

On TM-Algebras

Monisha, B
(12PMA012)

Thesis submitted to
Avinashilingam Institute for Home Science and Higher Education for Women,
Coimbatore-641 043

In Partial Fulfilment of the Requirements for the
Degree of Master of Science in Mathematics

March, 2014

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Signature of the

Head of the Department


Signature of the

Supervisor

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INTRODUCTION

INTRODUCTION

In 1966, Imai and Iseki [14] introduced the notion of a BCK-algebra. In the same year, Iseki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. In 1983, Hu and Li [12] introduced a wide class of abstract algebras: BCH-algebras.

In 2001, Neggers et al. [18] introduced Q-algebras, which is a generalization of BCH/BCI/BCK-algebras. In 2010, Megalai and Tamilarasi [23] introduced the notion of TM-algebras, which is a generalization of BCH/BCI/BCK/Q-algebras.

The main aim of this thesis is to discuss a few interesting articles on TM-algebras. The following articles are chosen for our discussion.

- ❖ TM-algebra- An introduction by Megalai, K. and Tamilarasi, A. [23]
- ❖ Classification of TM-algebra by Megalai, K. and A. Tamilarasi, A. [24]
- ❖ Quotient TM-algebras by Handam, A.H. [11]
- ❖ Derivations on TM-algebras by Ganeshkumar, T. and Chandramouleeswaran, M. [6]
- ❖ Generalized Derivation on TM-algebras by Ganeshkumar, T. and Chandramouleeswaran, M. [7]
- ❖ t-Derivations on TM-algebra by Ganeshkumar, T. and Chandramouleeswaran, M. [8]
- ❖ Symmetric Bi-derivation on TM-algebra by Ganeshkumar, T. and Chandramouleeswaran, M. [9]

This thesis is split into four chapters.

In **chapter I**, the notion of TM-algebra and its properties are presented. Also ideal of a TM-algebra, p-radical, p-semisimple of a TM-algebra are discussed due to Megalai and Tamilarasi [23].

The interesting results discussed in this chapter are given as follows.

- (1) Let $(X, *, 0)$ be a TM-algebra. Then $B(X) = \{x \in X / 0 \leq x\}$ is an ideal of X .
- (2) If $(X, *, 0)$ is a TM-algebra of order 3, then
 - (i) $G(X) = \{x \in X / 0 * x = x\} \neq X$ and
 - (ii) $G(X)$ is an ideal of X if $|G(X)| = 1$
- (3) Let $(X, *, 0), (Y, \Delta, 0')$ be TM-algebras and let B be an ideal of Y .
Let $f : X \rightarrow Y$ be a homomorphism. Then $f^{-1}(B)$ is an ideal of X .

Chapter II deals with the study of Classification of TM-algebras [24] and Quotient TM-algebras due to Handam, A.H. [11]. In this chapter, the concepts of positive implicative, implicative, 1-weakly positive implicative, 2-weakly positive implicative TM-algebras, right translation, left translation, weak right translation, weak left translation of TM-algebras and their related properties are characterized.

Also, the following interesting results are discussed.

- (1) Let I be an ideal of a TM-algebra. If we define a binary operation on the quotient set $X/I = \{I_x / x \in X\}$ by $I_x \otimes I_y = I_{x*y} \quad \forall x, y \in X$, then
 $(X/I, \otimes, I_0)$ is a TM-algebra, called the quotient algebra of X relative to I .
- (2) Let I be a closed proper ideal of a TM-algebra X . Then X/I is simple if and only if I is a maximal ideal of X .
- (3) Fundamental homomorphism theorem on TM-algebras.

Chapter III deals with the study of derivations on TM-algebras. In this chapter, derivations and generalized derivations on TM-algebras and their properties are discussed. Also generalized derivations on a torsion free TM-algebra are studied due to Ganeshkumar, T. and Chandramouleeswaran, M. [3,7].

In this chapter, the following important results are discussed.

- (1) Let $(X, *, 0)$ be a TM-algebra and d_1, d_2 be two derivations of X , then
$$d_1 * d_2 = d_2 * d_1$$
- (2) Let $(X, *, 0)$ be a 0-commutative TM-algebra and d be a derivation on X .
Then $d(x) * d(y) = x * y$

- (3) Let \mathcal{D} be a generalized (r, l) -derivation of a TM-algebra X . Then
- (i) $\mathcal{D}(a) \in G(X) \quad \forall a \in G(X)$
 - (ii) $\mathcal{D}(a) = a * \mathcal{D}(0) = a + \mathcal{D}(0) \quad \forall a \in X$
 - (iii) $\mathcal{D}(a+b) = \mathcal{D}(a) + \mathcal{D}(b) - \mathcal{D}(0) \quad \forall a, b \in X$
 - (iv) \mathcal{D} is the identity map on X iff $\mathcal{D}(0)=0$.
- (4) Let X be a Torsion free TM-algebra and let \mathcal{D}_1 and \mathcal{D}_2 be two generalized derivations. If $\mathcal{D}_1\mathcal{D}_2=0$ on X , then $\mathcal{D}_2=0$ on X .

In **Chapter IV**, the concepts of t -derivations and Symmetric bi-derivations on TM-algebras due to Ganeshkumar, T. and Chandramouleeswaran, M. [8,9] and their properties are studied.

In this chapter, the following interesting results are discussed.

- (1) Let X be a TM-algebra. Let d_t be a (r, l) - t -derivations of X and d_t' be a (l, r) - t -derivation of X . Then $d_t \circ d_t' = d_t' \circ d_t$.
- (2) $L_t Der(X)$, the set of all (l, r) - t -derivation of X is a semi-group under the binary operation \wedge defined by $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t''(x) \quad \forall x \in X$, and $d_t, d_t' \in L_t Der(X)$.
- (3) Let X be an associative TM-algebra. Then the symmetric map $D: X \times X \rightarrow X$ defined by $D(x, y) = x + y \quad \forall x, y \in X$ is a symmetric bi-derivation.
- (4) The binary composition \wedge defined on D_L , the set of all (l, r) -symmetric bi-derivation on a TM-algebra X is associative.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

Imai and Iseki [14] in 1966 introduced the notion of a BCK-algebra. In the same year, Iseki introduced the notion of a BCI-algebra [17]. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras. In 1983, Hu and Li [12] introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In 1998, Jun et al. [20] introduced a new notion, called a BH-algebra, which is a generalization of BCH/BCI/BCK-algebras, and they showed that there is a maximal ideal in bounded BH-algebra. In 1999, Neggers, J. and Kim, H.S. [29] introduced the notion of d-algebras which is a generalization of BCK-algebras.

In 2001, Neggers et al. [18] introduced Q-algebras, which is a generalization of BCH/BCI/BCK-algebras, and generalized some theorems from the theory of BCI-algebras. Motivated by the notion of derivations on rings and near-rings, in 2004, Jun and Xin [19] studied the notion of derivation of BCI-algebras. In 2011, Chandramouleeswaran et al. [5] introduced the notion of derivation of d-algebras. In 2011, Sabahattin et al. [31] have discussed the notion of symmetric bi-derivation on BCI-algebras. In 2010, Megalai and Tamilarasi [23] introduced the notion of TM-algebras, which is a generalization of BCH/BCK/Q-algebras.

Several other authors' have also contributed to the study of the concepts mention above. We give here a brief survey of some of the articles published on TM-algebras.

(1) An introduction to the theory of BCK-algebra

Iseki, K. and Tanaka, S. (1978) [16]

In this articles the definition of BCK-algebra and its fundamental properties are studied, various ideals in BCK-algebras are discussed in a detailed manner. Also, the homomorphism properties on BCK-algebra are discussed.

(2) On BH-algebras

Jun, Y.B., Roh, E.H. and Kim, H.S. (1998) [20]

In this article, the authors introduced a new notion called a BH-algebra, which is a generalization of BCH/BCI/BCK-algebras. The authors defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. Furthermore, the authors establish construct the quotient BH-algebras via translation ideals and obtained the fundamental theorem of homomorphism for BH-algebras as a consequence.

(3) On d-algebra

Neggers, J., Ahn, S.S. and Kim, H.S. (1999) [29]

In this article, the authors introduced the notion of d-algebras which is an other generalization of BCK-algebras and investigated several relations between d-algebras and BCK-algebras. Furthermore, the authors showed that the class of oriented di-graphs corresponds in a simple way to the class of edge d-algebras and that arbitrary d-algebra.

(4) On QS-algebra

Ahn, S.S. and Kim, H.S. (1999) [2]

In this article, the authors introduced a new notion called an QS-algebra, which is related to the areas of BCI/BCK-algebras and discussed the G-part of QS-algebras.

(5) On Q-algebras

Joseph Neggers, Sun Shin Ahn and Hee Sik Kim (2001) [18]

In this article, the authors introduced a new notion, called a Q-algebras, which is a generalization of idea of BCH/BCI/ BCK-algebras and they generalized some theorems, discussed in BCI-algebras. Moreover, they introduced the notion of “quadratic” Q-algebra, and showed that every quadratic Q-algebra $(X, *, e), e \in X$, has a product of the form $x * y = x - y + e$ where $x, y \in X$ when X is a field with $|X| \geq 3$.

(6) On Positive and Weakly Positive Implicative of BCI-algebras

Yisheng Huang (2002) [35]

In this article, the authors showed that the notion of positive implicative BCI-algebras coincides with that of weakly positive implicative BCI-algebras. From this reasons the entire conclusions in weakly positive implicative BCI-algebras will be valid in positive implicative BCI-algebras. In particular, using the quasi-commutativity, they obtained a definite answer to the first half of Meng's open problem: All of positive implicative BCI-algebras form a variety. As to the second half of the same problem, a number of further properties will be got.

(7) Generalizations of BCK-algebras

Sung Min Hong, Young Bae Jun and Mehmet Ali Ozturk (2003) [34]

As a generalization of positive implicative BCK-algebra, the notion of generalized BCK-algebras is introduced. A method to make BCK-algebra from a quasi-ordered set is provided. The notion of generalized BCK-ideals of generalized BCK-algebras is introduced, and then the connections between such ideals and congruences are considered. Characterizations of generalized BCK-ideals are given. A generalized, BCK-ideals generated by a set is established.

(8) On derivations of BCI-algebras

Young Bae Jun and Xiao Long Xin (2004) [19]

The notion of left-right (respectively right-left) derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular derivation the authors has given the characterizations of a p-semisimple BCI-algebra. The authors has also given a condition for a derivation to be regular.

(9) Congruences and Quotient Algebras of BCI-algebras

Yuzhong Ding and Zhiyong Pang (2007) [36]

The authors have formalized the BCI-algebras closely following the book [7] pp.16-19 and pp.58-65. Firstly, the article focuses on the properties of the element and then the definition and properties of congruences and quotient algebras are given. Quotient algebras are the basic tools for exploring the structures of BCI-algebras.

(10) Generalized derivations of BCI-algebras

Mehmet Ali Ozturk, Yilmaz cevev and Young Bae Jun (2009) [26]

The notion of generalized derivations of BCI-algebras are introduced and some related properties are investigated. Also, the concept of Torsion free BCI-algebra is introduced and some properties are discussed.

(11) On F-derivation of BCI-algebras

Farhat Nisar (2009) [5]

In this article the authors introduced the notions of right F-derivation and left F-derivation of a BCI-algebra and some related properties are explored.

(12) Quotient BCI-algebras induced by pseudo-valuation

Shokoofeh Ghorbani (2010) [33]

In this article, the author studied pseudo-valuation on a BCI-algebra and obtained some related results. The relation between pseudo-valuations and ideals is investigated. The authors used a pseudo-metric induced by a pseudo-valuation to introduce a congruence relation on a BCI-algebra. They defined the quotient algebra induced by this relation and proved that it is also a BCI-algebra and studied its properties.

(13) On L-Fuzzy Subalgebras of TM-Algebras

Chandramouleeswaran, M., Anusuya, R. and Muralikrishna, P. (2011) [3]

In 2010, Tamilarasi and Megalai introduced a class of abstract algebra - TM- algebra and claimed that their algebra is a generalization of BCH/BCI/BCK and Q-algebras. In this paper, they first given some counter examples to disprove their claim. Also the authors introduced the notion of L-fuzzy subalgebras and L-fuzzy ideals in TM-algebras and prove some simple but interesting results.

(14) Derivations On d– algebras

Chandramouleeswaran, M. and Kandaraj, N. (2011) [4]

Motivated by some results on derivations in rings, and the generalizations of BCK and BCI algebras, in this article, the authors defined derivations on d– algebras and investigated some important results.

(15) On Symmetric Bi-Derivations of BCI-Algebras

Sabahattin Ilbira, Alev Firat, Young Bae Jun (2011) [31]

The notion of left-right (resp. right-left) symmetric bi-derivation of BCI-algebras is introduced and some related properties are investigated.

(16) Fuzzy Subalgebras and Fuzzy T-ideals in TM-Algebras

Kandasamy Megalai and Angamuthu Tamilarasi (2011) [21]

In this study, the introduced the concepts of fuzzy subalgebras and fuzzy ideals in TM-algebras and investigated some of its properties.

(17) Fuzzy TM-ideals of TM-algebras

Samy M. Mostafa, Mokhtar A. Abdel Naby and Osama R. Elgendy (2011) [32]

The fuzzification of TM- ideals in TM-algebras is considered, and several properties are investigated. Characterizations of a fuzzy ideal are provided.

(18) On Intuitionistic Fuzzy T-ideals in TM-algebra

Megalai Kandasamy and Tamilarasi Angamuthu (2011) [25]

In this article, first the authors defined the notions of TM-algebra, T-ideals, fuzzy sets, intuitionistic fuzzy T-ideals and intuitionistic fuzzy closed T-ideals. Using the concept of level subsets, they proved some theorems which shows that there are some relationships between these notions. Finally the authors defined the homomorphism of TM-algebras and then the authors has given related theorem about the relationship between their images and intuitionistic fuzzy T-ideals.

(19) Anti Fuzzy T - Ideals Of TM- Algebras And Its Lower Level Cuts

Ramachandran, T., Priya, T. and Parimala, M. (2012) [30]

In this paper, the authors introduced the concept of Anti fuzzy T-ideals of TM-algebras, lower level cuts of a fuzzy set, lower level T-ideal and prove some results . The authors showed that a fuzzy subset of a TM-algebra is a T-ideal if and only if the complement of this fuzzy subset is an anti fuzzy T-ideal. Also they discussed few results of T-ideal of TM-algebra under homomorphism as well as anti homomorphism. Cartesian product of Anti fuzzy T-ideal also discussed.

(20) On t-Derivations of BCI-Algebras

Muhiuddin, G. and Abdullah M. Al-Roqi (2012) [27]

The authors introduced the notion of t-derivation of a BCI-algebra and investigated related properties. Moreover, they studied t-derivations in a p-semisimple BCI-algebra and established some results on t-derivations in a p-semisimple BCI-algebra.

(21) On (α, β) -Derivations in BCI-Algebras

Muhiuddin, G. (2012) [28]

The notion of (regular) (α, β) -derivations of a BCI-algebra X is introduced, some useful examples are discussed, and related properties are investigated. The condition for (α, β) -derivation to be a regular is provided. The concepts of a $d(\alpha, \beta)$ -invariant (α, β) -derivation and α -ideal are introduced, and their relations are discussed. Finally, some results on regular (α, β) -derivations are obtained.

(22) A New Kind of Derivations in BCI-algebras

Kyoung Ja Lee (2013) [22]

A new kind of derivation in BCI-algebras is introduced, and related properties are investigated. For a self map d_q^f of a BCI-algebra X, conditions for the kernel of d_q^f to be both a subalgebra and an ideal of X are provided.

(23) (α, β) Derivation On TM-algebras

Ganeshkumar, T. and Chandramouleeswaran, M., (2013) [10]

In this paper, the authors introduced the notion of derivation on TM-algebras. Also, they introduced the notion of (α, β) -derivation on a TM-algebra and studied its properties.

(24) On generalized (α, β) -derivations in BCI-algebras

Abdullah M. Al-Roqi (2014) [1]

The notion of generalized (regular) (α, β) -derivations of a BCI-algebra are introduced, some useful examples are discussed, and related properties are investigated. The condition for a generalized (α, β) -derivation to be regular is provided. The concepts of a generalized F-invariant (α, β) -derivation and α -ideal are introduced, and their relations are discussed. Moreover, some results on regular generalized (α, β) -derivations are proved.

