Discrete Structures and Graph Theory

Third Edition

Gajavelli S.S. Bhishma Rao,
M.Sc., M.Phil.(Applied Mathematics)
Former Professor and Head
Department of Mathematics,
Narasaropeta Engineering College
Kottappakonda Road, Yeamanda (Po)
Narasaropeta - 522 601

Publishing for future

SCITECH PUBLICATIONS (INDIA) PVT. LTD
CHENNAI

SCITECH PUBLICATIONS (INDIA) PVT. LTD.,
7/3c, Madley Road,
T. Nagar, Chennai-600 017
E-mail: scitech@md5.vsnl.net.in
visit us at : www.scitechpublications.com

81 8371 043 3

Published by V. Ramesh for Scitech Publications (India) Pvt. Ltd., No. 7/3c, Madley Road, T. Nagar, Chennai - 600 017.

Dedicated to
my beloved parents
Sri. Gajavelli Venkata Narayana
and
Smt. Annapurnamma
and other family members

Preface to Third Edition

The subject Discrete Structures and Graph Theory is gaining importance in the curriculum of Engineering especially Computer Science, Information Technology and Communication Engineering subjects. This book is the outcome of my teaching experience.

This book contains 6 Chapters. Chapter 1 contains Mathematical logic, theories of inference and related material. Chapter 2 covers Set, relations, posers, lattices, functions, recursive functions and their operations. Chapter 3, on Counting, deals with permutations, combinations and recurrence relations. Chapter 4 covers Graphs, their properties, basic definitions in an elementary way. Chapter 5 introduces Trees, minimal spanning trees and related applications with an elementary touch. Chapter 6 covers Graph theoretic Algorithms.

More examples and exercises are added in this book for better understanding. Suggestions for improvements of the book shall be gratefully acknowledged.

July 2009 **G.S.S. Bhishma Rao**

Acknowledgments

In writing this book, many people helped me. I would like to express thanks to:

- Our Narasaraopeta Engineering College Chairman Sri. M. V. Koteswara Rao &
- Respected Principal Prof. A. Kamasundara Rao for their constant encouragement
- My beloved teacher Prof. K. Panduranga Rao, Head, Department of Applied Mathematics, NUPGC, Nuzvid for his guidance.
- My beloved parents, without their untiring efforts, I could not have become what I am today
- My beloved brothers, G. Pavan Kumar, Lecturer in Mathematics, G. Raghava Rao for their support and encouragement
- My loving wife Uma and daughter Amoolya for not disturbing me while writing this book most of the time. Also to K. Venkataramaiah & K. Satyavathi.
- My colleague friends: P. Balakrishna Prasad, M.S.S. Chalapathi, L. Nataraj Kumar, G. P. Raju, N. Srinivasa Rao (M), V. Mahesh, Y. Swamy, N. Srinivasa Rao (C), S. Gopikrishna, K. Prasad, K. V. Ranga Reddy, Ravichand, S. Venkateswarlu
- My friends: D. Hanumantha Rao, A. Srinivasa Rao, M.N. Chari, D. Veeraswamy, B.V. Appa Rao, PVVS Rama Rao, Dr. D. Ramachandran, K.V. Ramana, M. Chandramouli, (DVREC), P. Durga Prasad, B.V.S.N. Hari Prasad, Ram Reddy, Krishna, Sai printers(NRT), O. Prabhakar
- My scientist friend D. Muthyal Reddy, BHEL (R&D), Hyderabad
- To our college teaching staff and nonteaching staff.
- The teachers of JNTU and other colleges affiliated to it, who have been teaching mathematics and this subject
- the students, who induced me to author this book
- The office and library staff of our college.
- My near and dear
- *M/S Scitech Publications (India) Pvt. Ltd., (especially M. R. Purushothaman)*, for readily accepting the proposal, and bringing out this book in a nice get up in a short period.

–G.S.S. Bhishma Rao

List of Symbols

Symbol	Used for
$\neg$	negation
$\wedge$	and
$\vee$	or
$\rightarrow$	conditional
$\rightleftarrows$	Biconditional
$\Leftrightarrow$	Equivalence
$\Rightarrow$	imply
$\uparrow$	NAND
$\downarrow$	NOR
$\sum$	Summation
$\prod$	Product
$\exists$	there exists
$\forall$	For all
$\in$	belongs to
$\notin$	not belongs to
$\subseteq$	inclusion
$\subset$	proper inclusion
$P(A)$	Power set of A
$\emptyset$	null set
$\cup$	Union
$\cap$	intersection
Δ	symmetric difference
$\times$	cross product
$'$	complement
$[x]$	Equivalence class 'x'
o	compositions of
$\sim$	converse
ψ	Si
$\lfloor x \rfloor$	floor of x
$\lceil x \rceil$	ceiling of x
$*$	GLB{a,b}
$\oplus$	LUB{a,b}
$\delta(G)$	Minimum degree
$V(G)$	Maximum degree
$\|V\|$	Number of vertices
$\|E\|$	Number of edges

Contents

1. Mathematical Logic

> [The Sentence in the Square Bracket is False]
>
> \- Liar Paradox

1. Introduction

Mathematical logic or logic is the discipline that deals with the methods of reasoning. It provides rules and techniques for determining whether a given argument or mathematical proof or conclusion in a scientific theory is valid or not.

Logic is concerned with studying arguments and conclusions.

Logic is used in mathematics to prove theorems, and to draw conclusions from experiments in physical sciences, and in our every day life to solve many types of problems.

Logic is used in computer science to verify the correctness of programs.

The rules of logic or techniques of logic are called *rules of inference*, because the main aim of logic is to draw conclusions, inferences from given set of hypotheses. At this context, the theory of inference needs a language in which these rules of inference can be stated. It is necessary to develop a formal language called the *object language*. A *formal language* is one in which the syntax is well defined. Apart from syntax, symbols will be used in the object languages. Thus, a systematic study of arguments by making extensive use of symbols is known as *Symbolic logic*. Our study of the object language requires the use of another language i.e., a natural language (English) called as *metalanguage*.

In this chapter, we introduce the building blocks of our object language viz., Statements, Truthvalues, Connectives etc., to state and apply rules of valid inference.

2. Statements and Notation

In any theory, assertions are made in the form of sentences. Sentences are usually classified as declarative, exclamatory, interrogative and imperative. In our study of logic, we will confine ourselves to declarative sentences only. i.e., we begin by assuming that the object language contains a set of declarative sentences. A *primary statement* is a declarative sentence which cannot be further broken down or analyzed into simpler sentences. These primary statements are the basic units of the object language. The declarative sentences will be admitted in the object language if they have one and only one of two possible values called "*truth values*". The two truth values are 'TRUE' and 'FALSE' and are denoted by the symbols T and F respectively. They are also denoted by the symbols 1 and 0. Our logic is called as *two-valued logic*, since we have only two possible truth values in our logic.

Declarative sentences in the object language are of two types. The first type includes those sentences which are considered to be *primitive* or *primary* in the object language. These will be denoted by distinct alphabetical capital letters A, B, C, ..., P, Q, ... while declarative sentences of the second type are obtained from the primitive ones by using certain symbols, called *connectives*, and certain punctuation marks, such as parentheses to join primitive sentences.

In any case, all the declarative sentences to which it is possible to assign one and only one of the two possible truth values are called *statements*. These statements which do not contain any of the connectives are called *atomic (primary, primitives)* statements.

Consider the following sentences:

1. The integer 5 is a prime number
2. The integer 25 is a prime number
3. The sum of the angles of a rectangle is 360.
4. MOSCOW is the capital of England
5. This Statement is false.
6. Close the Box
7. Today is Monday
8. Do you speak Telugu?
9. Mathematical logic is a dull subject
10. $1 + 101 = 110$.

The sentences (1), (2), (3), (4), (7), (9), (10) are declarative statements. The statements (1) and (3) have truth value 'TRUE', while the statements (2), (4) have truth value 'FALSE''. Statement (5) is not a statement according to our definition, because we cannot properly assign to it a definite truth value. If we assign the truth value true, then sentence (5) says that statement (5) is false. On the other hand if we assign to it the truth value false, then sentence (5) implies that statement (5) is true. This example illustrates a semantic paradox. Clearly (6) is not a statement, it is a command. The truth value of statement (7) depends upon the day in which the statement is made or said. If the sentence is uttered by some one on a Monday, the statement is true and if it is uttered on any otherday, the statement is false. Clearly (8) is not a statement, it is interrogative sentence. The truth value of the statement (9) depends on the person who utters this statement. Lostly, the truth value of statement (10) depends upon the context; viz. if we are talking about numbers in the decimal system, then it is a false statement. On the other hand, for numbers in binary, it is a true statement.

Definition 2.1 A statement or proposition is a declarative sentence to which it is possible to assign a truth value TRUE or FALSE, but not both simultaneously.

3. Connectives

Till now, we considered atomic or primary statements. But in practice, we often combine, simple (Primary) statements to form compound statements by using certain connecting words known as sentential connectives or simply *connectives*. Thus primary statements are combined by means of connectives: *and, or, if...then* and *if and only if,* lostly '*not*'. These five main types of connectives can be defined in terms of the three: *and, or* and *not.*

Note 3.1 To denote statements we use the capital letters $P, Q, \ldots P_1, P_2 \ldots$

Ex: P: It is raining today

Here, a statement "P" either denotes a particular statement or serves as a placeholder for the statement.

We proceed to give the definitions of the connectives.

3.1 Negation

The negation of a statement is generally formed by introducing the word "not" at a proper place in the statement or by prefixing the statement with the phrase, "it is not the case that (or) it is not true that."

If "P" denotes a statement, then the negation of "P" is written as "$\neg P$" and read as "not P". If the truth value of "P" is T, then the truth value of $\neg P$ is F. Also if the truth value of "P" is F, then the truth value of $\neg P$ is T. This definition of the negation is summarized as follows by a table:

The truth table of $\neg P$:

P	$\neg P$
T	F
F	T

We now illustrate the formation of the negation of a statement.

Example 3.1.1 Consider the statement

P : HYDERABAD is a city

Then $\neg P$ is the statement

$$\neg P : \text{It is not that case that HYDERABAD is a city.}$$

Simply $\neg P$ can be written as

$$\neg P : \text{HYDERABAD is not a city.}$$ □

Example 3.1.2 The negation of the statement

$$P : 2 + 2 > 1$$

$$\neg P : 2 + 2 \not> 1$$

or $$\neg P : 2 + 2 \leq 1$$

in words "it is not true that $2 + 2 > 1$. □

Example 3.1.3 The negation of the statement

P : I went to a movie yesterday is

$\neg$P : I did not go to movie yesterday □

Note 3.1.1 The negation $\neg P$ of P is also denoted by '$\sim P$' or '$\bar{P}$' or 'not P'.

3.2 Conjunction

The conjunction ie., joining of two statements P and Q is the statement $P \wedge Q$ which is read as "P and Q". The statement $P \wedge Q$ has the truth value T whenever both P and Q have the truth value T; otherwise it has the truthvalue F.

The conjunction is defined as follows:

Truth table for conjunction:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

We illustrate the usage of 'and' by the following examples:

Example 3.2.1 The conjunction of the statements

$$P : \text{It is raining.}$$

$$Q: 2+2=4.$$

is $P \wedge Q$: It is raining and $2+2=4$. □

Note 3.2.1 To form $P \wedge Q$, the statements P and Q need not be related to each other in one way or the other. We can form $P \wedge Q$ even if P and Q are totally unrelated to each other. The statements P and Q given in above example have no common property or any relation.

The truthvalue of $P \wedge Q$ is True only when the statements P and Q are both true. Let us consider the case when it is raining. Then P is true. The statement Q is always true. So in the case of raining, as both statements P and Q are true, $P \wedge Q$ is also true. When it is not raining, the statement P is false and by definition $P \wedge Q$ is also false.

Example 3.2.2 Translate the following statement into symbolic form

Ramu and Raghu went to school.

Solution: In order to write it as a conjunction of two statements, it is necessary first to paraphrase the statement as Ramu went to school and Raghu went to school.

Now write: P: Ramu went to school
Q: Raghu went to school

then the given statement can be written in symbolic form as $P \wedge Q$. □

Note 3.2.2 From the definition of $P \wedge Q$, it is clear that the truth value of the conjunction $P \wedge Q$ of two statements P and Q depends upon the truth values of P and Q. There are 2^2 possible combinations of truth values of P and Q that must be considered, because, each one of the statements P and Q can have any one of the two possible truth vlaues true and false. For each such possible combinations of truth values of P and Q, we determine the truth value of $P \wedge Q$. All possible truth values of $P \wedge Q$ can be shown by means of a table (discussed already).

3.3 Disjunction

The disjunction of two statements P and Q is the statement $P \vee Q$ which is read as "P or Q". The statement $P \vee Q$ has the truth value F only when both P and Q have the truth value F otherwise it is true. The disjunction is defined by the following table:

Truth table for disjunction:

Thus $P \vee Q$ is true if either P is true or Q is true (or both P and Q are true).

Note 3.3.1 The connective $\vee$ is not always the same as the word "or" because of the fact that the word "or" in English Language can be used in two different senses:

i) Inclusive OR (one or the other or both) and

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

ii) Exclusive OR (one or the other, but not both)

In logic we use $\vee$ ('or') as inclusive OR.

For example, Consider the following statements:

1. Ramu will take Mpc or Bi.p.c group in intermediate. The above statement says that only one of the group will be taken by Ramu as a main group of study in his intermediate. Here 'or' is used in the sense 'one or the other, but not both', (Exclusive OR).
2. "I will buy a computer or a car next year" The above statement indicates that the speaker may mean that he is trying to make up his mind so as to which one of the two to buy, but he could also mean that he will buy atleast one of them, possibly both. Here 'or' is used in the sense 'one or the other or both' (Inclusive OR).

Example 3.3.1 Consider the following statements

P : I will buy a computer

Q : I will buy a car

Then $P \vee Q$ is the following statement

$P \vee Q$: I will buy a computer or I will buy a car.

3.4 Conditional Statements

If P nd Q are any two statements, then the statement $P \to Q$ read as "If P, then Q" is called a *conditional* statement. The Statement $P \to Q$ has a truth value F when Q has the truth value F and P the truth value T; otherwise it has the truth value T. The conditional is defined by the following table:

Truth table for conditional: $P \to Q$

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

The Statement P is called the *antecedent* and Q the *consequent* in $P \rightarrow Q$. The sign "$\rightarrow$" is called the sign of implication we will also write $P \rightarrow Q$, for

P only if Q
Q if P
Q provided that P
P is sufficient condition for Q
Q is necessary condition for P
P implies Q
Q is implied by P

Note 3.4.1 According to the definition, it is not necessary that there be any kind of relation between P and Q in order to form $P \rightarrow Q$.

Note 3.4.2 In general, the use of "If ..., then ..." in English has only partial resemblance to the use of $\rightarrow$ in logic.

Note 3.4.3 The converse of $P \rightarrow Q$ is $Q \rightarrow P$ and the contrapositive of $P \rightarrow Q$ is $\neg Q \rightarrow \neg P$. The inverse of $P \rightarrow Q$ is $\neg P \rightarrow \neg Q$. Also $P \rightarrow Q$ and contrapositive $\neg Q \rightarrow \neg P$ have the same truth values.

Example 3.4.1 Let

P : Amulya works hard
Q : Amulya will pass the exam.

Then $P \rightarrow Q$: If Amulya works hard, then she will pass the exam. □

3.5 Biconditional Statements

If P and Q are any two statements, then the statement $P \rightleftarrows Q$ which is read as "P if and only if Q" and abbreviated as 'P iff Q' is called a biconditional statement. The statement $P \rightleftarrows Q$ has the truth value T whenver both P and Q have identical truth values. The biconditional $P \rightleftarrows Q$ is the conjunction of the conditionals $P \rightarrow Q$ and $Q \rightarrow P$. i.e., $(P \rightarrow Q) \wedge (Q \rightarrow P)$ is an alternate notation for $P \rightleftarrows Q$. The following table defines the biconditional:

Truth table for biconditional $P \rightleftarrows Q$

P	Q	$P \rightleftarrows Q$
T	T	T
T	F	F
F	T	F
F	F	T

Also Truth table for $(P \rightarrow Q) \wedge (Q \rightarrow P)$

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that both truth tables are identical.

Thus biconditional $P \rightleftarrows Q$ may be read by following way:

1. P if and only if Q.
2. P is equivalent to Q.
3. P is necessary and sufficient condition for Q.
4. Q is necessary and sufficient condition for P.

We also write '$P \rightleftarrows Q$' for '$P \leftrightarrow Q$'.

Example 3.5.1 1. $8 > 4$ if and only if $8 - 4$ is positive

2. $2 + 2 = 4$ if and only if it is raining
3. Two lines are parallel if and only if they have the same slope.

Example 3.5.2 Write the following statement in symbolic form.

If either Mr. Srinu takes calculus or Mr. Swamy takes Graph theory then Mr. Mahesh will take computer programming.

Solution: Denoting the statements as

S : Mr. Srinu takes calculus
W : Mr. Swamy takes Graph theory
M : Mr. Mahesh takes computer programming

the above statement can be symbolized as

$$(S \vee W) \rightarrow M$$

□

4. Statement Formulas

Statements which do not contain any connectives are called atomic or simple statements. On the other hand, the statements which contain one or more primary statements and at least one connective are called molecular or composite or compound statements.

For example, let P and Q be any two simple statements. Some of the compound statements formed by P and Q are

$$\neg P \quad P \vee Q \quad (P \wedge Q) \vee (\neg P) \quad P \wedge (\neg Q) \quad (P \vee \neg Q) \wedge P$$

The above compound statements are called *Statement formulas* derived from the *Statement variables* P and Q. Therefore P and Q are called as *Components* of the statement formulas.

A statement formula alone has no truth value. It has truth value only when the statement variables in the formula are replaced by definite statements and it depends on the truth values of the statements used in replacing the variables.

5. The Truth Table of a Statement Formula

A table showing all the possible truth values of a statement formula for each possible combination of the truth values of the component statements is called the *truth table* of the formula.

Truth tables have already been introduced in the definitions of the connectives.

In general, if there are 'n' distinct components in a statement formula, we need to consider 2^n possible combinations of truth values in order to obtain the truth table.

For example, if any statement formula have two component statements namely P and Q, $\because$ 2^2 possible combinations of truth values must be considered.

Example 5.1 Construct the truth table for $P \wedge \neg P$

Solution:

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Example 5.2 Construct the truth table for $P \vee \neg P$

Solution:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Example 5.3 Construct the truth table for $P \rightarrow (Q \rightarrow R)$.

Solution: P, Q and R are the three statement variables that occur in this formula $P \rightarrow (Q \rightarrow R)$. There are $2^3 = 8$ different sets of truth value assignments for the variables P, Q and R. They are

The following table is the truth table fo $P \rightarrow (Q \rightarrow R)$

P	Q	R	$Q \rightarrow R$	$P \rightarrow (Q \rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

□

Example 5.4 Construct the truth table for the formula

$$(P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \quad \text{(I)}$$

Solution:

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge \neg Q$	(I)
T	T	F	F	T	F	F	F	T
T	F	F	T	F	F	T	F	T
F	T	T	F	F	T	F	F	T
F	F	T	T	F	F	F	T	T

□

Exercise 1:

(I) Write the following statements in symbolic form with statements

P : Pavan is rich

Q : Raghav is happy

(a) Pavan is rich and Raghav is not happy

(b) Pavan is not rich and Raghav is happy

(II) Write the following statements in symbolic form with statements

R : Naveen is rich

H : Naveen is happy

(a) Naveen is poor but happy

(b) Naveen is rich or unhappy

(c) Naveen is neither rich nor happy

(d) Naveen is poor or he is both rich and unhappy

(III) Write the following statements in symbolic form with statements.

P : Naveen is smart

Q : Amal is smart

(a) Naveen is smart and Amal is not smart

(b) Naveen and Amal are both smart

(c) Neither Naveen nor Amal are smart

(d) It is not true that Naveen and Amal are both smart.

(IV) Let P, Q, R denote the following statements:

P : Triangle ABC is isosceles

Q : Triangle ABC is Equilateral

R : Triangle ABC is Equiangular

Translate each of the following into a statement of English

(a) $Q \rightarrow P$

(b) $\neg P \rightarrow \neg Q$

(c) $Q \rightleftarrows \mathrm{R}$

(d) $P \wedge \neg Q$

(e) $\mathrm{R} \rightarrow P$

(V) If P. Q, R are three statements with truth values 'true', 'true' and 'false' respectively, Find the truth values of the following

(i) $P \vee Q$ (ii) $P \wedge R$ (iii) $(P \vee Q) \wedge R$ (iv) $P \wedge (\neg R)$

(v) $(P \wedge \neg Q) \wedge (\neg R)$ (vi) $P \rightarrow R$ (vii) $P \rightarrow Q$ (viii) $R \rightarrow P$

(ix) $(R \wedge P) \rightarrow Q$ (x) $(P \wedge \neg Q) \rightarrow R$ (xi) $(P \vee Q) \longleftrightarrow (P \rightarrow \neg R)$

(xii) $(P \longleftrightarrow R) \rightarrow R$

(VI) If P, Q are statements with truth values 'true' and R and S are statements with truth value 'false'. Find the truth value of the following

(a) $(R \wedge P) \rightarrow S$
(b) $(P \wedge Q) \wedge R$
(c) $(P \rightleftarrows Q) \rightarrow (S \longrightarrow R)$
(d) $P \vee (Q \wedge S)$
(e) $(P \rightarrow \neg Q) \rightarrow (S \rightleftarrows R)$
(f) $(P \rightarrow \neg Q) \rightarrow (P \vee Q)$
(g) $P \rightarrow (Q \longleftrightarrow (R \rightarrow S))$
(h) $S \rightarrow P$

(VII) Construct the truth tables for the following formulas

(a) $\neg(\neg P \wedge \neg Q)$
(b) $(\neg P \vee Q) \wedge (\neg Q \vee P)$
(c) $(P \wedge Q) \rightarrow (P \vee Q)$

(VIII) Given the truth values of P and Q as T and those of R and S as F, find the truth values of the following

(a) $P \vee (Q \wedge R)$
(b) $(P \wedge (Q \wedge R)) \wedge \neg((P \vee Q) \wedge (R \vee S))$

Answers 1

(I) (a) $P \wedge \neg Q$
(b) $\neg P \wedge Q$

(II) (a) $\neg R \wedge H$
(b) $R \vee \neg H$
(c) $\neg R \wedge \neg H$
(d) $\neg R \vee (R \wedge \neg H)$

(III) (a) $P \wedge \neg Q$

(b) $P \wedge Q$

(c) $\neg P \wedge \neg Q$

(d) $\neg(P \wedge Q)$

(IV) (a) If triangle ABC is equilateral, then it is isosceles

(b) If the triangle ABC is not isosceles then it is not equilateral

(c) The triangle ABC is equilateral if and only if it is equiangular

(d) If the triangle ABC is equiangular, then it is isosceles.

(V) (i) T (ii) F (iii) F (iv) T

(v) F (vi) F (vii) T (viii) T

(ix) T (x) T (xi) T (xii) T

(VI) (a) T (b) F (c) T (d) T

(e) T (f) T (g) T (b) T

(VII) (a) The variables that occur in the formula are P and Q so we have to consider $2^2 = 4$ possible combinations of truthvalues of two statements P and Q

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

The entries in the last column are the truth values of the formula $\neg(\neg P \wedge \neg Q)$.

(b) The variables are P and Q, clearly there are 2^2 rows in the truth table of this formula

P	Q	$\neg P$	$\neg Q$	$\neg P \vee Q$	$\neg Q \vee P$	$(\neg P \vee Q) \wedge (\neg Q \vee P)$
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

(c)

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q \rightarrow (P \vee Q))$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

(VIII) (a)

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	F	F	T

(b) Check that

$$(P \wedge (Q \wedge R)) \wedge \neg((P \vee Q) \wedge (R \vee S)) \text{ is true}$$

6. Well-formed Formulae

Definition 6.1 A statement formula is an expression which is a string consisting of variables (capital letters with or without subscripts), parentheses and connective symbols ($\wedge$, $\vee$, $\rightarrow$ $\rightleftarrows$, $\neg$), which produces a statement when the variables are replaced by statements.

