CHAPTER – 3
ANTI FUZZY IDEALS AND ANTI FUZZY SUBALGEBRAS IN CI-ALGEBRAS

SECTION 3.1
ANTI FUZZY IDEALS OF CI-ALGEBRAS AND THEIR LOWER LEVEL SUBSETS

Definition : 3.1.1

A fuzzy set μ of a CI-algebra X is called an **anti fuzzy ideal** of X, if

(AFI 1) $\mu(x * y) \leq \mu(y)$, for all x, y $\in$ X.
(AFI 2) $\mu((x * (y * z)) * z) \leq \max\{\mu(x), \mu(y)\}$, for all x, y, z $\in$ X.

Theorem : 3.1.2

Every anti fuzzy ideal μ of a CI-algebra X satisfies the inequality $\mu(1) \leq \mu(x)$, for any x $\in$ X.

Proof

By definition of anti fuzzy ideal of a CI-algebra X, $\mu(1) = \mu(x * x) \leq \mu(x)$.

Theorem : 3.1.3

If μ is an anti fuzzy ideal of a CI-algebra X, then for all x, y $\in$ X, $\mu((x * y) * y) \leq \mu(x)$.

Proof

Let μ be an anti fuzzy ideal of a CI-algebra X.

Then $\mu((x * y) * y) = \mu[(x * (1 * y)) * y]$

$\leq \max\{\mu(x), \mu(1)\} = \mu(x)$

Theorem : 3.1.4

Every anti fuzzy ideal μ of a CI-algebra X is order reversing. That is, if $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in X$.

Proof

Let μ be an anti fuzzy ideal of a CI-algebra X and let $x, y \in X$ be such that $x \leq y$, then $x * y = 1$.

$\mu(y) = \mu(1 * y) = \mu((x * y) * y) \leq \mu(x)$

Hence every anti fuzzy ideal μ of a CI-algebra X is order reversing.

Theorem : 3.1.5

Let μ be a fuzzy set of a CI-algebra X which satisfies $\mu(1) \leq \mu(x)$ and $\mu(x * z) \leq \max\{\mu(x * (y * z)), \mu(y)\}$ for all $x, y, z \in X$. Then μ is order reversing.

Proof

Let $x, y \in X$ be such that $x \leq y$; then $x * y = 1$.

Let $\mu(y) = \mu(1 * y)$

$\leq \max\{\mu(1 * (x * y)), \mu(x)\} = \max\{\mu(1 * 1), \mu(x)\}$

$= \max\{\mu(1), \mu(x)\} = \mu(x)$

Therefore $\mu(x) \geq \mu(y)$

Hence μ is order reversing.

Theorem : 3.1.6

Let X be a transitive CI-algebra. If a fuzzy set μ in X is an anti fuzzy ideal of X then it satisfies condition

$$\mu(1) \leq \mu(x) \text{ and } \mu(x * z) \leq \max\{\mu(x * (y * z)), \mu(y)\} \ \forall\, x, y, z \in X.$$

Proof

Let μ be an anti fuzzy ideal of a transitive CI-algebra X. Then by theorem $\mu(1) \leq \mu(x)$. Since X is transitive, we have

$$((y * z) * z) * ((x * (y * z)) * (x * z)) = 1, \forall\, x, y, z \in X.$$

$$\begin{aligned} \text{Then } \mu(x * z) &= \mu(1 * (x * z)) \qquad \text{[by (CI 2)]} \\ &= \mu(\{((y * z) * z) * ((x * (y * z)) * (x * z))\} * (x * z)) \\ &\leq \max \{\mu((y * z) * z), \mu(x * (y * z))\} \\ &\leq \max \{\mu(y), \mu(x * (y * z))\} \end{aligned}$$

Theorem : 3.1.7

μ is a fuzzy ideal of a CI-algebra X if and only if μ^c is an anti fuzzy ideal of X.

Proof

Let μ be a fuzzy ideal of X. Let $x, y, z \in X$.