CHAPTER I

CHAPTER-I

IDEALS ON TM-ALGEBRAS

SECTION: 1.1 PRELIMINARIES ON TM-ALGEBRAS

Definition: 1.1.1[15]

A BCI- algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms for all $x, y, z \in X$

$$(BCI-1) \quad ((x * y) * (x * z)) * (z * y) = 0$$

$$(BCI-2) \quad (x * (x * y)) * y = 0$$

$$(BCI-3) \quad x * x = 0$$

$$(BCI-4) \quad x * y = 0 \text{ and } y * x = 0 \Rightarrow x = y$$

Definition: 1.1.2[16]

A BCI- algebra $(X, *, 0)$ is called a BCK-algebra if (BCK-1) $0 * x = 0 \forall x \in X$

Definition: 1.1.3[12]

A BCH-algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

$$(BCH-1) \quad x * x = 0$$

$$(BCH-2) \quad (x * y) * z = (x * z) * y$$

$$(BCH-3) \quad \text{If } x * y = 0 \text{ and } y * x = 0 \Rightarrow x = y \quad \forall x, y, z \in X$$

Definition: 1.1.4 [18]

A Q-algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

$$(Q-1) \quad x * x = 0$$

$$(Q-2) \quad x * 0 = x$$

$$(Q-3) \quad (x * y) * z = (x * z) * y \quad \forall x, y, z \in X$$

Note: Every Q-algebra x satisfies following conditions:

$$(Q-4) \quad (x * (x * y)) * y = 0 \quad \forall x, y, z \in X$$

Definition: 1.1.5[2]

A Q-algebra $(X, *, 0)$ is called QS-algebra if

$$(QS-1) \quad (x * y) * (x * z) = z * y \quad \forall x, y, z \in X$$

Definition: 1.1.6[29]

A d-algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

$$(d-1) \quad x * x = 0$$

$$(d-2) \quad 0 * x = 0$$

$$(d-3) \quad x * y = 0 \text{ and } y * x = 0 \Rightarrow x = y \quad \forall x, y \in X.$$

Note: (i)

In BCI/BCK/BCH/Q/QS/d-algebra $(X, *, 0)$, or simply denoted by X , define a binary relation “ \leq ” by $x \leq y$ iff $x * y = 0$.

Note: (ii)

- (i) Every BCK-algebra is a BCI-algebra but conversely.
- (ii) Every BCI-algebra is a BCH-algebra but conversely.
- (iii) Every BCH-algebra is a Q-algebra but conversely.
- (iv) Every Q-algebra satisfying the condition $(x * y) * (x * z) = z * y$ and $x * y = 0, y * x = 0 \Rightarrow x = y$ is a BCI-algebra.

Definition: 1.1.7[23]

A BCK- algebra $(X, *, 0)$ is called

$$(i) \quad \text{positive implicative if } (x * y) * z = (x * z) * (y * z) \quad \forall x, y, z \in X$$

$$(ii) \quad \text{implicative if } x * (y * x) = x$$

$$(iii) \quad \text{commutative if } x * (x * y) = y * (y * x)$$

Definition: 1.1.8[23]

A BCI- algebra $(X, *, 0)$ is called

$$(i) \quad \text{weakly positives implicative if}$$

$$(x * y) * z = ((x * z) * z) * (y * z) \quad \forall x, y, z \in X$$

$$(ii) \quad \text{weakly implicative if } (x * (y * x)) * (0 * (y * x)) = x \quad \forall x, y \in X$$

$$(iii) \quad \text{weakly commutative if } (x * (x * y)) * (0 * (x * y)) = y * (y * x) \quad \forall x, y \in X$$

Theorem: 1.1.9

A BCI-algebra $(X, *, 0)$ is called weakly positive implicative iff
 $x * y = ((x * y * y) * (0 * y))$.

Proof: Obvious

Definition: 1.1.10 [23]

A BCH-algebra $(X, *, 0)$ is called

- (i) weakly positive implicative if $x * y = ((x * y) * y) * (0 * y) \quad \forall x, y \in X$
- (ii) weakly implicative if $(x * (y * x)) * (0 * (y * x)) = x \quad \forall x, y \in X$
- (iii) weakly commutative if $(x * (x * y)) * (0 * (x * y)) = y * (y * x) \quad \forall x, y \in X$

Note:

- (1) Every weakly implicative BCI-algebra X is a weakly positive implicative BCI-algebra.
- (2) Every implicative BCK-algebra X is a positive implicative BCK-algebra.

Definition: 1.1.11

A TM-algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation “ $*$ ” satisfying the following axioms:

- (TM-1) $x * 0 = x$ for $x \in X$
- (TM-2) $(x * y) * (x * z) = z * y$ for $x, y, z \in X$

Note:

If X is a TM-algebra, a partial ordering “ \leq ” is defined as $x \leq y$ iff $x * y = 0$

Example: 1.1.12

Let Z be the set of all integers, and let $nZ = \{nx : x \in Z\}$, $n \in Z$. Then $(Z, -, 0)$ is a TM-algebra “ $-$ ” is the usual subtraction, since $x - 0 = x \quad \forall x \in Z$ and $(x - y) - (x - z) = z - y \quad \forall x, y, z \in Z$
 Similarly $(nZ, -, 0)$ is a TM-algebra.

Example: 1.1.13

Let $X = \{0,1,2,3\}$ be a set with cayley table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X,*,0)$ is a TM-algebra.

Proposition: 1.1.14

If $(X,*,0)$ is a TM-algebra, then

(P1) $x * x = 0$

(P2) $(x * y) * x = 0 * y$

(P3) $x * (x * y) = y$ for any $x, y \in X$

Proof:

Let $(X,*,0)$ be a TM-algebra.

(P1) $x * x = (x * 0) * (x * 0)$ (by TM-1)

$= 0 * 0$ (by TM-2)

$= 0$

(P2) $(x * y) * x = (x * y) * (x * 0)$ (by TM-1)

$= 0 * y$ (by TM-2)

(P3) $x * (x * y) = (x * 0) * (x * y)$ (by TM-1)

$= y * 0$ (by TM-1)

$= y$ (by TM-1)

Proposition: 1.1.15

Let $(X, *, 0)$ be a TM-algebra. Then

$$(P4) \quad (x * y) * z = (x * z) * y \quad \text{for any } x, y, z \in X$$

Proof:

Given $(X, *, 0)$ is a TM-algebra. Then

$$(x * y) * (x * z) = z * y \tag{1}$$

Put $z = x * y$ and $y = z$ in (1). Then

$$\begin{aligned} (x * y) * z &= (x * z) * (x * (x * y)) \\ &= (x * z) * y \quad \text{(by P3)} \end{aligned}$$

Proposition: 1.1.16

Let $(X, *, 0)$ be a TM-algebra. Then for any $x, y, z \in X$

$$(P5) \quad x * 0 = 0 \Rightarrow x = 0$$

$$(P6) \quad (x * z) * (y * z) \leq x * y$$

$$(P7) \quad x \leq y \Rightarrow x * z \leq y * z \quad \text{and} \quad z * y \leq z * x$$

$$(P8) \quad x * (x * (x * y)) = x * y$$

$$(P9) \quad 0 * (x * y) = y * x = (0 * x) * (0 * y)$$

$$(P10) \quad (x * (x * y)) * y = 0$$

$$(P11) \quad \text{If } x * y = 0, y * x = 0 \text{ then } x = y$$

Proof:

Let $(X, *, 0)$ be a TM-algebra.

$$(P5) \quad x * 0 = x \tag{by TM-1}$$

If $x * 0 = 0$, then $x = 0$, proves the result.

$$\begin{aligned} (P6) \quad ((x * z) * (y * z)) * (x * y) &= ((x * z) * (x * y)) * (y * z) && \text{(by P4)} \\ &= (y * z) * (y * z) && \text{(by TM-2)} \\ &= 0 && \text{(by P1)} \end{aligned}$$

Hence $(x * z) * (y * z) \leq x * y$.

(P7) To prove $x * z \leq y * z$

That is to prove $(x * z) * (y * z) = 0$

$$x \leq y \Rightarrow x * y = 0$$

By P6, $(x * z) * (y * z) \leq x * y$

Since $x * y = 0$, $(x * z) * (y * z) = 0$

Similarly, $(z * y) * (z * x) = 0$

Hence if $x \leq y$ then $(x * z) \leq (y * z)$ and $(z * y) \leq (z * x)$

$$\begin{aligned} \text{(P8)} \quad x * (x * (x * y)) &= (x * 0) * (x * (x * y)) && \text{(by TM-1)} \\ &= (x * y) * 0 && \text{(by TM-2)} \\ &= x * y && \text{(by TM-1)} \end{aligned}$$

$$\begin{aligned} \text{(P9)} \quad 0 * (x * y) &= (x * x) * (x * y) && \text{(by P1)} \\ &= y * x && \text{(by TM-2)} \\ &= (0 * x) * (0 * y) && \text{(by TM-2)} \end{aligned}$$

$$\text{(P10)} \quad (x * (x * y)) * y = (x * y) * (x * y) = 0 \quad \text{(by P1)}$$

$$\begin{aligned} \text{(P11)} \quad x &= x * 0 = x * (x * y) = y \quad \text{(or)} \\ y &= y * 0 = y * (y * x) = x \end{aligned}$$

Note:

A QS-algebra is obviously a TM-algebra, But a TM-algebra is said to be a QS-algebra if it satisfies the additional relations,

$$(x * y) * z = (x * z) * y \quad \text{and} \quad y * z = z * y \quad \forall x, y, z \in X$$

The relations between TM-algebra and other algebras are presented below.

Theorem: 1.1.17

Every BCK-algebra is a TM-algebra but the converse is not true.

Note:

The example in (1.1.13) is a TM-algebra but not BCK-algebra

Since $0 * x \neq 0 \quad \forall x = 1, 2, 3$

Theorem: 1.1.18

Every TM-algebra is a BH-algebra, but the converse is not true. Similarly, every TM-algebra is a Q-algebra, but the converse is not true, as shown in table.

Let, $X = \{0,1,2,3\}$

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X,*,0)$ is a Q-algebra

The condition $(x*y)*(x*z) = z*y$ is not satisfied as

$$(1*2)*(1*3) = 0*0 = 0 \neq 3 = 3*2$$

Therefore $(X,*,0)$ is not a TM-algebra.

Theorem: 1.1.19

Every TM-algebra is a BCH-algebra. Every BCH-algebra satisfying $(x*y)*(x*z) = z*y$ is a TM-algebra.

Proof: Obvious

Theorem: 1.1.20

Every TM-algebra is a BCI-algebra.

Proof: Obvious

Theorem: 1.1.21

Every TM-algebra X satisfying $x*z = z$ is a trivial algebra.

Proof:

Put $x = z$ in $x*z = z$, then $z*z = z$ which implies $0 = z$.

Hence X is a trivial algebra.

Lemma: 1.1.22

Let $(X, *, 0)$ be a TM-algebra. Then $a * b = a * c \quad \forall a, b, c \in X \Rightarrow 0 * b = 0 * c$.

Proof:

Let $(X, *, 0)$ be a TM-algebra.

Since $(x * y) * z = (x * z) * y$

$$(a * b) * a = (a * a) * b = 0 * b \quad \text{and} \quad (a * c) * a = (a * a) * c = 0 * c$$

Therefore, $a * b = a * c \Rightarrow 0 * b = 0 * c$.

Definition: 1.1.23

Define $G(X) = \{x \in X / 0 * x = x\}$

For any TM-algebra $(X, *, 0)$ the set $B(X) = \{x \in X / 0 \leq x\} = \{x \in X / 0 * x = 0\}$ is called the p-radial of X . If $B(X) = \{0\}$, then X is said to be p-semisimple TM-algebra.

Note: $G(X) \cap B(X) = \{0\}$

Proposition: 1.2.24

A TM-algebra X is p-semisimple iff $0 * (0 * x) = x \quad \forall x \in X$.

Proof:

Let X be p-semisimple. Then $B(X) = \{0\}$

That is $0 * 0 = 0$ and $0 * x \neq 0 \quad \forall x \neq 0$

Now, $0 * (0 * x) = (0 * 0) * (0 * x) = x * 0 = x$

Conversely, let $0 * (0 * x) = x$

Then $0 * (0 * x) = x \Rightarrow 0 * x \neq 0$

If $0 * x = 0$ for $x \neq 0$, then $0 * 0 = x \Rightarrow x = 0$

Hence X is p-semisimple.

Proposition: 1.1.25

If $(X, *, 0)$ be a TM-algebra and $x, y \in X$ then $y \in B(X) \Leftrightarrow (x * y) * x = 0$

Proof:

Let $y \in B(X)$. Then $0 * y = 0$

Now, $(x * y) * x = (x * y) * (x * 0)$

$$= 0 * y \quad (\text{by TM-2})$$

$$= 0, \text{ since } y \in B(X)$$

Conversely, let $(x * y) * x = 0$

Then $0 = (x * y) * x$

$$= (x * y) * (x * 0)$$

$$= 0 * y$$

$$\Rightarrow y \in B(X)$$

Proposition: 1.1.26

Let $(X, *, 0)$ be a TM-algebra. If $G(X) = X$, then X is p-semisimple

Proof:

Let $G(X) = X$. Also $G(X) \cap B(X) = \{0\}$

So $X \cap B(X) = \{0\}$

That is $B(X) = \{0\}$

Hence X is p-semisimple.

Note:

A TM-algebra in which $G(X) = X$ is a d-algebra.

SECTION: 1.2 IDEALS AND SUBALGEBRAS ON TM-ALGEBRA

Definition: 1.2.1

Let $(X, *, 0)$ be a TM-algebra. A non-empty subset I of X is called an ideal of X if it satisfies

$$(TM I1) \quad 0 \in I$$

$$(TM I2) \quad x * y \in I \text{ and } y \in I \Rightarrow x \in I \quad \forall x, y \in X$$

Note:

Any ideal I has the property that $y \in I$ and $x \leq y \Rightarrow x \in I$.

Example: 1.2.2

*	0	a	b	C
0	0	0	c	B
a	a	0	c	B
b	b	b	0	C
c	c	c	b	0

Then the set $I = \{0, a\}$ is an ideal of X .

Proposition: 1.2.3

Let $(X, *, 0)$ be a TM-algebra. Then $B(X)$ is an ideal of X .

Proof:

Let $(X, *, 0)$ be a TM-algebra.

Since $0*(0*0) = 0$, $0 \in B(X) = \{x \in X / 0 \leq x\}$

Let $x * y \in B(X)$ and $y \in B(X)$, then $0*(x*y) = 0$ and $0*y = 0$

Now, by proposition (1.1.24) $y \in B(X)$

$\Rightarrow (x*y) * x = 0$ and $x*y \in B(X)$

$\Rightarrow (x*(x*y)) * x = 0$

$\Rightarrow (x*(x*y)) * (x*0) = 0$

$\Rightarrow 0*(x*y) = 0$

$\Rightarrow (0*x) * (0*y) = 0$

$\Rightarrow (0*x) * 0 = 0$

$\Rightarrow (0*0) * x = 0$

$\Rightarrow 0*x = 0$

$\Rightarrow x \in B(X)$

Therefore, $B(X)$ is an ideal of X .

Definition: 1.2.4

A non-empty subset S of a TM-algebra $(X, *, 0)$ is said to be a subalgebra of X if $x * y \in S$ whenever $x, y \in S$.

Definition: 1.2.5

An ideal A of a TM-algebra $(X, *, 0)$ is said to be closed if $0 * x \in A \quad \forall x \in A$

Proposition: 1.2.6

Every closed ideal of a TM-algebra is a TM-sub-algebra.

Proof:

Let A be a closed ideal of TM-algebra $(X, *, 0)$

Let $x, y \in A$ then $0 * x, 0 * y \in A$

As $0 * (x * y) = (0 * x) * (0 * y)$, $0 * (x * y) \in A$

Hence $x * y \in A$. So A is a sub algebra of TM-algebra.

Note:

The converse of the above proposition is not true.

Definition: 1.2.7

An ideal A of a TM-algebra $(X, *, 0)$ is said to be translational ideal of X if whenever $x * y \in A$, $y * x \in A$ then $(x * z) * (y * z) \in A$ and $(z * x) * (z * y) \in A$ $\forall x, y, z \in X$.

Proposition: 1.2.8

If S is a sub-algebra of a TM-algebra $(X, *, 0)$ then $G(X) \cap S = G(S)$

Proof:

Obviously $G(X) \cap S \subseteq G(S)$

We know $G(S) = \{x \in S \subseteq X / 0 * x = x\}$

Let $x \in G(S)$. Then $0 * x = x$ and $x \in S \subseteq X \Rightarrow x \in G(X) \cap S$

Hence $G(S) \subseteq G(X) \cap S$, which proves the result.

Theorem: 1.2.9

If $(X, *, 0)$ is a TM-algebra of order 3, then

- (i) $G(X) = \{x \in X / 0 * x = x\} \neq X$
- (ii) $G(X)$ is an ideal of X if $|G(X)| = 1$

Proof:

- (i) Let $X = \{0, a, b\}$ be a TM-algebra.