But, every string of these symbols is not a formula. We now give a recursive definition of a statement formula, called a well-formed formula (wff).

Definition 6.2 A Well-formed formula (wff) can be generated by the following rules:

1. A statement variable standing alone (ie., a string of length one, consisting of a statement variable) is a well-formed formula.
2. If A is a well-formed formula, then $\neg A$ is a well-formed formula
3. If A and B are well-formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \rightleftarrows B)$ are well-formed formulas.
4. A string of symbols containing the statement variables, connectives and parentheses is a well-formed formula, if and only if it can be obtained by finitely many applications of rules (1), (2) and (3).

Similarly the set of all well-formed formulas can be defined as follows:

Definition 6.3 The set of all well-formed formulas is the smallest set of strings such that

i) Every statement variable is in the set
ii) If A and B are in the set, then so are $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \rightleftarrows B)$

Note 6.1 Now-onwards, formula means well-formed formula.

Example 6.1 *Formulae:* $\neg(P \wedge Q)$, $\neg(P \vee Q)$

$(P \wedge Q)$; $(\neg(P \wedge Q)) \rightarrow (R \wedge (\neg S))$;

$(\neg((\neg P) \wedge (\neg Q)))$; $(P \rightarrow (P \vee Q))$;

$(((P \vee Q) \wedge (P \vee \neg S)) \rightarrow (P \rightleftarrows R))$;

$((P \wedge Q) \rightarrow Q)$; $(P \rightarrow (Q \rightarrow R))$

Not formulae :

1. $\neg P \vee Q$, obviously P and Q are wffs, A wff would be either $(\neg P \vee Q)$ or $\neg(P \vee Q)$.
2. $((P \rightarrow Q) \rightarrow (\wedge Q))$ is not a formula, as $(\wedge Q)$ is not a wff.
3. $(P \rightarrow Q$ is not a wff as ')' is omitted. Note that $(P \rightarrow Q)$ is a formula.
4. $(P \wedge Q \rightarrow Q)$. The cause for this not being a wff is that one of the parentheses in the beginning is missing. $((P \wedge Q) \rightarrow Q)$ is a wff, while $(P \wedge Q) \rightarrow Q$ is still not a wff.

It is possible to introduce some conventions so that the number of parentheses used can be reduced.

1. For the sake of convenience, we shall omit the outer parentheses. Thus we write $P \vee Q$ for $(P \vee Q)$, $(P \vee Q) \rightarrow Q$ in place of $((P \vee Q) \rightarrow Q)$, $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightleftarrows (P \rightarrow R)$ instead of $(((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightleftarrows (P \rightarrow R).)$.
2. Also $(\neg(P \wedge Q) \rightarrow ((\neg P) \wedge (\neg R)))$ can be abbreviated to $\neg(P \wedge Q) \rightarrow (\neg P \wedge \neg R)$.

Note 6.2 It should be remembered that the above are just conventions, and the precise definitions must include the parentheses.

7. Tautology

Definition 7.1 A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a universally valid formula or a *tautology* or a logical truth.

i.e., If each entry in the final column of the truthtable of a statement formula is T alone, then it is called as *tautology*.

Similarly,

Definition 7.2 A statement formula which is false regardless of the truth values of the statements which replaces the variables in it is called a *contradiction*.

i.e., If each entry in the final column of the truth table of a statement formula is F alone, then it is called as *contradiction*.

Clearly, the negation of a contradiction is a tautology and vice-versa.

We can call a statement formula which is a tautology as identically true and a formula which is a contradiction as identically false.

Determining whether a given formula is a Tautology:

I) BY TRUTH TABLE

The first, straight forward method to determine whether a given formula is a tautology is to construct its *truth table*.

Example 7.1 i) Verify whether $P \vee \neg P$ is a tautology.

Solution:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Since, the entries in the last column of the truth table are T, thereore the given is a tautology.

ii) Verify whether $P \wedge \neg P$ is a tautology

Solution: Check that the last column of the truth table of $P \wedge \neg P$ contains false,

∴ The formula is not a tautology in particular, it is a contradiction

iii) Verify whether $(P \vee Q) \rightarrow P$ is a tautology

Solution:

P	Q	$P \vee Q$	$(P \vee Q) \rightarrow P$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Since the entries in the last column of the truth table $(P \vee Q) \rightarrow P$, contain one false, the formula is not a tautology. □

Example 7.2 Verify whether $(P \wedge (P \rightleftarrows Q)) \rightarrow Q$ is a tautology

Solution:

P	Q	$(P \rightleftarrows Q)$	$P \wedge (P \rightleftarrows Q)$	$(P \wedge (P \rightleftarrows Q)) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

As the entries in the last column are T, the given formula is a tautology.

Exercise 2:

I. Prove the following are tautologies (using truth tables):

(a) $\neg(P \vee Q) \vee (\neg P \wedge Q) \vee P$

(b) $((P \rightarrow Q) \wedge (R \rightarrow S) \wedge (P \vee R)) \rightarrow (Q \vee S)$

(c) $((P \rightarrow R) \wedge (Q \rightarrow R)) \rightarrow ((P \vee Q) \rightarrow R)$

(d) $((P \rightarrow (Q \vee R)) \wedge (\neg Q)) \rightarrow (P \rightarrow R)$

(e) $(((P \cup Q) \rightarrow R) \wedge (\neg P)) \rightarrow (Q \rightarrow R)$

(f) $(P \rightarrow Q) \rightleftarrows (\neg P \vee Q)$

(g) $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

II. Show that the truth values of the following formula is independent of its components.

$(P \rightarrow Q) \rightleftarrows (\neg P \vee Q)$

Answers 2:

I. Construct truth tables for all the given formulae.

II. Since the given formulae has truth value T for any truth values of its statement variables (or) components. It is independent of its components.

(II) Alternative Method:

Recall that the numbers of rows in a truth table is 2^n, where n is the number of distinct variables in the formula. Therefore, this process of determining whether a given formula is a tautology is tedious, particularly when the number of distinct variables is large (or) when the formula is complicated.

Keeping the above point in view, we now consider alternative methods to determine whether a statement formula is a tautology without constructing its truth table.

(i) It is very clear, that the conjunction of two tautologies is also a tautology. Let us denote by A and B two statement formulas which are tautologies. If we assign any truth values to the variables of A and B, then the truth values of both A and B will be T. Thus the truth value of $A \wedge B$ will be T, so that $A \wedge B$ will be a tautology.

(ii) A formula A is called a *substitution instance* of another formula B if A can be obtained from B by substituting formulas for some variables of B, with the condition that the same formula is substituted for the same variable each time it occurs.

It should be noted that in constructing substitution instances of a formula, substitutions are made for the atomic formula and never for the molecular formula. Thus $P \to Q$ is not a substitution instance of $P \to \neg R$ because it is R which must be replaced and not $\neg R$.

Example 7.3 Substitution instances of $P \to \neg Q$ are:

1. $(R \wedge S) \to \neg(J \vee M)$
2. $Q \to \neg(P \wedge \neg Q)$
3. $(R \wedge \neg S) \to \neg P$
4. $(P \vee Q) \to \neg R$

It should also be noted that in constructing substitution instances of a formula, substitutions should be made simultaneously, not one after the other.

Example 7.4 Consider the following formulas from $P \to \neg Q$

(i) Substitute $P \vee Q$ for P and R for Q to get the substitution instance $(P \vee Q) \to \neg R$.

(ii) First substitute $P \vee Q$ for P to obtain the substitution instance $(P \vee Q) \to \neg Q$. Next, substitute R for Q in $(P \vee Q) \to \neg Q$, and we get $(P \vee R) \to \neg R$. This formula is a substitution instance of $(P \vee Q) \to \neg Q$, but it is not a substitution instance of $P \to \neg Q$ under the substitution $(P \vee Q)$ for P and R for Q, because we did not substitute simultaneously as we did in (i).

The importance of substitution instance lies in the fact that any substitution of a tautology is a tautology. For example, consider the tautology $P \vee \neg P$. Regardless of what is substituted for P, the truth value of $P \vee \neg P$ is always T. Therefore, if we substitute any statement formula for P, the resulting formula will be a tautology. Hence The following substitution instances of $P \vee \neg P$ are tautologies

$$((P \vee Q) \wedge R) \vee \neg((P \vee Q) \wedge R)$$

$$(((P \vee \neg S) \to R) \rightleftarrows S) \vee \neg (((P \vee \neg S) \to R) \rightleftarrows S)$$

Thus, if it is possible to detect whether a given formula is a substitution instance of a tautology, then it is immediately known that the given formula is also a tautology.

8. Equivalence of Formulae

Definition 8.1 Two formulas A and B are said to be *equivalent* to each other if and only if $A \rightleftarrows B$ is a tautology.

If $A \rightleftarrows B$ is a tautology, we write $A \Leftrightarrow B$.

Note 8.1 $A \Leftrightarrow B$ if and only if the truth tables of A and B are the same.

8.1 Truth table method

One method to determine whether any two statement formulas are *equivalent* is to construct their truth tables.

Example 8.1.1 Prove

$$P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q)$$

Solution:

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$P \vee Q \rightleftarrows \neg(\neg P \wedge \neg Q)$
T	T	T	F	F	F	T	T
T	F	T	F	T	F	T	T
F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T

As $P \vee Q \rightleftarrows \neg(\neg P \wedge \neg Q)$ is a tautology, then $P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q)$.

Example 8.1.2 Prove $(P \rightarrow Q) \Longleftrightarrow (\neg P \vee Q)$.

Solution:

P	Q	$P \rightarrow Q$	$\neg P$	$\neg P \vee Q$	$(P \rightarrow Q) \rightleftarrows (\neg P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(P \rightarrow Q) \rightleftarrows (\neg P \vee Q)$ is a tautology.
Then $(P \rightarrow Q) \Longleftrightarrow (\neg P \vee Q)$.

Equivalent formulas:

$P \vee P \Longleftrightarrow P$	$P \wedge P \Longleftrightarrow P$	Idempotent laws
$(P \vee Q) \vee R \Longleftrightarrow P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \Longleftrightarrow P \wedge (Q \wedge R)$	Associative laws
$P \vee Q \Longleftrightarrow Q \vee P$	$P \wedge Q \Longleftrightarrow Q \wedge P$	Commutative laws
$P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$	$P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$	Distributive laws
$P \vee F \Longleftrightarrow P$	$P \wedge T \Longleftrightarrow P$	
$P \vee T, F \Longleftrightarrow T$	$P \wedge T, F \Longleftrightarrow F$	
$P \vee \neg P \Longleftrightarrow T$	$P \wedge \neg P \Longleftrightarrow F$	
$P \vee (P \wedge Q) \Longleftrightarrow P$	$P \wedge (P \vee Q) \Longleftrightarrow P$	Absorption laws
$\neg(P \vee Q) \Longleftrightarrow \neg P \wedge \neg Q$	$\neg(P \wedge Q) \Longleftrightarrow \neg P \vee \neg Q$	Demorgan's laws

Check the above formulas as an exercise by truth table technique

8.2 Replacement process

Consider the formula $A: P \to (Q \to R)$. The formula $Q \to R$ is a part of the formula A. If we replace $Q \to R$ by an *equivalent formula* $\neg Q \vee R$ in A, we get another formula $B: P \to (\neg Q \vee R)$. One can easily verify that the formulas A and B are equivalent to each other. This process of obtaining B from A is known as the replacement process.

Example 8.2.1 Prove that $P \to (Q \to R) \Longleftrightarrow P \to (\neg Q \vee R) \Longleftrightarrow (P \wedge Q) \to R$.

Solution: We know that $Q \to R \Longleftrightarrow \neg Q \vee R$
Replacing $Q \to R$ by $\neg Q \vee R$, we get $P \to (\neg Q \vee R)$, which is equivalent to $\neg P \vee (\neg Q \vee R)$ by the same rule,
Now

$$\neg P \vee (\neg Q \vee R) \Longleftrightarrow (\neg P \vee \neg Q) \vee R \Longleftrightarrow (\neg (P \wedge Q) \vee R) \Longleftrightarrow (P \wedge Q) \to R$$

by associativity of $\vee$, Demorgan's law and the previously used rule.

Example 8.2.2 $(P \to Q) \wedge (R \to Q) \Longleftrightarrow (P \vee R) \to Q$

Solution: $(P \to Q) \wedge (R \to Q) \Longleftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)$ replacing $P \to Q$ and $R \to Q$ by $\neg P \vee Q$ and $\neg R \vee Q$. respectively

$$\Longleftrightarrow (\neg P \wedge \neg R) \vee Q \left[\because (S_1 \vee S_2) \wedge (S_3 \vee S_2) \Leftrightarrow (S_1 \wedge S_3) \vee S_2 \right]$$
$$\Longleftrightarrow \neg (P \vee R) \vee Q \left[\text{replacing } \neg P \wedge \neg R \text{ by } \neg (P \vee R) \right]$$
$$\Longleftrightarrow P \vee R \to Q \left[\because \neg A \vee B \Leftrightarrow (A \to B) \right]$$

Example 8.2.3 Prove that

$$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Longleftrightarrow R.$$

Solution:

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Longleftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ (Associative Law & Distributive Law)

$\Longleftrightarrow (\neg (P \vee Q) \wedge R) \vee ((Q \vee P) \wedge R)$ (Demorgan's Laws)

$\Longleftrightarrow (\neg (P \vee Q) \vee (P \vee Q)) \wedge R$ (Distributive Law)

$\Longleftrightarrow \mathbf{T} \wedge R$ since $\neg S \vee S = \mathbf{T}$

$\Longleftrightarrow R$ as $\mathbf{T} \wedge R \leftrightarrow R$

Example 8.2.4 $P \to (Q \to P) \Longleftrightarrow \neg P \to (P \to Q)$

Solution: $P \to (Q \to P) \Longleftrightarrow \neg P \vee (Q \to P)$

$$\begin{aligned}
&\Longleftrightarrow \neg P \vee (\neg Q \vee P)\\
&\Longleftrightarrow (\neg P \vee P) \vee \neg Q\\
&\Longleftrightarrow T \vee (\neg Q)\\
&\Longleftrightarrow T
\end{aligned}$$

$$\begin{aligned}
\text{and } \neg P \to (P \to Q) &\Longleftrightarrow \neg(\neg P) \vee (P \to Q)\\
&\Longleftrightarrow P \vee (\neg P \vee Q)\\
&\Longleftrightarrow (P \vee \neg P) \vee Q\\
&\Longleftrightarrow T \vee Q\\
&\Longleftrightarrow T
\end{aligned}$$

So $P \to (Q \to P) \Longleftrightarrow T \Longleftrightarrow \neg P \to (P \to Q)$

Example 8.2.5 $(P \to Q) \wedge (R \to Q) \Longleftrightarrow (P \vee R) \to Q$

Solution:

$$\begin{aligned}
(P \to Q) \wedge (R \to Q) &\Longleftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)\\
&\Longleftrightarrow (\neg P \wedge \neg R) \vee Q\\
&\Longleftrightarrow \neg(P \vee R) \vee Q\\
&\Longleftrightarrow (P \vee R) \to Q
\end{aligned}$$

Example 8.2.6

(*i*) $\neg(P \rightleftarrows Q) \Longleftrightarrow (P \vee Q) \wedge \neg(P \wedge Q)$

(*ii*) $\neg(P \rightleftarrows Q) \Longleftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Solution:

$$\begin{aligned}
\neg(P \rightleftarrows Q) &\Longleftrightarrow \neg((P \to Q) \wedge (Q \to P))\\
&\Longleftrightarrow \neg((\neg P \vee Q) \wedge (\neg Q \vee P))\\
&\Longleftrightarrow \neg\left[((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)\right]\\
&\Longleftrightarrow \neg((\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) \vee (\neg P \wedge P) \vee (Q \wedge P))\\
&\Longleftrightarrow \neg(\neg(P \vee Q) \vee F \vee F \vee (Q \wedge P))\\
&\Longleftrightarrow \neg(\neg(P \vee Q) \vee (Q \wedge P))\\
&\Longleftrightarrow (P \vee Q) \wedge \neg(P \wedge Q) \qquad \text{(i)}\\
&\Longleftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)\\
&\Longleftrightarrow (P \wedge (\neg P \vee \neg Q)) \vee (Q \wedge (\neg P \vee \neg Q))\\
&\Longleftrightarrow \left[(P \wedge \neg P) \vee (P \wedge \neg Q)\right] \vee \left[(Q \wedge \neg P) \vee (Q \wedge \neg Q)\right]
\end{aligned}$$

$$\Longleftrightarrow F \vee (P \wedge \neg Q) \vee (Q \wedge \neg P) \vee F \qquad \text{(By associative law)}$$
$$\Longleftrightarrow (P \wedge \neg Q) \vee (Q \wedge \neg P)$$
$$\Longleftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q) \qquad \text{(ii)}$$

Thus (i) and (ii) are proved.

Example 8.2.7 Show that $((P\vee Q)\wedge\neg(\neg P\wedge(\neg Q\vee\neg R)))\vee(\neg P\wedge\neg Q)\vee(\neg P\wedge\neg R)$ is a tautology

Solution: By Demorgan's Laws, we have

$$\neg P \wedge \neg Q \Longleftrightarrow \neg(P \vee Q)$$
$$\neg P \wedge \neg R \Longleftrightarrow \neg(P \vee R)$$
$$\neg(P \wedge \neg Q) \vee (\neg P \wedge \neg R) \Longleftrightarrow \neg(P \vee Q) \vee \neg(P \vee R)$$
$$\Longleftrightarrow \neg((P \vee Q) \wedge (P \vee R))$$

Also

$$\neg(\neg P \wedge (\neg Q \vee \neg R)) \Longleftrightarrow \neg(\neg P \wedge \neg(Q \wedge R))$$
$$\Longleftrightarrow P \vee (Q \wedge R)$$
$$\Longleftrightarrow (P \vee Q) \wedge (P \vee R)$$
$$(P \vee Q) \wedge ((P \vee Q) \wedge (P \vee R)) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$$

Consequently, the given formula is equivalent to

$$((P \vee Q) \wedge (P \vee R)) \vee \neg((P \vee Q) \wedge (P \vee R))$$

which is a substitution instance of $P \vee \neg P$.

P→Q ¬P∨Q

¬P ∨Q - P→Q

(P∨P)

P→Q : ¬P∨Q

Exercises (3)

(I) Show the following equivalences (without using truth table).

(a) $P \to (Q \vee R) \Longleftrightarrow (P \to Q) \vee (P \to R)$

(b) $\neg(P \to Q) \Longleftrightarrow P \wedge \neg Q$

(c) $(P \rightleftarrows Q) \Longleftrightarrow (P \to Q) \wedge (Q \to P)$

(d) $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Longleftrightarrow (\neg P \vee Q)$

(e) $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Longleftrightarrow (\neg P \wedge \neg Q)$

(f) $P \to Q \Longleftrightarrow \neg Q \to \neg P$

(II) Show that P is equivalent to the following formulas

$\neg\neg P$, $P \wedge P$, $P \vee P$, $P \vee (P \wedge Q)$, $P \wedge (P \vee Q)$, $(P \wedge Q) \vee (P \wedge \neg Q)$, $(P \vee Q) \wedge (P \vee \neg Q)$

9. Law of Duality

Definition 9.1 Two formulas A and A^* are said to be *duals* of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called duals of eath other. If the formula A contains the special variables T or F, the A^*, its dual is obtained by replacing T by F and F by T in addition to the above-mentioned-interchanges

Example 9.1 If the formula A is given by

$$A : \neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg R))$$

The its dual A^* is given by $A^* : \neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg R))$

Result 9.1 Let A and A^* be dual formulas and let $P_1, P_2, \ldots, P_n$ be all the atomic variables that are in A and A^*.

i.e; $A : A(P_1, P_2, \ldots, P_n)$ and
$A^* : A^*(P_1, P_2, \ldots, P_n)$

Then using the Demorgan's Laws $P \wedge Q \Longleftrightarrow \neg(\neg P \vee \neg Q)$
$P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q)$

We can prove

$$\neg A(P_1, P_2, \ldots, P_n) \iff A^*(\neg P_1, \neg P_2, \ldots, \neg P_n) \tag{1}$$

Thus the negation of a formula is equivalent to its dual in which every variable is replaced by its negation.

Similarly

$$A(\neg P_1, \neg P_2, \ldots \neg P_n) \iff \neg A^*(P_1, P_2, \ldots P_n). \tag{2}$$

Example 9.2 Show that $\neg(\neg P \wedge \neg(Q \vee R)) \iff P \vee (Q \vee R)$

Solution: Let $A(P, Q, R)$ be $\neg P \wedge \neg(Q \vee R)$

Then $A^*(P, Q, R)$ is $\neg P \vee \neg(Q \wedge R)$ and

$$A^*(\neg P, \neg Q, \neg R) : \neg\neg P \vee \neg(\neg Q \wedge \neg R) \iff P \vee (Q \vee R)$$

On the other hand

$$\neg A(P, Q, R) \text{ is } \neg(\neg P \wedge \neg(Q \vee R)) \iff P \vee (Q \vee R)$$

Result 9.2 If any two formulas are equivalent, then their duals are also equivalent to each other.

$$\text{i.e., If } A \iff B \text{ then } A^* \iff B^*$$

(OR) Let $P_1, P_2, \ldots, P_n$ be all the atomic variables appearing in the formulas A and B. Given that $A \iff B$ means " $A \rightleftarrows B$ is a tautology", then the following are also tautologies.

$$A(P_1, P_2, \ldots, P_n) \rightleftarrows B(P_1, P_2, \ldots, P_n)$$
$$A(\neg P_1, \neg P_2, \ldots, \neg P_n) \rightleftarrows B(\neg P_1, \neq P_2, \ldots, \neq P_n)$$

Using (2) we get $\neg A^*(P_1, P_2, \ldots, P_n) \rightleftarrows \neg B^*(P_1, P_2, \ldots, P_n)$

Hence $A^* \iff B^*$.

Example 9.3 Prove that

(a) $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \iff (\neg P \vee Q)$

(b) $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \iff (\neg P \wedge Q)$

Solution:

(a) $\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q))$

$$\Longleftrightarrow (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q))$$
$$\Longleftrightarrow (P \wedge Q) \vee (\neg P \vee Q)$$
$$\Longleftrightarrow (P \wedge Q) \vee \neg P \vee Q$$
$$\Longleftrightarrow ((P \vee \neg P) \wedge (Q \vee \neg P)) \vee Q$$
$$\Longleftrightarrow (Q \vee \neg P) \vee Q$$
$$\Longleftrightarrow Q \vee \neg P$$
$$\Longleftrightarrow \neg P \vee Q$$

(b) From (I) $(P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \Longleftrightarrow \neg P \vee Q$
Writing dual,

$$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Longleftrightarrow \neg P \wedge Q.$$

10. Tautological Implications

Definition 10.1 A statement A is said to *tautologically imply* a statement B if and only if $A \rightarrow B$ is a tautology. In this case, we write $A \Longrightarrow B$, read as "A implies B"

Note 10.1 $\Longrightarrow$ is not a connective, $A \Longrightarrow B$ is not a statement formula.

i) Thus $A \Longrightarrow B$ states that $A \rightarrow B$ is a tautology or A tautologically implies B.
ii) Clearly $A \Longrightarrow B$ guarantees that B has the truthvalue T whenever A has the truth value T.
iii) By constructing the truth tables of A and B, we can determine whether $A \Longrightarrow B$.

Example 10.1 Prove that $(P \rightarrow Q) \Longrightarrow (\neg Q \rightarrow \neg P)$

Solution: We prove this by Using the truth table for $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$	$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since all the entries in the last column are true, $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$ is a tautology Hence $(P \rightarrow Q) \Longrightarrow (\neg Q \rightarrow \neg P)$.

(iv) In order to show any of the given implications, it is sufficient to show that an assignment of the truth value T to the antecedent of the corresponding conditional leads to the truth value T for the consequent. This procedure ensures that the conditional becomes a tautology, thereby proving the implication.