Then (i) $\mu^c(x * y) = 1 - \mu(x * y) \leq 1 - \mu(y) = \mu^c(y)$

i.e, $\mu^c(x * y) \leq \mu^c(y)$.

$$\begin{aligned} \text{(ii) } \mu^c((x * (y * z)) * z) &= 1 - \mu((x * (y * z)) * z) \\ &\leq 1 - \min \{\mu(x), \mu(y)\} \\ &= 1 - \min \{1 - \mu^c(x), 1 - \mu^c(y)\} \\ &= \max \{\mu^c(x), \mu^c(y)\} \end{aligned}$$

i.e, $\mu^c((x * (y * z)) * z) \leq \max \{\mu^c(x), \mu^c(y)\}$

Thus, μ^c is an anti fuzzy ideal of X. Similarly, the converse can be proved.

Theorem : 3.1.8

If μ is an anti fuzzy ideal of CI-algebra X, then $L(\mu ; t)$ is an ideal of X for every $t \in [0, 1]$.

Proof

Let μ be an anti fuzzy ideal of CI-algebra X.

(i) Let $x \in X$ and $y \in L(\mu ; t) \Rightarrow \mu(y) \le t$.

$\mu(x * y) \le \mu(y) \le t$

$\Rightarrow x * y \in L(\mu ; t)$

(ii) Let $x \in X$ and $a, b \in L(\mu ; t)$

$\Rightarrow \mu(a) \le t$ and $\mu(b) \le t$

$\mu((a * (b * x)) * x) \le \max \{\mu(a), \mu(b)\} \le \max \{t, t\} = t$

$\Rightarrow (a * (b * x)) * x \in L(\mu ; t)$

Hence $L(\mu ; t)$ is an ideal of X.

Theorem : 3.1.9

Let μ be a fuzzy set of CI-algebra X. If for each $t \in [0, 1]$, the lower level cut $L(\mu ; t)$ is an ideal of X, then μ is an anti fuzzy ideal of X.

Proof

Let $L(\mu ; t)$ be an ideal of X. If $\mu(x * y) > \mu(y)$ for some $x, y \in X$. Then $\mu(x * y) > t_0 > \mu(y)$ where $t_0 = \frac{1}{2} \{\mu(x * y) + \mu(y)\}$.

Hence $x * y \notin L(\mu ; t_0)$ and $y \in L(\mu ; t_0)$ which is a contradiction.

Therefore, $\mu(x * y) \le \mu(y)$.

Let $x, y, z \in X$ be such that $\mu((x * (y * z)) * z) > \max \{\mu(x), \mu(y)\}$. Taking $t_1 = \frac{1}{2} \{\mu((x * (y * z)) * z) + \max \{\mu(x), \mu(y)\}\}$ and $\mu((x * (y * z)) * z) > t_1 > \max \{\mu(x), \mu(y)\}$. Then $x, y \in L(\mu ; t_1)$ and $(x * (y * z)) * z \notin L(\mu ; t_1)$.

This is a contradiction. Hence $\mu((x * (y * z)) * z) \le \max \{\mu(x), \mu(y)\}$. Therefore, μ is an anti fuzzy ideal of X.

Theorem : 3.1.10

Let μ be a fuzzy set in CI-algebra X. Then μ is an anti fuzzy ideal of X iff μ satisfies the following condition.

$(a, b) \in L(\mu ; t) \Rightarrow A(a, b) \subseteq L(\mu ; t), \forall\ a, b \in X ; \forall\ t \in [0, 1]$

Proof

Assume that μ is an anti fuzzy ideal of X. Let a, b $\in L(\mu ; t)$. Then $\mu(a) \leq t$ and $\mu(b) \leq t$. Let $x \in A(a, b)$. Then $a * (b * x) = 1$.

$\mu(x) = \mu(1 * x) = \mu((a * (b * x)) * x) \leq \max\{\mu(a), \mu(b)\} \leq \max\{t, t\} = t$

$\Rightarrow \mu(x) \leq t$

$\Rightarrow x \in L(\mu ; t)$

Therefore $A(a, b) \subseteq L(\mu ; t)$.

Conversely, suppose that $A(a, b) \subseteq L(\mu ; t)$. Obviously, $1 \in A(a, b) \subseteq L(\mu ; t)$ for all a, b $\in$ X. Let x, y, z $\in$ X be such that $x * (y * z) \in L(\mu ; t)$ and $y \in L(\mu ; t)$.

Since $(x * (y * z)) * (y * (x * z)) = (x * (y * z)) * (x * (y * z)) = 1$. We have $x * z \in A(x * (y * z), y) \subseteq L(\mu ; t)$. Thus $L(\mu ; t)$ is an ideal of X. Hence μ is an anti fuzzy ideal of X.