Assume $G(X) = X$.

Then $0 * 0 = 0, 0 * a = a, 0 * b = b$

Also we know $x * x = 0$ and $x * 0 = x \quad \forall x \in X$

Therefore $a * 0 = a, a * a = 0$ and $b * b = 0$

Let $a * b = 0$. It is argued for $b * a = 0, a, b$

Now if $b * a = 0$, then $a * b = 0 = b * a$ and

$(a * b) * a = (a * b) * (a * 0) = 0 * b = b$ and $(b * a) * a = 0 * a = a$

Since $(a * b) * a = (b * a) * a$, it follows that $a = b$, a contradiction. so $b * a \neq 0$

If $b * a = a$, then $a = b * a = (0 * b) * a = (0 * a) * b = a * b = 0$, a contradiction

If $b * a = b$, then $b = b * a = (0 * b) * a = (0 * a) * b = a * b = 0$, a contradiction

Next, if $a * b = a$, then $(a * (a * b)) * b = (a * a) * b = 0 * b \neq 0$, contradicting P6

Finally, let $a * b = b$

It is argued for $b * a = 0, a, b$

If $b * a = 0$, then $b = a * b = (0 * a) * b = (0 * b) * a = b * a = 0$, a contradiction

If $b * a = a$, then $b = a * b = (0 * a) * b = (0 * b) * a = b * a = a$, a contradiction

If $b * a = b$, then $a = 0 * a = (b * b) * a = (b * a) * b = b * b = 0$, a contradiction

Thus it is concluded that there exists some other elements in $G(X)$, which is not in X .

- (ii) Let $X = \{0, a, b\}$ be a TM-algebra of order 3.

If the order of $G(X)$ is 1, then $G(X) = \{0\}$ is the trivial ideal of X

Conversely, assume $G(X)$ is an ideal of X

By (i) of this proposition $|G(X)| = 1$ or $|G(X)| = 2$

Suppose $|G(X)| = 2$. Then $G(X) = \{0, a\}$ or $G(X) = \{0, b\}$

If $G(X) = \{0, a\}$ since $G(X)$ is an ideal of X , $b * a \notin G(X)$ so $b * a = b$

Now, $a = 0 * a = (b * b) * a = (b * a) * b = b * b = 0$, a contradiction.

Hence $|G(X)| \neq 2$ and so $|G(X)| = 1$

Definition: 1.2.10

Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be TM-algebras. A mapping $f : X \rightarrow Y$ is called a homomorphism if $f(x * y) = f(x) \Delta f(y) \quad \forall x, y \in X$.

Note:

$f^{-1}(Y) = \{x \in X / f(x) = y \text{ for some } y \in Y\}$ and $f(X) = \{f(x) / x \in X\}$ is called the image of f .

Proposition: 1.2.11

Suppose $f : X \rightarrow Y$ is a homomorphism of TM-algebras. Then

- (i) $f(0) = 0'$
- (ii) If $x * y = 0 \quad \forall x, y \in X$ then $f(x) \Delta f(y) = 0'$

Proof:

- (i) $f(0) = f(0 * 0) = f(0) \Delta f(0) = 0'$.
- (ii) Let $x, y \in X$ and $x * y = 0$, then $f(x) \Delta f(y) = f(x * y) = f(0) = 0'$

Theorem: 1.2.12

Let $(X, *, 0)$, $(Y, \Delta, 0')$ be TM-algebras and let B be an ideal of Y .

Let $f : X \rightarrow Y$ be a homomorphism. Then $f^{-1}(B)$ is an ideal of X .

Proof:

We know $f^{-1}(B) = \{x \in X / f(x) = y \text{ for } y \in B\}$

Since $0' \in B$ and $f(0) = 0', 0 \in f^{-1}(B)$

Assume $x * y \in f^{-1}(B)$ and $y \in f^{-1}(B)$, then $f(x * y) \in B$ and $f(y) \in B$

Since f is a homomorphism, $f(x * y) = f(x) \Delta f(y) \in B$.

Since B is an ideal of Y $f(x) \in B$, so $x \in f^{-1}(B)$

Hence $f^{-1}(B)$ is an ideal of X .

Definition: 1.2.13

Let $(X, *, 0)$, $(Y, \Delta, 0')$ be TM-algebras. Let $f : X \rightarrow Y$ be a homomorphism. Then the set $\{x \in X / f(x) = 0'\}$ is called the kernel of f and is denoted by $Ker f$.

Theorem: 1.2.14

Let $f : X \rightarrow Y$ be a homomorphism of TM-algebras. Then $Ker f$ is a closed ideal of X .

Proof:

Obviously $0 \in \ker f$, since $f(0) = 0'$

Let $x * y \in \ker f$, and $y \in \ker f$

So $f(x * y) = 0'$ and $f(y) = 0'$

That is $f(x) \Delta f(y) = 0'$

$\Rightarrow f(x) \Delta 0' = 0'$

$\Rightarrow f(x) = 0'$

Hence $x \in \ker f$, so $\ker f$ is an ideal of X .

Let $x \in \ker f$ then $f(x) = 0'$

$f(0 * x) = f(0) \Delta f(x) = 0' \Delta 0' = 0'$

$\Rightarrow 0 * x \in \ker f$

Hence $\ker f$ is a closed ideal of X .

Theorem: 1.2.15

If $f : X \rightarrow Y$ be a homomorphism of a TM-algebras. If f is a surjective and I is an ideal of X , then $f(I)$ is an ideal of Y .

Proof:

Assume that f is a surjective and I is an ideal of X . Obviously, $0' \in f(I)$.

Let $x, y \in Y$ be such that $x \Delta y \in f(I)$ and $y \in f(I)$. Since f is a surjective, there exist $a \in I$ and $b \in X$ such that $f(a) = y$ and $f(b) = x$.

Hence $f(b * a) = f(b) \Delta f(a) = x \Delta y \in f(I)$, which imply that $b * a \in I$. Since I is an ideal of X , we get $b \in I$, and thus $x = f(b) \in f(I)$.

Therefore, $f(I)$ is an ideal of Y .

CHAPTER II

CHAPTER-II

CLASSIFICATION OF TM-ALGEBRAS AND QUOTIENT ALGEBRAS

SECTION: 2.1 CLASSIFICATION OF TM-ALGEBRAS

Definition: 2.1.1

A TM-algebra $(X, *, 0)$ is called positive implicative if $(x * y) * z = (x * z) * (y * z) \quad \forall x, y, z \in X$. It is called implicative if $x * (y * x) = x$.

Definition: 2.1.2

An ideal I of a TM-algebra $(X, *, 0)$ is said to be implicative if $(x * y) * z \in I$ and $y * z \in I \Rightarrow x * z \in I$ for any $x, y, z \in X$.

Theorem: 2.1.3

Let $(X, *, 0)$ be a TM-algebra and let I be an implicative ideal of X . Then $G(X) \subseteq I$.

Proof:

Let $x \in G(X)$. Then $0 * x = x$

Now, $0 = x * x = (0 * x) * x \in I$ and $x * x = 0 \in I$

Since I is an implicative ideal $0 * x \in I$

That is $x \in I$

Hence $G(X) \subseteq I$.

Definition: 2.1.4

A TM-algebra $(X, *, 0)$ is said to be 1-weakly positive implicative if $((x * y) * y) * (0 * y) = x * y \quad \forall x, y \in X$.

A TM-algebra $(X, *, 0)$ is said to be 2-weakly positive implicative if $(x * (x * y)) * (0 * (x * y)) = x * y \quad \forall x, y \in X$.

Definition: 2.1.5

Let $(X, *, 0)$ be a TM-algebra. For a fixed $x \in X$, the map $R_x : X \rightarrow X$ given by $R_x(y) = y * x, \forall y \in X$ is called a right translation of X .

Similarly, the map $R'_x : X \rightarrow X$ given by $R'_x(y) = x * y, \forall y \in X$ is called a left translation of X .

Definition: 2.1.6

Let $(X, *, 0)$ be a TM-algebra. For a fixed $x \in X$, the map $T_x : X \rightarrow X$ given by $T_x(y) = (y * x) * (0 * x), \forall y \in X$ is called a weak right translation of X .

Similarly, the map $T'_x : X \rightarrow X$ given by $T'_x(y) = (x * y) * (0 * y), \forall y \in X$ is called a weak left translation of X .

The following theorem characterizes the weakly positive implicative TM-algebra.

Theorem: 2.1.7

A TM-algebra $(X, *, 0)$ is 1-weakly positive implicative iff $R_z = T_z \circ R_z$ $\forall z \in X$ and “ \circ ” is the composition of functions.

Proof:

Let X be a TM-algebra and let $R_z = T_z \circ R_z \forall z \in X$

Then $y * z = R_z(y) = (T_z \circ R_z)(y)$

$$= T_z \circ (R_z(y))$$

$$= T_z(y * z)$$

$$= ((y * z) * z) * (0 * z), \forall y, z \in X$$

Hence X is 1-weakly positive implicative TM-algebra.

Conversely, if X is a weakly positive implicative TM-algebra then

$$(y * z) = ((y * z) * z) * (0 * z)$$

So $R_z(y) = y * z = ((y * z) * z) * (0 * z)$

$$= ((R_z(y) * z) * (0 * z) = T_z(R_z(y))$$

$$= (T_z \circ R_z)(y) \quad \forall y, z \in X$$

Hence $R_z = T_z \circ R_z$

Theorem: 2.1.8

A TM-algebra X is 2-weakly positive implicative iff $R'_z = T'_z \circ R'_z \quad \forall z \in X$

Proof:

Let X be a TM-algebra and let $R'_z = T'_z \circ R'_z$ for $z \in X$

Then, $z * y = R'_z(y)$

$$\begin{aligned} &= (T'_z \circ R'_z)(y) \\ &= T'_z(R'_z(y)) \\ &= T'_z(z * y) \\ &= (z * (z * y)) * (0 * (z * y)) \quad \forall y, z \in X \end{aligned}$$

Hence X is 2-weakly positive implicative TM-algebra.

Conversely,

If X is 2-weakly positive implicative TM-algebra

Then, $x * y = (x * (x * y)) * (0 * (x * y))$

Now, $R'_x(y) = x * y$

$$\begin{aligned} &= (x * (x * y)) * (0 * (x * y)) \\ &= (x * R'_x(y)) * (0 * R'_x(y)) \\ &= T'_x(R'_x(y)) \\ &= (T'_x \circ R'_x)(y) \quad \forall x, y \in X \end{aligned}$$

Hence $R'_x = T'_x \circ R'_x$.

Theorem: 2.1.9

Let X be a 1-weakly positive implicative TM-algebra.

Then $T_y = T_y \circ T_y = T_y^2$

Proof:

Let X be a 1-weakly positive implicative TM-algebra.

Then, $x * y = ((x * y) * y) * (0 * y)$ (1)

Now, $(x * y) * (0 * y) = ((x * y) * y) * (0 * y) * (0 * y)$
 $= (((x * y) * (0 * y)) * y) * (0 * y)$ (2)

$$T_y(x) = (x * y) * (0 * y)$$

$$\begin{aligned} T_y^2(x) &= T_y((x * y) * (0 * y)) \\ &= (((x * y) * (0 * y)) * y) * (0 * y) \\ &= (x * y) * (0 * y) \\ &= T_y(x), \quad \forall x, y \in X \end{aligned}$$

Note:

The converse of the above theorem is not true, which can be shown by the following example.

Example: 2.1.10

Let $X = \{0, a, b, c\}$ in which $*$ is defined by the following table.

*	0	a	B	c
0	0	0	B	b
a	a	0	B	b
b	b	b	0	0
c	c	b	A	0

Then $(X, *, 0)$ is a BCI-algebra, which in turn is a TM-algebra.

As $a = c * b \neq ((c * b) * b) * (0 * b) = (a * b) * (0 * b) = b * b = 0$,

X is not 1-weakly positive implicative.

But $T_0^2 = T_0, T_a^2 = T_a, T_b^2 = T_b, T_c^2 = T_c$.

Theorem: 2.1.11

Let X be a 2-weakly positive implicative TM-algebra. Let $T'_x(y) \in G(X)$ and $x \leq T'_x(y)$. Then $T_x'^2 = T_x'^2 \quad \forall x, y \in X$.

Proof:

Let X be a 2-weakly positive implicative TM-algebra.

Then $(x * (x * y)) * (0 * (x * y)) = x * y$

$$\begin{aligned}
\text{Now, } T_x'^2(y) &= T_x'(T_x'(y)) \\
&= T_x'((x * y) * (0 * y)) \\
&= (x * ((x * y) * (0 * y))) * (0 * ((x * y) * (0 * y))) \\
&= 0 * ((x * y) * (0 * y)) \quad \text{since } x \leq T_x'(y) \\
&= (x * y) * (0 * y) \quad \text{since } T_x'(y) \in G(X) \\
&= T_x'(y)
\end{aligned}$$

Converse need not be true.

SECTION: 2.2 QUOTIENT TM-ALGEBRAS

Proposition: 2.2.1

Let I be an ideal of a TM-algebra X and let $x \sim_I y$ if and only if $x * y \in I$ and $y * x \in I$. Then \sim_I is an equivalence relation on X .

Proof:

Let X be a TM-algebra and I be an ideals of X . Let $x \in X$, $x * x = 0$ and $0 \in I$ implies $x * x = 0 \in I$. Thus $x \sim_I x$. This means \sim_I is reflexive.

Clearly, \sim_I is symmetric.

Let $x, y, z \in X$. If $x \sim_I y$ and $y \sim_I z$, then $x * y, y * x \in I$ and $y * z, z * y \in I$. By TM-2, we obtain $(x * z) * (x * y) = y * z$ and $(z * x) * (z * y) = y * x$.

Since I be an ideal of X , by TM-2. Hence $x \sim_I z$.

Therefore, \sim_I is an equivalence relation on X .

Note:

Denote the equivalence class containing x by $I_x = \{y \in X / y \sim_I x\}$

Definition: 2.2.2

Let X be a TM-algebra. An equivalence relation \sim on X is called a congruence if $x \sim y$ and $u \sim v$ imply $x * u \sim y * v$ and $u * x \sim v * y$, where $x, y, u, v \in X$.

Proposition: 2.2.3

Let X be a TM-algebra and I be an ideal of X . Then the relation \sim_I is an congruence relation on X .

Proof:

If $x \sim_I y$ and $u \sim_I v$, then $x * y, y * x \in I$ and $u * v, v * u \in I$.

By (P6), it follows that $(x * u) * (y * u) \leq x * y$ and $(y * u) * (x * u) \leq y * x$.

Since $x * y \in I$ and $y * x \in I$, we have $(x * u) * (y * u) \in I$ and $(y * u) * (x * u) \in I$.

Hence $x * u \sim_I y * u$.

By P3 and P4, we obtain

$$(y * u) * (y * v) = (y * (y * v) * u) = v * u \in I,$$

and

$$(y * v) * (y * u) = (y * (y * u) * v) = u * v \in I$$

Thus $y * u \sim_I y * v$. Thus $x * u \sim_I y * v$ since \sim_I is transitive.

Similarly, $u * x \sim_I v * y$.

Therefore, \sim_I is an congruence relation on X .

Proposition: 2.2.4

Let I be an ideal of a TM-algebra X , then

- (i) $I_0 = \{x \in X / x \sim_I 0\}$ is an ideal of X .
- (ii) $y \in I_x$ if and only if $x * (x * y) \in I_x$

Proof:

- (i) Let I be an ideal of a TM-algebra X . Let $I_0 = \{x \in X / x \sim_I 0\}$.

Obviously $0 \in I_0$. If $x * y, y \in I_0$, then $x * y \sim_I 0$ and $y \sim_I 0$.

Since $x \sim_I x$ and $x * y \sim_I 0$, we have $x * (x * y) \sim_I x * 0$.

Thus $y \sim_I x$. By the symmetry of \sim_I , we obtain $x \sim_I y$.

Since $x \sim_I y$ and $y \sim_I 0, x \sim_I 0$.

Therefore, $x \in I_0$.

- (ii) Straightforward.

Proposition: 2.2.5

Let I be an ideal of a TM-algebra X . Then I is closed ideal of X if and only if $I_0 = I$.

Proof:

Assume that I is a closed ideal of a TM-algebra X . Let $x \in I$. Then $x = x*0 \in I$ and $0*x \in I$. Hence $x \sim_I 0$, and thus $x \in I_0$.

Now if $x \in I_0$, then $x = x*0 \in I$.

Therefore, $I_0 = I$. Conversely, let I be an ideal with $I_0 = I$.

For any $x \in I = I_0$, we have $0*x \in I$.

Hence I is a closed ideal of X .