Example 10.2 Prove that $\neg Q \wedge (P \rightarrow Q) \Longrightarrow \neg P$

Solution: Assume that the antecedent $\neg Q \wedge (P \rightarrow Q)$ has the true value T, then both $\neg Q$ and $P \rightarrow Q$ have the truth value T, which means that Q has the value F. $P \rightarrow Q$ has the truth value T, and hence P must have the value F.
Therefore the consequent $\neg P$ must have the value T.

(v) Another method to show $A \Longrightarrow B$ is to assume that the consequent B has the value F and then show that this assumption leads to A's having the value F. Then $A \rightarrow B$ must have the value T.

Example 10.3 Show that $\neg(P \rightarrow Q) \Longrightarrow P$

Solution: Assume that P is false (F), when P is false $P \rightarrow Q$ has T, then $\neg(P \rightarrow Q)$ has F.
Then $\neg(P \rightarrow Q) \rightarrow P$ has T.
$\therefore \quad \neg(P \rightarrow Q) \Longrightarrow P$.

Note 10.2 $A \Longleftrightarrow B$ if and only if $A \Longrightarrow B$ and $B \Longrightarrow A$ i.e., If each of two formulas A and B implies the other, then A and B are equivalent.

Observations 10.1 If a formula is equivalent to a tautology then it must be a tautology

Observations 10.2 If a formula is implied by a tautology then it is a tautology

Observations 10.3 Both Implication and equivalence are transitive.

i.e., if $A \Longleftrightarrow B$ and $B \Longleftrightarrow C$, then $A \Longleftrightarrow C$.

It follows from the definition of equivalence.

To show that the implication is transitive:

Assume that $A \Longrightarrow B$ and $B \Longrightarrow C$.

The $A \rightarrow B$ and $B \rightarrow C$ are tautologies.

Hence $(A \rightarrow B) \wedge (B \rightarrow C)$ is also a tautology.

But from $(P \rightarrow Q) \wedge (Q \rightarrow R) \Longrightarrow P \rightarrow R$

$$(A \rightarrow B) \wedge (B \rightarrow C) \Longrightarrow (A \rightarrow C)$$

Hence $A \rightarrow C$ is a tautology.

Hence Implication is transitive. □

(vi) In order to show that $A \Longrightarrow C$, it is convenient to introduce a series of formulas $B_1, B_2, \ldots, B_m$ such that $A \Longrightarrow B_1, B_1 \Longrightarrow B_2, \ldots, B_{m-1} \Longrightarrow B_m$ and $B_m \Longrightarrow C$.

(vii) Important property of implication:
If $A \Longrightarrow B$ and $A \Longrightarrow C$, then $A \Longrightarrow (B \wedge C)$.

Proof: Assume if A is true, then B and C are both true. Thus $B \wedge C$ is true and hence $A \rightarrow (B \wedge C)$ is true. □

Result 10.1 If $H_1, H_2, \ldots, H_m$ and P imply Q, then $H_1, H_2, \ldots, H_m$ imply $P \rightarrow Q$.

Proof: We have $(H_1 \wedge H_2 \wedge \ldots \wedge H_m \wedge P) \Longrightarrow Q$

This means $(H_1 \wedge H_2 \wedge \ldots \wedge H_m \wedge P) \rightarrow Q$ is a tautology.
Now from the equivalence $P_1 \rightarrow (P_2 \rightarrow P_3) \Longleftrightarrow (P_1 \wedge P_2) \rightarrow P_3$
We can conclude that

$$(H_1 \wedge H_2 \wedge \ldots \wedge H_m) \rightarrow (P \rightarrow Q)$$

is a tautology.
Hence the theorem. □

Implications

$P \wedge Q \Longrightarrow P$
$P \wedge Q \Longrightarrow Q$
$P \Longrightarrow P \vee Q$
$\neg P \Longrightarrow P \rightarrow Q$
$Q \Longrightarrow P \rightarrow Q$
$\neg(P \rightarrow Q) \Longrightarrow P$
$\neg(P \rightarrow Q) \Longrightarrow \neg Q$
$P \wedge (P \rightarrow Q) \Longrightarrow Q$
$\neg Q \wedge (P \rightarrow Q) \Longrightarrow \neg P$
$\neg P \wedge (P \vee Q) \Longrightarrow Q$
$(P \rightarrow Q) \wedge (Q \rightarrow R) \Longrightarrow P \rightarrow R$
$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \Longrightarrow R$

Check the above implications.

Exercise 4

(I) Show the following implications

(a) $(P \to (Q \to R)) \Longrightarrow (P \to Q) \to (P \to R)$
(b) $Q \Longrightarrow P \to R$
(c) $(P \wedge Q) \Longrightarrow P \to Q$

(II) show the following implications without Constructing the truth tables

(a) $\neg Q \wedge (P \to Q) \Longrightarrow \neg P$
(b) $(P \vee Q) \wedge (\neg P) \Longrightarrow Q$
(c) $P \to Q \Longrightarrow P \to (P \wedge Q)$
(d) $(P \to Q) \to Q \Longrightarrow P \vee Q$
(e) $((P \vee \neg P) \to Q) \to ((P \vee \neg P) \to R) \Longrightarrow (Q \to R)$
(f) $(Q \to (P \wedge \neg P)) \to (R \to (P \wedge \neg P)) \Longrightarrow (R \to Q)$

Answers (4)

(I) (a) we prove this by Using the truth table for

$$(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$$

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$	$P \to (Q \to R)$	$(P \to Q) \to (P \to R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

as the columns of $P \to (Q \to R)$ and $(P \to Q) \to (P \to R)$ are identical $(P \to (Q \to R)) \to ((P \to) \to (P \to R))$ is a tautology

Therefore $(P \to (Q \to R)) \Longrightarrow ((P \to Q) \to (P \to R))$.

(II) (a). To prove that $\neg Q \wedge (P \to Q) \Longrightarrow \neg P$, it is enough to show that the assumption that $\neg Q \wedge (P \to Q)$ has the truth value T guarentees the truth value T for $\neg P$.

Now assume that $\neg Q \wedge (P \rightarrow Q)$ has the truth value T. Then both $\neg Q$ and $P \rightarrow Q$ have the truth value T. Since $\neg Q$ has truth value T, Q has the truth value F. As Q has the truth value F and $P \rightarrow Q$ has the truth value T, it follows that the truth value of P is F and the truth value of $\neg P$ is T. Thus we have proved that $\neg Q \wedge (P \rightarrow Q) \Longrightarrow \neg P$.

11. Formulas containing n Variables

A statement formula containing n variables must have as its truth table one of the 2^{2^n} possible truth tables each of them having 2^n rows. This fact suggests that there are many formulas which may look very different from one another but are equivalent.

For the case $n = 1$, any formula involving only one variable will have one of these four truth tables:

	1	2	3	4
P	P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	T	F	T	F
F	F	T	T	F

Every other formula depending upon P alone would then be equivalent to one of these four formulas.

In case of $n = 2$, the number of distinct truth tables for formulas involving two variables is $2^{2^2} = 2^4 = 16$. Clearly there are 2^2 rows in the truth table and since each row will have any of the two entries T or F, we have 16 possible tables.

P	Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

Distinct Formulae with two variables

12. Functionally Complete Sets of Connectives

1. We have already defined the connectives $\wedge, \vee, \neg, \rightarrow, \rightleftarrows$. Now we introduce some *other connectives* namely NAND and NOR which have useful applications in the design of computers. The word 'NAND' is a combination of 'NOT' and 'AND', while the word 'NOR' is a combination of 'NOT' and 'OR', where

'NOT' stands for negation, 'AND' stands for conjunction and 'OR' stands for the disjunction. The connective 'NAND' is denoted by the symbol $\uparrow$.
For any two formulas P and Q

$$P \uparrow Q \Longleftrightarrow \neg(P \wedge Q)$$

The connective 'NOR' is denoted by the symbol $\downarrow$. For any two formulae

$$P \downarrow Q \Longleftrightarrow \neg(P \vee Q)$$

The connectives $\uparrow$ and $\downarrow$ have been defined interms of the connectives $\wedge$, $\vee$ and $\neg$. Therefore, for any formula containing the connectives $\uparrow$ or $\downarrow$, one can obtain an equivalent formula containing the connectives $\wedge$, $\vee$ and $\neg$ only. Note that $\uparrow$ and $\downarrow$ are duals of each other. Therefore in order to obtain the dual of a formula which includes $\uparrow$ or $\downarrow$, we should interchange $\uparrow$ and $\downarrow$ in addition to the other interchanges mentioned earlier.

Definition 12.1 A set of connectives is said to be *functionally complete set of connectives* if every formula can be expressed interms of an equivalent formula containing the connectives only from this set.

Note 12.1 Functionally complete set should not contain any redundant connectives i.e., a connective which can be expressed interms of the other connectives.

Already, we have

(1) $P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q)$

(2) $P \wedge Q \Longleftrightarrow \neg(\neg P \vee \neg Q)$

(3) $P \rightarrow Q \Longleftrightarrow \neg P \vee Q$

(4) $P \rightleftarrows Q \Longleftrightarrow (\neg P \vee Q) \wedge (P \vee \neg Q)$

Hence by first replacing all bi conditionals, then the conditionals and finally all the conjuctions or all the disjunctions in any formula, we can obtain an equivalent formula which contains either the negation and disjunction only or the negation and conjunction only. In other words, for every formula, we can find an equivalent formula containing the connectives $\vee$ and $\neg$ only or $\wedge$ and $\neg$ only. From the definition of functionally complete set of connectives, the sets of connectives $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ are functionally complete sets.

Note 12.2 The set $\{\wedge, \vee\}$ is not functionally complete, as for the formula $\neg P$, it is no possible to find an equivalent formula containing connectives only from the set $\{\wedge, \vee\}$.

Result 12.1 $\{\uparrow\}$, $\{\downarrow\}$ are functionally complete.

Proof: In order to prove, it is sufficient to show that the sets of connectives $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ can be expressed either interms of $\uparrow$ alone or interms of $\downarrow$ alone.

To show that $\{\vee, \neg\}$ is functionally complete, it is enough to show that $\neg$ and $\vee$ can be expressed interms of $\downarrow$ alone.

We have

$$\neg P \Longleftrightarrow \neg P \wedge \neg P \Longleftrightarrow \neg(P \vee P) \Longleftrightarrow P \downarrow P \quad \text{and}$$
$$P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q) \Longleftrightarrow \neg P \downarrow \neg Q \Longleftrightarrow (P \downarrow P) \downarrow (Q \downarrow Q)$$

Then $\{\downarrow\}$ is a functionally complete set.

To show that $\{\neg, \wedge\}$ is functionally complete, we have to express $\neg$ and $\wedge$ in terms of $\uparrow$ alone. The following valid equalities help us in this direction.

$\neg P \Longleftrightarrow \neg P \vee \neg P \Longleftrightarrow \neg(P \wedge P) \Longleftrightarrow P \uparrow P$ and

$P \wedge Q \Longleftrightarrow \neg(P \uparrow Q) \Longleftrightarrow (P \uparrow Q) \uparrow (P \uparrow Q)$

Then $\{\uparrow\}$ is a functionally complete set.

Thus we proved that each of the sets $\{\uparrow\}$ and $\{\downarrow\}$ are functionally complete. □

Note 12.3

(a)
$$\neg P \Longleftrightarrow \neg P \vee \neg P \Longleftrightarrow \neg(P \wedge P) \Longleftrightarrow P \uparrow P$$
$$P \wedge Q \Longleftrightarrow \neg(P \uparrow Q) \Longleftrightarrow (P \uparrow Q) \uparrow (P \uparrow Q)$$
$$P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q) \Longleftrightarrow \neg P \uparrow \neg Q \Longleftrightarrow (P \uparrow P) \uparrow (Q \uparrow Q)$$

(b)
$$\neg P \Longleftrightarrow \neg(P \vee P) \Longleftrightarrow P \downarrow P$$
$$P \vee Q \Longleftrightarrow \neg(P \downarrow Q) \Longleftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q)$$
$$P \wedge Q \Longleftrightarrow \neg P \downarrow \neg Q \Longleftrightarrow (P \downarrow P) \downarrow (Q \downarrow Q).$$

We call each of the sets $\{\uparrow\}$ and $\{\downarrow\}$ a *minimal functionally complete set* or inshort a *minimal set.*

Note (c):

(i) $P \uparrow Q \Longleftrightarrow Q \uparrow P$; $P \downarrow Q \Longleftrightarrow Q \downarrow P$ (Commutative)

(ii) The connectives $\uparrow$ and $\downarrow$ are not associative

since
$$P \uparrow (Q \uparrow R) \Longleftrightarrow P \uparrow \neg(Q \wedge R) \Longleftrightarrow (P \wedge \neg(Q \wedge R))$$
$$\Longleftrightarrow P\neg(Q \wedge R)$$
$$(P \uparrow Q) \uparrow R \Longleftrightarrow (P \wedge Q) \wedge \neg R$$

Similarly

$$P \downarrow (Q \downarrow R) \Longleftrightarrow \neg P \wedge (Q \vee R)$$
$$(P \downarrow Q) \downarrow R \Longleftrightarrow (P \vee Q) \wedge \neg R$$

(iii) $P \uparrow Q \uparrow R \Longleftrightarrow \neg(P \wedge Q \wedge R)$

However $P \uparrow Q \uparrow R$ is not equivalent to any of

$$P \uparrow (Q \uparrow R), \quad (P \uparrow Q) \uparrow R, \quad Q \uparrow (P \uparrow R)$$

$$P \uparrow Q \Longleftrightarrow \neg(P \wedge Q) \Longleftrightarrow \neg P \vee \neg Q \Longleftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$$

$$P \downarrow Q \Longleftrightarrow \neg(P \vee Q) \Longleftrightarrow \neg P \wedge \neg Q \Longleftrightarrow (\neg P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee \neg Q)$$

13. Normal Forms

By constructing and comparing truth tables, we can determine whether two statement formulas A and B are equivalent. But this is very tedious and difficult to follow even on a computer because the number of entries increases very rapidly as n increases.

A better method is to transform the statement formulas A and B to some standards forms A' and B' such that a simple comparison of A' and B' shows whether $A \Longleftrightarrow B$. The standard forms are called canonical forms or normal forms.

Let $A(P_1, P_2, \ldots P_n)$ be a statement formula where $P_1, P_2, \ldots P_n$ are the primitive variables.

If A has the truthvalue T for atleast one combination of truth values assigned to $P_1, P_2, \ldots, P_n$ then A is said to be *satisfiable*.

The problem of determining, in a finite number of steps, whether a given statement formula is a tautology or a contradiction or at least satisfiable is known as a *decision problem*.

However, the solution of the decision problem may not be simple as we mentioned earlier, the construction of truth tables may not be practical, even with the help of a computer. Therefore we are in need of other procedures known as reduction to normal forms.

Note 13.1 It will be convenient to use the word "product" in place of "conjunction" and "sum" inplace of "disjunction" in our discussion throughout.

Definition 13.1 A product of (statement) variables and their negations is called an elementary product.

Similarly, a sum of the variables and their negations is called an elementary sum.

Example 13.1 The formulae $P, \neg P, \neg P \wedge Q, \neg Q \wedge P \wedge \neg P, P \wedge \neg P, Q \wedge \neg P \neg P \wedge \neg Q$ are some examples for elementary products. The formulae $P, \neg P, \neg P \vee Q, \neg Q \vee P \vee \neg P, P \vee \neg P, Q \vee \neg P$ are examples for elementary sums.

Definition 13.2 Any part of an elementary sum or product which is itself an elementary sum or product is called a *factor* of the original elementary sum or product.

Example 13.2 $\neg Q, P \wedge \neg P$ and $\neg Q \wedge P$ are some of the factors of $\neg Q \wedge P \wedge \neg P$.

Observations 13.1 A necessary and sufficient condition for an elementary product to be identically false is that it contain atleast one pair of factors in which one is the negation of the other.

Explanation 1 For any variable P, $P \wedge \neg P$ is identically false. Hence if $P \wedge \neg P$ appears in the elementary product, then the product is identically false.

Assume that if an elementary product is identically false and does not contain at least one factor of this type then we can assign truth values T and F to variables and negated variables respectively that appear in the product. But this assignment says that the elementary product has the truth value T. This is contrary to our assumption. Hence the observation follows.

Similarly,

Observations 13.2 A necessary and sufficient condition for an elementary sum to be identically true is that it contain at least one pair of factors in which one is the negation of the other.

The explanation of the observation 2 will follow along the similar lines of observation 1.

13.1 Disjunctive Normal Forms (d.n.f)

Definition 13.1.1 A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

Procedure to obtain a disjunctive normal form of a given formula

Step 1: If the connectives '$\rightarrow$' and $\rightleftarrows$ appear in the given formula, obtain an equivalent formula in which '$\rightarrow$' and '$\rightleftarrows$' do not appear. i.e., an equivalent formula can be obtained in which '$\rightarrow$' and '$\rightleftarrows$' do not appear. i.e., an equivalent formula can be obtained in which '$\rightarrow$' and '$\rightleftarrows$' are replaced by $\wedge$, $\vee$ and $\neg$.

Example 13.1.1 $P \rightarrow Q$ is replaced by $\neg P \vee Q$ and $P \rightleftarrows Q$ is replaced $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ or $(\neg P \vee Q) \wedge (\neg Q \vee P)$.

Therefore, there is no loss of generality in assuming that the given formula contains the connectives $\wedge$, $\vee$ and $\neg$ only.

Step 2: If the negation is applied to the formula or to a part of the formula and not to the variables appearing in it, (i.e., formula which is not a statement variable). Then by using DeMorgan's laws an equivalent formula can be obtained in which the negation is applied to the statement variables only.

Step 3: Now apply the distributive law until a sum of elementary products is obtained. This will be a disjunctive normal form, after application of the Idempotent Law and suitable reordering. In this normal form, the elementary products which are equivalent to 'F' (false), if any, can be omitted.

Note 13.1.1 Extended Distributive Law

$$(P \vee Q) \wedge (R \vee S) \Longleftrightarrow (P \wedge R) \vee (P \wedge S) \vee (Q \wedge R) \vee (Q \wedge S)$$

This is as follows:

$$\begin{aligned}(P \vee Q) \wedge (R \vee S) &\Longleftrightarrow [(P \vee Q) \wedge R] \vee [(P \vee Q) \wedge S] \\ &\Longleftrightarrow (P \wedge R) \vee (Q \wedge R) \vee (P \wedge S) \vee (Q \wedge S)\end{aligned}$$

Example 13.1.2 obtain a disjuctive normal form of $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

Solution: $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

$$\begin{aligned}&\Longleftrightarrow \neg P \vee ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)) \\ &\Longleftrightarrow \neg P \vee ((\neg P \vee Q) \wedge \neg(\neg Q \vee \neg P)) \\ &\Longleftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P)) \\ &\Longleftrightarrow \neg P \vee [(\neg P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P))] \\ &\Longleftrightarrow \neg P \vee [(P \wedge \neg P \wedge Q)] \vee (Q \wedge P) \\ &\Longleftrightarrow \neg P \vee \boldsymbol{F} \vee (P \wedge Q) \\ &\Longleftrightarrow \neg P \vee (P \wedge Q)\end{aligned}$$

which is the required d.n.f

Example 13.1.3 obtain a d.n.f of $P \wedge (P \rightarrow Q)$

Solution:

$$P \wedge (P \rightarrow Q) \Longleftrightarrow P \wedge (\neg P \vee Q) \Longleftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$$

Example 13.1.4 obtain a d.n.f of $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

Solution:

$$\begin{aligned}&\neg(P \vee Q) \rightleftarrows (P \wedge Q) \\ &\Longleftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))\end{aligned}$$

(Elimination of biconditional)

$$\begin{aligned}&\Longleftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q)) \\ &\Longleftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \\ &\quad \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q) \qquad \text{[by Extended distributive law]}\end{aligned}$$

Then the required d.n.f.: $(P \wedge \neg Q) \vee (\neg P \vee Q)$, by ingnoring '$\boldsymbol{F}$'

Example 13.1.5 Find a d.n.f of $(P \wedge \neg(Q \vee R)) \vee (((P \wedge Q) \vee \neg R) \wedge P)$.

Solution:

$$\begin{aligned}&(P \wedge \neg(Q \vee R)) \vee (((P \wedge Q) \wedge \neg R) \wedge P) \\ &\Longleftrightarrow (P \wedge (\neg Q \wedge \neg R)) \vee (((P \wedge Q) \vee \neg R) \wedge P) \\ &\Longleftrightarrow (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge P) \vee (\neg R \wedge P) \\ &\Longleftrightarrow (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q) \vee (\neg R \wedge P).\end{aligned}$$

Example 13.1.6 obtain a d.n.f of $(Q \vee (P \wedge R)) \wedge \neg((P \vee R) \wedge Q)$

Solution: $(Q \vee (P \wedge R)) \wedge \neg((P \vee R) \wedge Q)$

$$\Longleftrightarrow (Q \vee (P \wedge R)) \wedge (\neg(P \vee R) \vee \neg Q)$$
$$\Longleftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg P \wedge \neg R) \vee \neg Q)$$

$$\Longleftrightarrow (Q \wedge (\neg P \wedge \neg R)) \vee (Q \wedge \neg Q) \vee [(P \wedge R) \wedge (\neg P \wedge \neg R)] \vee ((P \wedge R) \wedge \neg Q)$$

(By Extended distributive law)

$$\Longleftrightarrow (\neg P \wedge Q \wedge \neg R) \vee \boldsymbol{F} \vee (\boldsymbol{F} \wedge \boldsymbol{F}) \vee (P \wedge \neg Q \wedge R)$$
$$\Longleftrightarrow (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$$

Example 13.1.7 Find a d.n.f of $[(P \wedge Q) \vee (P \wedge \neg Q)] \wedge [(P \wedge \neg Q) \vee (\neg P \wedge \neg Q)]$

Solution:

$$\Longleftrightarrow [(P \wedge Q) \wedge (P \wedge \neg Q)] \vee [(P \wedge Q) \wedge (\neg P \wedge \neg Q]$$
$$\vee [(P \wedge \neg Q) \wedge (P \wedge \neg Q)] \vee [(P \wedge \neg Q) \wedge (\neg P \wedge \neg Q)]$$

[By Extended distributive Law]

$$\Longleftrightarrow \boldsymbol{F} \vee \boldsymbol{F} \vee (P \wedge \neg Q) \vee \boldsymbol{F}$$
$$\Longleftrightarrow P \wedge \neg Q.$$

Example 13.1.8 obtain a d.n.f of $\neg(P \to (Q \wedge R))$

Solution:

$$\neg(P \to (Q \wedge R)) \Longleftrightarrow \neg(\neg P \vee (Q \wedge R))$$
$$\Longleftrightarrow P \wedge (\neg Q \vee \neg R)$$
$$\Longleftrightarrow (P \wedge \neg Q) \vee (P \wedge \neg R).$$

Note 13.1.2 The d.n.f of a given formula is not unique. For example consider the formula $P \vee (Q \wedge R)$. This is already in the d.n.f. However, we may write

$$P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$$
$$\Longleftrightarrow (P \wedge P) \vee (P \wedge Q) \vee (P \wedge R) \vee (Q \wedge R)$$

The last equivalent formula is another equivalent d.n.f. Infact, different d.n.fs can be obtained for a given formula, of course, these different d.n.fs of the same formula are equivalent.

Exercise 5:

(I) obtain a d.n.f of the following:

(a) $(P \wedge \neg(Q \wedge R)) \vee (P \rightarrow Q)$

(b) $P \vee (\neg P \rightarrow (Q \vee (Q \rightarrow R)))$

(c) $(\neg P \vee \neg Q) \rightarrow (\neg P \wedge R)$

(d) $P \vee (\neg P \wedge \neg Q \vee R)$

(e) $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

Miscellneous Example (Truth table method to find d.n.f) 13.1.9

We know how to construct the truth table for a given formula. The truth table can also be used in opposite mode. That is given a table with truth values only we can determine from the table a formula for which the given table is the truth table.

We remark about d.n.f, that a given formula is identically false if every elementary product appearing in its d.n.f is identically false.

