)$ iff there exists a t-prefilter $\mathcal{G}$ finer than and $\mathcal{G} \overset{t}{\rightarrow} p(x, t)$.

Consequently, if $\mathcal{F} \sqsubset \mathcal{G}$, then $\mathcal{G} \overset{t}{\infty} p(x, t)$ implies $\mathcal{F} \overset{t}{\infty} p(x, t)$.

Definition 2.2.2

Let $\mathcal{G}$ be a prefilter on X ;

We define $\mathrm{Lim}(\mathcal{G}) : X \rightarrow I$ by

$$\mathrm{Lim}(\mathcal{G})\ (x) = \mathrm{Sup}\{\, t : \mathcal{G} \overset{t}{\rightarrow} p(x, t)\,\}$$

and $\mathrm{Adh}(G) : X \to I$ by

$$\mathrm{Adh}(G)(x) = \mathrm{Sup}\{ t : G^{t} \infty p(x, t) \}$$

Remark

If $G \xrightarrow{t} p(x, t)$, then $t \leq \mathrm{Lim}(G)(x)$.

Theorem 2.2.2

Let (X, δ) be a t-topologically generated fuzzy topological space. i.e., $\delta = w_t(\tau)$ for some $t \in [0, 1)$. Then

i) $\mathrm{Lim}(w_t(F)) = t.\chi\, \mathrm{Lim}(F)$

ii) $\mathrm{Adh}(w_t(F)) = t.\chi\, \mathrm{Adh}(F)$.

Proof

It follows immediately from the fact that $w_t(F) \xrightarrow{t} p(x, t)$ iff $F \to x$ in $\tau = i_t(w_t(\tau))$.

CHAPTER III

CHAPTER - III

FUZZY CONVERGENCE VERSUS CONVERGENCE

INTRODUCTION

A new approach to fuzzy convergence of prefilters through the convergence of some family of filters on the collection of its level topologies is discussed. For this purpose with every filter the characteristic set Δ_{F} is defined. The important problem studied here is as follows.

"Suppose that Δ is an interval in $[0, 1]$ of type $[0, t]$ for $t \in [0, 1)$ or $[0, t)$ for $t \in (0, 1)$ and $\{ F_t / t \in \Delta \}$ is a family of filters on X. Verifying the inclusions $F_s \subset F_t$ if $s \leq t$, then the question is "Is there any prefilteration X such that $\Delta = \Delta_{F}$ for each $t \in \Delta$ we have $i_t(F) = F_t$?" This problem is solved in the theorem 3.2.

Definition 3.1

A prefilter F is maximal in $P^t(X)$ if it is not stirctly included in any other t-prefilter. Let $M^t(X)$ denote the family of all maximal t-prefilters on X.

Definition 3.2

With every t-prefilter F, we define

$$\ell_t(F) = \{ \mu^{-1}(t, 1] \, / \, \mu \in F \}$$

Claim

$\ell_t(F)$ is a filter.

i) $\ell_t(F)$ is nonempty since F is nonempty.

$F = \{ \mu \in I^X \, / \, \mu(x_0) \neq 0 \}$

$\Delta_f = [0, t], \; t \in [0, 1)$

$= [0, t], \; t \in [0, 1)$

$t \neq 0, \; t \in \Delta_f$

$t > 0, \; C_t \in F$

$C_t(x_0) \neq 0.$

$F_2 = \{ \mu \in I^X \, / \, \mu(x_0) > \frac{1}{2} \}$

$\Delta F_2 = [0, \frac{1}{2}]$

$t < \frac{1}{2}$

$C_t \notin F_2, \; t < \frac{1}{2}$

$C_t \in F_2, \; t \geq \frac{1}{2}$

$C_t(x_0) = t > \frac{1}{2} \Longrightarrow C_t \in F_2$

$C_t(x_0) = t \leq \frac{1}{2} \Longrightarrow C_t \in F_2$

STRONG EQUIVALENCE OF PREFILTERS

Let $\mathcal{F} \in P(X)$.

We note $\Delta_{\mathcal{F}} = \{ t \in [0, 1) : \mathcal{F} \in P^t(X) \}$

Lemma 3.1

Let $\mathcal{F} \in p(X)$. If $t \in \Delta_{\mathcal{F}}$ and $s \leq t$, then $s \in \Delta_{\mathcal{F}}$

Proof

Let $t \in \Delta_{\mathcal{F}}$

$\Longrightarrow \mathcal{F} \in P^t(X)\ P^s(X)$

$\Longrightarrow s \in \Delta_{\mathcal{F}}$

Lemma 3.2

Let $\mathcal{F} \in P(X)$. We have the following possibilities.

i) $\Delta_{\mathcal{F}} = [0, t]$ for $t \in [0, 1)$ or

ii) $\Delta_{\mathcal{F}} = [0, t)$ for $t \in (0, 1)$.