Theorem: 2.2.6

Let I be an ideal of a TM-algebra X . If we define a binary operation on the quotient set $X/I = \{I_x / x \in X\}$ by $I_x \otimes I_y = I_{x*y} \quad \forall x, y \in X$, then $(X/I, \otimes, I_0)$ is a TM-algebra, called the quotient algebra of X relative to I .

Proof:

Let I be an ideal of a TM-algebra X . If we define a binary operation on the quotient set $X/I = \{I_x / x \in X\}$ by $I_x \otimes I_y = I_{x*y} \quad \forall x, y \in X$.

First we show that “ \otimes ” is well defined. Let $x, y, z, w \in X$ be such that $I_x = I_y$ and $I_z = I_w$, then $x \sim_I y$ and $z \sim_I w$. Hence $x*z \sim_I y*w$, since \sim_I is a congruence relation.

Therefore, $I_x \otimes I_z = I_{x*z} = I_{y*w} = I_y \otimes I_w$. Hence “ \otimes ” is well defined on X/I .

Claim:

$(X/I, \otimes, I_0)$ is a TM-algebra. Let $I_x, I_y, I_z \in X/I$.

$$I_x \otimes I_0 = I_{x*0} = I_x \tag{1}$$

$$\text{and } (I_x \otimes I_y) \otimes (I_x \otimes I_z) = I_{x*y} \otimes I_{x*z} = I_{(x*y)*(x*z)} = I_{z*y} = (I_z \otimes I_y) \tag{2}$$

By (1) and (2), $(X/I, \otimes, I_0)$ is a TM-algebra.

Therefore, $(X/I, \otimes, I_0)$ is a TM-algebra.

Example: 2.2.7

Let $X = \{0, a, b, c\}$ be the set with the following cayley table.

*	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

Then X is a TM-algebra and $I = \{0, a\}$ is an ideal of X . The quotient algebra of X relative to I is $X/I = \{I_0, I_b, I_c\}$ where $I_0 = \{0, a\}$, $I_b = \{b\}$, $I_c = \{c\}$.

Theorem: 2.2.8

If I and J be any two ideals of a TM-algebra X and $I \subseteq J$, then J/I is an ideal of the quotient algebra X/I .

Proof:

Let I and J be any two ideals of a TM-algebra X and $I \subseteq J$.

$$J/I = \{I_x / x \in J\} \subset \{I_x / x \in X\} = X/I.$$

Since J is an ideal, $0 \in J$ and so $I_0 \in J/I$.

If $I_y \in J/I$ and $I_x \otimes I_y \in J/I$, then $y \in J$ and $x * y \in J$.

Since J is an ideal of X , we get $x \in J$.

Hence $I_x \in J/I$.

Definition: 2.2.9

A maximal ideal of a TM-algebra X is an ideal M different from X such that there is no proper ideal N of X properly containing M .

A TM-algebra without proper ideals is called a simple TM-algebra.

Theorem: 2.2.10

Let I be a closed proper ideal of a TM-algebra X . Then X/I is simple if and only if I is a maximal ideal of X .

Proof:

Assume that X/I is simple.

To prove:

I is a maximal ideal of X . If I is not maximal ideal of X , then there exists a proper ideal C such that $I \subset C \subset X$. It follows from theorem (2.2.8) that C/I is an ideal of X/I .

If $x \in C - I$, then $x \in I_x \in C/I$. Obviously, $I = I_0 \in C/I$. Thus C/I is properly contained in X/I and has atleast two elements. Thus C/I is a proper ideal of X/I , so X/I is not simple, a contradiction. Therefore I is maximal ideal of X . Conversely, assume that I is maximal ideal of X .

To Prove:

X/I is simple.

Suppose X/I is not simple, then there exists a proper ideal D of X/I .

Let $f : X \rightarrow X/I$ be the canonical homomorphism from X onto X/I .

It follows from theorem (1.2.12) that $f^{-1}(D)$ is an ideal of X .

Moreover, $I = I_0 \subset f^{-1}(D)$ and $I \neq f^{-1}(D)$, a contradiction.

Therefore, X/I is simple.

Theorem: 2.2.11

If $X \neq B(X)$ is not a p-semisimple TM-algebra X , then $X/B(X)$ is a p-semisimple.

Proof:

By proposition (1.2.3), $B(X)$ is an ideal of X .

We shall show that the p-radical of $X/B(X)$ is trivial.

If $B_0(X) \oplus B_x(X) = B_0(X)$, then $B_{0*x}(X) = B_0(X)$, and so $0*x \sim_{B(X)} 0$.

It follows that, $0*x = (0*x)*0 \in B(X)$ and $x = 0*(0*x) \in B(X)$ thus $x = 0$.

Hence $B_x(X) = B_0(X)$.

This implies that $X/B(X)$ is a p-semisimple.

Theorem: 2.2.12

Let I and J be two ideals of a TM-algebra X and $I \subseteq J$. Then J is a closed ideal of X if and only if J/I is a closed ideal of X/I .

Proof:

Let I and J be two ideals of a TM-algebra X and $I \subseteq J$.

Let J be a closed ideal of X .

By theorem (2.2.8), J/I is an ideal of X/I .

Let $I_y \in J/I$, for some $y \in J$.

It follows from hypothesis that $0 * y \in J$. So that $I_0 \otimes I_y = I_{0*y} \in J/I$.

Consequently, J/I is a closed ideal of X/I .

Conversely, suppose J/I is a closed ideal of X/I and let $x \in X$ be such that $x \in J$. Then $I_x \in J/I$, and so $I_0 \otimes I_x = I_{0*x} \in J/I$. Hence $0 * x \in J$.

Consequently, J is a closed ideal of X .

Theorem: 2.2.13

Let I and J be two ideals of a TM-algebra X and $I \subseteq J$. Then J is an implicative ideal of X if and only if J/I is an implicative ideal of X/I .

Proof:

Let I be an ideal of a TM-algebra X .

Assume that J is an implicative ideal of X and let $x, y, z \in X$ be such that

$$(I_x \otimes I_y) \otimes I_z \in J/I \text{ and } I_y \otimes I_z \in J/I.$$

Then $I_{(x*y)*z} \in J/I$ and $I_{y*z} \in J/I$ and so $(x * y) * z \in J$ and $y * z \in J$.

Since J is an implicative ideal of X , we have $x * z \in J$ and so

$$(I_x \otimes I_z) = I_{x*z} \in J/I.$$

Hence J/I is an implicative ideal of X/I .

Conversely, suppose that J/I is an implicative ideal of X/I and

let $x, y, z \in X$ be such that $(x * y) * z \in J$ and $y * z \in J$.

$$\text{Then } (I_x \otimes I_y) \otimes I_z = I_{(x*y)*z} \in J/I \text{ and } (I_y \otimes I_z) = I_{y*z} \in J/I.$$

Since J/I is an implicative ideal of X/I , we have $(I_x \otimes I_z) \in J/I$.

Hence $x * z \in J$. Therefore, J is an implicative ideal of X .

Notation:

Let I be an ideal of X . Denote by $\mathfrak{I}(X, I)$ the set of all ideals on X containing I , and $\mathfrak{I}(X/I)$ the set of all ideals of X/I .

Theorem: 2.2.14

If I is an ideal of a TM-algebra X , then there is a bijection from $\mathfrak{I}(X, I)$ to $\mathfrak{I}(X/I)$.

Proof:

Let I be an ideal of a TM-algebra X .

Define a mapping $f : \mathfrak{I}(X, I) \rightarrow \mathfrak{I}(X/I)$ by $f(J) = J/I$.

By theorem (2.2.8), f is well defined. Clearly, f is onto.

Let $A, C \in \mathfrak{I}(X, I)$ and $A \neq C$. Without loss of generality, we may assume that there is an $x \in C - A$. If $f(A) = f(C)$, then $I_x \in f(C) = C/I$ and $I_x \in f(A) = A/I$. Thus, $I_x = I_y$ for some $y \in A$, since $I_x \in A/I$.

Hence $x \sim_I y$, that is, $x * y \in I$ and $y * x \in I$. Since $I \subseteq A$, we have $x * y \in A$. Since $x * y \in A$ and $y \in A$, $x \in A$ by (TM I2).

This is a contradiction.

Therefore, f is one to one.

Theorem: 2.2.15 (The Fundamental Homomorphism Theorem)

If $f : X \rightarrow Y$ is a homomorphism from a TM-algebra $(X, *, 0)$ onto a TM-algebra $(Y, \Delta, 0')$, then $X / \text{Ker}(f) \cong Y$.

Proof:

Let $f : (X, *, 0) \rightarrow (Y, \Delta, 0')$ be a homomorphism of TM-algebras.

Let $K = \text{Ker}(f)$. By proposition (1.2.14), K is an ideal of X .

Define a mapping $g : X/K \rightarrow Y$ by $g(K_x) = f(x)$ where $x \in X$ and $K_x = \{w \in X / w \sim_k x\}$.

For any $K_x, K_y \in X/K$,

$$\begin{aligned}
 \text{we have } (K_x = K_y) &\Leftrightarrow x * y \in K, y * x \in K \\
 &\Leftrightarrow f(x * y) = 0', f(y * x) = 0' \\
 &\Leftrightarrow f(x)\Delta f(y) = 0', f(y)\Delta f(x) = 0' \\
 &\Leftrightarrow f(x) = f(y) \quad (\text{by P11}) \\
 &\Leftrightarrow g(x) = g(y)
 \end{aligned}$$

This means that g is well defined. Let $K_x, K_y \in X/K$ with $K_x \neq K_y$.

Then $x * y \notin K$ or $y * x \notin K$. Without loss of generality, assume that $x * y \notin K$.

It follows that $f(x)\Delta f(y) = f(x * y) \neq 0'$, and hence $f(x) \neq f(y)$.

This means that g is one to one.

$$\begin{aligned}
 \text{Since } g(K_x \otimes K_y) &= g(K_{x * y}) \\
 &= f(x * y) \\
 &= f(x)\Delta f(y) \\
 &= g(K_x)\Delta g(K_y)
 \end{aligned}$$

g is a homomorphism. Hence $X/K = X/\text{Ker}(f) \cong Y$.

CHAPTER III

CHAPTER-III

DERIVATIONS AND GENERALIZED ON TM-ALGEBRA

Section: 3.1 DERIVATIONS ON TM-ALGEBRA

Definition: 3.1.1

Let $(X, *, 0)$ be a TM-algebra. A self map $d : X \rightarrow X$ is said to be a (l, r) -derivation on X if $d(x * y) = (d(x) * y) \wedge (x * d(y))$

Example: 3.1.2

Let $(X, *, 0)$ be a TM-algebra with the following cayley table.

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

A self map $d : X \rightarrow X$ be defined by $d(0) = 1, d(1) = 2, d(2) = 0$. Then d is a (l, r) -derivation.

Note:

If $d : X \rightarrow X$ is a (l, r) -derivation on X , then $d(x * y) = d(x) * y$

Example: 3.1.3

Let $(X, *, 0)$ be a TM-algebra. A self map $d : X \rightarrow X$ is said to be a (r, l) -derivation on X if $d(x * y) = (x * d(y)) \wedge (d(x) * y)$

Note:

If the self map $d : X \rightarrow X$ is a (r, l) -derivation on X , then

$$d(x * y) = x * d(y)$$

Definition: 3.1.4

Let $d : X \rightarrow X$ be a self map on TM-algebra $(X, *, 0)$. The map d is said to be a derivation on X if d is both a (l, r) -derivation and a (r, l) -derivation on X .

Note:

If d is a derivation on X then, $d(x * y) = d(x) * y = x * d(y)$

Example: 3.1.5

Let $(X, *, 0)$ be a TM-algebra with the following cayley table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

A self map $d : X \rightarrow X$ be defined by $d(0) = 3, d(1) = 2, d(2) = 1, d(3) = 0$ is a derivation.

Definition: 3.1.6

Let X be a TM-algebra. A self map $d : X \rightarrow X$ is said to be regular if $d(0) = 0$

Proposition: 3.1.7

Let $(X, *, 0)$ be a TM-algebra. If $d : X \rightarrow X$ is a regular (r, l) - derivation on X then $x \leq d(x) \quad \forall x \in X$.

Proof:

Let X be a TM-algebra and $d : X \rightarrow X$ be a regular (r, l) - derivation on X
Then $d(0) = 0$. But $x * x = 0$

Therefore $d(x * x) = 0$, (by P1)

$$\Rightarrow x * d(x) = 0$$

$$\Rightarrow x \leq d(x)$$

Proposition: 3.1.8

Let $(X, *, 0)$ be a TM-algebra. Let $d : X \rightarrow X$ is a derivation.

- (1) If $x * d(x) = 0 \forall x \in X$, then d is regular.
- (2) If $d(x) * x = 0 \forall x \in X$, then d is regular.

Proof:

Let X be a TM-algebra, and $d : X \rightarrow X$ be a derivation on X . Then d is both a (l, r) -derivation and (r, l) -derivation.

- (1) Given $x * d(x) = 0 \forall x \in X$.

$$\text{Now, } d(0) = d(x * x) = x * d(x) = 0$$

$\Rightarrow d$ is regular.

- (2) Given $d(x) * x = 0$

$$\text{Now, } d(0) = d(x * x) = d(x) * x = 0$$

$\Rightarrow d$ is regular.

Proposition: 3.1.9

Let d be a self map of a TM-algebra X .

- (1) If d is regular (l, r) -derivation on X , then $d(x) = d(x) \wedge x$
- (2) If d is regular (r, l) -derivation on X , then $d(x) = x \wedge d(x)$

Proof:

Let d be a self map of a TM-algebra X .

- (1) Given d is regular (l, r) -derivation on X . Therefore $d(0) = 0$.

$$\text{Now, } x = x * 0$$

$$\Rightarrow d(x) = d(x * 0)$$

$$= (d(x) * 0) \wedge (x * d(0))$$

$$= d(x) \wedge (x * 0) = d(x) \wedge x$$

- (2) Given d is regular (r, l) -derivation on X .

$$d(x) = d(x * 0)$$

$$= (x * d(0)) \wedge (d(x) * 0)$$

$$= (x * 0) \wedge d(x) = x \wedge d(x)$$

Definition: 3.1.10

Let d_1, d_2 be self maps on a TM-algebra X .

Define $d_1 \circ d_2$ as follows, $(d_1 \circ d_2)(x) = d_1(d_2(x)) \forall x \in X$

Lemma: 3.1.11

Let $(X, *, 0)$ be a TM-algebra. Let d_1, d_2 be two (l, r) – derivations on X .

Then $(d_1 \circ d_2)$ is also a (l, r) – derivation on X .

Proof:

Let X be a TM-algebra. Let d_1, d_2 be two (l, r) – derivations on X .

Then $d_1(x * y) = d_1(x) * y$ and $d_2(x * y) = d_2(x) * y \forall x, y \in X$.

Now, $(d_1 \circ d_2)(x * y) = d_1(d_2(x * y)) = d_1(d_2(x) * y)$
 $= (d_1(d_2(x))) * y = (d_1 \circ d_2)(x) * y$

Therefore $(d_1 \circ d_2)$ is also a (l, r) – derivation on X .

Lemma: 3.1.12

Let $(X, *, 0)$ be a TM-algebra. Let d_1, d_2 be two (r, l) – derivation on X , then $(d_1 \circ d_2)$ is also a (r, l) – derivation on X .

Proof:

Let X be a TM-algebra. Let d_1, d_2 be two (r, l) – derivations on X .

Then $d_1(x * y) = x * d_1(y)$ and $d_2(x * y) = x * d_2(y) \forall x, y \in X$.

Now, $(d_1 \circ d_2)(x * y) = d_1(d_2(x * y)) = d_1(x * d_2(y))$
 $= x * (d_1(d_2(y))) = x * ((d_1 \circ d_2)(y))$

Hence $(d_1 \circ d_2)$ is also a (r, l) – derivation on X .

Theorem: 3.1.13

Let $(X, *, 0)$ be a TM-algebra and d_1, d_2 be derivations on X then $(d_1 \circ d_2)$ is also a derivation on X .

Proof:

Proof of the theorem follows by lemma (3.1.11) and lemma (3.1.12).

Theorem: 3.1.14

Let $(X, *, 0)$ be a TM-algebra. Let d_1, d_2 be two derivations on X , then
 $(d_1 \circ d_2) = (d_2 \circ d_1)$.

Proof:

Let X be a TM-algebra and d_1, d_2 be two derivations on X , then d_1, d_2 are both (l, r) and (r, l) -derivations on X .

$$\begin{aligned} \text{Now, } (d_1 \circ d_2)(x * y) &= d_1(d_2(x * y)) = d_1(d_2(x) * y) \\ &= d_2(x) * d_1(y) \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Also, } (d_2 \circ d_1)(x * y) &= d_2(d_1(x * y)) = d_2(x * d_1(y)) \\ &= d_2(x) * d_1(y) \end{aligned} \quad (2)$$

From (1) and (2), $(d_1 \circ d_2)(x * y) = (d_2 \circ d_1)(x * y)$

This implies $(d_1 \circ d_2) = (d_2 \circ d_1)$.