Theorem : 3.1.11

Let μ be a fuzzy set in CI-algebra X. If μ is an anti fuzzy ideal of X then $\mu^t \neq \Phi \Rightarrow L(\mu ; t) \bigcup_{a, b \in \mu^t} A(a, b), \ \forall\ t \in [0, 1]$.

Proof

Let $t \in [0, 1]$ be such that $L(\mu ; t) \neq \Phi$. Since $1 \in L(\mu ; t)$ we have

$L(\mu ; t) \subseteq \bigcup_{a \in L(\mu ; t)} A(a, 1) \subseteq \bigcup_{a, b \in L(\mu ; t)} A(a, b)$ Let $x \in \bigcup_{a, b \in L(\mu ; t)} A(a, b)$

Then there exists, u, v $\in L(\mu ; t)$ such that $x \in A(u, v) \subseteq L(\mu ; t)$

Thus $\bigcup_{a, b \in L(\mu ; t)} A(a, b) \subseteq L(\mu ; t)$. This completes the proof.

SECTION 3.2
ANTI FUZZY SUBALGEBRAS AND HOMOMORPHISM OF CI-ALGEBRAS

Definition : 3.2.1

A fuzzy set μ in a CI-algebra X is called a **fuzzy subalgebra** of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all x, y $\in$ X.

Definition : 3.2.2

A fuzzy set μ in a CI-algebra X is called an **Anti fuzzy subalgebra** of X if $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$, for all x, y $\in$ X.

Note

Every anti fuzzy ideal of CI-algebra X is an anti fuzzy subalgebra if x = y for any x, y $\in$ X.

Theorem : 3.2.3

If μ is an anti fuzzy subalgebra of a CI-algebra X, then $\mu(1) \leq \mu(x)$, for any x $\in$ X.

Proof

Let μ be an anti fuzzy subalgebra of a CI-algebra X. Since $x * x = 1$ for any x $\in$ X, then $\mu(1) = \mu(x * x) \leq \max\{\mu(x), \mu(x)\} = \mu(x)$.

$\mu(1) \leq \mu(x)$.

Theorem : 3.2.4

A fuzzy set μ of a CI-algebra X is an anti fuzzy subalgebra if and only if for every t $\in$ [0, 1], L(μ ; t) is either empty or a subalgebra of X.

Proof

Assume that μ is an anti fuzzy subalgebra of X and $L(\mu\,;\,t) \neq \Phi$. Then for any $x, y \in L(\mu\,;\,t)$ we have $\mu(x * y) \leq \max\{\mu(x), \mu(y)\} \leq t$.

Therefore $x * y \in L(\mu\,;\,t)$. Hence $L(\mu\,;\,t)$ is a subalgebra of X.

Now let $x, y \in X$. Take $t = \max\{\mu(x), \mu(y)\}$.

Then by assumption $L(\mu\,;\,t)$ is a subalgebra of X implies $x * y \in L(\mu\,;\,t)$. Therefore $\mu(x * y) \leq t = \max\{\mu(x), \mu(y)\}$.

Hence μ is an anti fuzzy subalgebra of X.

Theorem : 3.2.5

Any subalgebra of a CI-algebra X can be realized as a level subalgebra of some anti fuzzy subalgebra of X.

Proof

Let A be subalgebra of a given CI-algebra X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

where $t \in [0, 1]$ is fixed. Then $L(\mu\,;\,t) = A$. To prove that such defined μ is an anti fuzzy subalgebra of X, let $x, y \in X$. If $x, y \in A$, then $x * y \in A$. Hence $\mu(x) = \mu(y) = \mu(x * y) = t$ and $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$. If $x, y \notin A$, then $\mu(x) = \mu(y) = 0$ and $\mu(x * y) \leq \max\{\mu(x), \mu(y)\} = 0$.

If atmost one of $x, y \in A$, then atleast one of $\mu(x)$ and $\mu(y)$ is equal to 0. Therefore, $\max\{\mu(x), \mu(y)\} = 0$ so that $\mu(x * y) \leq 0$, which completes the proof.

Theorem : 3.2.6

Two level subalgebras $L(\mu\,;\,s)$, $L(\mu\,;\,t)$ $(s < t)$ of an anti fuzzy subalgebra are equal iff there is no $x \in X$ such that $s \leq \mu(x) < t$.