Q : Determine a formula having the truth values as shown in the table:

P	Q	R	f(P,Q,R)
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Solution:

Let $f(P, Q, R)$ be a formula whose truth table is the above table. The formula 'f' has the truth value T in first, third, fourth and seventh rows of the table and has the value F in all other rows. The assigned truth values for the variables P, Q, R in these rows are T, T, T; T, F, T; T, F, F and F, F, T respectively. The formulae $P \wedge Q \wedge R$, $P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$ and $\neg P \wedge \neg Q \wedge R$ have their truth values T only in the first, third, fourth and seventhrows respectively in their truth tables.

Thus the formula $f(P, Q, R)$ and

$$(P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R)$$

have the same truth table

Hence

$$f(P, Q, R) \Longleftrightarrow (P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R)$$

Clearly, the formula obtained is a disjunction of terms, each of which is a conjunction of statement variables and their negation. This particular form is known as a "disjunctive normal form" of the formula. Since the procedure used in the above problem is completely general, we conclude that every satisfiable formula can be expressed in a d.n.f.

Example 13.1.10 Find the d.n.f of the form $(\neg P \to R) \wedge (P \rightleftarrows Q)$

Solution: The truth of $(\neg P \to R) \wedge (P \rightleftarrows Q)$ is

P	Q	R	$(\neg P \to R) \wedge (P \rightleftarrows Q)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

The statement formula has the truthvalue T in first, second and seventh rows, and has the value F in all other rows. The assigned truth values for the variables P, Q, R in these rows are T, T, T; T, T, F; F, F, T; resphy.

Then the required d.n.f is

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R).$$

13.2 Conjuctive Normal Forms (c.n.f)

Definition 13.2.1 A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive normal form of the given formula.

The procedure of obtaining a c.n.f of a given formula is similar to the one given for d.n.f. Now, it will be illustrated by following examples.

Example 13.2.1 Obtain a c.n.f of each of the formulas

(a) $P \wedge (P \rightarrow Q)$

Solution:

$$P \wedge (P \rightarrow Q) \Longleftrightarrow P \wedge (\neg P \vee Q)$$

Hence $P \wedge (\neg P \vee Q)$ is a required c.n.f.

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

Solution:

$$\begin{aligned}
&\neg(P \vee Q) \rightleftarrows (P \wedge Q) \\
&\Longleftrightarrow (\neg(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg(P \vee Q)) \\
&\qquad\qquad (\because A \rightleftarrows B \Longleftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)) \\
&\Longleftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee (\neg P \wedge \neg Q)) \\
&\Longleftrightarrow [(P \vee Q \vee P) \wedge (P \vee Q \vee Q)] \wedge [(\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)] \\
&\Longleftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q).
\end{aligned}$$

which is the required c.n.f

(c) Find a c.n.f of $[Q \vee (P \wedge R)] \wedge \neg[(P \vee R) \wedge Q]$

Solution:

$$\begin{aligned}
&[Q \vee (P \wedge R)] \wedge \neg[(P \vee R) \wedge Q] \\
&\qquad \Longleftrightarrow [Q \vee (P \wedge R)] \wedge [\neg(P \vee R) \vee \neg Q] \\
&\qquad \Longleftrightarrow [Q \vee (P \wedge R)] \wedge [(\neg P \wedge \neg R) \vee \neg Q] \\
&\qquad \Longleftrightarrow (Q \vee P) \wedge (Q \vee R) \wedge (\neg P \vee \neg Q) \wedge (\neg R \wedge \neg Q).
\end{aligned}$$

Note 13.2.1 The c.n.f is also not unique. Furthermore, a given formula is tautology i every elementary sum in its c.n.f is tautology

Example 13.2.2 Show that the formula

$$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \text{ is a tautology}$$

Solution: First we obtain a c.n.f of the given formula

$$\begin{aligned}
Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) &\Longleftrightarrow Q \vee ((P \vee \neg P) \wedge \neg Q) \\
&\Longleftrightarrow [Q \vee (P \vee \neg P)] \wedge (Q \vee \neg Q) \\
&\Longleftrightarrow (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)
\end{aligned}$$

Since each of the elementary sums is a tautology, the given formula is a tautology.

Example 13.2.3 The truthtable for a formula A is given in the following table. Determine its c.n.f.

P	Q	R	A
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	T

The formula A has the truthvalue F in first, second, fourth and seventh rows of the table and has the value T in all other rows. The assigned truthvalues for the P, Q, R in these rows are T, T, T; T, T, F; T, F, F; F, F, T respectively. The formulas $\neg P \vee \neg Q \vee \neg R$, $\neg P \vee \neg Q \vee R$, $\neg P \vee Q \vee R$, and $P \vee Q \vee \neg R$ have their truthvalues F only in first, second, fourth and seventh rows respectively in their truth tables.

Therefore

$$A \Longleftrightarrow (\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee Q \vee \neg R)$$

Note 13.2.2 The d.n.f or c.n.f of a statement formula is not unique.

In order to arrive at a unique normal form of a given formula, we introduce the principal disjunctive (conjunctive) normal form.

Exercise 6

Obtain a c.n.f of the following formula:

(a) $P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$

(b) $[P \wedge \neg(Q \vee R)] \vee [((P \wedge Q) \vee \neg R) \wedge P]$

13.3 Principal Disjunctive Normal Forms (PDNF)

Definition 13.3.1 A *minterm* consists of conjunctions in which each statement variable or its negation, but not both, appears only once.

For example, for two variables P and Q, there are 2^2 minterms given by

$$P \wedge Q,\ P \wedge \neg Q,\ \neg P \wedge Q \quad \text{and} \quad \neg P \wedge \neg Q$$

For example, Minterms for the three variables P, Q and R are

$$P \wedge Q \wedge R,\quad P \wedge Q \wedge \neg R \quad P \wedge \neg Q \wedge R \quad P \wedge \neg Q \wedge \neg R$$
$$\neg P \wedge Q \wedge R \quad \neg P \wedge Q \wedge \neg R \quad \neg P \wedge \neg Q \wedge R \quad \neg P \wedge \neg Q \wedge \neg R$$

From the truth tables of these minterms of P and Q;

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

(i) It is clear that no two minterms are equivalent.

(ii) Each minterm has the truth value T for exactly one combination of the truth values of the variables P and Q.

The minterms of P, Q and R satisfy properties similar to those for two variables P and Q.

We conclude that if the truth table of any formula is known, then one can easily obtain an equivalent formula which consists of a disjunction of some of the minterms.

Note 13.3.1 For any formula containing n variables, an equivalent d.n.f can be obtained by selecting appropriate minterms out of its 2^n possible minterms.

Definition 13.3.2 An equivalent formula consisting of disjunctions of minterms only is known as its principal disjunctive normal form (sum-of-products canonical form), simply p.d.n.f.

Methods to obtain p.d.n.f of a given formula

(I) **By Truth table**: For every truth value T of the given formula, select the minterm which also has the value T for the same combination of the truth values of the statement variables.

Example 13.3.1 obtain the PDNF of $P \to Q$

Solution: From the truth table of $P \to Q$:

P	Q	$P \to Q$	Minterm
T	T	T	$P \wedge Q$
T	F	F	$P \wedge \neg Q$
F	T	T	$\neg P \wedge Q$
F	F	T	$\neg P \wedge \neg Q$

and from the previous truthtable of the minterms of P and Q:

The p.d.n.f of $P \to Q$ is

$$(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$

$$\therefore P \to Q \Longleftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$

Example 13.3.2 obtain the PDNF for $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

Solution: Constructing the truth table:

P	Q	R	Minterm	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R)$ $\vee (Q \wedge R)$
T	T	T	$P \wedge Q \wedge R$	T	F	T	T
T	T	F	$P \wedge Q \wedge \neg R$	T	F	F	T
T	F	T	$P \wedge \neg Q \wedge R$	F	F	F	F
T	F	F	$P \wedge \neg Q \wedge \neg R$	F	F	F	F
F	T	T	$\neg P \wedge Q \wedge R$	F	T	T	T
F	T	F	$\neg P \wedge Q \wedge \neg R$	F	F	F	F
F	F	T	$\neg P \wedge \neg Q \wedge R$	F	T	F	T
F	F	F	$\neg P \wedge \neg Q \wedge \neg R$	F	F	F	F

The p.d.n.f of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ is

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R).$$

Note 13.3.2 The no. of minterms appearing in the normal form is the same as the number of entries with the truth value T in the truth table of the given formula. Thus every formula (which is not a contradiction) has an equivalent p.d.n.f.

Further, such a normal form is unique.

(II) without constructing truth table:

In order to obtain the p.d.n.f of a given formula without constructing its truthtable:

Step1 First replace the conditionals and biconditionals by their equivalent formulas containing only $\wedge$, $\vee$ and $\neg$.

Step2 Next, the negations are applied to the variables by using De Morgan's laws followed by the application of distributive laws (as that in obtaining d.n.f or c.n.f).

Step3 Any elementary product which is a contradiction is dropped. Minterms are obtained in the disjunctions by introducing the missing factors. Identical minterms appearing in the disjunctions are deleted.

Example 13.3.3 By not using the truth table, find PDNF for
(a) $P \rightleftarrows Q$ (b) $\neg P \vee Q$

Solution:

(a)
$$\begin{aligned} P \rightleftarrows Q &\Longleftrightarrow (\neg P \vee Q) \wedge (P \vee \neg Q) \\ &\Longleftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \vee (\neg Q \wedge Q) \\ &\Longleftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q) \end{aligned}$$

Hence $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ is the PDNF of $P \rightleftarrows Q$.

(b)
$$\begin{aligned} \neg P \vee Q &\Longleftrightarrow (\neg P \wedge \mathbf{T}) \vee (\mathbf{T} \wedge Q) \\ &\Longleftrightarrow [\neg P \wedge (Q \vee \neg Q)] \vee [(P \vee \neg P) \wedge Q] \\ &\Longleftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \vee (\neg P \wedge Q) \\ &\Longleftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \end{aligned}$$

is p.d.n.f of $\neg P \vee Q$.

Observation 13.3.1 As we know that p.d.n.f is unique for a given statement formula. In that case, if two given formulas are equivalent, then both of them must have identical p.d.n.f. Therefore, by the second method, it can be determined whether two given formulas are equivalent.

Example 13.3.4 Show the following

$$(a) \quad P \vee (P \wedge Q) \Longleftrightarrow P \quad (b) \quad P \vee (\neg P \wedge Q) \Longleftrightarrow P \vee Q$$

Solution: We can show by comparing the p.d.n.fs of two formulas.

(a) $P \vee (P \wedge Q) \Longleftrightarrow [P \wedge (Q \vee \neg Q)] \vee (P \wedge Q) \Longleftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$

$$P \Longleftrightarrow P \wedge (Q \vee \neg Q) \Longleftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$$

(b) $P \vee (\neg P \wedge Q) \Longleftrightarrow [P \wedge (Q \vee \neg Q)] \vee [\neg P \wedge Q]$

$$\Longleftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

$$P \vee Q \Longleftrightarrow [P \wedge (Q \vee \neg Q)] \vee (Q \wedge (P \vee \neg P))$$

$$\Longleftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

Observation 13.3.2 If a formula is a tautology, then clearly all the minterms in its p.d.n.f. Therefore it is also possible to determine whether a givenformula is a tautology by determining its p.d.n.f.

Example 13.3.5 Find the p.d.n.f of $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

Solution:

$$\Longleftrightarrow \neg P \vee ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$$

$$\Longleftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P))$$

$$\Longleftrightarrow \neg P \vee ((\neg P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P))$$

$$\Longleftrightarrow \neg P \vee \boldsymbol{F} \vee (P \wedge Q)$$

$$\Longleftrightarrow \neg P \vee (P \wedge Q)$$

$$\Longleftrightarrow [\neg P \vee (Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\Longleftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$

Hence $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ is the required p.d.n.f

Example 13.3.6 Find the minterm normal form of $[\neg((P \vee Q) \wedge R)] \wedge [(P \wedge R)]$

Solution:

$$\Longleftrightarrow [\neg(P \vee Q) \vee \neg R] \wedge (P \wedge R)$$

$$\Longleftrightarrow ((\neg P \wedge \neg Q) \vee \neg R) \wedge (P \wedge R)$$

$$\Longleftrightarrow (\neg P \wedge \neg Q \wedge P) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P) \vee (\neg R \wedge R)$$

$$\Longleftrightarrow \mathbf{F} \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P) \vee \mathbf{F}$$

$$\Longleftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P)$$

$$\Longleftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)$$

$$(\because \neg R \wedge P \Longleftrightarrow (\neg R \wedge P \wedge Q) \vee (\neg R \wedge P \wedge \neg Q))$$

Hence $(\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)$ is the minterm normal form of the given formula.

Exercise 7:

(I) obtain the p.d.n.f of the following formulas

(a) $P \vee (\neg P \wedge \neg Q \wedge R)$

(b) $(Q \wedge \neg R \wedge \neg S) \vee (R \wedge S)$

Answers 7

(*a*) $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee$
$\vee (\neg P \wedge \neg Q \wedge \neg R)$.

(*b*) $(Q \wedge \neg R \wedge \neg S) \vee (Q \wedge R \wedge S) \vee (\neg Q \wedge R \wedge S)$.

13.4 Principal Conjunctive Normal Forms

Definition 13.4.1 A *maxterm* consists of disjunctions in which each variable or its negation, but not both, appears only once.

For example, For two variables P and Q, there are 2^2 maxterms given by

$$P \vee Q,\ P \vee \neg Q,\ \neg P \vee Q \text{ and } \neg P \vee \neg Q$$

Maxterms for the three variables P, Q and R are

$P \vee Q \vee R$	$P \vee Q \vee \neg R$	$P \vee \neg Q \vee R$	$P \vee \neg Q \vee \neg R$
$\neg P \vee Q \vee R$	$\neg P \vee Q \vee \neg R$	$\neg P \vee \neg Q \vee R$	$\neg P \vee \neg Q \vee \neg R$

clearly the maxterms are the duals of minterms.

Either from the duality principle or directly from the truth tables it can be concluded that each of the maxterms has the truth value F for exactly one combination of the truth values of the variables. Different maxterms have the truth value F for different combinations of the truth values of the variables.

Definition 13.4.2 An equivalent formula consisting of conjunctions of maxterms only is known as principal conjunctive normal form (product-of-sums cronical form), simply p.c.n.f.

- Every formula (which is not a tautology) has an equivalent p.c.n.f which is unique except for the rearrangement of the factors in the maxterms and conjunctions.
- By duality principle, All the assertions made for the pdnfs can also be made for the pcnfs.

Methods to obtain p.c.n.f of a given formula

(I) The method for obtaining the p.c.n.f for a given **formula is similar** to the one studied previously for the principal disjunctive **normal form (p.d.n.f)**

(II) If the PD(C)NF of a given formula A containing n **variables** is known, then the PD(C)NF of $\neg A$ will consist of the disjunction (conjunction) of the remaining minterms (maxterms) which do not appear in the PD(C)NF of A.
From $A \iff \neg\neg A$ one can obtain the PC(D)NF of A by repeated applications of Demorgan's laws to the PD(C)NF of $\neg A$.

Example 13.4.1 obtain the PCNF of the formula $(\neg P \rightarrow R) \wedge (Q \rightleftarrows P)$

Solution:

$\iff [\neg(\neg P) \vee R] \wedge [(Q \rightarrow P) \wedge (P \rightarrow Q)]$

$\iff (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$

$\iff [(P \vee R) \vee (Q \wedge \neg Q) \wedge [(P \vee \neg Q) \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \wedge (R \wedge \neg R)]$

$\iff (P \vee R \vee Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge$
$(\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

$\iff (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$
$\wedge (\neg P \vee Q \vee \neg R)$

which is the p.c.n.f. of $(\neg P \rightarrow R) \wedge (Q \rightleftarrows P)$.

Example 13.4.2 Given the PDNF of A: $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$. Find PCNF

Solution: PDNF of A: $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$
Now PDNF of $\neg A$ is the disjunction of the remaining minterms
i.e., PDNF of $\neg A$: $(P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$.
Therefore

$\neg A \iff (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

Now

$\neg\neg A \iff \neg[(P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)]$
$\iff (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee R)$

is the p.c.n.f of A.

Example 13.4.3 Find PDNF and PCNF for A : $(P \wedge Q) \vee (\neg P \wedge Q) \vee (Q \wedge R)$

Solution: From the problems of PDNF

PDNF of A : $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$

Now PDNF of $\neg A$: $(\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$.

Now

$$\begin{aligned} \neg\neg A &\iff \neg((\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)) \\ &\iff (P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \end{aligned}$$

is the p.c.n.f of the formula A.

Example 13.4.4 Find PDNF from PCNF of $S : P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R)))$

Solution:

$$\begin{aligned} S &\iff P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R))) \\ &\iff P \vee (\neg(\neg P) \vee (Q \vee (\neg(\neg Q) \vee R))) \\ &\iff P \vee (P \vee (Q \vee (Q \vee R))) \\ &\iff P \vee (P \vee Q \vee R) \\ &\iff P \vee Q \vee R \quad \text{is the p.c.n.f of S} \end{aligned}$$

Now PCNF of $\neg S$ is the conjunction of the remaining maxterms so

$$\begin{aligned} \text{PCNF of} \neg S : &(P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \\ &\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \\ &\wedge (\neg P \vee \neg Q \vee R) \end{aligned}$$

Hence the PDNF of S is $\neg$(PCNF of$\neg S$):

$$\begin{aligned} &(\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \\ &\vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R) \end{aligned}$$

Exercise 8 Show that the PCNF of $(\neg P \rightarrow R) \wedge (Q \rightleftarrows P)$ is $(P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$.

Observation 13.4.2 Any of the principal normal forms can be used to determine whether two given formulas A and B are equivalent. It is not necessary to assume that both formulas have the same variables. In fact, each formula can be assumed to depend upon all the variables that appear in both formulas, by introducing the missing variables and then reducing them to their principal normal forms.

14. Normal Forms: Order, Uniqueness

Normal forms: PCNF, PDNF are unique except for the rearrangements of the factors in the disjunctions / conjunctions as well as in each of the minterms / maxterms.

Now, we can get a unique normal form by imposing a certain order in which the variables appear in the minterms (maxterms) as well as a definite order in which minterms (maxterms) appear in the disjunction (conjunction).

(I) Let us assume that n variables are given and are arranged in a particular order. The 2^n minterms corresponding to the n variables can be designated by $m_0, m_1, \ldots m_{(2^n)-1}$.

Thus each of $m_0, m_1, \ldots, m_{(2^n)-1}$ corresponds to a unique minterm, which can be determined from the binary representation of its subscript (the number of digits in the subscript is exactly n). We can obtain minterm is the following manner:

If 1 appears in the i^{th} location from the left, then the i^{th} variable appears in the conjunction.

If 0 appears in the i^{th} location from the left, then the negation of the i^{th} variable appears in the conjunction forming the minterm.

Conversely, given any minterm, one can find which of $m_0, m_1, \ldots, m_{(2^n)-1}$ designates it.

Example 14.1 Let P, Q and R be three variables arranged in that order. The corresponding minterms are denoted by $m_0, m_1, \ldots, m_7 (2^3 - 1 = 7)$.

Consider m_5, the subscript 5 in binary as 101 and the minterm m_5 is $P \wedge \neg Q \wedge R$. Similarly m_0 corresponds to $\neg P \wedge \neg Q \wedge \neg R$. To obtain the minterm m_3, we write the subscript 3 in binary as 11 and append a zero on the left to get 011 and m_3 is $\neg P \wedge Q \wedge R$.

With the above notation for the representation of the minterms we designate the disjunction (Sum) of minterms by the compact notation Σ.

Using such a notation, the sum-of-products canonical form representing the disjunction of m_i, m_j and m_k can be written down as $\Sigma i, j, k$.

Example 14.2 The p.d.n.f of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ is $(\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$, From the previous notation: we denote the p.d.n.f as $\sum 1, 3, 6, 7$.

(II) The 2^n *maxterms* corresponding to the n statement variables can be designated by $M_0, M_1 \ldots, M_{(2^n)-1}$. Here also the maxterm corresponding to M_j is obtained by writing j in binary and appending the required number of zeros to the left in order to get n digits.
If 0 appears in the i^{th} location from the left of this binary number, then the i^{th} variable appears in the disjunction. If 1 appears in the i^{th} location, then the negation of the i^{th} variable appears.
Thus the binary representation of the subscript uniquely determines the maxterm. Conversely every binary representation of numbers between 0 and $2^n - 1$ determines a maxterm.
Note that the convention regarding 1 and 0 here is the opposite of what was used for minterms.

Example 14.3 The maxterms $M_0, M_1, \ldots, M_7$ corresponding to three variables P, Q and R are

$$P \vee Q \vee R \qquad P \vee Q \vee \neg R \qquad P \vee \neg Q \vee R \qquad P \vee \neg Q \vee \neg R$$
$$\neg P \vee Q \vee R \qquad \neg P \vee Q \vee \neg R \qquad \neg P \vee \neg Q \vee R \qquad \neg P \vee \neg Q \vee \neg R$$

with the above notation for the representation of the maxterms we designate the conjunction (product) of maxterms by the compact notation Π.
Thus $\Pi i, j, k$ represents the conjunction of maxterms M_i, M_j, M_k.

Example 14.4 The PCNF of $(P \wedge Q) \vee (\neg P \wedge R)$: $(\neg P \wedge Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R)$ it can be represented as $\prod 0, 2, 4, 5$.

15. Statement Calculus: Theory of Inference

Definition 15.1 The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with rules of inference is known as *inference theory*.

Definition 15.2 If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a *deduction* or a *formal proof*, and the arguement or conclusion is called a *valid arguement or valid conclusion*.

Note 15.1 Premises $\approx$ assumptions, axioms, hypotheses.

(I) The method to determine whether the conclusion logically follows from the given premises by constructing the relevant truth table is called "**truth table technique**"

Definition 15.3 Let A and B be two statement formulas we say that "B logically follows from A" or "B is a valid conclusion (consequence) of the premise A" iff $A \rightarrow B$ tautology i.e., $A \Rightarrow B$.

By extending the above definition, we say that from a set of premises $\{H_1, H_2, \ldots, H_m\}$ a conclusion C follows logically iff

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow C. \tag{1}$$

(I) By Truthtable

(i) Let $P_1, P_2, \ldots, P_n$ be all the atomic variables appearing in the premises $H_1, H_2, \ldots, H_m$ and the conclusion C. If all possible combinations of truth values are assigned to $P_1, P_2, \ldots, P_n$ and if the truth values of $H_1, H_2, \ldots, H_m$ and C are entered in a table, then it is easy to see from such a table whether (1) is true.

(ii) Look for the rows in which all $H_1, H_2, \ldots, H_m$ have the value T. If for every such row, C also has the value **T**, *then (1) holds.*

(iii) Atternatively, look for the rows in which C has the value F. If in every such row, atleast one of the values of $H_1, H_2, \ldots, H_m$ is **F**, *then (1) also holds.*

Example 15.1 Determine whether the conclusion C follows logically from the hypotheses H_1 and H_2.

(i) $H_1 : P \rightarrow Q \quad H_2 : P \quad C : Q$
(ii) $H_1 : P \rightarrow Q \quad H_2 : Q \quad C : P$
(iii) $H_1 : \neg P \quad H_2 : P \rightleftarrows Q \quad C : (P \wedge Q)$

P	Q	$P \rightarrow Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$P \rightleftarrows Q$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	F
F	F	T	T	T	T	T

For

(i) we observe that the first row is the only row in which both the premises have the value T. The conclusion also has the value T in that row. Hence it is valid.
(ii) The conclusion does not follow logically from the premises $P \rightarrow Q$ and Q.
(iii) Similarly, we can show that the conclusions are valid in (d).

Exercise 9 Determine whether the conclusion C is valid in the following the premises (By Truth table technique).

(a) $H_1 : P \rightarrow Q \quad H_2 : \neg P \quad C : Q$
(b) $H_1 : \neg Q \quad H_2 : P \rightarrow Q \quad C : \neg P$
(c) $H_1 : R \quad H_2 : P \vee \neg P \quad C : R$

Answers 9 (a) not valid (b) valid (c) valid

(II) Without Using Truth Table

The truth table technique becomes tedious when the number of atomic variables present in all the formulae representing the premises and the conclusion is large. To overcome this disadvantage, we need to investigate other possible methods, without using the truth table.