Proof

Since $P^{\circ}(X) = P(X)$, we have $0 \in \Delta_{\mathcal{F}}$, for each $\mathcal{F} \in P(X)$.

$\therefore$ By lemma 2.1, $\Delta_{\mathcal{F}}$ must be an internal in $[0, 1]$.

Examples

Let X be a set and x_0 a fixed point in X.

i) Let $\mathcal{F}_1 = \{ \mu \in I^X ; \mu(x_0) \neq 0 \}$. Then $\Delta_{\mathcal{F}_1} = \{ 0 \}$

ii) Let $F_2 = \{ \mu \in I^X : \mu(x_0) > \frac{1}{2} \}$. Then $\Delta_{F_2} = [0, \frac{1}{2}]$

iii) Let $F_3 = \{ \mu \in I^X : \mu(x_0) \geq \frac{1}{2} \}$. Then $\Delta_{F_3} = [0, \frac{1}{2})$.

Define $K_F = \text{Sup } \Delta_F$

K is not in general contained in Δ_F.

We define on P(X), the following equivalence relation.

If $F_1, F_2 \in P(X)$, we say that F_1 and F_2 are strong equivalent prefilters if they fulfil the following conditions :

i) $\Delta_{F_1} = \Delta_{F_2} = \Delta$

ii) For each $t \in \Delta$, we have $\ell_t(F_1) = \ell_t(F_2)$

Let $F \in P(X)$. We denote by $[F]$, the equivalence class of F by this equivalence relation.

Lemma 3.3

Let $F, G \in P(X)$ such that $F \subset G$. Then $\Delta_G \subset \Delta_F$

Proof

Let $t \in \Delta_G$

i.e. $G \in P^t(X)$

i.e. $C_t \notin G$

Then $C_t \notin F$

So $t \in \Delta_F$

Theorem 3.1

Let $F \in P(X)$. The prefilter $F^{\#} = \bigcap_{t \in \Delta_F} w_t \circ \ell_t(F)$ is comparable by inclusion with each prefilter in the class $[F]$. And $F^{\#}$ is the finest element in this class.

Proof

Since $F \sqsubset F^{\#}$, $\Delta_{F^{\#}} \sqsubset \Delta_F$

Let $s \in \Delta_F$. Then

$C_s \notin w_s \circ \ell_s(F)$

$\Longrightarrow$ $C_s \notin F^s$. Then

$s \in \Delta_{F^s}$

$\Longrightarrow$ $\Delta_F \sqsubset \Delta_{F^{\#}}$

Clearly, $\ell_t(F) = \ell_t(F^{\#})$ for each $t \in \Delta_F$. For,

i) $F \sqsubset F^{\#} \Longrightarrow \ell_t(F) \sqsubset \ell_t(F^{\#})$

ii) $\ell_t(F^s) = \{ \mu^{-1}(t, 1] : \mu^{-1}(s, 1] \in \ell_s(F)$ for each $s \in \Delta_F \} \ \ell_t(F)$

So, $F^s \in [F]$

Let $G \in [F]$ and $\gamma \in G$

For each $t \in \Delta_F$, we have $\gamma^{-1}(t, 1] \in \ell_t(F) = \ell_t(F)$. Then $\gamma \in w_t \circ \ell_t(F)$ for each $t \in \Delta_F$.

So $\gamma \in F^{\#}$, i.e. $G \sqsubset F^{\#}$

Proposition 3.1

Let $F, G \in P(X)$ such that $F \subset G$. Then $F^{\#} \subset G^{\#}$.

Proof

We know that $\Delta_G \subset \Delta_F$

If $t \in \Delta_G$, we have the inclusion $w_t \circ \ell_t(F) \subset w_t \circ \ell_t(G)$,

$$\text{Then, } F^{\#} = \bigcap_{t \in \Delta_F} w_t \circ \ell_t(F) \subset \bigcap_{t \in \Delta_G} w_t \circ \ell_t(F)$$

$$= \bigcap_{t \in \Delta_G} w_t \circ \ell_t(G) = G^{\#}$$

Proposition 3.2

Let $F \in P(X)$. If $t \in \Delta_F$, we have the following chain inclusions :

$$F^{\#} \subset (F^{\#})^t \subset \bigcap_{\substack{s \in \Delta_F \\ s \geq t}} w_s \circ \ell_s(F)$$

Proof

Let $V \in (F^{\#})^t$

There is $\mu \in F^{\#}$ such that $\gamma \supset \mu \wedge \psi_{\mu^{-1}(t,1]}$

Then, if $s \geq t$, we have

$$\gamma^{-1}(s, 1] \supset \mu^{-1}(s, 1] \cap \mu^{-1}(t, 1] = \mu^{-1}(s, 1] \in \ell_s(F).$$

So $\gamma^{-1}(s, 1] \in \ell_s(F)$ and then $\gamma \in w_s \circ \ell_s(F)$, $\forall\, s \geq t$.