Definition: 3.1.15

Let $(X, *, 0)$ be a TM-algebra. Let d_1, d_2 be two self maps on X .

Define $(d_1 * d_2): X \rightarrow X$ as $(d_1 * d_2)(x) = d_1(x) * d_2(x) \quad \forall x \in X$.

Theorem: 3.1.16

Let $(X, *, 0)$ be a TM-algebra and d_1, d_2 be two derivations on X , then
 $d_1 * d_2 = d_2 * d_1$.

Proof:

Let d_1, d_2 be two derivations on a TM-algebra X .

$$\begin{aligned} (d_1 \circ d_2)(x * y) &= d_1(d_2(x * y)) \\ &= d_1(d_2(x) * y) \\ &= d_2(x) * d_1(y) \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Again, } (d_1 \circ d_2)(x * y) &= d_1(d_2(x * y)) \\ &= d_1(x * d_2(y)) \\ &= d_1(x) * d_2(y) \end{aligned} \quad (2)$$

$$\text{Combining (1) and (2), we get } d_2(x) * d_1(y) = d_1(x) * d_2(y) \quad (3)$$

Substituting $y = x$ in (3) we get,

$$d_2(x) * d_1(x) = d_1(x) * d_2(x)$$

$$(d_2 * d_1)(x) = (d_1 * d_2)(x)$$

This is true for all elements $x \in X$. Therefore, $d_2 * d_1 = d_1 * d_2$

Lemma: 3.1.17

In a TM-algebra both right and left cancellation holds good.

Proof:

Let $(X, *, 0)$ be a TM-algebra.

Assume that $x * y = x * z \quad \forall x, y, z \in X$.

Now, $y = x * (x * y) = x * (x * z) = z$

This proves that the left cancellation law holds in X .

Assume that $y * x = z * x$

Consider, $x * y = (y * y) * (y * x)$

$$= 0 * (z * x)$$

$$= (z * z) * (z * x)$$

Thus $x * y = x * z$

Therefore $y = z$ (By left cancellation law)

Hence the right cancellation law holds in X .

Theorem: 3.1.18

Let d be a (l, r) -derivation of TM-algebra X , then

- (1) $d(0) = d(x) * x$
- (2) d is 1-1
- (3) If d is regular then d is the identity map.
- (4) If there is an element $x \in X \ni d(x) = x$, then d is the identity map.
- (5) If there is an element $x \in X \ni d(y) * x = 0$ or $x * d(y) = 0 \quad \forall y \in X$, then $d(y) = x$, (i.e.) d is a constant map.

Proof:

Let d be a (l, r) -derivation of TM-algebra X .

- (1) $x * x = 0$, therefore $d(0) = d(x * x) = d(x) * x$

(2) Let $x, y \in X$ and $d(x) = d(y)$

$$\text{Now } d(0) = d(x * x) = d(x) * x \quad (1)$$

$$\text{Again } d(0) = d(y * y) = d(y) * y = d(x) * y \quad (2)$$

From (1) and (2), $d(x) * x = d(x) * y$

$$\Rightarrow x = y \quad (\text{By left cancellation law})$$

(3) Given d is regular. Therefore, $d(0) = 0$.

$$d(0) = d(x) * x \quad (\text{By (1)})$$

$$0 = d(x) * x$$

$$x * x = d(x) * x$$

Applying right cancellation law in a TM-algebra.

We get $x = d(x)$. Hence d is the identity map.

(4) Let $x \neq y$, $x, y \in X$.

Given that there is an element $x \in X \ni d(x) = x$

Now, $y = x * (x * y)$

$$d(y) = d(x) * (x * y) \quad (\text{Since } d \text{ is } (l, r)\text{-derivation})$$

$$= x * (x * y) = y$$

Therefore d is the identity map.

(5) Given $d(y) * x = 0$

$$d(y) * x = x * x$$

$$\Rightarrow d(y) = x \quad (\text{By right cancellation law})$$

Again if $x * d(y) = 0$

$$x * d(y) = x * x$$

$$\Rightarrow d(y) = x \quad (\text{By left cancellation law})$$

Hence $d(y) = x, \forall y \in X$.

Therefore d is a constant map.

Theorem: 3.1.19

Let d be (r, l) -derivation of TM-algebra X , then

$$(1) \quad d(0) = x * d(x)$$

$$(2) \quad d(x) = d(x) \wedge x \quad \forall x \in X.$$

(3) d is 1-1.

- (4) If d is regular then d is the identity map.
- (5) If there is an element $x \in X \ni d(x) = x$, then d is the identity map.
- (6) If there is an element $x \in X \ni d(y)^*x = 0$ or $x^*d(y) = 0 \quad \forall y \in X$ then $d(y) = x$ (i.e.) d is a constant map.

Proof:

Let d be the (r, l) -derivation of TM-algebra X .

(1), (3), (4), (5) and (6) are analogous to results (1) to (5) of above theorem (3.1.18)

Hence we prove only the property (2)

Since $x^*(x^*y) = y$, $d(x) \wedge x = x^*(x^*d(x)) = d(x) \quad \forall x \in X$.

Theorem: 3.1.20

Let X be a TM-algebra and d_1, d_2, \dots, d_n be derivations on X , then

$$d_n(d_{n-1}(d_{n-2}(d_{n-3} \dots (d_2(d_1(x)))))) \leq x.$$

Proof:

$$\begin{aligned} d_n(d_{n-1}(d_{n-2}(d_{n-3} \dots (d_2(d_1(x)))))) &= d_n(d_{n-1}(d_{n-2}(d_{n-3} \dots (d_2(d_1(x)) \dots))) \\ &\leq d_{n-1}(d_{n-2}(\dots (d_2(d_1(x)) \dots))) \\ &\cdot \\ &\cdot \\ &\cdot \\ &\leq d_1(x) \\ &\leq x \end{aligned}$$

Definition: 3.1.21

Let $LDer(X)$ denote the set of all (l, r) -derivations on X . Define the binary operation \wedge on $LDer(X)$ as follows. For $d_1, d_2 \in LDer(X)$, define

$$(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x) \quad \forall x \in X.$$

Lemma: 3.1.22

If d_1 and d_2 are (l, r) -derivations on a TM-algebra X , then $(d_1 \wedge d_2)$ is also a (l, r) -derivation.

Proof:

Let d_1 and d_2 be (l, r) -derivations on a TM-algebra X .

To Prove: $(d_1 \wedge d_2)(x * y) = (d_1 \wedge d_2)(x) * y \quad \forall x, y \in X$

$$\begin{aligned}
(d_1 \wedge d_2)(x * y) &= d_1(x * y) \wedge d_2(x * y) \\
&= (d_1(x) * y) \wedge (d_2(x) * y) \\
&= (d_2(x) * y) * ((d_2(x) * y) * (d_1(x) * y)) \\
&= d_1(x) * y
\end{aligned} \tag{1}$$

$$\begin{aligned}
(d_1 \wedge d_2)(x) * y &= (d_1(x) \wedge d_2(x)) * y \\
&= (d_2(x) * (d_2(x) * d_1(x))) * y \\
&= d_1(x) * y
\end{aligned} \tag{2}$$

From (1) and (2), $(d_1 \wedge d_2)(x * y) = (d_1 \wedge d_2)(x) * y$

Therefore $(d_1 \wedge d_2)$ is a (l, r) -derivation.

Lemma: 3.1.23

The binary composition \wedge defined on $LDer(X)$ is associative.

Proof:

Let X be a TM-algebra.

Let d_1, d_2, d_3 are (l, r) -derivations.

$$\begin{aligned}
\text{Now, } ((d_1 \wedge d_2) \wedge d_3)(x * y) &= (d_1 \wedge d_2)(x * y) \wedge d_3(x * y) \\
&= (d_1(x) * y) \wedge (d_3(x) * y) \\
&= (d_3(x) * y) * ((d_3(x) * y) * (d_1(x) * y)) \\
&= d_1(x) * y
\end{aligned} \tag{1}$$

$$\begin{aligned}
\text{Again, } (d_1 \wedge (d_2 \wedge d_3))(x * y) &= d_1(x * y) \wedge ((d_2 \wedge d_3)(x * y)) \\
&= (d_1(x) * y) \wedge (d_2(x) * y) \\
&= (d_2(x) * y) * ((d_2(x) * y) * (d_1(x) * y)) \\
&= d_1(x) * y
\end{aligned} \tag{2}$$

Combining (1) and (2) we get, $(d_1 \wedge d_2) \wedge d_3 = d_1 \wedge (d_2 \wedge d_3)$

Theorem: 3.1.24

$LDer(X)$ is a semi-group under the binary composition \wedge defined by
 $(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x) \quad \forall x \in X$ and $d_1, d_2 \in LDer(X)$.

Proof:

Proof follows by above two lemmas (3.1.22) and (3.1.23)

Theorem: 3.1.25

$RDer(X)$ is a semi-group under the binary operation \wedge defined by
 $(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x) \quad \forall x \in X$ and $d_1, d_2 \in RDer(X)$.

Proof: Obvious**Theorem: 3.1.26**

If $Der(X)$ denotes the set of all derivations on X , it is a semi-group under the binary operation \wedge defined by $(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x) \quad \forall x \in X$ and $d_1, d_2 \in Der(X)$.

Proof:

Proof follows by the above two theorems (3.1.24) and (3.1.25)

Definition: 3.1.27

A TM-algebra $(X, *, 0)$ is said be 0-commutative if
 $x*(0*y) = y*(0*x) \quad \forall x, y \in X$.

Example: 3.1.28

Let $(X, *, 0)$ be TM-algebra with cayley table.

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ form the 0-commutative TM-algebra.

Lemma: 3.1.29

If $(X, *, 0)$ is a 0-commutative TM-algebra then

- (1) $(0 * x) * (0 * y) = y * x$
- (2) $(z * y) * (z * x) = x * y$
- (3) $(x * y) * z = (x * z) * y$
- (4) $(x * (x * y)) * y = 0$
- (5) $(x * z) * (y * t) = (t * y) * (z * x) \quad \forall x, y, z, t \in X$
- (6) $x * (x * y) = y$

Proof:

Let X be a 0-commutative TM-algebra. Results 1-4 follows easily.

To Prove: (5) $(x * z) * (y * t) = (0 * (y * t)) * (0 * (x * z))$ (by TM-2)

$$= ((0 * y) * (0 * t)) * ((0 * x) * (0 * z))$$

$$= (t * (0 * (0 * y))) * (z * (0 * (0 * x)))$$

$$= (t * y) * (z * x)$$

To Prove: (6) $x * (x * y) = (x * 0) * (x * y) = y * 0 = y$

Theorem: 3.1.30

Let $(X, *, 0)$ is a 0-commutative TM-algebra and d be a derivation on X .

Then $d(x) * d(y) = x * y$

Proof:

Let X be a 0-commutative TM-algebra.

Then $x * (0 * y) = y * (0 * x) \quad \forall x, y \in X$

$$d(x * (0 * y)) = d(y * (0 * x))$$

$$d(x) * (0 * y) = d(y) * (0 * x)$$

$$(d(x) * (0 * y)) * y = (d(y) * (0 * x)) * y$$

$$(d(x) * y) * (0 * y) = (d(y) * y) * (0 * x) \quad (\text{by P4})$$

$$= 0 * (0 * x)$$

$$= x \quad (\text{by P3})$$

$$\text{(i.e.) } (d(x) * y) * (0 * y) = x \quad (1)$$

Interchanging x and y in (1) we have

$$(d(y) * x) * (0 * x) = y \quad (2)$$

$$\begin{aligned} \text{From (1) and (2), } (x * y) &= ((d(x) * y) * (0 * y)) * ((d(y) * x) * (0 * x)) \\ &= ((y * 0) * (y * d(x))) * ((x * 0) * (x * d(y))) \\ &= (y * (y * d(x))) * (x * (x * d(y))) \\ &= d(x) * d(y) \quad (\text{by P3}) \end{aligned}$$

SECTION: 3.2 GENERALIZED DERIVATIONS ON A TM-ALGEBRA

Definition: 3.2.1

Let $(X, *, 0)$ be a TM-algebra. In X , define a binary composition “+” as follows: $x + y = x * (0 * y) \quad \forall x, y \in X$

Theorem: 3.2.2

In any TM-algebra $(X, *, 0)$ if “+” defined as $x + y = x * (0 * y) \quad \forall x, y \in X$, then the following hold:

- (1) $x + 0 = x = 0 + x$
- (2) Addition is associative
- (3) If $x + y = 0$, then $x = 0 * y$
- (4) Addition is commutative.
- (5) Additive inverse of $x \in X$ is $0 * x$

Proof:

- (1) $x + 0 = x * (0 * 0) = x * 0 = x = 0 * (0 * x) = 0 + x$
- (2) Addition is associative by repeated application of the definition of “+”
- (3) Applying left cancellation law on $x + y = 0 \Rightarrow x * (0 * y) = 0 = x * x$, we get $0 * y = x$.

- (4) Addition is commutative. For ,

$$\begin{aligned} x + y &= 0 + (x + y) = (y * y) + (x * (0 * y)) \\ &= (y * y) * (0 * (x * (0 * y))) \\ &= (y * y) * ((0 * x) * (0 * (0 * y))) \quad (\text{by P9}) \\ &= (y * y) * ((0 * x) * y) \quad (\text{by P3}) \\ &= y * (0 * x) = y + x \end{aligned}$$

$$(5) \quad x + (0 * x) = x * (0 * (0 * x)) = x * x = 0$$

Therefore the additive inverse of x is $-x = 0 * x$

Definition: 3.2.3

Let X be a TM-algebra. If we define an addition $+$ as $x + y = x * (0 * y)$ $\forall x, y \in X$, then $(X, +)$ is an abelian group with identity 0 and the additive inverse denoted by $-x = 0 * x \quad \forall x \in X$.

Note:

If we have a TM-algebra $(X, *, 0)$ it follows from the above definition that $(X, +)$ is an abelian group with $-y = 0 * y \quad \forall y \in X$. Then we have $x - y = x * y \quad \forall x, y \in X$. On the other hand if we choose an abelian group $(X, +)$ with an identity 0 and define $x * y = x - y$, we get a TM-algebra $(X, *, 0)$ where $x + y = x * (0 * y) \quad \forall x, y \in X$.

Example:3.2.4

Let $(X, *, 0)$ be a TM-algebra with the cayley table.

*	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Define $x + y = x * (0 * y)$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Then $(X, +)$ is an abelian group.

Definition: 3.2.5

Let X be a TM – algebra. A mapping $\mathcal{D} : X \rightarrow X$ is called generalized (l, r) -derivation of X if there exist an (l, r) -derivation $d : X \rightarrow X$ such that $\mathcal{D}(x * y) = (\mathcal{D}(x) * y) \wedge (x * d(y)) \quad \forall x, y \in X$.

Examples: 3.2.6

For the TM-algebra defined in example (3.2.4) if we defined a map $d : X \rightarrow X$ by $d(0) = 3, d(1) = 4, d(2) = 0, d(3) = 1, d(4) = 2$. Then d is a (l, r) -derivation of X .

But d is not a (r, l) -derivation of X . Since $d(x * y) \neq (x * d(y)) \wedge (d(x) * y)$ for $x, y \in X$. The map $\mathcal{D} : X \rightarrow X$ defined as $\mathcal{D}(0)=1, \mathcal{D}(1)=2, \mathcal{D}(2)=3, \mathcal{D}(3)=4, \mathcal{D}(4)=0$ is a generalized (l, r) -derivation of X .

Remark:

In a TM-algebra, $x \wedge y = y * (y * x) = x \quad \forall x, y \in X$ if \mathcal{D} is a generalized (l, r) -derivation of X , then $\mathcal{D}(x * y) = \mathcal{D}(x) * y \quad \forall x, y \in X$. Hence for every (l, r) -derivation d of X and any self map $\mathcal{D} : X \rightarrow X$ is a generalized (l, r) -derivation of X .

Analogously we define a generalized (r, l) -derivation on a TM-algebra $(X, *, 0)$ as follows:

Definition: 3.2.7

Let X be a TM-algebra. A mapping $\mathcal{D}: X \rightarrow X$ is called generalized (r,l) -derivation of X if there exist an (r,l) -derivation $d: X \rightarrow X$ such that $\mathcal{D}(x * y) = (x * \mathcal{D}(y)) \wedge (d(x) * y) \quad \forall x, y \in X$.