Now, we discuss the process of derivation by which one demonstrate that a particular formula is a valid consequence of a given set of premises. Before we go to actual process of derivation, we give three rules of inference, which are called Rule P, Rule T and Rule CP respectively. For the moment we consider only two of these rules. One permits us to introduce premises when needed and the other permits piecemeal use of tautological implications.

Before we proceed with the actual process of derivation, we list some important implications and equivalences that will be referred to frequently.

Implications

I_1	$P \wedge Q \Rightarrow P$	Simplification
I_2	$P \wedge Q \Rightarrow Q$	
I_3	$P \Rightarrow P \vee Q$	addition
I_4	$Q \Rightarrow P \vee Q$	
I_5	$\neg P \Rightarrow P \rightarrow Q$	
I_6	$Q \Rightarrow P \rightarrow Q$	
I_7	$\neg(P \rightarrow Q) \Rightarrow P$	
I_8	$\neg(P \rightarrow Q) \Rightarrow \neg Q$	
I_9	$P, Q \Rightarrow P \wedge Q$	
I_{10}	$\neg P, P \vee Q \Rightarrow Q$	(disjunctive syllogism)
I_{11}	$P, P \rightarrow Q \Rightarrow Q$	(modus ponens)
I_{12}	$\neg Q, P \rightarrow Q \Rightarrow \neg P$	(modus tollens)
I_{13}	$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$	hypothetical syllogism
I_{14}	$P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$	dilemma

Equivalences

E_1	$\neg\neg P \Leftrightarrow P$	double negation
E_2	$P \wedge Q \Leftrightarrow Q \wedge P$	Communtative laws
E_3	$P \vee Q \Leftrightarrow Q \vee P$	
E_4	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	Associative laws
E_5	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	
E_6	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	distributive laws
E_7	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	
E_8	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	Demorgan's laws
E_9	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	
E_{10}	$P \vee P \Leftrightarrow P$	
E_{11}	$P \wedge P \Leftrightarrow P$	
E_{12}	$R \vee (P \wedge \neg P) \Leftrightarrow R$	
E_{13}	$R \wedge (P \vee \neg P) \Leftrightarrow R$	
E_{14}	$R \vee (P \vee \neg P) \Leftrightarrow T$	
E_{15}	$R \wedge (P \wedge \neg P) \Leftrightarrow F$	
E_{16}	$P \rightarrow Q \Leftrightarrow \neg P \vee Q$	
E_{17}	$\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$	
E_{18}	$P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$	
E_{19}	$P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$	
E_{20}	$\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$	
E_{21}	$P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$	
E_{22}	$(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$	

We now give the first two rules

Rule P:	We may introduce a premise at any point in the derivation.
Rule T:	We may introduce a formula S in a derivation if S is tautologically implied by any one or more of the preceding formulae in the derivation.

We now consider an example to show how these rules of inference are used. It is better to indicate the reason for each step of the derivation.

Example 15.2 Demonstrate that S is a valid inference from the premises: $P \longrightarrow \neg Q$, $Q \vee R$, $\neg S \longrightarrow P$ and $\neg R$.

Solution:

[1]	(1)	$Q \vee R$	Rule P
[2]	(2)	$\neg R$	Rule P
[1, 2]	(3)	Q	Rule T, (1), (2) and I_{10}
[4]	(4)	$P \rightarrow \neg Q$	Rule P
[1, 2, 4]	(5)	$\neg P$	Rule, (3), (4) and I_{12}
[6]	(6)	$\neg S \rightarrow P$	Rule P
[1, 2, 4, 6]	(7)	S	Rule T, (5), (6) and I_{12}

Hence S is a valid inference.

There are seven lines in this derivation. The second column of numbers designate the formula and the line of derivation in which it occurs. The introduction of each line is justified by one of two rules Rule P and Rule T. The lines (1), (2), (4) and (6) are just premises of the argument. The other three lines are obtained by showing that they are tautological implications of preceding lines.

For example, in the case of line (5), it is easy to see that the formula in that line ie., $\neg P$ is tautologically implied by the conjunction of the formula in lines (3) and (4). That is $(P \rightarrow \neg Q) \wedge Q \Rightarrow \neg P$. In case of line (7), the conjunction of (5) and (6) tautologically implies S.

All these reasons for each step are indicated in the last column of the derivation. The set of numbers in braces (the first column) in each line shows the premises on which the formula in that depends on the other hand, the numerals in the last column simply indicate the lines from which the statement in the 3rd column is inferred.

The argument is usually given in a condensed form. The letter 'P' and 'T' are used for Rule P (premise) and Rule T (tautology) respectively. Some important implications and equivalences are listed, which will be referred frequently. The tautology $(Q \vee R) \wedge (\neg R) \Rightarrow Q$ can be indicated by either I_{10} or disjunctive syllogism.

Example 15.3 Show that R$\vee$S follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow (R \vee S)$.

Solution:

{1}	(1)	$(C \vee D) \rightarrow \neg H$	Rule P
{2}	(2)	$\neg H \rightarrow (A \wedge \neg B)$	Rule P
{1, 2}	(3)	$(C \vee D) \rightarrow (A \wedge \neg B)$	Rule T, (1), (2) and I_{13}

{4}	(4) $(A \wedge \neg B) \to (R \vee S)$	Rule P
{1, 2, 4}	(5) $(C \vee D) \to (R \vee S)$	Rule T, (3), (4) and I_{13}
{6}	(6) $C \vee D$	Rule P
{1, 2, 4, 6}	(7) $R \vee S$	Rule T, (5), (6) and I_{11}

Note 15.2 I_{13}: hypothetical syllogism. I_{11} : modus ponens.

Example 15.4 Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

Solution:

{1}	(1)	$P \to M$	Rule P
{2}	(2)	$\neg M$	Rule P
{1, 2}	(3)	$\neg P$	Rule T, (1), (2) and I_{12}
{4}	(4)	$P \vee Q$	Rule P
{1, 2, 4}	(5)	Q	Rule T, (3), (4) and I_{10}
{6}	(6)	$Q \to R$	Rule P
{1, 2, 4, 6}	(7)	R	Rule T, (5), (6) and I_{11}
{1, 2, 4, 6}	(8)	$R \wedge (P \vee Q)$	Rule T, (4), (7) and I_9

Example 15.5 Show that

$$\{(P \to Q) \wedge (R \to S), (Q \to T) \wedge (S \to U), \neg(T \wedge U), P \to R\} \Rightarrow \neg P$$

Solution:

[1]	(1)	$(P \to Q) \wedge (R \to S)$	Rule P
[1]	(2)	$P \to Q$	Rule T, (1)
[1]	(3)	$R \to S$	Rule T, (1)
[4]	(4)	$(Q \to T) \wedge (S \to U)$	Rule P
[4]	(5)	$Q \to T$	Rule T, (4)
[4]	(6)	$S \to U$	Rule T, (4)
[1, 4]	(7)	$P \to T$	Rule T, (2), (5), hypothetical syllogism
[1, 4]	(8)	$\neg T \to \neg P$	Rule T, (7) and E_{16}
[1, 4]	(9)	$R \to U$	Rule T, (3), (6), hypothetical syllogism
[10]	(10)	$P \to R$	Rule P
[1, 10]	(11)	$P \to U$	Rule T, (10), (3) hypothetical syllogism
[1, 10]	(12)	$\neg U \to \neg P$	Rule T, (11) and E_{18}
[1, 4, 10]	(13)	$(\neg T \vee \neg U) \to \neg P$	Rule T, (8), (12) and I_{14}
[14, 10]	(14)	$\neg(T \wedge U) \to \neg P$	Rule T, (13) Demorgan's law
[15]	(15)	$\neg(T \wedge U)$	Rule P
[1, 4, 10, 15]	(16)	$\neg P$	Rule T (14), (15), modus ponens.

We shall now introduce the third and last rule of inference is the rule of conditional proof, which we call *Rule CP*.

The general idea of this rule is that we may introduce a new premise R conditionally and use it in conjunction with the original premises to derive a conclusion S, and then assert that the implication $R \to S$ follows from the original premises alone. If S is a valid inference from premises $P_1, P_2, \ldots, P_n$ and R, then $R \to S$ is a valid inference from premises $P_1, P_2, \ldots, P_n$.

> **Rule CP**: If we can derive S from R and *a* set of premises then we can derive $R \to S$ from the set of premises alone.

Simply, $(P \wedge R) \to S \Leftrightarrow P \to (R \to S)$

where P denote the conjunction of the set of premises say $P_1, P_2, \ldots, P_n$.

The above equivalence states that if R is included as an additional premise and S is derived from $P \wedge R$, then $R \to S$ can be derived from the premises P alone.

Rule CP is also called the deduction theorem.

Example 15.6 Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$ and Q

Solution: It is enough to include R as an additional premise and derive S.

[1]	(1)	$\neg R \vee P$	Rule P
[2]	(2)	R	Rule P (additional premise)
[1, 2]	(3)	P	Rule T, (1), (2) and I_{10}
[4]	(4)	$P \to (Q \to S)$	Rule P
[1, 2, 4]	(5)	$Q \to S$	Rule T, (3), (4) and I_{11}
[6]	(6)	Q	Rule P
[1, 2, 4, 6]	(7)	S	Rule T, (5), (6) and I_{11}
[1, 4, 6]	(8)	$R \to S$	Rule CP

Example 15.7 Show that $P \to S$ can be derived from the premises $\neg P \vee Q$, $\neg Q \vee R$, $R \to S$.

Solution: we include P as an additional premise and derive S.

[1]	(1)	$\neg P \vee Q$	Rule P
[2]	(2)	P	Rule P (additional premise)
[1, 2]	(3)	Q	Rule T, (1), (2) and I_{10}
[4]	(4)	$\neg Q \vee R$	Rule P

[1, 2, 4]	(5)	R	Rule T, (3), (4) and I_{10}
[6]	(6)	$R \to S$	Rule P
[1, 2, 4, 6]	(7)	S	Rule T, (5), (6) and I_{11}
[1, 4, 6]	(8)	$P \to S$	Rule CP

Example 15.8 Derive $P \to (Q \to S)$ using the rule CP if necessary from $P \to (Q \to R)$, $Q \to (R \to S)$

Solution:

[1]	(1)	$P \to (Q \to R)$	Rule P
[2]	(2)	P	Rule P (additional premise)
[1, 2]	(3)	$Q \to R$	Rule T, (1), (2) and I_{11}
[4]	(4)	$Q \to (R \to S)$	Rule P
[1, 2]	(5)	$\neg Q \vee R$	Rule T, (3), E_{16}
[4]	(6)	$\neg Q \vee (R \to S)$	Rule T, (4), E_{16}
[1, 2, 4]	(7)	$\neg Q \vee (R \vee (R \to S))$	Rule T, (5), (6) distributive law
[1, 2, 4]	(8)	$\neg Q \vee S$	Rule T, (7), I_{11}
[1, 2, 4]	(9)	$Q \to S$	Rule T, (8), E_{16}
[1, 4]	(10)	$P \to (Q \to S)$	Rule CP

Definition 15.4 A set of formulas $H_1, H_2, \ldots, H_m$ is said to be *consistent* if their conjunction has the truth value **T** for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, \ldots, H_m$.

If for every assignment of the truth values to the atomic variables, atleast one of the formulas $H_1, H_2, \ldots, H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, \ldots, H_m$ are called *inconsistent*.

In other words, a set of formulas $H_1, H_2, \ldots, H_m$ is inconsistent if their conjunction implies a contradiction, that is

$$H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow R \wedge \neg R$$

where R is any formula.

Note 15.3 $R \wedge \neg R$ is a contradiction, and it is necessary and sufficient for the implication that $H_1 \wedge H_2 \wedge \ldots \wedge H_m$ be a contradiction.

16. Indirect Method of Proof

The method of using the rule of conditional proof and the notion of an inconsistent set of premises is called the indirect *method of proof* or *proof by contradiction* or *reduction ad absurdum*.

The technique of indirect method of proof is as follows:

i) Introduce the negation of the desired conclusion as a new premise.
   That is, assume the conclusion C is false and consider $\neg C$ as an additional premise (or new premise)
ii) From the additional or new premise, together with the given premises, derive a contradiction.
   That is, if the new set of premises is inconsistent, then they imply a contradiction. Therefore C is true whenever, $H_1 \wedge H_2 \wedge \ldots \wedge H_m$ is true.
iii) Assert the desired conclusion as a logical inference from the premises.
   Thus C follows logically from the premises $H_1, H_2, \ldots, H_m$.

Example 16.1 Using indirect method of proof, derive $P \rightarrow \neg S$ from $P \rightarrow Q \vee R$, $Q \rightarrow \neg P$, $S \rightarrow \neg R$, P.

Solution: The desired result is $P \rightarrow \neg S$. Its negation is $P \wedge S$. Since $P \wedge S \rightleftarrows \neg(\neg P \vee \neg S) \rightleftarrows \neg(P \rightarrow \neg S)$ is a tautology, from the law of negation for implication. We include $P \wedge S$ as an additional premise

[1]	(1)	$P \rightarrow (Q \vee R)$	Rule P
[2]	(2)	P	Rule P
[1, 2]	(3)	$Q \vee R$	Rule T, (1), (2), modus ponens (I_{11})
[4]	(4)	$S \rightarrow \neg R$	Rule P
[5]	(5)	$P \wedge S$	Rule P (new premise)
[5]	(6)	S	Rule T, (5)
[4, 5]	(7)	$\neg R$	Rule T, (4), (6) modus ponens (I_{11})
[1, 2, 4, 5]	(8)	Q	Rule T, (3), (7), I_{10}
[9]	(9)	$Q \rightarrow \neg P$	Rule P
[1, 2, 4, 5, 9]	(10)	$\neg P$	Rule T, (8), (9), modus ponens
[1, 2, 4, 5, 9]	(11)	$P \wedge \neg P$	Rule T, (2), (10), contradiction

Thus additional premise $P \wedge S$ and the given premises together lead to a contradiction. So $\neg(P \wedge S)$ is derivable from $P \rightarrow Q, \vee R$, $Q \rightarrow \neg P$, $S \rightarrow \neg R$, P.

Example 16.2 Prove by indirect method that

$$(\neg Q),\ P \rightarrow Q,\ P \vee R \Rightarrow R$$

Solution: The desired result is R. Include its negation as a new premise.

[1]	(1)	$P \vee R$	Rule P
[2]	(2)	$\neg R$	Rule P (additional premise)

[1, 2]	(3)	P	Rule T, (1), (2)
[4]	(4)	$P \to Q$	Rule P
[1, 2, 4]	(5)	Q	Rule T, (3), (4), modus ponens
[6]	(6)	$\neg Q$	Rule P
[1, 2, 4, 6]	(7)	$Q \wedge \neg Q$	Rule T, (5), (6), Contradiction

The new premise, together with the given premises, leads to a contradiction. Thus

$$(\neg Q), P \to Q, \ P \vee R \Rightarrow R.$$

Example 16.3 By indirect proof, show that

$$P \to Q, \ Q \to R, \ \neg(P \wedge R), \ P \vee R \Rightarrow R$$

Solution: The desired result is R. Include $\neg R$ as a new premise.

[1]	(1)	$Q \to R$	Rule P
[2]	(2)	$\neg R$	Rule P (additional premise)
[1, 2]	(3)	$\neg Q$	Rule T, (1), (2)
[4]	(4)	$P \to Q$	Rule P
[1, 2, 4]	(5)	$\neg P$	Rule T, (3), (4), modus ponens
[6]	(6)	$P \vee R$	Rule P
[1, 2, 4, 6]	(7)	R	Rule T, (5), (6)
[1, 2, 4, 6]	(8)	$R \wedge \neg R$	Rule T, (2), (7), Contradiction

Thus we get a contradiction. Therefore we get

$$P \to Q, \ Q \to R, \ P \vee R, \ R.$$

But, the other premise $\neg(P \wedge R)$ will not yield a contradiction with R.

We shall now give some examples of derivation involving statements in English. In everyday life, we come across arguement expressed in (English) sentences. We can represent the premises in symbols and verity the validity as in earlier examples.

Example 16.4 Determine the validity of the following arguement: If 7 is a prime number, then 7 does not divide 35, 7 divides 35, implies 7 is not a prime number.

Solution: Let us indicate the statements as follows:

P : 7 is a prime number

Q : 7 divides 35

The given argument is of the form: $\{P \to \neg Q, Q\} \Longrightarrow \neg P$. Construct the truth table

P	Q	$\neg Q$	$P \to \neg Q$	Q	$\neg P$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	T	T
F	F	T	T	F	T

$P \to \neg Q$ and Q are both true only in the third row and in that row $\neg P$ is also true. Hence $P \to \neg Q, Q \Longrightarrow \neg P$. Hence the arguement is true.

Note 16.1 The above arguement is valid by I_{12} i.e., modustollens.

Example 16.5 Determine the validity of the following arguement. If two sides of a triangle are equal, then two opposite angles are equal.

Two sides of a triangle are not equal

Therefore, the opposite angles are not equal.

Solution: Let us indicate the statements as follows:

P : Two sides of a triangle are equal

Q : The two opposite angle are equal

The given argument is of the form

$$P \to Q, \neg P \Longrightarrow \neg Q$$

Let us now construct the truth table for $[(P \to Q) \wedge (\neg P)] \to \neg Q$.

P	Q	$P \to Q$	$\neg P$	$(P \to Q) \wedge (\neg P)$	$\neg Q$	$((P \to Q) \wedge (\neg P)) \to \neg Q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

This shows that $(P \to Q) \wedge (\neg P) \to \neg Q$ is not a tautology. Hence the conclusion $\neg Q$ is not valid.

Example 16.6 Determine the validity of the following arguement. My father praises me only if I can be proud of myself Either I do well insports or I can't be pround of myself. If I study hard, then I can't do well in sports. Therefore, if father praises me, then I do not study well.

Solution: Let us indicate the statements as follows:

P : My father praises me

Q : I can be purd of myself

R : I do well in sports

S : I study hard

The given argument is of the form

$$P \to Q, R \vee \neg Q, S \to \neg R \Longrightarrow P \to \neg S$$

As the desired result is $P \to \neg S$, assume that P is true.

It is enough to verify the validity of

$$P, P \to Q, R \vee \neg Q, S \to \neg R \Longrightarrow \neg S.$$

From P and $P \to Q$, we have Q.

From Q and $R \vee \neg Q$, we have R.

From R and $S \to \neg R$, we have $\neg S$.

Thus the arguement is valid.

Example 16.7 Show that the following set of premises is in consistent.

If the contract is valid, then john is liable for penalty. If john is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid and the bank will loan him money.

Solution: We indicate the given statements as follows:

V : The contract is valid

L : John is liable for penalty.

M : Bank will loan him money

B : He will go bankrupt.

Then the given premises are

$$V \to L, L \to B, M \to \neg B, V \wedge M$$

[1]	(1)	$V \to L$	Rule P
[2]	(2)	$L \to B$	Rule P
[1, 2]	(3)	$V \to B$	Rule T, (1), (2) law of hypo. syllogism
[4]	(4)	$V \wedge M$	Rule P
[4]	(5)	V	Rule T, (4)
[4]	(6)	M	Rule T, (4)
[1, 2, 4]	(7)	B	Rule T, (3), (5), modus ponens
[8]	(8)	$M \to \neg B$	Rule P
[4, 8]	(9)	7B	Rule T, (6), (8)
[1, 2, 4, 8]	(10)	$B \wedge \neg B$	Rule T, (7), (9), contradiction

Thus the given set of premises leads to a contradiction and hence it is inconsistent.

Note 16.2 on Notation: Let A, B and C be three statement formulas.

The form $\begin{array}{l} A \\ B \\ \hline \therefore C \end{array}$ is equivalent to $A \wedge B \rightarrow C$ is a tautology.

The statement formulas above the horizontal line are called premises (or hypotheses). The statement formula below the line is called the conclusion.

The most common rules of inference

1. Modus ponens: $\begin{array}{c} P \\ P \rightarrow Q \\ \hline Q \end{array}$

2. Modus tollens: $\begin{array}{c} P \rightarrow Q \\ 7Q \\ \hline 7P \end{array}$

3. Disjunctive syllogism: $\begin{array}{c} P \vee Q \\ 7P \\ \hline Q \end{array}$

4. Chain rule: $\begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ \hline P \rightarrow R \end{array}$

5. Resolution: $\begin{array}{c} P \vee R \\ Q \vee (7R) \\ \hline P \vee Q \end{array}$

Example 16.8 Show that the following argument is valid

$$\begin{array}{c} (P \vee Q) \rightarrow (S \wedge T) \\ [7((7S) \vee (7T))] \rightarrow [(7R) \vee Q] \\ \hline (P \vee Q) \rightarrow (R \rightarrow Q) \end{array}$$

Solution: Clearly

$$7((7S) \vee (7T)) \Longleftrightarrow (S \wedge T)$$
$$(7R) \vee Q \Longleftrightarrow R \rightarrow Q$$

Thus the given argument can be rewritten as

$$\begin{array}{r} (P \vee Q) \rightarrow (S \wedge T) \\ (S \wedge T) \rightarrow (R \rightarrow Q) \\ \hline (P \vee Q) \rightarrow (R \rightarrow Q) \end{array}$$

by The chain rule, the argument is valid.

Exercise 10:

(I) Show by direct proof

(i) $R \to (S \to Q), \neg P \vee R$ and $S \Longrightarrow P \to Q$

(ii) $P, P \to (Q \to (R \wedge S)) \Longrightarrow Q \to S$

(iii) If $P \to Q, R \to \neg Q, R$ then $\neg Q$.

(II) Show by indirect proof

(i) $P \to Q, R \to S, P \vee R \Longrightarrow Q \vee S$

(ii) $E \to S, S \to H, A \to \neg H \Longrightarrow \neg(E \wedge A)$

(iii) If $P \to (Q \wedge R), (Q \vee S) \to T$ and $(P \vee S)$ then T.

(III) Determine whether each of the following inference patterns is valid or invalid

(i) If I like Applied mathematics, then I will study Either I study or I fail. If I fail, then I do not like applied mathematics

(ii) If today is Sunday, then yesterday was saturday yesterday was saturday Today is Sunday

Answers 10

III (i) Valid (ii) not valid.

---

17. Automatic Theorem Proving

The method of derivation just discussed provides only a partial solution to the decisior problem, because if an arguement is valid, then it is possible to show by this methoc that the arguement is valid. On the other hand, if an arguement is not valid, then it i: very difficult to decide, after a finite number of steps, that this is the case. Also the formulation or process of derivation given earlier could not be used for this purposc because the construction of derivation depends upon the skill, experience and ingenuity of the person to make the right decision at every step. Let us first examine the limitation: of the earlier procedure of derivation:

Rule P permits the introduction of a premise at any point in the derivation, but doe. not suggest either the premise or the step at which it should be introduced.

Rule T allows us to introduce any formula which follows from the previous steps However, there is no definite choice of such a formula nor is there any guidance for the use of any particular equivalence.

Similarly, Rule CP does not tell anything about the stages at which an antecedent is to be introduced as an assumed premise, nor does it indicate the stage at which it is agaii incorporated into the conditional.

At every step, such decisions are taken from a large number of alternatives, with th ultimate aim of reaching the conclusion. Such a procedure is far from mechanical.

Now we describe a set of rules and a procedure which allow one to construct each step of derivation in a specified manner without recourse to any ingenuity and finally to arrive at a last step which clearly indicates whether a given conclusion follows from the premises. Our procedure is mechanical, also it is a full decision process for validity.

Our system of derivation consisting of 10 rules an axiom schema, and rules of well formed sequents and formulas:

1. Variables: The capital letters $A, B, C \ldots P, Q, R \ldots$ are used as statement variables and statement formulas.
2. Connectives: The connectives $\neg, \wedge, \vee, \rightarrow$ and $\rightleftarrows$ appear in the formulas with the order of precedence as given.
3. String of formulas: A string of formula is defined as follows:
   (a) Any formula is a string of formulas
   (b) If α and β are strings of formulas, then α, β and β, α are strings of formulas
   (c) Only those strings which are obtained by steps (a) and (b) are strings of formulas, with the exception of the empty string which is also a string of formulas
   Note: The order in which the formulas appear in any string is not important and so the strings A, B, C; B, C, A; A, C, B; etc., are the same.
4. Sequents: If α and β are strings of formulas, then $\alpha \xrightarrow{s} \beta$ is called a sequent in which α is denoted the antecedent and β the consequent of the sequent.