Corollary

Let $\mathcal{F} \in P(X)$. If $\Delta_{\mathcal{F}} = \{ 0 \}$, then

$\mathcal{F}^{\#} = (\mathcal{F}^{\#})^{\circ} = w_0 \circ \ell_0(\mathcal{F})$.

Proposition 3.3

Let $\mathcal{F} \in P(X)$. $\mathcal{F}$ is a prime prefilter iff $\mathcal{F}^{\#}$ is a prime prefilter.

Proof

If $\mathcal{F}$ is prime, then $\ell_t(\mathcal{F}) = \ell_t(\mathcal{F}^{\#})$ is an ultrafilter on X for each $t \in \Delta_{\mathcal{F}}$. Then $(\mathcal{F}^{\#})^t$ is prime for each $t \in \Delta_{\mathcal{F}}$. Then particularly $\mathcal{F}^0 = (\mathcal{F}^{\#})^0$ is prime. The converse is similar.

Remark

If $\mathcal{F}$ is prime, there is a unique ultrafilter $\mathcal{U}$ such that $\ell_t(\mathcal{F}) = \mathcal{U}$ for each $t \in \Delta_{\mathcal{F}}$ and we have

$$\mathcal{F} = \bigcap_{t \in \Delta_{\mathcal{F}}} w_t(\mathcal{U})$$

Theorem 3.2

Let $F \in F(X)$. For each $t \in [0, 1)$, we have $w_t(F) = (w_t(F))^{\#}$.

Proof

$\Delta_{w_t(F)} = [0, t]$ since $C_t \notin w_t(F)$.

$$(w_t(F))^{\#} = \bigcap_{s \in [0,\, t]} w_s \circ \ell_s(w_t(F)) = \bigcap_{s \in [0,t]} w_s(F) = w_t(F)$$

Corollary

Let $\mathcal{F} \in M_{\mathcal{F}}^{t}(X)$. Then $\mathcal{F} = \mathcal{F}^{\#}$ and $\Delta_{\mathcal{F}} = [0, t]$. Let X and Y be two sets and an application $f : X \rightarrow Y$. If $\mathcal{F} \in P(X)$, then $f(\mathcal{F}) \in P(Y)$.

Lemma 3.4

$$- \Delta_{\mathcal{F}} = \Delta_{f(\mathcal{F})}$$

Proof

If $t \notin \Delta_{\mathcal{F}}$, then $C_t \in \mathcal{F}$

Since $f(C_t) \sqsubset C_t$, we have $C_t \in f(\mathcal{F})$ and so $t \notin \Delta_{f(\mathcal{F})}$.

Conversely, let $t \notin \Delta_{f(\mathcal{F})}$.

Then, $C_t \in f(\mathcal{F})$.

i.e. There is $\mu \in \mathcal{F}$ such that $f(\mu) \sqsubset C_t$.

So, for each $x \in X$, we have $\mu(x) \leq f(\mu) \,.\, f(x) \leq t$,

So, $\mu \sqsubset C_t$. Thus $C_t \in \mathcal{F}$ and we conclude that $t \notin \Delta_{\mathcal{F}}$.

Lemma 3.5

If $\mathcal{F} \in P(X)$, then $f(\mathcal{F}^{\#}) = (f(\mathcal{F}))^{\#}$.

Proof

Let $\gamma \in f(\mathcal{F}^{\#})$. Then there is $\mu \in \mathcal{F}^{\#}$ such that $\gamma \sqsupset f(\mu)$.

But for $s \in \Delta_{\mathcal{F}}$, we have

$$(f(\mu))^{-1}[s, 1] = f(\mu^{-1}(s,1]) \in f(\ell_s(\mathcal{F})) = \ell_s(f(\mathcal{F})).$$

Then for each $s \in \Delta_{\mathcal{F}}$, we have $\gamma^{-1}(s, 1] \in \ell_s(f(\mathcal{F}))$.

i.e. $\gamma \in (f(\mathcal{F}))^{\#}$

Conversely,

let $\gamma \in (f(\mathcal{F}))^{\#}$

Then for each $s \in \Delta_{\mathcal{F}}$, we have $\gamma \in f(w_s \circ \ell_s(\mathcal{F}))$

$= w_s \circ \ell_s(f(\mathcal{F}))$.