Remark:

In a TM-algebra, it is observed that for every (r,l) -derivation d of X and any map $\mathcal{D}: X \rightarrow X$, is a generalized (r,l) -derivation of X .

Definition: 3.2.8

Let X be a TM-algebra. A mapping $\mathcal{D}: X \rightarrow X$ is called generalized derivation of X if there exist a derivation $d: X \rightarrow X$ such that \mathcal{D} is both a (l,r) -generalized derivation and (r,l) -generalized derivation.

Example: 3.2.9

Let $(X, *, 0)$ be a TM-algebra with the following cayley table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$d: X \rightarrow X$ be defined by $d(0) = 3, d(1) = 2, d(2) = 1, d(3) = 0$ is a derivation on X .

The map $\mathcal{D}: X \rightarrow X$ defined as $\mathcal{D}(0)=2, \mathcal{D}(1)=3, \mathcal{D}(2)=0, \mathcal{D}(3)=1$, is a generalized derivation of X .

Lemma: 3.2.10

Let \mathcal{D} be a self map of a TM-algebra X . Then

- (1) If \mathcal{D} is a generalized (l,r) -derivation of X , then $\mathcal{D}(x) = \mathcal{D}(x) \wedge x \quad \forall x \in X$.
- (2) If \mathcal{D} is a generalized (r,l) -generalized derivation of X , then $\mathcal{D}(0)=0$ iff $\mathcal{D}(x) = x \wedge d(x) \quad \forall x \in X$ and for some (r,l) -generalized derivation d of X .

Proof:

(1) If \mathcal{D} is a generalized (l, r) -derivation of X , then there exists a

(l, r) -derivation d of X such that $\mathcal{D}(x * y) = (\mathcal{D}(x) * y) \wedge (x * d(y)) \quad \forall x, y \in X$

$$\begin{aligned} \mathcal{D}(x) &= \mathcal{D}(x * 0) = (\mathcal{D}(x) * 0) \wedge (x * d(0)) \\ &= \mathcal{D}(x) \wedge (x * d(0)) \\ &= (x * d(0)) * ((x * d(0)) * \mathcal{D}(x)) \\ &= (x * d(0)) * ((x * \mathcal{D}(x) * d(0))) \\ &= x * (x * \mathcal{D}(x)) \\ &= \mathcal{D}(x) \wedge x \quad \forall x \in X \end{aligned}$$

(2) Let \mathcal{D} be a generalized (r, l) -derivation of X such that $\mathcal{D}(0) = 0$. Then,

$$\mathcal{D}(x * y) = (x * \mathcal{D}(y)) \wedge (d(x) * y) \quad \forall (r, l)\text{-derivation } d \quad (1)$$

Putting $y = 0$ in (1), we get

$$\mathcal{D}(x * 0) = (x * \mathcal{D}(0)) \wedge (d(x) * 0)$$

$$\text{(i.e.) } \mathcal{D}(x) = (x * 0) \wedge d(x) = x \wedge d(x) \quad \forall x \in X$$

Conversely, if $\mathcal{D}(x) = x \wedge d(x)$, then

$$\begin{aligned} \mathcal{D}(0) &= 0 \wedge d(0) = d(0) * (d(0) * 0) \\ &= d(0) * d(0) \\ &= 0 \end{aligned}$$

Lemma: 3.2.11

Let \mathcal{D} be a generalized (l, r) -derivation of a TM-algebra X . Then

$$(1) \quad \mathcal{D}(a) = \mathcal{D}(0) + a \quad \forall a \in X$$

$$(2) \quad \mathcal{D}(a + x) = \mathcal{D}(a) + x \quad \forall a, x \in X$$

$$(3) \quad \mathcal{D}(a + b) = \mathcal{D}(a) + b = a + \mathcal{D}(b) \quad \forall a, b \in X$$

Proof:

$$\begin{aligned} (1) \quad \mathcal{D}(a) &= \mathcal{D}(0 * (0 * a)) = (\mathcal{D}(0) * (0 * a)) \wedge (0 * d(0 * a)) \\ &= \mathcal{D}(0) * (0 * a) \end{aligned}$$

$$\text{(i.e.), } \mathcal{D}(a) = \mathcal{D}(0) + a \quad \forall a \in X$$

$$(2) \quad \mathcal{D}(a + x) = \mathcal{D}(a * (0 * x)) = \mathcal{D}(a) * (0 * x) = \mathcal{D}(a) + x$$

(3) Since $(X, +)$ is an abelian group, the result follows from

$$\mathcal{D}(a) + b = \mathcal{D}(a + b) = \mathcal{D}(b + a) = \mathcal{D}(b) + a$$

Theorem: 3.2.12

Let \mathcal{D} be a generalized (r, l) -derivation of a TM-algebra X . Then

- (1) $\mathcal{D}(a) \in G(X) \quad \forall a \in G(X)$
- (2) $\mathcal{D}(a) = a * \mathcal{D}(0) = a + \mathcal{D}(0) \quad \forall a \in X$
- (3) $\mathcal{D}(a+b) = \mathcal{D}(a) + \mathcal{D}(b) - \mathcal{D}(0) \quad \forall a, b \in X$
- (4) \mathcal{D} is the identity map on X iff $\mathcal{D}(0) = 0$.

Proof:

Let \mathcal{D} be a generalized (r, l) -derivation of a TM-algebra X .

- (1) For $a \in G(X)$, $\mathcal{D}(a) = \mathcal{D}(0 * a) = (0 * \mathcal{D}(a)) \wedge (d(0) * a)$
 $= 0 * \mathcal{D}(a)$
 $\Rightarrow \mathcal{D}(a) \in G(X)$

- (2) Now, $\mathcal{D}(a) = \mathcal{D}(a * 0) = (a * \mathcal{D}(0)) \wedge (d(a) * 0)$

Therefore $\mathcal{D}(a) = a * \mathcal{D}(0) = a * \mathcal{D}(0 * 0) = a * (0 * \mathcal{D}(0))$

$$= a + \mathcal{D}(0)$$

- (3) By result (2) above, we have

$$\mathcal{D}(a+b) = (a+b) + \mathcal{D}(0)$$

Since $(X, +)$ is an abelian group, on simplifying the right hand side, we get

$$\mathcal{D}(a+b) = \mathcal{D}(a) + \mathcal{D}(b) - \mathcal{D}(0)$$

- (4) If $\mathcal{D}(0) = 0$, then $\mathcal{D}(a) = \mathcal{D}(a * 0) = a * \mathcal{D}(0) = a * 0 = a \quad \forall a \in X$
 $\Rightarrow \mathcal{D}$ is the identity map.

Conversely, if \mathcal{D} is the identity map on X , then $\mathcal{D}(a) = a \quad \forall a \in X$.

In particular $\mathcal{D}(0) = 0$

Definition: 3.2.13

A TM-algebra X is said to be torsion free if it satisfies,

$$x + x = 0 \Rightarrow x = 0 \quad \forall x \in X.$$

If there exists a non-zero element $x \in X$ such that $x + x = 0$, then X is not Torsion free.

Example: 3.2.14

The TM-algebra X given by the cayley table in example (3.2.4) is a Torsion-free TM-algebra. For,

$$0+0=0, 1+1=1*(0*1)=1*4=2$$

$$2+2=2*(0*2)=2*3=4$$

$$3+3=3*(0*3)=3*2=1$$

$$4+4=4*(0*4)=4*1=3$$

The TM-algebra X given by the cayley table in example (3.2.9) is not a Torsion-free TM-algebra. For, $1+1=1*(0*1)=1*1=0$

(i.e.) There exists a non-zero element $x \in X$ such that $1+1=0$.

Theorem: 3.2.15

Let X be a torsion free TM-algebra and let \mathcal{D}_1 and \mathcal{D}_2 be two generalized derivations. If $\mathcal{D}_1\mathcal{D}_2=0$ on X , then $\mathcal{D}_2=0$ on X .

Proof:

Let $x \in X$, then $x+x=X$

$$\begin{aligned} 0 &= (\mathcal{D}_1\mathcal{D}_2)(x+x) = \mathcal{D}_1(\mathcal{D}_2(x+x)) \\ &= \mathcal{D}_1(0) + \mathcal{D}_2(x+x) \\ &= \mathcal{D}_1(0) + \mathcal{D}_2(x) + \mathcal{D}_2(x) - \mathcal{D}_2(0) \\ &= \mathcal{D}_1(0) - \mathcal{D}_2(0) + \mathcal{D}_2(x) + \mathcal{D}_2(x) \\ &= (\mathcal{D}_1(0) * \mathcal{D}_2(0)) + \mathcal{D}_2(x) + \mathcal{D}_2(x) \\ &= (\mathcal{D}_1(0) * (0 * \mathcal{D}_2(0))) + \mathcal{D}_2(x) + \mathcal{D}_2(x) \\ &= \mathcal{D}_1(0) + \mathcal{D}_2(0) + \mathcal{D}_2(x) + \mathcal{D}_2(x) \\ &= \mathcal{D}_1(\mathcal{D}_2(0)) + \mathcal{D}_2(x) + \mathcal{D}_2(x) \\ &= (\mathcal{D}_1\mathcal{D}_2)(0) + \mathcal{D}_2(x) + \mathcal{D}_2(x) \\ &= 0 + \mathcal{D}_2(x) + \mathcal{D}_2(x) \\ &= \mathcal{D}_2(x) + \mathcal{D}_2(x) \end{aligned}$$

Since X is Torsion-free, $\mathcal{D}_2=0$ on X .

In the above theorem, if we replace both the generalized derivation \mathcal{D}_1 and \mathcal{D}_2 by a generalized derivation \mathcal{D} itself, we get the following theorem

Corollary: 3.2.16

Let X be a Torsion free TM-algebra and \mathcal{D} be a generalized derivation.
If $\mathcal{D}^2=0$ on X , then $\mathcal{D}=0$ on X .

Proof:

Let $\mathcal{D}^2=0$ on X . That is $\mathcal{D}^2(x)=0 \forall x \in X$.

Now for any $x \in X$, by using theorem (3.2.12)

$$\begin{aligned} 0 &= \mathcal{D}^2(x+x) \\ &= \mathcal{D}(\mathcal{D}(x+x)) \\ &= \mathcal{D}(0) + \mathcal{D}(x+x) \\ &= \mathcal{D}(0) + \mathcal{D}(x) + \mathcal{D}(x) - \mathcal{D}(0) \\ &= \mathcal{D}(x) + \mathcal{D}(x) \end{aligned}$$

Since X is torsion free, $\mathcal{D}(x)=0 \forall x \in X$ proving that $\mathcal{D}=0$ on X .

CHAPTER IV

CHAPTER –IV

t- DERIVATIONS AND SYMMETRIC BI-DERIVATIONS ON TM-ALGEBRA

SECTION: 4.1 t- DERIVATIONS ON TM-ALGEBRAS

Definition: 4.1.1

A TM-algebra X is said to be associative if
 $(x * y) * z = x * (y * z) \forall x, y, z \in X$.

Example: 4.1.2

Let $(X, *, 0)$ be a TM-algebra with the cayley table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then X is an associative TM-algebra.

Definition: 4.1.3

Let X be a TM-algebra. Then for any $t \in X$, we define a self
 $d_t : X \rightarrow X$ by $d_t(x) = x * t \forall x \in X$.

Definition: 4.1.4

Let X be a TM-algebra. Then for any $t \in X$, a self map $d_t : X \rightarrow X$ is called
a $(l, r) - t$ -derivation of X if it satisfies the condition

$$d_t(x * y) = (d_t(x) * y) \wedge (x * (d_t(y))) \forall x, y \in X.$$

Example: 4.1.5

Let $(X, *, 0)$ be a TM-algebra with the following cayley table.

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Define, When $t = 0$, $d_t(x) = x \quad \forall x \in X$

When $t = 1$, $d_t(0) = 2, d_t(1) = 0, d_t(2) = 1$

When $t = 2$, $d_t(0) = 1, d_t(1) = 2, d_t(2) = 0$

For each $t \in X$, d_t is a $(l, r) - t$ -derivation of X .

Remark:

In a TM-algebra $x \wedge y = y * (y * x) = x \quad \forall x, y \in X$. By using the above property, if d_t is a $(l, r) - t$ -derivation of X then $d_t(x * y) = d_t(x) * y$.

Definition: 4.1.6

Let X be a TM-algebra. Then for any $t \in X$, a self map $d_t : X \rightarrow X$ is called a $(r, l) - t$ -derivation of X if it satisfies the condition

$$d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y) \quad \forall x, y \in X.$$

Remark:

If d_t is a $(r, l) - t$ -derivation of X , then $d_t(x * y) = x * d_t(y) \quad \forall x, y \in X$.

Definition: 4.1.7

Let X be a TM-algebra. Then for any $t \in X$, a self map $d_t : X \rightarrow X$ is called a t -derivation on X if d_t is both a $(l, r) - t$ -derivation and $(r, l) - t$ -derivation of X .

Example: 4.1.8

Consider the TM-algebra $(X, *, 0)$ in example (4.1.2)

Define the mapping d_t as follows:

When $t = 0$, $d_t(x) = x \quad \forall x \in X$

When $t = 1$, $d_t(0) = 1, d_t(1) = 0, d_t(2) = 3, d_t(3) = 2$

When $t = 2$, $d_t(0) = 2, d_t(1) = 3, d_t(2) = 0, d_t(3) = 1$

When $t = 3$, $d_t(0) = 3, d_t(1) = 2, d_t(2) = 1, d_t(3) = 0$

For each $t \in X$, d_t is a t -derivation of X .

Remark:

Any self map d_t of a TM-algebra X is a (l, r) - t -derivation of X .

Proposition: 4.1.9

Let d_t be a self map of an associative TM-algebra X . Then d_t is a (r, l) - t -derivation of X .

Proof:

Let d_t be a self map of an associative TM-algebra X .

Then we have,

$$\begin{aligned}
 d_t(x * y) &= (x * y) * t \\
 &= (x * t) * y && \text{(by P4)} \\
 &= ((x * t) * y) * 0 \\
 &= ((x * t) * y) * (((x * t) * y) * ((x * t) * y)) && \text{(by P1)} \\
 &= (((x * t) * y) * (((x * t) * y) * ((x * y) * t))) && \text{(by P4)} \\
 &= ((x * t) * y) * (((x * t) * y) * (x * (y * t))) \\
 &= (x * (y * t)) \wedge ((x * t) * y) \\
 &= (x * d_t(y)) \wedge (d_t(x) * y)
 \end{aligned}$$

Therefore d_t is a (r, l) - t -derivation of X .

Theorem: 4.1.10

Let X be an associative TM-algebra. For any $t \in X$, a self map d_t is a t -derivation of X .

Proof: Obvious.

Definition: 4.1.11

A self map d_t of a TM-algebra X is said to be t -regular if $d_t(0) = 0$

Example: 4.1.12

In example (4.1.8) d_t is a regular t -derivation on X when $t = 0$. However, $t = 1$ or $t = 2$ or $t = 3$, d_t is not a regular t -derivation of X .

Proposition: 4.1.13

For any self map d_t of a TM-algebra X , the following holds.

- (1) If d_t is a (l, r) - t -derivation of X , $d_t(x) = d_t(x) \wedge x \quad \forall x \in X$.
- (2) If d_t is a (r, l) - t -derivation of X , $d_t(x) = x \wedge d_t(x) \quad \forall x \in X$ iff d_t is t -regular.

Proof:

Let d_t be a self map of a TM-algebra X .

- (1) Let d_t be a (l, r) - t -derivation of X . Then we have,

$$\begin{aligned}
 d_t(x) &= d_t(x * 0) \\
 &= (d_t(x) * 0) \wedge (x * d_t(0)) \\
 &= d_t(x) \wedge (x * d_t(0)) \\
 &= (x * d_t(0)) * ((x * d_t(0)) * d_t(x)) \\
 &= (x * d_t(0)) * ((x * d_t(x)) * d_t(0)) && \text{(by P4)} \\
 &= x * (x * d_t(x)) && \text{(by TM-2)} \\
 &= d_t(x) \wedge x
 \end{aligned}$$

Therefore d_t is a (l, r) - t -derivation of X .

- (2) Let d_t be a (r, l) - t -derivation of X and $d_t(x) = x \wedge d_t(x)$ (1)

Put $x = 0$ in (1), we have

$$\begin{aligned}
d_t(0) &= 0 \wedge d_t(0) = d_t(0) * (d_t(0) * 0) \\
&= d_t(0) * d_t(0) = 0
\end{aligned}$$

Therefore d_t is t -regular.

Conversely, suppose that d_t is t -regular (r, l) - t -derivation of X . Then

$$\begin{aligned}
d_t(x) &= d_t(x * 0) \\
&= (x * d_t(0)) \wedge (d_t(x) * 0) \\
&= (x * 0) \wedge d_t(x) = x \wedge d_t(x)
\end{aligned}$$

Hence the proof.