   Thus $A, B, C \xrightarrow{s} D, E, F$ is true iff $A \wedge B \wedge C \rightarrow D \vee E \vee F$ is true. i.e., A sequent $\alpha \xrightarrow{s} \beta$ is true iff either at least one of the formulas of the antecedent is false or at least one of the formulas of the consequent is true.

   In this sense, the symbol "$\xrightarrow{s}$" is a generalization of the connection $\rightarrow$ to strings of formulas.

   Similarly, we use the symbol "$\xRightarrow{s}$" applied to strings of formulas as a generalization of the symbol "$\Longrightarrow$". Thus $A \Longrightarrow B$ means "A implies B" or $A \rightarrow B$ is a tautology while $\alpha \xRightarrow{s} \beta$ means that $\alpha \xrightarrow{s} \beta$ is true. Ex: $P, Q, R \xRightarrow{s} P, N$

   The empty antecedent is interpreted as the logical constant "true" (T) and the empty consequent is interpreted as the logical constant "false" (F).
5. *Axiom Schema*: If α and β are strings of formulas such that every formula in both α and β is a variable only, then the sequent $\alpha \xrightarrow{s} \beta$ is an axiom iff α and β have at least one variable in common.

   Ex: $A, B, C \xrightarrow{s} P, B, R$ is an axiom, where $A, B, C, P \& R$ are variables

   Note that if $\alpha \xrightarrow{s} \beta$ is an axiom, then $\alpha \xRightarrow{s} \beta$
6. *Theorem*: The following sequents are theorems of our system

(a) Every axiom is a theorem.

(b) If a sequent α is a theorem and a sequent β results from α through the use of one of the 10 rules of the system which are given below, then β is a theorem

(c) Sequents obtained by (a) and (b) are the only theorems.

7. Rules: The following rules are used to combine formulas within strings by introducing connectives. Corresponding to each of the connectives there are two rules, one for the introduction of the connective in the antecedent and the other for its introduction in the consequent. In the description of these rules, $\alpha, \beta, \gamma, \ldots$ are strings of formulas while X and Y are formulas to which the connectives are applied.

Antecedent Rules

Rule $\neg \Longrightarrow$:If $\alpha, \beta \overset{s}{\Longrightarrow} X, \gamma$, then $\alpha, \neg X, \beta \overset{s}{\Longrightarrow} \gamma$.

Rule $\wedge \Longrightarrow$:If $X, Y, \alpha, \beta \overset{s}{\Longrightarrow} \gamma$, then $\alpha, X \wedge Y, \beta \overset{s}{\Longrightarrow} \gamma$.

Rule $\vee \Longrightarrow$:If $X, \alpha, \beta \overset{s}{\Longrightarrow} \gamma$,and $Y, \alpha, \beta \overset{s}{\Longrightarrow} \gamma$, then $\alpha, X \vee Y, \beta \overset{s}{\Longrightarrow} \gamma$.

Rule $\rightarrow \Longrightarrow$:If $Y, \alpha, \beta \overset{s}{\Longrightarrow} \gamma$,and $\alpha, \beta \overset{s}{\Longrightarrow} X, \gamma$, then $\alpha, X \rightarrow Y, \beta \overset{s}{\Longrightarrow} \gamma$.

Rule $\rightleftarrows \Longrightarrow$:If $X, Y, \alpha, \beta \overset{s}{\Longrightarrow} \gamma$,and $\alpha, \beta \overset{s}{\Longrightarrow} X, Y, \gamma$, then $\alpha, X \rightleftarrows Y, \beta \overset{s}{\Longrightarrow} \gamma$.

Consequent Rules

Rule $\Longrightarrow \neg$:If $X, \alpha, \overset{s}{\Longrightarrow} \beta, \gamma$, then $\alpha \overset{s}{\Longrightarrow} \beta, \neg X, \gamma$.

Rule $\Longrightarrow \wedge$:If $\alpha, \overset{s}{\Longrightarrow} X, \beta, \gamma$, and $\alpha \overset{s}{\Longrightarrow} Y, \beta, \gamma$, then $\alpha \overset{s}{\Longrightarrow} \beta, X \wedge Y, \gamma$.

Rule $\Longrightarrow \vee$:If $\alpha, \overset{s}{\Longrightarrow} X, Y, \beta, \gamma$, then $\alpha \overset{s}{\Longrightarrow} \beta, X \vee Y, \gamma$.

Rule $\Longrightarrow \rightarrow$:If $X, \alpha, \overset{s}{\Longrightarrow} Y, \beta, \gamma$, then $\alpha, \overset{s}{\Longrightarrow} \beta, X \rightarrow Y, \gamma$.

Rule $\Longrightarrow \rightleftarrows$:If $X, \alpha, \overset{s}{\Longrightarrow} Y, \beta, \gamma$, and $Y, \alpha, \overset{s}{\Longrightarrow} X, \beta, \gamma$, then $\alpha, \overset{s}{\Longrightarrow} \beta, X \rightleftarrows Y$,

The system described here is equivalent to the one described earlier except that the procedures and techniques of derivation are different. This difference does not affect the validity of an arguement.

The description of our system is complete.

Now, we describe the procedure used in practice:

In order to show that the conclusion C follows from $H_1, H_2 \cdots H_m$, we establish that

$$\overset{s}{\rightarrow} H_1 \rightarrow (H_2 \rightarrow (H_3 \cdots (H_m \rightarrow C) \cdots)) \qquad \text{(I)}$$

is a theorem. We must show that

$$\overset{s}{\Longrightarrow} H_1 \rightarrow (H_2 \rightarrow (H_3 \cdots (H_m \rightarrow C) \cdots)) \qquad \text{(II)}$$

Our procedure involves showing (I) to be a theorem.

For this, we first assume (II) and then show that this assumption is or is not justified.

This is accomplished by working backward from (II), using the rules and showing that (II) holds if some simpler sequent is a theorem (By a simpler sequent we mean a sequent in which some connective is eliminated in one of the formulas appearing in the antecedent or the consequent). We continue working backward until we arrive at the simplest possible sequents i.e., those which do not have any connectives. If these sequente are axioms, then we have justified our assumption of (II). If at least one of the simplest sequents is not an axiom, then the assumption of (II) is not justified and C does not follow from $H_1, H_2, \cdots, H_m$. In case when C follows from $H_1, H_2, \cdots, H_m$ the derivation of (II) is easily constructed by simply working through the same steps, starting from the axioms obtained.

Example 17.1 Prove that $P \vee Q$ follows from P.

Solution: We need to show that

(1) $\overset{s}{\Longrightarrow} P \rightarrow (P \vee Q)$

(1) if, (2) $P \overset{s}{\Longrightarrow} P \vee Q$ $(\Longrightarrow\rightarrow)$

(2) if, (3) $P \overset{s}{\Longrightarrow} P, Q$ $(\Longrightarrow \vee)$

We first eliminate the connective $\rightarrow$ in (1).

Using the rule $\Longrightarrow\rightarrow$ we have "if $P \overset{s}{\Longrightarrow} P \vee Q$ then $\overset{s}{\Longrightarrow} P \rightarrow (P \vee Q)$" Here we have named $P \overset{s}{\Longrightarrow} P \vee Q$ by (2). Each line of derivation thus introduces the name as well as gives a rule. Note also that "(1) if (2)" means "if (2) then (1)". The chain of arguements is then given by (1) holds if (2) and (2) holds if (3). Finally (3) is a theorem because it is an axiom. The actual derivation is simply a reversal of those steps in which (3) is an axiom that leads to $\overset{s}{\Longrightarrow} P \rightarrow (P \vee Q)$ as shown

(a) $P \overset{s}{\Longrightarrow} P, Q$ axiom

(b) $P \overset{s}{\Longrightarrow} P \vee Q$ Rule $(\Longrightarrow \vee)$, (a)

(c) $\overset{s}{\Longrightarrow} P \rightarrow (P \vee Q)$ Rule $(\Longrightarrow\rightarrow)$, (b).

□

Example 17.2 Show that $\overset{s}{\Longrightarrow}(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$.

Solution:

(1) $\overset{s}{\Longrightarrow}(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$

(1) if (2) $\neg Q \wedge (P \rightarrow Q) \overset{s}{\Longrightarrow} \neg P$ $(\Longrightarrow\rightarrow)$

(2) if (3) $\neg Q, P \rightarrow Q \overset{s}{\Longrightarrow} \neg P$ $(\wedge \Longrightarrow)$

(3) if (4) $P \rightarrow Q \overset{s}{\Longrightarrow} \neg P, Q$ $(\neg \Longrightarrow)$

(4) if (5) $Q \overset{s}{\Longrightarrow} \neg P, Q$ and (6) $\overset{s}{\Longrightarrow} P, \neg P, Q$ $(\rightarrow\Longrightarrow)$

(5) if (7) $P, Q \overset{s}{\Longrightarrow} Q$ $(\Longrightarrow \neg)$

(6) if (8) $P \overset{s}{\Longrightarrow} P, Q$ $(\Longrightarrow \neg)$

Now (7) and (8) are axioms, hence the theorem (1) follows. Derivation, easily obtained by starting with the axioms (7) and (8) and retracing the steps.

Example 17.3 Does P follow from $P \vee Q$?

Solution: We investigate whether $\overset{s}{\rightarrow}(P \vee Q) \rightarrow P$ is a theorem, Assume

(1) $\overset{s}{\Longrightarrow}(P \vee Q) \rightarrow P$

(1) if (2) $P \vee Q \overset{s}{\Longrightarrow} P$ $(\Longrightarrow\rightarrow)$

(2) if (3) $P \overset{s}{\Longrightarrow} P$ and (4) $Q \overset{s}{\Longrightarrow} P$ $(\vee \Longrightarrow)$

here (3) is an axiom, but (4) is not. Hence P does not follow from $P \vee Q$.

Note 17.1 In some cases the derivation is longer if this procedure is used. The reason is that for every connective appearing there is at least one step introduced. In some steps a bracnching appears and then we have to pursue two steps.

Example 17.4 Show that S ∨R is tautologically implied by

$$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S).$$

Solution: To Show

(1) $\underset{\Longrightarrow}{s}((P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)) \rightarrow S \vee R$

(1) if (2) $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S) \underset{\Longrightarrow}{s} (S \vee R)$ $(\Longrightarrow\rightarrow)$

(2) if (3) $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S) \underset{\Longrightarrow}{s} S, R$ $(\Longrightarrow V)$

(3) if (4) $(P \vee Q), (P \rightarrow R), (Q \rightarrow S) \underset{\Longrightarrow}{s} S, R$ $(\wedge \Longrightarrow$ twice)

(4) if (5) $P, P \rightarrow R, Q \rightarrow S \underset{\Longrightarrow}{s} S, R$ and $(v \Rightarrow)$
(6) $Q, P \rightarrow R, Q \rightarrow S \underset{\Longrightarrow}{s} S, R$
$(\vee \Longrightarrow)$

(5) if (7) $P, R, Q \rightarrow S \underset{\Longrightarrow}{s} S, R$ and (8) $P, Q \rightarrow S \underset{\Longrightarrow}{s'} P, S, R$ $(\rightarrow\Longrightarrow)$

(7) if (9) $P, R, S \underset{\Longrightarrow}{s} S, R$ and (10) $P, R \underset{\Longrightarrow}{s} S, R, Q$ $(\rightarrow\Longrightarrow)$

(8) if (11) $P, S \underset{\Longrightarrow}{s} P, S, R$ and (12) $P \underset{\Longrightarrow}{s} P, S, R, Q$ $(\rightarrow\Longrightarrow)$

(6) if (13) $Q, R, Q \to S \overset{s}{\Longrightarrow} S, R$ and (14) $Q, Q \to S \overset{s}{\Longrightarrow} S, R, P$ $(\to \Longrightarrow)$

(13) if (15) $Q, R, S \overset{s}{\Longrightarrow} S, R$ and (16) $Q, R \overset{s}{\Longrightarrow} S, R, Q$ $(\to \Longrightarrow)$

(14) if (17) $Q, S \overset{s}{\Longrightarrow} S, R, P$ and (18) $Q \overset{s}{\Longrightarrow} S, R, P, Q$ $(\to \Longrightarrow)$

Now, (9) to (12) and (15) to (18) are all axioms; therefore the result follows. The nesting of steps shows how the branching occurs.

Exercise 11 Show the following

(a) $P \Longrightarrow (\neg P \rightarrow Q)$
(b) $R \Longrightarrow P \vee \neg P \vee Q$
(c) $\neg(P \wedge Q) \Longrightarrow \neg P \vee \neg Q$

18. The Predicate Calculus

18.1 Basic Terminology and Notation

We now turn our attention to the part of logic which deals with the content of statements. Mainly, we introduce the concept of a predicate in an atomic statement. The logic based upon the analysis of predicates in any statement is called *predicate logic*.

The statement "x is a student" has two parts. The first part, the variable x is the subject of the statement. The second part the *predicate* "is a student"-refers to a property that the subject of the statement can have. We can denote the statement 'x is a student" by $S(x)$ where S denotes the predicate " is a student" and x is the variable. Once a value has been assigned to the variable x the statement $S(x)$ has a truth value.

In general, any statement of the type "p is Q" where Q is a predicate and p is the subject can be denoted by $Q(p)$.

Further, the connectives described earlier can now be used to form compound statements such as "Amulya is a student and this painting is Blue" which can be written as $S(a) \wedge B(p)$ other connectives can also be used to form statements such as

$$S(a) \rightarrow B(p) \quad \neg B(p) \quad S(a) \vee B(p) \quad \text{etc.,}$$

A predicate requiring $m (m > 0)$ names or objects is called an *m-place predicate*. Consider the examples:

(i) Amulya is a student
The predicate S: is a student is a 1-place predicate because it is related to one object: Amulya
(ii) Naveen is tallen than Amal
The predicate "is taller than" is a 2-place predicate

The above statement can be represented as $T(n, a)$. Note that the order in which the names or objects appear in the statement as well as in the predicate is important.

In general, an n-place predicate requires n names of objects to be inserted in fixed positions in order to obtain a statement. The position of these names is important. If S is an n-place letter and $a_1, a_2, \ldots, a_n$ are the names of objects, then $S(a_1, a_2, \ldots a_n)$ is a statement.

If we write $S(x)$ for "x is a student", then $S(a)$, $S(b)$, $S(c)$ and others having the same form can be obtained from $S(x)$, by replacing x by an appropriate name. Note that $S(x)$

is not a statement, but it results in a statement when x is replaced by the name of an object. The letter x used here is a place holder. We use small letters as individual or object variables as well as names of objects.

A simple statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable. Such a statement function becomes a statement when the variable is replaced by the name of any object. We can form *compound statement functions* by combining one or more simple statement functions and the logical connectives.

Example 18.1.1 i) $M(x) \wedge N(x)$, $M(x) \rightarrow R(x)$, $\neg M(x)$, etc.,

ii) $G(x, y)$: x is greater than y, if both x and y are replaced by the names of objects, we get a statement.

If $x = 3$, $y = 2$, then, $G(3, 2) : 3$ is greater than 2.

Some restriction can be introduced by limiting the class of objects under consideration. This limitation means that the variables which are mentioned stand for only those objects which are members of a particular set or class. Such a restricted class is called the *Universe of discourse* or the *domain of* individuals or simply the *universe*. If the discussion refers to human beings only, then the universe of discourse is the class of human beings. In elementary algebra or number theory, the universe of discourse could be numbers.

Example 18.1.2 Consider the statement "Given any positive integer, there is a greater positive integer'. In this case the universe of discourse is the set of positive integers.

The universe of discourse, if any, must be explicitly stated, because the truth value of a statement depends upon it.

18.2 Quantifiers

Certain statements involve words that indicate quantity such as 'all', 'some', 'none' or 'one'. They answer the question 'How many?'. Since such words indicate quantity they are called *Quantifiers*. An analysis of mathematical sentences involving quantifiers indicates that the main two quantifiers are "all" and "some", where 'some' is interpreted to mean 'atleast one'

Consider the following examples:

1. *some* men are tall
2. *All* birds have wings
3. *No* air balloon is perfectly round
4. There is a real number less than 11.

Example (1) Uses 'some', it can be restated as

'there is *atleast one* tall man'

Example (3) means that

'all air balloons fail to be perfectly round'

Example (4) can be restated as

'atleast one real number is less than 11'

The quantifier 'all' is the *Universal quantifier*. We denote it by the symbol '$(\forall x)$'. The symbol $(\forall x)$ represents each of the following phrases, since all have essentially the same meaning:

For all x

For every x

For each x

Every thing x is such that

Each thing x is such that

The quantifier 'some' is *existential quantifier*. We denote it by the symbol $(\exists x)$, The symbol $(\exists x)$ represents each of the following phrases, since all have essentially the same meaning:

For some x,

Some x such that

There exists an x such that

There is an x such that

There is atleast one x such that

We now write the following statements in symbolic form:

(5) Something is good
(6) Everything is good
(7) Nothing is good
(8) Something is not good

Statement (5) means "there is atleast one x such that x is good".

Statement (6) means "for all x, x is good".

Statement (7) means "for all x, x is not good".

Statement (8) means "there is atleast one x such that x is not good".

Thus if $G(x)$: x is good, then

Statement (5) can be denoted by $(\exists x)(G(x))$

Statement (6) can be denoted by $(\forall x)(G(x))$

Statement (7) can be denoted by $(\forall x)(\neg G(x))$

Statement (8) can be denoted by $(\exists x)(\neg G(x))$

We observe that $G(x) : x$ is good

Now, we have two ways of obtaining statements with truth values from a statement function like $G(x)$.

1. by replacing each symbol by an object which it represents or
2. by prefixing either the existential quantifier or the universal quantifier.

The truth of a Quantified statement depends upon the universe of that statement.

Example 18.2.1 Let $Q(x) : x$ is greater than 10
Consider the following universes for this statement:

a. $\{11, 15, 20, 25, 30\}$
b. $\{5, 15, 20, 25, 30\}$
c. $\{5, 7, 8, 9\}$

With universe (a), we can produce the statement

$Q(11) : 11$ is greater than 10
$Q(15) : 15$ is greater than 10

Similarly we can produce $Q(20)$, $Q(25)$ and $Q(30)$.

All these statements are true. Therefore $(\forall x)(Q(x))$ is a true statement in this Universe. There does not exist a number in this universe which is not greater than 10. So $(\exists x)(\neg Q(x))$ is false. The negation of this statement is $\neg(\exists x)(\neg Q(x))$. We observe that $(\forall x)(Q(x))$ and $\neg(\exists x)(\neg Q(x))$ make the same assertion.

With universe (b), $Q(5)$ is a false statement, while $Q(15)$, $Q(20)$, $Q(25)$ and $Q(30)$ are true statements. So with Universe (b) $(\exists x)(Q(x))$ is a true statement, while $(\forall x)(Q(x))$ is a false statement. $(\forall x)(Q(x))$ is false means $(\exists x)(\neg Q(x))$ is a true statement.

With universe (c), all the statements $Q(5)$, $Q(7)$, $Q(8)$ and $Q(9)$ are false and hence $(\forall x)(\neg Q(x))$ is true. It follows that $\neg(\exists x)(Q(x))$ is true.

The above example also illustrates that

'all true' means the same as 'none false'
'all false' means the same as 'none true'
'not all true' means the same as 'atleast one false'
'not all false' means the same as 'atleast one true'

Infact, the following quantified statements are equivalences:

$$(\forall x)(P(x)) \rightleftarrows \neg(\exists x)(\neg P(x))$$
$$(\forall x)(\neg P(x)) \rightleftarrows \neg(\exists x)(P(x))$$
$$\neg(\forall x)(P(x)) \rightleftarrows (\exists x)(\neg P(x))$$

$$\neg(\forall x)(\neg P(x)) \rightleftarrows (\exists x)(P(x))$$

Now we have the rule for negating a statement covered by one quantifier.

To negate a statement covered by one quantifier, change the quantifier from universal to existential or from existential to universal and negate the statement which it quantifies.

Statement	Its Negation
$(\forall x)(P(x))$	$(\exists x)(\neg P(x))$
$(\exists x)(P(x))$	$(\forall x)(\neg P(x))$

Now consider the following four statements:

(9) All monkeys have tails
(10) No monkey has a tail
(11) Some monkeys have tails
(12) Some monkeys have no tails.

If the universe for the statement All monkeys have tails consists only of monkeys–then this statement is merely

$$(\forall x)(P(x)), \text{ where } P(x) : \text{x has a tail.}$$

However, if the universe Consists of objects some of which are not monkeys, a furthe refinement is necessary.

Let M(x) : x is a monkey
and P(x) : x has a tail.

All monkeys have tails makes no statement about objects in the universe which univers which are monkeys. If the object is a monkey then it has, or does not have, a tail. If ther is one monkey which does not have a tail, the statement is false, otherwise it is true. Th statement (9) can be rephrased as follows:

"For all x, if x is a monkey, then x has a tail" and it can be written as $(\forall x)[M(x) - P(x)]$

The statement (10) means "for all x if x is a monkey, then x has no tail" and it can b written as

$$(\forall x)(M(x) \rightarrow \neg P(x))$$

The statement (11) means "there is an x such that x is a monkey and x has a tail" an can be written as

$$(\exists x)(M(x) \wedge P(x))$$

The statement (12) "there is an x such that x is a monkey and x has no tail" and can be written as

$$(\exists x)(M(x) \wedge \neg p(x))$$

Thus we have the following table:

Statement	Symbolic form
All monkeys have tails	$(\forall x)(M(x) \rightarrow P(x))$
No monkey has a tail	$(\forall x)(M(x) \rightarrow \neg P(x))$
Some monkeys have tails	$(\exists x)(M(x) \wedge P(x))$
Some monkeys have no tails	$(\exists x)(M(x) \wedge \neg P(x))$

Where M(x): x is a monkey and
P(x): x has a tail.

Example 18.2.2 (I) Write each of the following in symbolic form.

(a) All men are good
(b) No men are good
(c) Some men are good
(d) Some men are not good

Solution: We assume that the universe consists of objects some of which are not men.

Let M(x): x is a man and
G(x): x is good

Statement (a) means 'for all x, if x is a man, then x is good'. So (a) is $\forall x)[M(x) \rightarrow G(x)]$

Statement (b) means "For all x, if x is a man, then x is not good" and it is represented by

$$(\forall x)[M(x) \rightarrow \neg G(x)]$$

Statement (c) means "there is an x, such that x is a man and x is good". It is written as

$$(\exists x)(M(x) \wedge G(x))$$

Statement (d) means, "there is an x, such that x is a man and x is not good". So it is $\exists x)(M(x) \wedge \neg G(x))$

Example 18.2.2 Write the following sentences in the closed form or symbolic form.

(a) Some people who trust others one rewarded.
(b) If any one is good then john is good.
(c) He is ambitious or no one is ambitious.

(d) Some one is teasing

(e) It is not true that all roads lead to Rome.

Solution: Let

$P(x) : x$ is a person

$T(x) : x$ trusts others

$R(x) : x$ is rewarded

$G(x) : x$ is good

$A(x) : x$ is ambitious

$Q(x) : x$ is teasing

$S(x) : x$ is a road

$L(x) : x$ lead to Rome

Then

(a) "Some people who trust others one rewarded" can be rephrased as
"There is one x such that x is a person, x trusts others and x is rewarded".
Symbolic form: $(\exists x)[P(x) \wedge T(x) \wedge R(x)]$

(b) If any one is good, then John is good' can be worded as
"If there is one x such that x is a person and x is good, then john is good"
Symbolic form: $(\exists x)[P(x) \wedge G(x)] \rightarrow G(\text{John})$.