So, for each $s \in \Delta_{\mathcal{F}}$, there is $\mu_s \in w_s \circ \ell_s(\mathcal{F})$ such that $\gamma \sqsupset f(\mu_s)$.

Let $\mu = \bigvee_{s \in \Delta_{\mathcal{F}}} \mu_s$

Then $\gamma \sqsupset \bigvee_{s \in \Delta_{\mathcal{F}}} f(\mu_s) = f(\mu)$

where $\mu \in w_s \circ \ell_s(\mathcal{F})$ for each $s \in \Delta_{\mathcal{F}}$

Then $\gamma \in f(\mathcal{F}^{\#})$

We conclude by resolving the following problem :

Suppose that Δ is an interval in $[0, 1]$ of type (i) or (ii) in lemma 3.2. Let $\{ F_t : t \in \Delta \}$ be a family of filters on X verifying the inclusions $F_s \sqsubset F_t$ if $s \leq t$. Is there any prefilter $\mathcal{F}$ on X such that $\Delta = \Delta_{\mathcal{F}}$ and for each $t \in \Delta$ we have $\ell_t(\mathcal{F}) = F_t$?

Theorem 3.2

In the previous hypothesis, the prefilter $\mathcal{F} = \bigcap_{t \in \Delta} w_t(F_t)$ verifies the property that $\Delta = \Delta_{\mathcal{F}}$ and $i_t(\mathcal{F}) = F_t$ for $t \in \Delta$.

Moreover, it is the finest prefilter which satisfies these conditions.

Proof

Let $s \in \Delta_{\mathcal{F}}$, i.e., $C_s \notin \mathcal{F}$

There is $t \in \Delta$ such that $C_s \notin w_t(F_t)$.

Thus $s \leq t$ and so $s \in \Delta$.

Conversely, if $s \in \Delta$ (since $C_s \notin w_s(F_s)$), we have $C_s \notin \mathcal{F}$ and so $s \in \Delta_{\mathcal{F}}$.

Thus, $\Delta = \Delta_{\mathcal{F}}$. Let $s \in \Delta$ and $A \in i_s(\mathcal{F})$. There is $\mu \in \mathcal{F}$ such that $A = \mu^{-1}(s, 1]$. Since $\mu \in w_t(F_t)$ for each $t \in \Delta$, particularly, we have $\mu \in w_s(F_s)$ and thus, $A \in F_s$. Conversely, let $A \in F_s$.

Let $\mu \in \chi_{\Lambda} V_s \chi_{\Lambda c}$. Then we have

$$\mu^{-1}(t, 1] = \begin{cases} A \text{ if } s \leq t \\ X \text{ if } s > 1 \end{cases}$$

So $\mu^{-1}(t, 1] \in F_t$ for each $t \in \Delta_{\mathcal{F}}$. Then $\mu \in \mathcal{F}$ and consequently, $A = \mu^{-1}(s, 1] \in i_s(\mathcal{F})$. This proves that $F_s = i_s(\mathcal{F})$, for each $s \in \Delta$. Let $\mathcal{G} \in P(X)$ such that $\Delta = \Delta_{\mathcal{G}}$ and $i_t(\mathcal{G}) = F_t$ for each $t \in \Delta$. Then for each $t \in \Delta$, we have the inclusion

$$\mathcal{G} \sqsubset w_t \circ i_t(\mathcal{G}) = w_t(F_t)$$

$$\text{So } \mathcal{G} \sqsubset \bigcap_{t \in \Delta} w_t(F_t) = \mathcal{F}$$

The previous result allows us to study the fuzzy convergence of prefilters on a fuzzy topological space (X, δ) through the study of the convergence of collections of usual filters on X, $(F_t : t \in \Delta)$, (where Δ is a set of type (i) or (ii) in lemma 3.2 which satisfies the chain of inclusions $F_s \sqsubset F_t$ if $s \in [0, t)$), on the family of topological spaces associated $\{ (X, i_t(\delta) : t \in \Delta \}$

SUMMARY AND CONCLUSION

i) $i_t \circ w_t(\tau) = \tau$

ii) w_t, i_t are respectively isotone injection and isotone surjection for each $t \in [0, 1)$.

iii) $\tau \sqsubset i_s \circ w_t(\tau)$

iv) $\delta \sqsubset w_t \circ i_t(\tau)$

Lastly the relation between t-convergence and convergence is obtained.

A new approach to fuzzy convergence is studied in Chapter III. Here the fuzzy convergence of free filters on fuzzy topological space is described through the convergence of some family of filters on the collection of its level topologies.

The theory of topological convergenece is developed in a number of directions. It is a good problem to generalize some of these theories to fuzzy situation.

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