Proof

Let $L(\mu ; s)$, $L(\mu ; t)$ for some $s < t$.

If there exist $x \in X$ such that $s \leq \mu(x) < t$, then $L(\mu ; t)$ is a proper subset of $L(\mu ; s)$ which is a contradiction.

Conversely, assume that there is no $x \in X$ such that $s \leq \mu(x) < t$.

If $x \in L(\mu ; s)$ then $\mu(x) \leq s$ and $\mu(x) \leq t$, since $\mu(x)$ does not lie between s and t. Thus $x \in L(\mu ; t)$ which gives $L(\mu ; s) \subseteq L(\mu ; t)$. Also $L(\mu ; t) \subseteq L(\mu ; s)$. Therefore $L(\mu ; s) = L(\mu ; t)$.

Definition : 3.2.7

Let $(X ; * ; 1)$ and $(Y ; \Delta ; 1')$ be CI-algebras. A mapping $f : X \to Y$ is said to be a **homomorphism** of CI-algebras (or CI-homomorphism) if $f(x * y) = f(x) \Delta f(y)$ for all $x, y \in X$.

Definition : 3.2.8

Let $f : X \to X$ be an **endomorphism** of CI-algebra X (or CI-endomorphism) and μ be a fuzzy set in X. We define a new fuzzy set in X by μ_f in X as $\mu_f(x) = \mu(f(x))$ for all x in X.

Definition : 3.2.9

For any homomorphism $f : (X, *, 1) \to (Y, \Delta, 1')$ of CI-algebras the set $\{x \in X \ / \ f(x) = 1'\}$ is called the **kernel of f**, denoted by **ker(f)** and the set $\{f(x) \ / \ x \in X\}$ is called the **image of f**, denoted by **Im(f)**.

Theorem : 3.2.10

Let f be an endomorphism of a CI-algebra X. If μ is an anti fuzzy ideal of X, then so is μ_f.

Proof

$\mu_f(x * y) = \mu(f(x * y)) = \mu(f(x) * f(y)) \leq \mu(f(y)) = \mu_f(y)$, for all $x, y \in X$.

Let $x, y, z \in X$.

$$\begin{aligned} \text{Then } \mu_f((x * (y * z)) * z) &= \mu(f((x * (y * z)) * z)) \\ &= \mu(f(x * (y * z)) * f(z)) \\ &= \mu((f(x) * f(y * z)) * f(z)) \\ &= \mu((f(x) * (f(y) * f(z))) * f(z)) \\ &\leq \max\{\mu(f(x)), \mu(f(y))\} \\ &= \max\{\mu_f(x), \mu_f(y)\} \end{aligned}$$

$\therefore \mu_f((x * (y * z)) * z) \leq \max\{\mu_f(x), \mu_f(y)\}$

Hence μ_f is an anti fuzzy ideal of X.

Theorem : 3.2.11

Let $f : (X, *, 1) \rightarrow (Y, \Delta, 1')$ be an epimorphism of CI-algebras. If μ_f is an anti fuzzy ideal of X, then μ is an anti fuzzy ideal of Y.

Proof

Let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$. Let $y_1, y_2, y_3 \in Y$.

$$\begin{aligned} \mu(y_1 \Delta y_2) &= \mu(f(x_1) \Delta f(x_2)) \\ &= \mu(f(x_1 * x_2)) \\ &= \mu_f(x_1 * x_2) \leq \mu_f(x_2) \\ &= \mu(f(x_2)) = \mu(y_2) \end{aligned}$$

$\therefore \mu(y_1 \Delta y_2) \leq \mu(y_2)$

$$\begin{aligned} \text{Then } \mu((y_1 \Delta (y_2 \Delta y_3)) \Delta y_3) &= \mu([f(x_1) \Delta (f(x_2) \Delta f(x_3))] \Delta f(x_3)) \\ &= \mu([f(x_1) \Delta f(x_2 * x_3)] \Delta f(x_3)) \\ &= \mu(f[x_1 * (x_2 * x_3)] \Delta f(x_3)) \\ &= \mu(f([x_1 * (x_2 * x_3)] * x_3)) \\ &= \mu_f([x_1 * (x_2 * x_3)] * x_3) \\ &\leq \max\{\mu_f(x_1), \mu_f(x_2)\} \end{aligned}$$

$= \max\{\mu(f(x_1)), \mu(f(x_2))\}$

$= \max\{\mu(y_1), \mu(y_2)\}$

$\therefore\ \mu((y_1 \Delta (y_2 \Delta y_3)) \Delta y_3) \leq \max\{\mu(y_1), \mu(y_2)\}$

Hence μ is an anti fuzzy ideal of Y.