(c) 'He' represents a particular person. Let that person be y. So the statement is
'y is ambitious or for all x, if x is a person then x is not ambitious'.
So $A(y) \vee ((\forall x)[P(x) \rightarrow \neg A(x)]$

(d) 'Some one is teasing' can be written as
'There is one x such that x is a person and x is teasing' and it is

$$(\exists x)[P(x) \wedge Q(x)]$$

(e) The statement can be written as

$$\neg(\forall x)[S(x) \rightarrow L(x)]$$

$$\text{or} \quad (\exists x)[S(x) \wedge \neg L(x)].$$

18.3 Statement Formulas in Predicate Calculus.

$P(x_1, x_2, \ldots, x_n)$ denotes an n-place predicate formula in which the letter P is an n-place predicate an $x_1, x_2, \ldots, x_n$ are object or individual variables. In general $P(x_1, x_2, \ldots, x_n)$ will be called an atomic formula of predicate calculus.

The following are some examples of atomic formulas

$$R \quad Q(x) \quad P(x, y) \quad A(x, y, z) \quad P(a, y) \text{ and } A(x, a, z)$$

A well-formed formula of predicate calculus is obtained by using the following rules

1. An atomic formula is a well-formed formula
2. If A is a well-formed formula, then $\neg A$ is a well-formed formula
3. If A and B are well formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \rightleftarrows B)$ are also well-formed formulas.
4. If A is a well-formed formula x is any variable, then $(x)A$ and $(\exists x)A$ are well formed formulas
5. Only those formulas obtained by using rules (1) to (4) are well- formed formulas.

18.4 Bound and Free Variables

Generally predicate formulas contain a part of the form $(\forall x)P(x)$ or $(\exists x)P(x)$. Such a part is called x-bound part of the formula. Any variable appearing in an x bound part of the formula is called *bound variable*. Otherwise it is called *free variable*. The smallest formula immediately following $(\forall x)$ or $(\exists x)$ is called the *scope* of the quantifier.

Consider the following formulae:

1. $(\forall x)P(x, y)$
2. $(\forall x)[P(x) \rightarrow Q(x)]$
3. $(\forall x)[P(x) \rightarrow (\exists y)Q(x, y)]$
4. $(\exists x)p(x) \wedge Q(x)$

In (1). $P(x, y)$ is the scope of the quantifier, and occurrence of x is bound occurrence, while the occurrence of y is free occurrence. In (2), the scope of the $(\forall x)$ is $P(x) \rightarrow Q(x)$, and all occurrences of x are bound. In (3), the scope of $(\forall x)$ is $P(x) \rightarrow (\exists y)Q(x, y)$, while the scope of $(\exists y)$ is $Q(x, y)$, all occurrences of both x and y are bound occurrences. However, in (4) the scope of $(\exists x)$ is $P(x)$ and the last occurrence of x in $Q(x)$ is free.

18.5 Valid Formulae and Equivalence

Let A and B be any two predicate formulae defined over a common universe E. If, for ever assignment of object names from the universe E to each of the variables appearing in A and B, the resulting statements have the same truth values, then the predicate formulae A and B are said to be equivalent to each other over E. This idea is symbolized by writing $A \Longleftrightarrow B$ over E. If E is arbitrary, we say A and B are equivalent and write $A \Longleftrightarrow B$.

Similarly, a formula A is said to be valid in E if, for every assignment of object names from E to the corresponding variables in A and for every assignment of statements to statement variables, the resulting statements have the truth value T.

We write $\models A$ in E. If A is valid for an arbitrary E, then we write $\models A$.

In this discussion we derive several valid formulae, which will be useful in the inference theory of predicate logic.

Formulae of predicate calculus that involve quantifiers and no free variables are also formulae of the statement calculus. Therefore substitution instances of all the tautologies by these formulae yield any number of special tautologies.

For example, if in the tautology $P \to Q \Longleftrightarrow \neg P \vee Q$, we substitute the formulae $(\forall x)R(x)$ and $(\exists x)S(x)$ for P and Q respectively, the following tautology is obtained.

$$((\forall x)R(x)) \to ((\exists x)S(x)) \Longleftrightarrow \neg((\forall x)R(x)) \vee ((\exists x)S(x)).$$

Now we consider the substitution $R(x)$ and $S(x)$ for P and Q in $P \to Q \Longleftrightarrow \neg P \vee Q$. Let E be any arbitrary universe. Let b be an object in the Universe. When b replaces x in the statement $(R(x) \to S(x)) \rightleftarrows (\neg R(x) \vee S(x))$, the statement $(R(b) \to S(b)) \rightleftarrows (\neg R(b) \vee S(b))$ is obtained. Clearly, it is a true statement, Since $P \to Q \Longleftrightarrow \neg P \vee Q$ is a tautology. This general argument shows that $(P(x) \to Q(x)) \rightleftarrows (\neg P(x) \vee Q(x))$ is always a true statement regardless of what open statements $P(x)$ and $Q(x)$ represent and regardless of what universe is involved. We therefore conclude that $(P(x) \to Q(x)) \rightleftarrows (\neg P(x) \vee Q(x))$ is a logically valid statement.

Actually our argument was general enough for us to conclude that if elementary statements such as P, Q, R of a tautology are replaced by predicate statements, then the resulting formula is logically valid.

Let us consider $(P(x) \to (\forall x)Q(x)) \wedge \neg(\forall x)(Q(x)) \to \neg P(x)$ which we obtain from $((P \to Q) \wedge \neg Q) \to \neg P$ by replacing P by $P(x)$ and Q by $(\forall x)(Q(x))$. This statement has a universe. With respect to this universe $(\forall x)(Q(x))$ has a fixed truth value and $\neg(\forall x)(Q(x))$ has the opposite truth value. For each 't' of the universe, $P(t)$ and $\neg P(t)$ have opposite truth- values. Therefore for this specific universe and a specific 't' of this universe $P(t)$ and $(\forall x)(Q(x))$ produce a combination of truth values. The result produced in the given statement by this combination of truth-values is exactly the same as the result produced in the logically valid statement pattern $((P \to Q) \wedge \neg Q) \to \neg P$ by the truth value combination. Where P has the same truth value as $P(t)$ and Q has the same truth value as $(\forall x)(Q(x))$.

However, this logically valid statement pattern is true for all combinations of truth-values of the component parts. Therefore it is true for this particular combination. A different choice for t may produce a different truth-value for $P(t)$, but the truth value of $(\forall x)(Q(x))$ remains the same for this specific universe. Thus a change in the truth-value of $P(t)$ merely causes, $P(t)$ and $(\forall x)(Q(x))$ to produce a different combination of truth values, but $(P(t) \to (\forall x)Q(x)) \wedge \neg(\forall x)(Q(x)) \to \neg P(t)$ is always true. Thus the formula

$(P(x) \rightarrow (\forall x)(Q(x)) \wedge \neg(\forall x)(Q(x)) \rightarrow \neg P(x))$ is a valid formula.

Thus from a tautology (of statement calculus) we can derive a lot of logically valid (predicate) formulae.

There are logically valid statement patterns other than those obtained from tautologies. We shall now introduce some of them.

We have already proved the validity of the following formulae

$$(\forall x)(P(x)) \rightleftarrows \neg(\exists x)(\neg P(x))$$

$$(\forall x)(\neg P(x)) \rightleftarrows \neg(\exists x)(P(x))$$

$$\neg(\forall x)(P(x)) \rightleftarrows (\exists x)(\neg P(x))$$

$$\neg(\forall x)(\neg P(x)) \rightleftarrows (\exists x)(P(x))$$

In addition to the implications and equivalence of statement formulas. We give some more identifies

$(\exists x)(A(x) \vee B(x))$	$\Longleftrightarrow$	$(\exists x)A(x) \vee (\exists x)B(x)$	E_{23}
$(x)(A(x) \wedge B(x))$	$\Longleftrightarrow$	$(x)A(x) \wedge (x)B(x)$	E_{24}
$\neg(\exists x)A(x)$	$\Longleftrightarrow$	$(x)\neg A(x)$	E_{25}
$\neg(x)A(x)$	$\Longleftrightarrow$	$(\exists x)\neg A(x)$	E_{26}
$(x)(A \vee B(x))$	$\Longleftrightarrow$	$A \vee (x)B(x)$	E_{27}
$(\exists x)(A \wedge B(x))$	$\Longleftrightarrow$	$A \wedge (\exists x)(A(x) \rightarrow B)$	E_{28}
$(x)A(x) \rightarrow B$	$\Longleftrightarrow$	$(\exists x)(A(x) \rightarrow B)$	E_{29}
$(\exists x)A(x) \rightarrow B$	$\Longleftrightarrow$	$(x)(A(x) \rightarrow B)$	E_{30}
$A \rightarrow (x)B(x)$	$\Longleftrightarrow$	$(x)(A \rightarrow B(x))$	E_{31}
$A \rightarrow (\exists x)B(x)$	$\Longleftrightarrow$	$(\exists x)(A \rightarrow B(x))$	E_{32}
$(x)A(x) \vee (x)B(x)$	$\Longleftrightarrow$	$(x)(A(x) \vee B(x))$	I_{15}
$(\exists x)(A(x) \wedge B(x))$	$\Longleftrightarrow$	$(\exists x)A(x) \wedge (\exists x)B(x)$	I_{16}

Example 18.5.1 Show that $(\forall x)(P(x)) \rightarrow (\exists x)(P(x))$ is a logically valid statement

Solution: If $\forall x(P(x))$ is true in some particular universe, then the universe has atleast one object t in it and $P(t)$ is a true statement for every t in the universe. In particular $P(t)$ must be true. Thus $(\exists x)(P(x))$ is true. Therefore

$$(\forall x)(P(x)) \rightarrow (\exists x)(P(x)) \quad \text{is a valid statement.}$$

Note 18.5.1 If all men are giants, then some men are giants is not a valid statement. If the universe contains no men, then $(\forall x)(M(x) \to G(x))$ is true, while $(\exists x)(M(x) \wedge G(x))$ is false, when $M(x)$: x is a man, $G(x)$: x is a giant. Thus

$$(\forall x)(M(x) \to G(x)) \to (\exists x)(M(x) \wedge G(x)) \quad \text{is not logically valid}$$

Example 18.5.2 Show that

$$(\forall x)(P(x) \wedge Q(x)) \rightleftarrows ((\forall x)(P(x) \wedge (\forall x)(Q(x))$$

is a logically valid statement.

(The universal quantifier "distributes through a conjunction". The universal quantifier can be 'factored out of a conjunction')

Solution: If $(\forall x)(P(x) \wedge Q(x))$ is true, then for every t in the universe, $P(t) \wedge Q(t)$ is true. Therefore, for each t, $P(t)$ is true and, for each t, $Q(t)$ is true. Thus $(\forall x)((P(x) \wedge (\forall x)Q(x)$ is true. This show that

$$(\forall x)(P(x) \wedge Q(x)) \to ((\forall x)P(x) \wedge (\forall x)(Q(x)) \quad \text{is valid.}$$

Conversely, if $(\forall x)(P(x) \wedge (\forall x)(Q(x))$ is true, then for each t in the universe $P(t)$ is true; and for each t in the universe, $Q(t)$ is true. Therefore $P(t) \wedge Q(t)$ is true for each object t in the universe. Thus $(\forall x)(P(x) \wedge Q(x))$ is true and

$$((\forall x)(P(x) \wedge (\forall x)Q(x)) \to (\forall x)(P(x) \wedge Q(x))$$

is valid and hence

$$(\forall x)(P(x) \wedge (\forall x)Q(x)) \to (\forall x)(P(x) \wedge Q(x))$$

is logically valid statement.

Example 18.5.3 Show that

$$(\exists x)(P(x) \vee Q(x)) \rightleftarrows (\exists x)P(x) \vee (\exists x)Q(x).$$

is valid statement. (It shows that the existential quantifier "distributes through disjunction" and can be "factored out of disjunction")

Solution: This can be proved in a similar way as the previous example.

Also, it can be proved by considering the negations of the two statements involved. we know that

$$(\forall x)(P(x) \wedge Q(x)) \rightleftarrows ((\forall x)(P(x) \wedge (\forall x)Q(x))$$

is a valid statement.

Hence $(\forall x)(\neg P(x) \wedge \neg Q(x)) \rightleftarrows ((\forall x)(\neg P(x)) \wedge (\forall x)(\neg Q(x))$ is valid.

If two statements always have the same truth value then their negations always have the same truth value.

Hence
$$\neg(\forall x)(\neg P(x) \wedge \neg Q(x)) \rightleftarrows \neg((\forall x)((\neg p(x)) \wedge (\forall x)(\neg Q(x)))$$

is valid

i.e., $(\exists x)(P(x) \vee Q(x)) \rightleftarrows \neg(\forall x)(\neg P(x)) \wedge \neg(\forall x)(\neg Q(x))$

i.e., $(\exists x)(P(x) \vee Q(x)) \rightleftarrows (\exists x)P(x) \vee (\exists x)Q(x)$

is a valid statement.

Example 18.5.4 Prove that (a) $(\forall x)(P(x)) \vee (\forall x)(Q(x)) \rightarrow (\forall x)(P(x) \vee Q(x))$ is logically valid.

(b) Also show by counterexample

$$(\forall x)(P(x) \vee Q(x)) \rightarrow (\forall x)(P(x)) \vee (\forall x)(Q(x))$$

is not valid.

Solution: (a) Consider the case when $(\forall x)(P(x)) \vee (\forall x)(Q(x))$ is true since this is a disjunction of statements, one of the statements $(\forall x)(P(x))$ and $(\forall x)(Q(x))$ must be true. If $(\forall x)(P(x))$ is true, then for every object 't' in the universe, $P(t)$ is true and hence $P(t) \vee Q(t)$ is true. Similarly when $(\forall x)(Q(x))$ is true $P(t) \vee Q(t)$ is true for every object t. In both the cases $P(t) \vee Q(t)$ is true for all t in the universe. Therefore $(\forall x)(P(x) \vee Q(x))$ is true and

$$(\forall x)(P(x)) \vee (\forall x)(Q(x)) \rightarrow (\forall x)(P(x) \vee Q(x))$$

is a valid statement

(b) Now consider the following statement

$$(\forall x)(P(x) \vee Q(x))$$

Where $P(x)$: x is an even integer

$Q(x)$: x is a prime integer and

the Universe is {2, 4, 6, 3, 7}

For this Universe the statement $(\forall x)(P(x) \vee Q(x))$ is true, but both $(\forall x)(P(x))$ and $(\forall x)(Q(x))$ are not true. So $(\forall x)(P(x) \vee Q(x))$ is true, while $(\forall x)(P(x)) \vee (\forall x)(Q(x))$ is not true. Thus

$$(\forall x)(P(x) \vee Q(x)) \rightarrow (\forall x)(P(x)) \vee (\forall x)(Q(x))$$

is not a valid statement.

Now, we are going to prove the validity of two important statements. They are needed for deriving many logical conclusions.

Example 18.5.5 Prove that the statements

(a) $(\forall x)(P(x)) \to P(y)$

(b) $P(y) \to (\exists x)(P(x))$ are valid statements.

(y represents any one of the objects in the given universe)

Solution:

(a) The logical ability of the first statement follows immediately from the fact that if $(\forall x)(P(x))$ is true, then $P(t)$ is true for every t in the universe and hence it is true for any specific object 'y' in the universe.

(b) The logical validity of the second statement is a consequence of the meaning of the existential quantifier. The statement $(\exists x)(P(x))$ is true if and only if there exists atleast one object in the Universe for which $P(x)$ is true. Therefore if $P(y)$ is true, then $(\exists x)(P(x))$ is true.

Similarly we will have some logically valid statements:

LS1 : $(P(x)) \rightleftarrows \neg(\exists x)(\neg P(x))$

LS2 : $\forall x(P(x)) \rightleftarrows (\exists x)(P(x))$

LS3 : $\forall x(P(x)) \rightleftarrows (P(y))$

LS4 : $P(y) \to (\exists x)(P(x))$

LS5 : $(\forall x)(P(x) \wedge Q(x)) \rightleftarrows ((\forall x)(P(x)) \wedge (\forall x)(Q(x))$

LS6 : $(\exists x)(P(x) \vee Q(x)) \rightleftarrows ((\exists x)(P(x)) \vee (\exists x)(Q(x))$

LS7 : $((\forall x)(P(x)) \vee (\forall x)Q(x)) \to (\forall x)(P(x) \vee Q(x))$

LS8 : $(\exists x)(P(x) \wedge Q(x)) \to ((\exists x)(P(x)) \wedge (\exists x)(Q(x))$

LS9 : $(\forall x)(P(x) \to Q(x)) \to ((\forall x)(P(x)) \to (\forall x)(Q(x))$

19. Predicate Calculus :: Theory of Inference

We are going to discuss the methods of derivation involving predicate formulae. In order to draw conclusions from quantified premises, we need to know how to remove the quantifiers properly, argue with the resulting statements and then properly prefix or add the correct quantifiers.

We can use the rules of information given for the statement calculus. The Rules P and T regarding the introduction of a premise at any stage of derivation and the introduction of any formula which follows logically from the formulae already introduced, remain the same. If the conclusion is given in the form of a conditional, we shall also use the Rule CP. Certain additional rules are required to deal with the formulae involving quantifiers. The elimination of quantifiers can be done by rules of specification called US and ES. To prefix the correct quantifier, we need the rules of generalization called UG and EG.

Now we give the rules of generalization and specification.

Rule US: (Universal Specification)

If a statement of the form $(\forall x)(P(x))$ is assumed to be true, then the universal quantifier can be dropped to obtain $P(t)$ is true for an arbitrary object 't' in the universe.

In symbols, this rule is:

$$\frac{(\forall x)(P(x))}{\therefore \quad P(t) \text{ for all } t}$$

Rule UG (Universal Generalization):

If a statement $P(t)$ is true for each element t of the universe, then the universal quantifier may be prefixed to obtain $(\forall x)(P(x))$.

In symbols, this rule is:

$$\frac{P(t) \text{ for all } t}{\therefore \quad (\forall x)(P(x))}$$

This rule holds, provided we know $P(t)$ is true for each element t in the universe.

Rule ES (Universal specification):

If $(\exists x)(P(x))$ is assumed to be true, then there is an element t in the universe such that $P(t)$ is true.

In symbols, this rule is:

$$\frac{(\exists x)(P(x))}{\therefore \quad P(t) \text{ for some } t}$$

Note that the element t is not arbitrary (as it was in US), but must be one for which $P(x)$ is true.

It follows from the truth of $(\exists x)(P(x))$ that atleast one such element must exist, but nothing *more* is guaranteed.

Rule EG (Existential Generalization):

It $P(t)$ is true for some element t in the universe, then $\exists x.\ P(x)$ is true.
In symbols, we have

$$\frac{P(t) \text{ for some } t}{\therefore \quad (\exists x)(P(x))}$$

Now we give several examples to explain the method of derivation.

Example 19.1 Verify the validity of the following arguement.

Every living thing is a plant or an animal. John's gold fish is alive and it is not a plant. All animals have hearts. Therefore John's gold fish has a heart.

Solution: Let the Universe consist of all living things. Let

$P(x)$: x is a plant

$A(x)$: x is an animal

$H(x)$: x has a heart

g : John's gold fish

Then the inference pattern is

$$\begin{array}{c} (\forall x)(P(x) \vee A(x)) \\ \neg P(g) \\ (\forall x)(A(x) \rightarrow H(x)) \\ \hline H(g) \end{array}$$

Argument

[1]	(1)	$(\forall x)(P(x) \vee A(x))$	Rule P
[2]	(2)	$\neg P(g)$	Rule P
[1]	(3)	$P(g) \vee A(g)$	Rule US, (1)
[1, 2]	(4)	$A(g)$	Rule T, (2), (3)
[5]	(5)	$(\forall x)(A(x) \rightarrow H(x))$	Rule P
[5]	(6)	$A(g) \rightarrow H(g)$	Rule US, (5)
[1, 2, 5]	(7)	$H(g)$	Rule T, (4), (6)

Thus the conclusion is valid

Example 19.2 Verify the validity of the following arguement:

Tigers are dangerous animals. There are Tigers. Therefore there are dangerous animals.

Solution:

Let $T(x)$: x is a tiger

$D(x)$: x is a dangerous animal

Then the inference pattern is

$$\frac{\begin{array}{c}(\forall x)(T(x) \rightarrow D(x)) \\ (\exists x)(T(x))\end{array}}{(\exists x)(D(x))}$$

Argument

[1]	(1)	$(\exists x)(T(x))$	Rule P
[2]	(2)	$T(b)$	Rule ES, (1)
[3]	(3)	$(\forall x)(T(x) \rightarrow D(x))$	Rule P
[1, 3]	(4)	$T(b) \rightarrow D(b)$	Rule US, (3)
[1, 3]	(5)	$D(b)$	Rule T, (2), (4)
[1, 3]	(6)	$(\exists x)(D(x))$	Rule EG, (5)

Thus the inference is valid.

Example 19.3 Verify the validity of the following arguement.

All men are mortal

Socrates is a man

Therefore Socrates is a mortal

Solution:

We denote

$H(x)$: x is a man

$M(x)$: x is a mortal

S : Socrates

We need to show $(\forall x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$.

Arguement:

[1]	(1)	$(\forall x)(H(x) \rightarrow M(x))$	Rule P
[1]	(2)	$H(s) \rightarrow M(s)$	Rule US, (1)

[3]	(3)	$H(s)$	Rule P
[1, 3]	(4)	$M(s)$	Rule T, (2), (3), I_{11}

Thus the inference is valid

Example 19.4 Given an argument which will establish the validity of the following inference:

All integers are rational numbers

Some integers are powers of 3

Therefore, Some vational numbers are powers of 3

Solution:

Let $P(x)$: x is an integer

$R(x)$: x is a rational number

$S(x)$: x is a power of 3

Then the given inference pattern is $(\forall x)(P(x) \rightarrow R(x))$

$$\frac{(\exists x)(P(x) \wedge S(x))}{\therefore (\exists x)(R(x) \wedge S(x))}$$

Arguement:

[1]	(1)	$(\exists x)(P(x) \wedge S(x))$	Rule P
[1]	(2)	$P(b) \wedge S(b)$	Rule ES, (1)
[1]	(3)	$P(b)$	Rule T, (2)
[1]	(4)	$S(b)$	Rule T, (2)
[2]	(5)	$(\forall x)(P(x) \rightarrow R(x))$	Rule P
[2]	(6)	$P(b) \rightarrow R(b)$	Rule US, (5)
[1, 2]	(7)	$R(b)$	Rule T, (3), (6)
[1, 2]	(8)	$R(b) \wedge S(b)$	Rule T, (7),(4)
[1, 2]	(9)	$(\exists x)(R(x) \wedge S(x))$	Rule EG, (8)

Example 19.5 Show that from

(a) $(\exists x)(F(x) \wedge S(x)) \rightarrow (\forall y)(M(y) \rightarrow W(y))$

(b) $(\exists y)(M(y) \wedge \neg W(y))$

the conclusion

$(\forall x)(F(x) \rightarrow \neg S(x))$ follows

Solution:

[1]	(1)	$(\exists y)(M(y) \wedge \neg W(y))$	Rule P
[1]	(2)	$M(z) \wedge \neg W(z)$	Rule ES, (1)
[1]	(3)	$\neg(M(z) \rightarrow W(z))$	Rule T, (2), E_{17}
[1]	(4)	$(\exists y)\neg(M(y) \rightarrow W(y))$	Rule EG, (3)
[1]	(5)	$\neg(\forall y)(M(y) \rightarrow W(y))$	(4), E_{26}
[6]	(6)	$(\exists x)(F(x) \wedge S(x) \rightarrow (\forall y)(M(y) \rightarrow W(y))$	Rule P
[1, 6]	(7)	$(\exists x)(F(x) \wedge S(x))$	Rule T, (5), (6) I_{12}
[1, 6]	(8)	$(\forall x)\neg(F(x) \wedge S(x))$	Rule T, (7), E_{25}
[1, 6]	(9)	$\neg(F(x) \wedge S(x))$	Rule US, (8)
[1, 6]	(10)	$F(x) \rightarrow \neg S(x)$	Rule T, (9), E_9, E_{16}
[1, 6]	(11)	$(\forall x)(F(x) \rightarrow \neg S(x))$	Rule UG, (10)

Example 19.6 Show that

$$(\forall x)(P(x) \vee Q(x)) \Rightarrow (\forall x)P(x) \vee (\exists x)Q(x)$$

Solution: We shall use the indirect method of proof by assuming $\neg((\forall x)P(x) \vee (\exists x)Q(x))$

[1]	(1)	$\neg((\forall x)P(x) \vee (\exists x)Q(x))$	Rule P (assumed)
[1]	(2)	$\neg(\forall x)P(x) \wedge \neg(\exists x)Q(x)$	Rule T, (1), E_9
[1]	(3)	$\neg(\forall x)P(x)$	Rule T, (2), I_1
[1]	(4)	$(\exists x)(\neg P(x))$	Rule T, (3), E_{26}
[1]	(5)	$\neg(\exists x)(Q(x))$	Rule T, (2), I_2
[1]	(6)	$(\forall x)(\neg Q(x))$	Rule T, (5), E_{25}
[1]	(7)	$\neg P(y)$	Rule ES, (4)
[1]	(8)	$\neg Q(y)$	Rule US, (6)
[1]	(9)	$\neg P(y) \wedge \neg Q(y)$	Rule T, (7), (8), I_9
[1]	(10)	$\neg(P(y) \vee Q(y))$	Rule T, (9), E_9

[11]	(11)	$(\forall x)(P(x) \vee Q(x))$	Rule P
[1, 11]	(12)	$P(y) \vee Q(y)$	Rule US (11)
[1, 11]	(13)	$(\neg(P(y) \vee Q(y))) \wedge (P(y) \vee Q(y))$	Rule T, (10), (12), I_9 contradiction

Example 19.7 Is the following conclusion validly derivable from the premises given?

$$\text{If}(\forall x)(P(x) \rightarrow Q(x)); (\exists y)P(y), \text{then}(\exists z)Q(z)$$

Solution: We use the indirect method, by assuming that the conclusion $(\exists z)Q(z)$ is false.