Theorem: 4.1.14

Let d_t be a (l, r) - t -derivation of a TM-algebra. Then the following hold.

- (1) $d_t(0) = d_t(x) * x \quad \forall x \in X$
- (2) d_t is one-one
- (3) d_t is t -regular then it is the identity map.
- (4) If there is an element $x \in X \ni d_t(x) = x$, then d_t is the identity map.
- (5) If $x \leq y$ then $d_t(x) \leq d_t(y) \quad \forall x, y \in X$

Proof:

- (1) Let d_t be a (l, r) - t -derivation of a TM-algebra X .

Then we have $d_t(0) = d_t(x * x) = d_t(x) * x$ (since d_t is a (l, r) - t -derivation)

- (2) Let $d_t(x) = d_t(y) \quad \forall x, y \in X$

Then $x * t = y * t$ and by applying the right cancellation law we have, $x = y$

- (3) Let d_t be t -regular and $x \in X$.

Now, $x * x = 0 = d_t(0) = d_t(x * x) = d_t(x) * x$

Hence by right cancellation law, $d_t(x) = x \quad \forall x \in X$, showing that d_t is the identity map.

- (4) Let $d_t(x) = x$ for some $x \in X$, $0 = x * x = d_t(x) * x = d_t(x * x) = d_t(0)$

showing that d_t is t -regular. Hence by (3) d_t is the identity map.

- (5) Since $x \leq y$, $d_t(x) * d_t(y) = (x * t) * (y * t) = (x * y) = 0$

thus proving $d_t(x) \leq d_t(y)$.

Theorem: 4.1.15

Let X be a TM-algebra and d_t be a t -derivation of X . If $x \leq y$ and $d_t(x * y) = d_t(x) * d_t(y) \quad \forall x, y \in X$. Then $d_t(x) = d_t(y)$.

Proof:

Let d_t be a t -derivation of a TM-algebra X .

$$\begin{aligned}
 d_t(x) &= d_t(x * 0) \\
 &= d_t(x * (x * y)) \\
 &= d_t(x) * d_t(x * y) \\
 &= d_t(x) * (d_t(x) * d_t(y)) \\
 &= d_t(y) \quad \text{(by P3)}
 \end{aligned}$$

Theorem: 4.1.16

Let d_t be a t -regular (r, l) - t -derivation of a TM-algebra.

Then the following hold.

- (1) $d_t(x) = x$
- (2) $d_t(x) * y = x * d_t(y) \quad \forall x, y \in X$
- (3) $d_t(x * y) = d_t(x) * y = d_t(x) * d_t(y) = x * d_t(y)$
- (4) $\ker(d_t) = \{x \in X / d_t(x) = 0\}$ is a sub algebra of X .

Proof:

(1) Since d_t is t -regular (r, l) - t -derivation of a TM-algebra X , for any $x \in X$, we have $d_t(x) = d_t(x * 0) = x * d_t(0) = x * 0 = x$.

(2) If d_t is t -regular (r, l) - t -derivation of X then by (1), $d_t(x) = x \quad \forall x \in X$

Thus, $d_t(x) * y = x * y = x * d_t(y)$

(3) If d_t is t -regular (r, l) - t -derivation of X then by (1), $d_t(x) = x \quad \forall x \in X$

For $x, y \in X$, $d_t(x * y) = x * y = d_t(x) * d_t(y)$

If d_t is a (r, l) - t -derivation of X then $d_t(x * y) = x * d_t(y)$

$d_t(x * y) = x * y = d_t(x) * y$

Hence $d_t(x * y) = d_t(x) * y = x * d_t(y) = x * y$.

(4) Since d_t is t -regular, $d_t(0) = 0$. Then $0 \in \ker(d_t)$ showing that $\ker(d_t)$ is a non-empty set. Let $x, y \in \ker(d_t)$, then $d_t(x) = 0, d_t(y) = 0$.

Now $d_t(x * y) = x * y = d_t(x) * d_t(y) = 0 * 0 = 0$.

Therefore, $(x * y) \in \ker(d_t)$, proving that $\ker(d_t)$ is a sub-algebra of X .

Proposition: 4.1.17

Let X be a TM-algebra. Then $\ker(d_t) = \{0\}$ iff d_t is t -regular.

Proof:

Let X be a TM-algebra. Obviously when $\ker(d_t) = \{0\}$, $d_t(0) = 0$ showing that d_t is t -regular.

On the other hand, if $x \in \ker(d_t)$, d_t is t -regular shows that,

$0 = d_t(0) = d_t(x * x) = d_t(x) * x = 0 * x$. Thus, $x = 0$ showing that $\ker(d_t) = \{0\}$.

Definition: 4.1.18

Let X be a TM-algebra and let d_t, d'_t be two self maps of X . Then we define $d_t \circ d'_t : X \rightarrow X$ by $(d_t \circ d'_t)(x) = d_t(d'_t(x)) \quad \forall x \in X$.

Example: 4.1.19

Consider the TM-algebra given in example (4.1.2)

The self-maps $d_t, d'_t : X \rightarrow X$ given by,

$$d_t(0) = 1, d_t(1) = 0, d_t(2) = 3, d_t(3) = 2$$

$$d'_t(0) = 2, d'_t(1) = 3, d'_t(2) = 0, d'_t(3) = 1 \text{ are } t\text{-derivations on } X.$$

Now, define a self-map $(d_t \circ d'_t) : X \rightarrow X$ by

$$(d_t \circ d'_t)(0) = 3, (d_t \circ d'_t)(1) = 2, (d_t \circ d'_t)(2) = 1, (d_t \circ d'_t)(3) = 0$$

Then $(d_t \circ d'_t)(x) = d_t(d'_t(x)) \quad \forall x \in X$ is also a t -derivation on X .

Proposition: 4.1.20

Let X be a TM-algebra and let d_t, d_t' be a $(l, r) - t$ -derivation of X . Then $(d_t \circ d_t')$ is also a $(l, r) - t$ -derivation of X .

Proof:

Let X be a TM-algebra and let d_t, d_t' be a $(l, r) - t$ -derivation of X . Then $\forall x, y \in X$. We have,

$$\begin{aligned}(d_t \circ d_t')(x * y) &= d_t(d_t'(x * y)) = d_t(d_t'(x) * y) \\ &= (d_t(d_t'(x)) * y) = (d_t \circ d_t')(x) * y\end{aligned}$$

Therefore $(d_t \circ d_t')$ is a $(l, r) - t$ -derivation of X .

Proposition: 4.1.21

Let X be a TM-algebra and let d_t, d_t' be a $(r, l) - t$ -derivations of X . Then $(d_t \circ d_t')$ is also a $(r, l) - t$ -derivations of X .

Proof:

Let X be a TM-algebra and let d_t, d_t' be a $(r, l) - t$ -derivation of X . Then $\forall x, y \in X$. We have,

$$\begin{aligned}(d_t \circ d_t')(x * y) &= d_t(d_t'(x * y)) = d_t(x * d_t'(y)) \\ &= x * d_t(d_t'(y)) = x * (d_t \circ d_t')(y)\end{aligned}$$

Therefore $(d_t \circ d_t')$ is a $(r, l) - t$ -derivation of X .

Theorem: 4.1.22

Let X be a TM-algebra and let d_t, d_t' be a t -derivation of X . Then $(d_t \circ d_t')$ is also a t -derivations of X .

Proof:

Proof follows by above two propositions (4.1.20) and (4.1.21)

Theorem: 4.1.23

Let X be a TM-algebra. Let d_t be a $(r, l) - t$ -derivations of X and d'_t be a $(l, r) - t$ -derivation of X . Then $d_t \circ d'_t = d'_t \circ d_t$.

Proof:

Let d'_t be a $(l, r) - t$ -derivation of a TM-algebra X . Then we have,

$$d'_t(x * y) = d'_t(x) * y$$

$$\begin{aligned} \text{Now } (d_t \circ d'_t)(x * y) &= d_t(d'_t(x * y)) \\ &= d_t(d'_t(x) * y) \\ &= d'_t(x) * d_t(y) \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Again } (d'_t \circ d_t)(x * y) &= d'_t(d_t(x * y)) \\ &= d'_t(x * d_t(y)) \\ &= d'_t(x) * d_t(y) \end{aligned} \quad (2)$$

From (1) and (2), $(d_t \circ d'_t)(x * y) = (d'_t \circ d_t)(x * y)$

This is true $\forall x, y \in X$. In particular this is true $\forall x$ and $y = 0$.

$$\text{Put } y = 0, \quad (d_t \circ d'_t)(x * 0) = (d'_t \circ d_t)(x * 0)$$

$$(d_t \circ d'_t)(x) = (d'_t \circ d_t)(x) \quad \forall x \in X$$

Hence $d_t \circ d'_t = d'_t \circ d_t$

Theorem: 4.1.24

Let X be a TM-algebra and let d_t, d'_t be two t -derivations of X , then

$$d_t \circ d'_t = d'_t \circ d_t$$

Proof: Obvious.

Definition: 4.1.25

Let X be a TM-algebra and let d_t, d'_t be two self maps of X . Then define

$$d_t * d'_t : X \rightarrow X \text{ as } (d_t * d'_t)(x) = d_t(x) * d'_t(x) \quad \forall x \in X.$$

Example: 4.1.26

Consider the TM-algebra $(X, *, 0)$ given in example (4.1.2)

Define $d_t : X \rightarrow X$ by

$$d_t(0) = 1, d_t(1) = 0, d_t(2) = 3, d_t(3) = 2 \text{ be a } t\text{-derivations on } X.$$

Define $d'_t : X \rightarrow X$ by

$$d'_t(0) = 2, d'_t(1) = 3, d'_t(2) = 0, d'_t(3) = 1 \text{ be a } t\text{-derivations on } X.$$

$$\text{Now, } (d_t * d'_t)(0) = 3 = d_t(0) * d'_t(0)$$

$$(d_t * d'_t)(1) = 3 = d_t(1) * d'_t(1)$$

$$(d_t * d'_t)(2) = 3 = d_t(2) * d'_t(2)$$

$$(d_t * d'_t)(3) = 3 = d_t(3) * d'_t(3)$$

Theorem: 4.1.27

Let X be a TM-algebra and let d_t, d'_t be two t -derivations of X , then

$$d_t * d'_t = d'_t * d_t.$$

Proof:

Let X be a TM-algebra and let d_t, d'_t be two t -derivations of X .

$$\text{Then we have, } (d_t \circ d'_t)(x * y) = d_t(d'_t(x * y))$$

$$= d_t(d'_t(x) * y)$$

$$= d_t(d'_t(x) * y)$$

$$= d'_t(x) * d_t(y) \tag{1}$$

$$\text{Again, } (d_t \circ d'_t)(x * y) = d_t(d'_t(x * y))$$

$$= d_t(x * d'_t(y))$$

$$= d_t(x) * d'_t(y) \tag{2}$$

$$\text{From (1) and (2), } d_t(x) * d'_t(y) = d'_t(x) * d_t(y) \tag{3}$$

Put $x = y$ in (3), $d_t(x) * d_t'(x) = d_t'(x) * d_t(x)$

$$(d_t * d_t')(x) = (d_t' * d_t)(x)$$

Hence $d_t * d_t' = d_t' * d_t$

Definition: 4.1.28

Let $L_tDer(X)$ denote the set of all $(l, r) - t -$ derivations of X . Define the binary operation \wedge on $L_tDer(X)$ as follows: For $d_t, d_t' \in L_tDer(X)$,

Define, $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x) \quad \forall x \in X$.

Lemma: 4.1.29

If d_t and d_t' are $(l, r) - t -$ derivations on a TM-algebra X . Then $(d_t \wedge d_t')$ is also a $(l, r) - t -$ derivation on X .

Proof:

Let d_t, d_t' be $(l, r) - t -$ derivation of a TM-algebra X .

Then we have, $(d_t \wedge d_t')(x * y) = d_t(x * y) \wedge d_t'(x * y)$

$$\begin{aligned} &= (d_t(x) * y) \wedge (d_t'(x) * y) \\ &= (d_t'(x) * y) * ((d_t'(x) * y) * (d_t(x) * y)) \\ &= d_t(x) * y \end{aligned} \tag{1}$$

Again, $(d_t \wedge d_t')(x) * y = (d_t(x) \wedge d_t'(x)) * y$

$$\begin{aligned} &= (d_t'(x) * (d_t'(x) * (d_t(x)))) * y \\ &= d_t(x) * y \end{aligned} \tag{2}$$

From (1) and (2), $(d_t \wedge d_t')(x * y) = (d_t \wedge d_t')(x) * y$

Hence $(d_t \wedge d_t')$ is also a $(l, r) - t -$ derivation on X .

Lemma: 4.1.30

The binary composition \wedge defined on $L_t Der(X)$ is associative.

Proof:

Let X be a TM-algebra. Let d_t, d_t', d_t'' be (l, r) - t -derivation on X .

Now by lemma (4.1.29)

$$\begin{aligned}
((d_t \wedge d_t') \wedge d_t'')(x * y) &= (d_t \wedge d_t')(x * y) \wedge d_t''(x * y) \\
&= (d_t(x) * y) \wedge (d_t'(x) * y) \\
&= (d_t''(x) * y) * ((d_t'(x) * y) * (d_t(x) * y)) \\
&= d_t(x) * y
\end{aligned} \tag{1}$$

$$\begin{aligned}
\text{Again, } (d_t \wedge (d_t' \wedge d_t''))(x * y) &= d_t(x * y) \wedge (d_t' \wedge d_t'')(x * y) \\
&= d_t(x * y) \wedge (d_t'(x) * y) \\
&= (d_t(x) * y) \wedge (d_t'(x) * y) \\
&= (d_t'(x) * y) * ((d_t'(x) * y) * (d_t(x) * y)) \\
&= d_t(x) * y
\end{aligned} \tag{2}$$

From (1) and (2), $((d_t \wedge d_t') \wedge d_t'')(x * y) = (d_t \wedge (d_t' \wedge d_t''))(x * y)$

Put $y = 0$, we get $((d_t \wedge (d_t' \wedge d_t''))(x)) = (d_t \wedge (d_t' \wedge d_t''))(x) \quad \forall x \in X$

Hence $(d_t \wedge d_t') \wedge d_t'' = d_t \wedge (d_t' \wedge d_t'')$

This proves the binary operation \wedge is associative.

Theorem: 4.1.31

$L_t Der(X)$ the set of all (l, r) - t -derivations of X is a semi-group under the binary operation \wedge defined by

$$(d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x) \quad \forall x \in X, \text{ and } d_t, d_t' \in L_t Der(X)$$

Proof:

Proof follows by above two lemmas (4.1.29) and (4.1.30)

Definition: 4.1.32

Let $R_t Der(X)$ denote the set of all $(r,l)-t$ -derivations of X . Define the binary operation \wedge on $R_t Der(X)$ as follows: For $d_t, d_t' \in R_t Der(X)$, Define $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x)$

Analogously we prove the following theorem.

Theorem: 4.1.33

$R_t Der(X)$ is a semi-group under the binary operation \wedge defined by $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x) \quad \forall x \in X$, and $d_t, d_t' \in R_t Der(X)$.

Proof: Obvious.

Theorem: 4.1.34

If $_t Der(X)$ denote the set of all t -derivations on X then it is a semi-group under the binary operation \wedge defined by $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x) \quad \forall x \in X$, and $d_t, d_t' \in _t Der(X)$.

Proof:

Proof follows by above two theorems (4.1.31) and (4.1.33)

SECTION: 4.2 SYMMETRIC BI-DERIVATIONS ON TM-ALGEBRA

Definition: 4.2.1

Let X, Y be TM-algebras. An operation $*$ on the Cartesian product $X \times Y$ of X, Y is defined as follows.

- (1) $(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)$
- (2) $(0, 0) = 0$

Lemma: 4.2.2

Cartesian product of two TM-algebras is again a TM-algebra.

Proof:

Let X and Y be two TM-algebras. Consider the Cartesian product $X \times Y$.

$$(x, y) * (0, 0) = (x * 0, y * 0) = (x, y)$$

$$\begin{aligned} ((x_1, y_1) * (x_2, y_2)) * ((x_1, y_1) * (x_3, y_3)) &= (x_1 * x_2, y_1 * y_2) * ((x_1 * x_3, y_1 * y_3)) \\ &= ((x_1 * x_2) * (x_1 * x_3), (y_1 * y_2) * (y_1 * y_3)) \\ &= ((x_3 * x_2), (y_3 * y_2)) \\ &= (x_3, y_3) * (x_2, y_2) \end{aligned}$$

Therefore $(X \times Y, *, 0)$ is a TM-algebra.