Theorem : 3.2.12

Let $f : (X ; *, 1) \rightarrow (Y ; \Delta, 1')$ be a homomorphism of CI-algebras. If μ is an anti fuzzy ideal of Y then μ_f is an anti fuzzy ideal of X.

Proof

Let $x, y, z \in X$.

$\mu_f(x * y) = \mu(f(x * y))$

$= \mu(f(x) \Delta f(y))$

$\leq \mu(f(y))$

$= \mu_f(y)$

$\therefore\ \mu_f(x * y) \leq \mu_f(y)$.

Then $\mu_f((x * (y * z)) * z) = \mu(f((x * (y * z)) * z))$

$= \mu(f(x * (y * z)) \Delta f(z))$

$= \mu((f(x) \Delta f(y * z)) \Delta f(z))$

$= \mu((f(x) \Delta (f(y) \Delta f(z))) \Delta f(z))$

$\leq \max\{\mu(f(x)), \mu(f(y))\}$

$= \max\{\mu_f(x), \mu_f(y)\}$

$\therefore\ \mu_f((x * (y * z)) * z) \leq \max\{\mu_f(x), \mu_f(y)\}$

Hence μ_f is an anti fuzzy ideal of X.

Theorem : 3.2.13

Let $(X ; *, 1)$ and $(Y, \Delta, 1')$ be CI-algebras. A mapping $f : X \rightarrow Y$ is a homomorphism of CI-algebra then ker(f) is an ideal.

Proof

Let f : (X ; $*$, 1) $\rightarrow$ (Y, Δ, 1') be a homomorphism of CI-algebras.

Clearly 1 $\in$ ker(f).

To prove : Ker(f) is an ideal.

To prove ker(f) is an ideal, by Lemma (2.2.12) it is enough to prove that $(x * y) * z \in \ker(f) \Rightarrow x * z \in \ker f$, for all x, z $\in$ X and y $\in$ ker f.

Let $(x * y) * z \in \ker(f)$ and $y \in \ker(f)$.

Then $f((x * y) * z) = 1'$ and $f(y) = 1'$

Since $1' = f((x * y) * z)$

$$= f(x)\ \Delta\ f(y * z)$$

$$= f(x)\ \Delta\ (f(y)\ \Delta\ f(z))$$

$$= f(x)\ \Delta\ (1'\ \Delta\ f(z))$$

$$= f(x)\ \Delta\ f(z)$$

$$= f(x * z)$$

$$\Rightarrow x * z \in \ker(f)$$

Hence ker(f) is an ideal.

SECTION 3.3

ANTI CARTESIAN PRODUCT OF ANTI FUZZY IDEALS OF CI-ALGEBRAS

Definition : 3.3.1

Let μ and δ be the **fuzzy sets** in X. The anti Cartesian product $\mu \times \delta : X \times X \to [0, 1]$ is defined by $(\mu \times \delta)(x, y) = \max\{\mu(x), \delta(y)\}$, for all $x, y \in X$.

Theorem : 3.3.2

If μ and δ are anti fuzzy ideals in a CI-algebra X, then $\mu \times \delta$ is an anti fuzzy ideal in X x X.

Proof

Let μ and δ be anti fuzzy ideals in a CI-algebra X.

Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$.

$$(\mu \times \delta)((x_1, x_2) * (y_1, y_2)) = (\mu \times \delta)(x_1 * y_1, x_2 * y_2)$$
$$= \max\{\mu(x_1 * y_1), \delta(x_2 * y_2)\}$$
$$\leq \max\{\mu(y_1), \delta(y_2)\}$$
$$= (\mu \times \delta)(y_1, y_2)$$

$$\therefore (\mu \times \delta)((x_1, x_2) * (y_1, y_2)) \leq (\mu \times \delta)(y_1, y_2)$$

$$(\mu \times \delta)\{((x_1, x_2) * ((y_1, y_2)) * (z_1, z_2))) * (z_1, z_2)\}$$
$$= (\mu \times \delta)\{[(x_1, x_2) * (y_1 * z_1, y_2 * z_2)] * (z_1, z_2)\}$$
$$= (\mu \times \delta)\{[(x_1 * (y_1 * z_1)), x_2 * (y_2 * z_2)] * (z_1, z_2)\}$$
$$= (\mu \times \delta)\{(x_1 * (y_1 * z_1)) * z_1, (x_2 * (y_2 * z_2)) * z_2\}$$
$$= \max\{\mu((x_1 * (y_1 * z_1)) * z_1), \delta((x_2 * (y_2 * z_2)) * z_2)\}$$
$$\leq \max\{\max\{\mu(x_1), \mu(y_1)\}, \max\{\delta(x_2), \delta(y_2)\}\}$$
$$= \max\{\max\{\mu(x_1), \delta(x_2)\}, \max\{\mu(y_1), \delta(y_2)\}\}$$
$$= \max\{(\mu \times \delta)(x_1, x_2), (\mu \times \delta)(y_1, y_2)\}$$

$$\therefore (\mu \times \delta)\{((x_1, x_2) * ((y_1 * y_2) * (z_1, z_2))) * (z_1, z_2)\}$$
$$\leq \max\{(\mu \times \delta)(x_1, x_2), (\mu \times \delta)(y_1, y_2)\}$$

Hence $\mu \times \delta$ is an anti fuzzy ideal in X x X.

Theorem : 3.3.3

Let μ and δ be fuzzy sets in a CI-algebra X such that $\mu \times \delta$ is an anti fuzzy ideal of X x X. Then

(i) Either $\mu(1) \leq \mu(x)$ (or) $\delta(1) \leq \delta(x)$ for all $x \in X$.

(ii) If $\mu(1) \leq \mu(x)$ for all $x \in X$, then either $\delta(1) \leq \mu(x)$ (or) $\delta(1) \leq \delta(x)$.

(iii) If $\delta(1) \leq \delta(x)$ for all $x \in X$, then either $\mu(1) \leq \mu(x)$ (or) $\mu(1) \leq \delta(x)$.

(iv) Either μ or δ is an anti fuzzy ideal of X.

Proof

Let μ and δ be fuzzy sets in a CI-algebra X. Let $\mu \times \delta$ be an anti fuzzy ideal of X x X. Therefore $(\mu \times \delta)\,((x_1, x_2) * (y_1, y_2)) \leq (\mu \times \delta)\,(y_1, y_2)$ and $(\mu \times \delta)$ $\{((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))) * (z_1, z_2)\} \leq \max\{(\mu \times \delta)\,(x_1, x_2), (\mu \times \delta)\,(y_1, y_2)\}$ for all (x_1, x_2), (y_1, y_2) and $(z_1, z_2) \in X \times X$.

To Prove (i)

Suppose that $\mu(1) > \mu(x)$ and $\delta(1) > \delta(x)$ for some $x, y \in X$.

Then $(\mu \times \delta)\,(x, y) = \max\{\mu(x), \delta(y)\}$

$< \max\{\mu(1), \delta(1)\}$

$= (\mu \times \delta)\,(1, 1)$ which is a contradiction.

Therefore $\mu(1) \leq \mu(x)$ and $\delta(1) \leq \delta(x)$, $\forall\, x \in X$.

To Prove (ii)

Assume that there exists $x, y \in X$ such that

$\delta(1) > \mu(x)$ and $\delta(1) > \delta(x)$.

Then $(\mu \times \delta)\,(1, 1) = \max\{\mu(1), \delta(1)\} = \delta(1)$.

And hence $(\mu \times \delta)\,(x, y) = \max\{\mu(x), \delta(y)\} < \delta(1) = (\mu \times \delta)\,(1, 1)$

which is a contradiction.

Hence if $\mu(1) \leq \mu(x)$ for all $x \in X$, then either $\delta(1) \leq \mu(x)$ (or) $\delta(1) \leq \delta(x)$.

Similarly, we can prove that if $\delta(1) \leq \delta(x)$, for all $x \in X$, then either $\mu(1) \leq \mu(x)$ (or) $\mu(1) \leq \delta(y)$, which yields (iii).

To Prove (iv)

First we prove that δ is an anti fuzzy ideal of X. Since by (i) either $\mu(1) \leq \mu(x)$ (or) $\delta(1) \leq \delta(x)$, for all $x \in X$.

Assume that $\delta(1) \leq \delta(x)$, for all $x \in X$. It follows from (iii) that either $\mu(1) \leq \mu(x)$ or $\mu(1) \leq \delta(x)$.

If $\mu(1) \leq \delta(x)$ for any $x \in X$, then

$$\delta(x) = \max\{\mu(1), \delta(x)\} = (\mu \times \delta)(1, x)$$

$$\begin{aligned}
\delta(x * y) = \max\{\mu(1), \delta(x * y)\} &= (\mu \times \delta)(1, x * y) \\
&= (\mu \times \delta)(1 * 1, x * y) \\
&= (\mu \times \delta)((1, x) * (1, y)) \\
&\leq (\mu \times \delta)(1, y) \\
&= \delta(y) \\
&\therefore\ \delta(x * y) \leq \delta(y)
\end{aligned}$$

$$\begin{aligned}
\delta((x * (y * z)) * z) &= \max\{\mu(1), \delta((x * (y * z)) * z)\} \\
&= (\mu \times \delta)(1, (x * (y * z)) * z) \\
&= (\mu \times \delta)\{1 * 1, (x * (y * z)) * z\} \\
&= (\mu \times \delta)\{(1, x * (y * z)) * (1, z)\} \\
&= (\mu \times \delta)\{(1 * 1, x * (y * z)) * (1, z)\} \\
&= (\mu \times \delta)\{[(1, x) * (1, y * z)] * (1, z)\} \\
&= (\mu \times \delta)\{[(1, x) * (1 * 1, y * z)] * (1, z)\} \\
&= (\mu \times \delta)\{[(1, x) * ((1, y) * (1, z))] * (1, z)\} \\
&\leq \max\{(\mu \times \delta)(1, x), (\mu \times \delta)(1, y)\} \\
&= \max\{\delta(x), \delta(y)\}
\end{aligned}$$

$$\delta((x * (y * z)) * z) \leq \max\{\delta(x), \delta(y)\}.$$

Hence δ is an anti fuzzy ideal of X.

Next, we prove that μ is an anti fuzzy ideal of X.

Let $\mu(1) \leq \mu(x)$.

Since by (ii), either $\delta(1) \leq \mu(x)$ or $\delta(1) \leq \delta(x)$.

Assume that $\delta(1) \leq \mu(x)$, then

$\mu(x) = \max\{\mu(x), \delta(1)\}$

$= (\mu \text{ x } \delta)(x, 1)$

$\mu(x * y) = \max\{\mu(x * y), \delta(1)\}$

$= (\mu \text{ x } \delta)(x * y, 1)$

$= (\mu \text{ x } \delta)(x * y, 1 * 1)$

$= (\mu \text{ x } \delta)((x, 1) * (y, 1))$

$\leq (\mu \text{ x } \delta)(y, 1)$

$= \mu(y)$.

$\mu((x * (y * z)) * z) = \max\{\mu((x * (y * z)) * z), \delta(1)\}$

$= (\mu \text{ x } \delta)\{(x * (y * z)) * z, 1\}$

$= (\mu \text{ x } \delta)\{(x * (y * z)) * z), 1 * 1\}$

$= (\mu \text{ x } \delta)\{(x * (y * z), 1) * (z, 1)\}$

$= (\mu \text{ x } \delta)\{(x * (y * z), 1 * 1) * (z, 1)\}$

$= (\mu \text{ x } \delta)\{(x, 1) * (y * z, 1)] * (z, 1)\}$

$= (\mu \text{ x } \delta)\{[(x, 1) * (y * z, 1 * 1)] * (z, 1)\}$

$= (\mu \text{ x } \delta)\{[x, 1) * ((y, 1) * (z, 1))] * (z, 1)\}$

$\leq \max\{(\mu \text{ x } \delta)(x, 1), (\mu \text{ x } \delta)(y, 1)\}$

$= \max\{\mu(x), \mu(y)\}$

$\mu((x * (y * z)) * z) \leq \max\{\mu(x), \mu(y)\}$

Hence μ is an Anti fuzzy ideal of X.