Definition: 4.2.3

Let X be a TM-algebra. A mapping $D: X \times X \rightarrow X$ is a symmetric map if $D(x, y) = D(y, x)$ holds for all pairs of elements $x, y \in X$.

Example: 4.2.4

Let $(X, *, 0)$ be a TM-algebra with the cayley table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The map $D: X \times X \rightarrow X$ defined by $D(x, y) = x * (0 * y)$ is a symmetric map.

Definition: 4.2.5

Let X be a TM-algebra and $D: X \times X \rightarrow X$ be a symmetric mapping. A mapping $d: X \rightarrow X$ defined by $d(x) = D(x, x)$ is called trace of D .

Example: 4.2.6

Let $(X, *, 0)$ be a TM-algebra with the cayley table.

*	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

The map $D: X \times X \rightarrow X$ defined by $D(x, y) = x * (0 * y) = x + y$ is a symmetric map.

If $x = 0, D(0,0) = 0 + 0 = 0$. $x = 1, D(1,1) = 1 + 1 = 3$. $x = 2, D(2,2) = 2 + 2 = 3$.

$x = 3, D(3,3) = 3 + 3 = 0$

Thus the mapping $d: X \rightarrow X$ given by

$$d(x) = D(x, x) = \begin{cases} 0 & \text{if } x = 0, 3 \\ 3 & \text{if } x = 1, 2 \end{cases} \text{ is the trace of the symmetric mapping } D.$$

Definition: 4.2.7

Let X be a TM-algebra and $D: X \times X \rightarrow X$ be a symmetric mapping. If D satisfies the identity, $D(x * y, z) = (D(x, z) * y) \wedge (x * D(y, z)) \quad \forall x, y, z \in X$, then D is called left-right symmetric bi-derivation. ((l, r) - symmetric bi-derivation)

If D satisfied the identity, $D(x * y, z) = (x * D(y, z)) \wedge (D(x, z) * y) \quad \forall x, y, z \in X$, then D is called right-left symmetric bi-derivation.

((r, l) -symmetric bi-derivation)

If D is both an (l, r) symmetric bi-derivation and an (r, l) -symmetric bi-derivation then D is called a symmetric bi-derivation.

Example: 4.2.8

Consider in example (4.2.6). Define a mapping $D: X \times X \rightarrow X$ by $D(x, y) = x * (0 * y) \quad \forall x, y \in X$ is a symmetric map. Then D is symmetric bi-derivation.

Example: 4.2.9

Consider in example (4.2.4). Define $D(x, y) = x * (0 * y) \quad \forall x, y \in X$ is a symmetric map. Then D is a symmetric bi-derivation.

Example: 4.2.10

Consider the TM-algebra with the cayley table as in example (4.2.4).

Define the symmetric map $D: X \times X \rightarrow X$ such that

$$D(x, x) = 3 \text{ if } x = 0, 1, 2, 3$$

$$D(0, 3) = D(3, 0) = D(1, 2) = D(2, 1) = 0$$

$$D(0, 2) = D(2, 0) = D(1, 3) = D(3, 1) = 1$$

$$D(0, 1) = D(1, 0) = D(2, 3) = D(3, 2) = 2$$

Then D is a symmetric bi-derivation.

Proposition: 4.2.11

Let X be a TM-algebra. Define a symmetric map $D: X \times X \rightarrow X$ by $D(x, y) = x + y \quad \forall x, y \in X$. Then D is a (l, r) symmetric bi-derivation.

Proof:

$$\begin{aligned} D(x * y, z) &= (x * y) + z \quad \forall x, y, z \in X \\ &= (x * y) * (0 * z) \\ &= (x * (0 * z)) * y && \text{(by P4)} \\ &= (x + z) * y \\ &= (x * (y + z)) * ((x * (y + z)) * ((x + z) * y)) && \text{(by P3)} \\ &= ((x + z) * y) \wedge (x * (y + z)) \\ &= (D(x, z) * y) \wedge (x * D(y, z)) \end{aligned}$$

This proves that D is a (l, r) symmetric bi-derivation.

Theorem: 4.2.12

Let X be an associative TM-algebra. Then the symmetric map $D: X \times X \rightarrow X$ defined by $D(x, y) = x + y \quad \forall x, y \in X$ is a symmetric bi-derivation.

Proof:

Let X be an associative TM-algebra and $D: X \times X \rightarrow X$ be the symmetric map. By the above proposition, D is a (l, r) symmetric bi-derivation.

$$\begin{aligned}
D(x * y, z) &= (x * y) + z = (x * y) * (0 * z) \\
&= (x * (0 * z)) * y \\
&= ((x * 0) * z) * y \\
&= (x * z) * y = (x * y) * z
\end{aligned} \tag{1}$$

Since $x \wedge y = y * (y * x) = x$,

$$\begin{aligned}
(x * D(y, z)) \wedge (D(x, z) * y) &= x * D(y, z) \\
&= x * (y + z) \\
&= x * (y * (0 * z)) \\
&= x * ((y * 0) * z) \\
&= x * (y * z) \\
&= (x * y) * z
\end{aligned} \tag{2}$$

From (1) and (2), $D(x * y, z) = (x * D(y, z)) \wedge (D(x, z) * y) \quad \forall x, y, z \in X$.

This proves that D is (r, l) -symmetric bi-derivation and hence a symmetric bi-derivation.

Proposition: 4.2.13

Let X be a TM-algebra and $D : X \times X \rightarrow X$ be a symmetric map. Then

- (1) If D is a (l, r) symmetric bi-derivation then $D(x, y) = D(x, y) \wedge x \quad \forall x, y \in X$
- (2) If D is a (r, l) symmetric bi-derivation then $D(x, y) = x \wedge D(x, y) \quad \forall x, y \in X$
iff $D(0, y) = 0 \quad \forall x \in X$

Proof:

Let X be a TM-algebra and $D : X \times X \rightarrow X$ be a symmetric map.

- (1) Let D be a (l, r) symmetric bi-derivation.

$$\begin{aligned}
D(x, y) &= D(x * 0, y) \quad \forall x, y \in X \\
&= (D(x, y) * 0) \wedge (x * D(0, y)) \\
&= D(x, y) \wedge (x * D(0, y)) \\
&= (x * D(0, y)) * ((x * D(0, y)) * D(x, y)) \\
&= (x * D(0, y)) * ((x * D(x, y)) * D(0, y)) && \text{(by P4)} \\
&= x * (x * D(x, y)) && \text{(by TM-2)} \\
&= D(x, y) \wedge x
\end{aligned}$$

(2) Let D be a (r, l) – symmetric bi-derivation and $D(0, y) = 0 \forall y \in X$.

$$\begin{aligned} D(x, y) &= D(x * 0, y) \\ &= (x * D(0, y)) \wedge (D(x, y) * 0) \\ &= (x * 0) \wedge D(x, y) \\ &= x \wedge D(x, y) \end{aligned}$$

Conversely, if $D(x, y) = x \wedge D(x, y) \forall x, y \in X$.

$$\begin{aligned} \text{Then } D(0, y) &= 0 \wedge D(0, y) = D(0, y) * (D(0, y) * 0) \\ &= D(0, y) * D(0, y) = 0 \end{aligned}$$

Proposition: 4.2.14

Let X be a TM-algebra and $D : X \times X \rightarrow X$ be a (l, r) – symmetric map.

Then

- (1) $D(a, y) = D(0, y) * (0 * a) = D(0, y) + a \forall a, y \in X$
- (2) $D(a + b, y) = D(a, y) + D(b, y) - D(0, y) \forall a, b, y \in X$
- (3) $D(a, y) = a \forall a, y \in X$ iff $D(0, y) = 0$

Proof:

Let X be a TM-algebra and $D : X \times X \rightarrow X$ be a symmetric map.

(1) Let $a = 0 * (0 * a)$

$$\begin{aligned} D(a, y) &= D(0 * (0 * a), y) \\ &= (D(0, y) * (0 * a)) \wedge (0 * D(0 * a, y)) \\ &= D(0, y) * (0 * a) \\ &= D(0, y) + a \end{aligned}$$

- (2) By (1) $D(a + b, y) = D(0, y) + a + b$

$$\begin{aligned} &= D(0, y) + a + D(0, y) + b - D(0, y) \\ &= D(a, y) + D(b, y) - D(0, y) \end{aligned}$$

(3) $D(a, y) = a \forall a, y \in X$

Put $a = 0$, $D(0, y) = 0 \forall y \in X$

Conversely, if $D(0, y) = 0$, then $D(a, y) = D(0, y) + a = 0 + a = a$.

Proposition: 4.2.15

Let X be a TM-algebra and $D: X \times X \rightarrow X$ be a (r,l) -symmetric bi-derivation. Then

$$(1) D(a, y) \in G(X) \quad \forall a \in G(X)$$

$$(2) D(a, y) = a * D(0, y) = a + D(0, y) \quad \forall a, y \in X$$

$$(3) D(a+b, y) = D(a, y) + D(b, y) - D(0, y) \quad \forall a, y \in G(X) \quad \forall a, b, y \in X$$

$$(4) D(a, y) = a \quad \forall a, y \in X \quad \text{iff} \quad D(0, y) = 0$$

Proof:

Let X be a TM-algebra and $D: X \times X \rightarrow X$ be a (r,l) -symmetric bi-derivation.

$$(1) \text{ Let } a \in G(X). \text{ Then } 0 * a = a$$

$$\begin{aligned} D(a, y) &= D(0 * a, y) \quad \forall a, y \in X \\ &= (0 * D(a, y)) \wedge (D(0, y) * a) \\ &= (D(0, y) * a) * ((D(0, y) * a) * (0 * D(a, y))) \\ &= 0 * D(a, y) \quad \text{(by P3)} \end{aligned}$$

This shows that $D(a, y) \in G(X)$.

$$\begin{aligned} (2) D(a, y) &= D(a * 0, y) \quad \forall a, y \in X \\ &= (a * D(0, y)) \wedge (D(a, y) * 0) \\ &= (a * D(0, y)) \wedge D(a, y) \\ &= D(a, y) * (D(a, y) * (a * D(0, y))) \\ &= a * D(0, y) \end{aligned}$$

Again $D(a, y) = a * D(0 * y)$

$$\begin{aligned} &= a * D(0 * 0, y) \\ &= a * ((0 * D(0, y)) \wedge (D(0, y) * 0)) \\ &= a * (0 * D(0, y)) \\ &= a + D(0, y) \end{aligned}$$

$$\begin{aligned} (3) D(a+b, y) &= a + b + D(0, y) = a + D(0, y) + b + D(0, y) - D(0, y) \\ &= D(a, y) + D(b, y) - D(0, y) \end{aligned}$$

(4) If $D(0, y) = 0$, then by(2)

$$\begin{aligned} D(a, y) &= D(a * 0, y) = a * D(0, y) \\ &= a * 0 = a \end{aligned}$$

Conversely, if $D(a, y) = a \quad \forall a \in X$, $D(0, y) = 0$.

Definition: 4.2.16

Let D_L denote the set of all (l, r) symmetric bi-derivation on X . Define the binary operation \wedge on D_L as follows: For $D_1, D_2 \in D_L$ define

$$(D_1 \wedge D_2)(x, y) = D_1(x, y) \wedge D_2(x, y) \quad \forall x, y \in X.$$

Proposition: 4.2.17

Let D_1 and D_2 are (l, r) symmetric bi-derivation on X , then $(D_1 \wedge D_2)$ is also a (l, r) symmetric bi-derivation on X .

Proof:

We will prove the following implication

$$\begin{aligned} (D_1 \wedge D_2)(x * y, z) &= ((D_1 \wedge D_2)(x, z) * y) \wedge (x * ((D_1 \wedge D_2)(y, z))) \\ (D_1 \wedge D_2)(x * y, z) &= D_1(x * y, z) \wedge D_2(x * y, z) \\ &= D_2(x * y, z) * (D_2(x * y, z) * D_1(x * y, z)) \\ &= D_1(x * y, z) \\ &= (D_1(x, z) * y) \wedge (x * D_1(y, z)) \\ &= (x * D_1(y, z)) * ((x * D_1(y, z)) * (D_1(x, z) * y)) \\ &= D_1(x, z) * y \end{aligned} \tag{1}$$

$$\begin{aligned} ((D_1 \wedge D_2)(x, z) * y) \wedge (x * (D_1 \wedge D_2)(y, z)) &= (x * (D_1 \wedge D_2)(y, z)) * \\ &((x * (D_1 \wedge D_2)(y, z)) * ((D_1 \wedge D_2)(x, z) * y)) \\ &= (D_1 \wedge D_2)(x, z) * y \\ &= (D_1(x, z) \wedge D_2(x, z)) * y \\ &= (D_2(x, z) \wedge D_2(x, z) * D_1(x, z)) * y \\ &= D_1(x, z) * y \end{aligned} \tag{2}$$

Combining (1) and (2), we get $(D_1 \wedge D_2)$ is also a (l, r) symmetric bi-derivation.

Proposition: 4.2.18

The binary composition \wedge defined on D_L , the set of all (l, r) – symmetric bi-derivation on a TM-algebra X is associative.

Proof:

Let X be a TM-algebra. Let D_1, D_2, D_3 are (l, r) – symmetric bi-derivation.

$$\begin{aligned}
((D_1 \wedge D_2) \wedge D_3)(x * y, z) &= ((D_1 \wedge D_2)(x * y, z)) \wedge D_3(x * y, z) \\
&= (D_1(x, z) * y) \wedge (D_3(x * y, z)) \\
&= D_3(x * y, z) * (D_3(x * y, z) * D_1(x, z) * y) \\
&= D_1(x, z) * y \tag{1}
\end{aligned}$$

$$\begin{aligned}
(D_1 \wedge (D_2 \wedge D_3))(x * y, z) &= (D_1(x * y, z)) \wedge ((D_2 \wedge D_3)(x * y, z)) \\
&= (D_1(x * y, z)) \wedge (D_2(x, z) * y) \\
&= (D_2(x, z) * y) * ((D_2(x, z) * y) * (D_1(x * y, z))) \\
&= D_1(x * y, z) \\
&= (D_1(x, z) * y) \wedge (x * D_1(y, z)) \\
&= (x * D_1(y, z)) * ((x * D_1(y, z)) * D_1(x, z) * y) \\
&= D_1(x, z) * y \tag{2}
\end{aligned}$$

Combining (1) and (2) we get, $(D_1 \wedge D_2) \wedge D_3 = D_1 \wedge (D_2 \wedge D_3)$

This proves that, \wedge is associative.

Theorem: 4.2.19

D_L is a semi-group under the binary composition \wedge defined by

$$(D_1 \wedge D_2)(x, y) = D_1(x, y) \wedge D_2(x, y) \quad \forall x, y \in X \text{ and } D_1, D_2 \in D_L$$

Proof:

The proof of the theorem follows by combining the above two propositions (4.2.17) and (4.2.18)

Theorem: 4.2.20

D_R is a semi-group under the binary composition \wedge defined by

$$(D_1 \wedge D_2)(x, y) = D_1(x, y) \wedge D_2(x, y) \quad \forall x, y \in X \text{ and } D_1, D_2 \in D_R \text{ where}$$

D_R is the set of all (r, l) -symmetric bi-derivation.

Proof: Obvious

Theorem: 4.2.21

If \mathcal{D} denotes the set of all symmetric bi-derivation on X , it is a semi-group under the binary operation \wedge defined by $(D_1 \wedge D_2)(x, y) = D_1(x, y) \wedge D_2(x, y)$

$\forall x, y \in X$ and $D_1, D_2 \in \mathcal{D}$.

Proof: Obvious

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

In 1966, Imai, Y. and Iseki, K. [14, 17] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras.

In this thesis, we have made an attempt to study of TM-algebras, ideal theory of TM-algebras, Quotient TM-algebras and different types of derivations on TM-algebras.

In **chapter I**, the notion of TM-algebra and its properties are presented. Also ideal of a TM-algebra, p-radical, p-semisimple of a TM-algebra are discussed due to Megalai and Tamilarasi [23].

In **Chapter II**, Classification of TM-algebras due to Megalai and Tamilarasi [24] are presented and the concept of Quotient TM-algebras and their properties are discussed due to Handam, A.H. [11].

In **Chapter III**, we have made an attempt to the detailed study of derivations and generalized derivations on TM-algebras due to Ganeshkumar, T. and Chandramouleeswaran, M. [3,7].

In **Chapter IV**, the concepts of t -derivations and symmetric bi-derivations on TM-algebras due to Ganeshkumar, T., and Chandramouleeswaran, M., [8,9] and their properties are studied.

A deep study of TM-algebras can be extended to different types of algebras and Fuzzy TM-algebras. So it provides a lot of scope for further research.

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