[1]	(1)	$\neg(\exists z)Q(z)$	Rule P (assumed)
[1]	(2)	$(\forall z)\neg Q(z)$	Rule T, (1)
[3]	(3)	$(\exists y)P(y)$	Rule P
[3]	(4)	$P(a)$	Rule ES, (3)
[1]	(5)	$\neg Q(a)$	Rule US, (2)
[1, 3]	(6)	$P(a) \wedge \neg Q(a)$	Rule T, (4), (5)
[1, 3]	(7)	$\neg(P(a) \rightarrow Q(a))$	Rule T, (6)
[8]	(8)	$(\forall x)(P(x) \rightarrow Q(x))$	Rule P
[8]	(9)	$P(a) \rightarrow Q(a)$	Rule US, (8)
[1, 3, 8]	(10)	$(P(a) \rightarrow Q(a)) \wedge \neg(P(a) \rightarrow Q(a))$	Rule T, (7), (9) contradictior

Example 19.8 Using CP or otherwise obtain the following implication

$$(\forall x)(P(x) \rightarrow Q(x)), (\forall x)(R(x) \rightarrow \neg Q(x)) \Rightarrow (\forall x)(R(x) \rightarrow \neg P(x))$$

Solution:

[1]	(1)	$(\forall x)(P(x) \rightarrow Q(x))$	Rule P
[2]	(2)	$(\forall x)(R(x) \rightarrow \neg Q(x))$	Rule P
[2]	(3)	$R(x) \rightarrow \neg Q(x)$	Rule US, (2)
[4]	(4)	$R(x)$	Rule P (assumed)
[2, 4]	(5)	$\neg Q(x)$	Rule T, (3), (4)
[1]	(6)	$P(x) \rightarrow Q(x)$	Rule US, (1)
[1, 2, 4]	(7)	$\neg P(x)$	Rule T, (5), (6)
[1, 2, 4]	(8)	$R(x) \rightarrow \neg P(x)$	Rule CP, (4), (7)
[1, 2]	(9)	$(\forall x)(R(x) \rightarrow \neg P(x))$	Rule UG, (9)

Hence the arguement is valid.

Example 19.9 There is a mistake in the following derivation. Find it. Is the conclusion valid? If so, obtain a correct derivation.

[1]	(1)	$(\forall x)(P(x) \rightarrow Q(x))$	Rule P
[1]	(2)	$P(y) \rightarrow Q(y)$	Rule US, (1)
[3]	(3)	$(\exists x)P(x)$	Rule P
[3]	(4)	$P(y)$	Rule ES, (3)
[1, 3]	(5)	$Q(y)$	Rule T, (2), (4), I_{11}
[1, 3]	(6)	$(\exists x)Q(x)$	Rule EG, (5)

Solution: The y introduced in step(2) is free, it should not be introduced again by ES in fourth step. We try to eliminate this mistake by obtaining a correct derivation

[1]	(1)	$(\exists x)P(x)$	Rule P
[1]	(2)	$P(y)$	Rule ES, (1)
[3]	(3)	$(\forall x)(P(x) \rightarrow Q(x))$	Rule P
[3]	(4)	$P(y) \rightarrow Q(y)$	Rule US, (3)
[1, 3]	(5)	$Q(y)$	Rule T, (2), (4), I_{11}
[1, 3]	(6)	$(\exists x)Q(x)$	Rule EG, (5)

Thus, $(\exists x)P(x), (\forall x)(P(x) \rightarrow Q(x)) \rightarrow (\exists x)Q(x)$ is a valid statement.

Exercise 12 (I) Show that the following statements are logically valid Statements

(a) $(\exists x)(P(x) \wedge Q(x)) \rightarrow ((\exists x)P(x) \wedge (\exists x)Q(x))$

(b) $(\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\forall x)(P(x)) \rightarrow (\forall x)(Q(x)))$

(c) $\neg(P(x) \vee Q(x)) \rightleftarrows (\neg P(x) \wedge \neg Q(x))$

(d) $(P(x) \rightarrow Q(y)) \rightleftarrows (\neg P(x) \vee Q(y))$

(e) $(P(x) \wedge Q(y)) \rightarrow (P(x) \rightarrow Q(y))$

(II) Show that each of the following statements is logically valid

(a) $(R(x) \wedge S(x)) \rightarrow R(x)$

(b) $(\forall x)(P(x) \vee Q(x)), (\forall x)\neg P(x) \Rightarrow (\forall x)(Q(x))$

(c) $(\forall x)(P(x) \rightarrow Q(x)) \wedge (\forall x)(Q(x) \rightarrow R(x)) \Rightarrow (\forall x)(P(x) \rightarrow R(x))$

(d) $P \rightarrow (\exists x)Q(x) \Leftrightarrow (\exists x)(P(x) \rightarrow Q(x))$

(e) $(\forall x)(P(x) \rightarrow Q(x)) \wedge (\forall x)(Q(x) \rightarrow R(x)) \Rightarrow (\forall x)(P(x) \rightarrow R(x))$

20. Statements Involving more than one Quantifier

If a predicate formulae involves more than one different variable, then more than one quantifier is needed to produce a closed sentence (symbolic sentence)

Consider the statement: $P(x, y) : x$ likes y

The statement $(\exists x)(P(x, y))$ represents the statement:

'there is an x, such that x likes y'

This can be rephrased as 'there is some one who likes y' or simply 'some one likes y'

In the same way $(\forall x)(P(x, y))$ means 'everyone likes y'

There are eight possible combinations of $(\forall x)$, $(\forall y)$, $(\exists x)$ and $(\exists y)$, for the statement $P(x, y) : x$ likes y, these are given below:

$(\exists y)(\exists x)P(x, y)$	:	There is someone whom some one likes
$(\exists y)(\forall x)P(x, y)$	:	There is someone whom everybody likes
$(\forall y)(\exists x)P(x, y)$	:	Everybody is liked by some one
$(\forall y)(\forall x)P(x, y)$	:	Everybody is liked by every one
$(\exists x)(\exists y)P(x, y)$	:	Some one likes somebody
$(\exists x)(\forall y)P(x, y)$	:	Some one likes every one
$(\forall x)(\exists y)P(x, y)$	:	Every one likes some one
$(\forall x)(\forall y)P(x, y)$	:	Every one likes everybody

Note that $(\exists y)(\exists x)P(x, y)$ and $(\exists x)(\exists y)P(x, y)$ have the same meaning. The statements $(\forall x)(\forall y)P(x, y)$ and $(\forall y)(\forall x)P(x, y)$ have the same meaning. Infact

$$(\exists y)(\exists x)P(x, y) \rightleftarrows (\exists x)(\exists y)P(x, y)$$

and $(\forall x)(\forall y)P(x, y) \rightleftarrows (\forall y)(\forall x)P(x, y)$ are logically valid equivalences.

But the statements $(\forall x)(\exists y)P(x, y)$ and $(\exists y)(\forall x)P(x, y)$ are not the same. The first statement means "every one likes some body" and the second statement means "there is someone whom everyone like". If the second statement is true, then the first statement is also true, but the converse is not true.

Now we shall prove that $(\exists y)(\forall x)P(x, y) \to (\forall x)(\exists y)P(x, y)$

[1]	(1)	$(\exists y)(\forall x)(P(x, y))$	Rule P
[1]	(2)	$(\forall x)P(x, b)$	Rule ES, (1)
[1]	(3)	$P(x, b)$	Rule US, (2)
[1]	(4)	$(\exists y)P(x, y)$	Rule EG, (3)
[1]	(5)	$(\forall x)(\exists y)P(x, y)$	Rule UG, (4)

Note 20.1 One can be tempted to derive $(\exists y)(\forall x)P(x, y)$ from $(\forall x)(\exists y)P(x. y)$ as follows:

[1]	(1)	$(\forall x)(\exists y)P(x, y)$	Rule P
[1]	(2)	$(\exists y)P(x, y)$	Rule US, (1)
[1]	(3)	$P(x, b)$	Rule ES, (2)
[1]	(4)	$(\forall x)P(x, b)$	Rule UG, (3)
[1]	(5)	$(\exists y)(\forall x)P(x, y)$	Rule EG, (4)

But there is a mistake in this derivation. In the third line existential object 'b' was ntroduced. But in the fourth line Rule UG was applied. As per Rule UG. $(\forall x)$ should iot be prefixed to obtain $(\forall x)P(x, b)$ as the existential object 'b' in $P(x, b)$, which lepends on x, is not covered by any Quantifier.

One can prove the following implications and equivalences using the method of lerivation

$$(\forall y)(\forall x)P(x, y) \Rightarrow (\exists x)(\forall y)P(x, y)$$

$$(\exists x)(\forall y)P(x, y) \Rightarrow (\forall y)(\exists x)P(x, y)$$

$$(\forall x)(\exists y)P(x, y) \Rightarrow (\exists y)(\exists x)P(x, y)$$

$$(\forall y)(\exists x)P(x, y) \Rightarrow (\exists x)(\exists y)P(x, y)$$

:xample 20.1 Write the following statement in the symbolix form "Every one who kes fun will enjoy each of these plays"

Solution:

We write $L(x)$: x likes fun

$P(x)$: x is a play

$E(x, y)$: x will enjoy y

The statement can be represented as "for each x, if x likes fun and for each y, if y is a play, then x enjoys y", in symbolic form:

$$(\forall x)(\forall y)\,[L(x) \wedge P(y) \rightarrow E(x, y)].$$

Example 20.2 Write in the symbolic form and negate the following statements

(a) Every one who is healthy can do all kinds of work
(b) Some people are not admired by every one
(c) Every one should help his neighbors, or his neighbors will not help him.

Solution:

(a) The given statement is
"every one who is healthy can do all kinds of work"

Let $H(x)$: x is a healthy person

$W(y)$: y is a kind of work

D(x,v) : x cando y

The statement is
"for all x, if x is healthy, and for all y, if y is a kind of work, then x can do y".
So a symbolic form is

$$(\forall x)(\forall y)\,[H(x) \wedge H(y) \rightarrow D(x, y)]$$

Its negation is given by

$$\neg((\forall x)(\forall y)(H(x) \wedge H(y) \rightarrow D(x, y))$$

$$\text{i.e.,}(\exists x)(\neg(\forall y)(H(x) \wedge H(y) \rightarrow D(x. y))$$

$$\text{i.e.,}(\exists x)(\exists y)(\neg(H(x) \wedge H(y) \rightarrow D(x, y))$$

$$\text{i.e.,}(\exists x)(\exists y)(H(x) \wedge H(y) \wedge \neg D(x, y))$$

i.e., there exists a healthy person and there exists a kind of work such that x cannot do y

i.e., there is some healthy person who cannot do some kind of work.

(b) In the universe of people, let $A(x, y)$: x admires y

Then the given statement is "there is a person who is not admired by some person". So it is $(\exists y(\exists x)(\neg A(x, y))$

Its negation is $\neg(\exists y((\exists x)(\neg A(x, y))$

i.e., $(\forall y)\neg((\forall x(A(x, y)))$

$\Rightarrow$ every person is admired by everyone.

In the Universe, which consists of everything, let

$P(x)$: x is a person and $A(x, y)$: x admires y

Then the given statement is $(\exists y)(\exists x)(P(x) \wedge P(y) \rightarrow \neg A(x, y))$

Its negation is $(\forall y)(\forall x)(A(x, y) \wedge P(x) \wedge P(y))$.

(c) In the universe of people, let

$N(x, y)$: x and y are neighbors

$H(x, y)$: x should help y

$P(x, y)$: x will help y

The given statement is: for every person x and every person y, if x and y are neighbors, then either x should help y or y will not help x
So the symbolic form is

$$(\forall x)(\forall y)(N(x, y) \rightarrow (H(x, y) \vee \neg P(y, x)))$$

Its negation is

$$\neg((\forall x)(\forall y)(N(x, y)) \rightarrow (H(x, y) \vee \neg P(y, x)))$$

$$\text{i.e.,}(\exists x)\neg((\forall y)(\neg N(x, y)) \vee H(x, y) \vee \neg P(y, x)))$$

$$\text{i.e.,}(\exists x)(\exists y)\neg(\neg(N(x, y)) \vee H(x, y) \vee \neg P(y, x)))$$

$$\text{i.e.,}(\exists x)(\exists y)(N(x, y)) \wedge \neg H(x, y) \wedge P(y, x))$$

There are some people who should not help (one of) their neighbors but their neighbors will help them.

Note 20.2 The negation of the formulas can be obtained by repeated applications of the equivalences E_{25} and E_{26}

Example 20.3 Verify the validity of the following inference

If one person is more successful than another, then he has worked harder to deserve success. Naveen has not worked harder than Amal. Therefore, Naveen is not more successful then Amal.

Solution: Let the universe consists of all persons

Let $S(x, y)$: x is more successful than y

$W(x, y)$: x has worked harder than y to derserve success

a : Naveen

b : Amal

Then the inference pattern is:

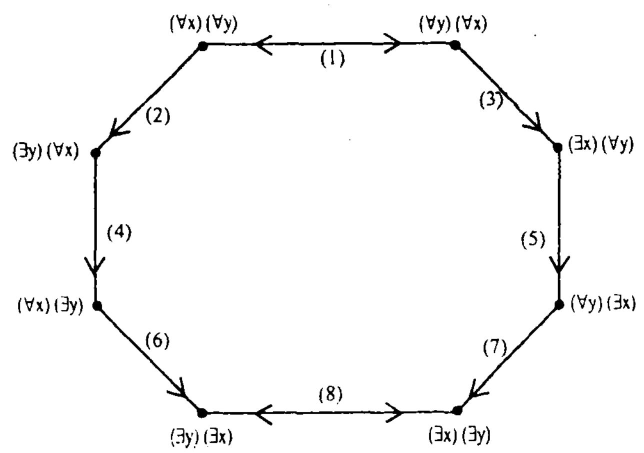
$$\frac{\begin{array}{l}(\forall x)(\forall y)\,[S(x, y) \rightarrow W(x, y)] \\ \neg W(a, b)\end{array}}{\neg S(a, b)}$$

Argument

[1]	(1)	$\neg W(a, b)$	Rule P
[2]	(2)	$(\forall x)(\forall y)\,[S(x, y) \rightarrow W(x, y)]$	Rule P
[2]	(3)	$(\forall y)\,[S(a, y) \rightarrow W(a, y)]$	Rule US, (2)
[2]	(4)	$S(a, b) \rightarrow W(a, b)$	Rule US, (3)
[1, 2]	(5)	$\neg S(a, b)$	Rule T, (1), (4)

Thus the inference is valid one.

Note 20.3 The implications and equivalences can be shown by the



Exercise 13 (I) Show that each of the following is a logically valid statement

(a) $(\forall x)(\forall y)P(x, y) \rightarrow (\exists x)(\forall y)P(x, y)$

(b) $(\forall y)(\exists y)P(x, y) \rightarrow (\exists x)(\exists y)P(x, y)$

(c) $(\forall x)(H(x) \rightarrow A(x)) \rightarrow (\forall x)\big[(\exists y)(H(y) \wedge N(x, y)) \rightarrow (\exists y)(A(y) \wedge N(x, y))\big]$

(II) Show that $\neg P(a, b)$ follows logically from

$$(\forall x)(\forall y)(p(x, y) \rightarrow W(x, y)) \text{and} \neg W(a, b).$$

Solved problems

Problem 1 Let P be "It is cold" and Let Q be "It is raining". Give a simple verbal sentence which describes each of the following statemetns

(a) $\neg P$
(b) $P \wedge Q$
(c) $P \vee Q$
(d) $Q \vee \neg P$

Solution: In each case translate $\wedge$, $\vee$ and $\neg$ to read "and", "or" and "it is false that" or "not" respectively and then simplify the English sentence

(a) It is not cold
(b) It is cold and raining
(c) It is cold or it is raining
(d) It is raining or it is not cold

Problem 2 Give the English verbal sentences which describes

(a) $\neg P \wedge \neg Q$
(b) $\neg(\neg Q)$

Solution:

(a) It is not cold and it is not raining (or) in other words
It is neither cold nor raining
(b) It is not true that it is not raining (or) in other words
It is false that it is not raining

Problem 3 Let P be 'He is tall' and Let Q be 'He is Handsome' write each of the following statements in symbolic form using P and Q: (Assume that 'He is short' means 'He is not tall' i.e., $\neg P$)

(a) He is tall and handsome
(b) He is tall but not handsome
(c) It is false that he is short or handsome
(d) He is neither tall or handsome

Solution:

(a) $P \wedge Q$
(b) $P \wedge \neg Q$
(c) $\neg(\neg P \vee Q)$
(d) $\neg P \wedge \neg Q$

Problem 4 From the problem (3), Repeat for each of the following statements

(a) He is tall, or he is short and handsome
(b) It is not true that he is short or not handsome

Solution:

(a) $P \vee (\neg P \wedge Q)$
(b) $\neg(\neg P \vee \neg Q)$

Problem 5 Let P: Ramu reads Newsweek Let Q: Ramu reads the new yorker, and R: Ramu reads Time write each of the following in symbolic form:

(a) Ramu reads Newsweek or the New Yorker, but not Time
(b) Ramu reads Newsweek and The New Yorker or he does not read Newsweek and Time
(c) It is not true that Ramu reads Time or The New Yorker but not Newsweek

Solution:

(a) $(P \vee Q) \wedge \neg R$
(b) $(P \wedge Q) \vee \neg(P \wedge R)$
(c) $\neg(P \wedge \neg R)$
(d) $\neg((R \wedge Q) \vee \neg P)$

Problem 6 Let P: Anand speaks Telegu and Q: Anand speaks Hindi. Give a simple verbal sentence which describes each of the following

(a) $P \vee Q$
(b) $P \wedge Q$
(c) $P \wedge \neg Q$
(d) $\neg P \vee \neg Q$

Solution:

(a) Anand speaks Telegu or Hindi
(b) Anand speaks Telegu and Hindi
(c) Anand speaks Telegu but not Hindi
(d) Anand does not speak Telegu or she does not speak Hindi.

Problem 7 Repeat the problem (6) for (a) $\neg(\neg p)$ and (b) $\neg(\neg P \wedge \neg Q)$

Solution:

(a) It is not true that Anand does not speak Telegu
(b) It is not true that Anand speaks neither Telegu nor Hindi

Problem 8 Find the truth tables of the following

(a) $P \wedge (Q \vee R)$
(b) $(P \wedge Q) \vee (P \wedge R)$

Solution:

(a)

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

(b)

$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T
T	F	T
F	T	T
F	F	F
F	F	F
F	F	F
F	F	F
F	F	F

Problem 9 Verify that proposition $P \vee \neg(P \wedge Q)$ is a tautology

Solution:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$P \vee \neg(P \wedge Q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Problem 10 Verify that the proposition $(P \wedge Q) \wedge \neg(P \vee Q)$ is a contradiction

P	Q	$P \wedge Q$	$p \vee Q$	$\neg(P \vee Q)$	$(P \wedge Q) \wedge \neg(P \vee Q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Problem 11 Construct the truth table of $(P \to Q) \to (P \wedge Q)$

Solution:

P	Q	$P \to Q$	$P \wedge Q$	$(P \to Q) \to (P \wedge Q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Problem 12 Construct the truth table of $\neg P \to (Q \to P)$

Solution:

P	Q	$\neg P$	$Q \to P$	$\neg P \to (Q \to P)$
T	T	F	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Problem 13 Verify that $(P \wedge Q) \to (P \vee Q)$ is a tautology

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q) \to (P \vee Q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Problem 14 Give the truth tables of Converse, inverse and contrapositive of the proposition $P \to Q$.

Only the contrapositive $\neg Q \to \neg P$ is logically equivalent to the $P \to Q$.

Problem 15 The given statement is equivalent to "If Raju passes the test, then he studied". Give the contrapositive

Solution: It Raju does not study, then he will not pass the test.

Problem 16 Find the truth table for $(P \rightleftarrows \neg Q) \rightleftarrows (Q \to P)$

P	Q	$P \to Q$	$Q \to P$	$\neg P \to \neg Q$	$\neg Q \to \neg P$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

P	Q	$\neg Q$	$P \rightleftarrows \neg Q$	$Q \to P$	$(P \rightleftarrows \neg Q) \rightleftarrows (Q \to P)$
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

Problem 17 Verify that the proposition
$(P \wedge \neg Q) \vee \neg(P \wedge \neg Q)$ is a tautology

Solution: Already we knew that $R \vee \neg R$ is a tautology thus by substitution instance, $P \wedge \neg R$ substituted for R gives the above formula and is a tautology

Problem 18 The propositional connective $\veebar$ is called the exclusive disjunction $P \nabla Q$ is read P or Q but not both.

Construct the truth table for $P \nabla Q$

Solution: $P \veebar Q$ is true if P is true or if Q is true but not if both are true.

P	Q	$P \nabla Q$
T	T	F
T	F	T
F	T	T
F	F	F

Problem 19 The connective $\downarrow$ is called the joint denial $P \downarrow Q$ is read 'Neither P nor Q' (or NAND)

construct a truth table for $P \downarrow Q$

Solution: $P \downarrow Q$ is true only in the case that P is not true and Q is not true

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

Problem 20 Show by truth table

(i) $\neg p \Leftrightarrow P \downarrow P$
(ii) $P \wedge Q \Leftrightarrow (P \downarrow P) \downarrow (Q \downarrow Q)$
(iii) $P \vee Q \Leftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q)$

Solution:

P	Q	$P \vee Q$	$P \downarrow Q$	$(P \downarrow Q) \downarrow (P \downarrow Q)$
T	T	T	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

Problem 21 Simplify each of the following statements

(a) It is not true that his mother is English or his father is Telegu
(b) It is not true that he studies physics but not mathematics.

Solution:

(a) P: His mother is English
Q: His father is Telegu
Then given statement is $\neg(P \vee Q)$
But $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$.
Hence the given statement is logically equivalent to 'His mother is not English and his father is not French'.

(b) P: He studies physics
Q: He studies mathematics
Then the given statement is $\neg(P \wedge \neg Q)$
But $\neg(P \wedge \neg Q) \Leftrightarrow \neg P \vee Q$
Hence the given statement is logically equivalent to the statement. 'He does not study physics or he studies mathematics'.

Problem 22 Let P: It is cold and Let Q: It rains. Write the following statements in symbolic form.

(a) It rains only if it is cold
(b) A necessary condition for it to be cold is that it rain
(c) A sufficient condition for it to be cold is that it rain.

Solution: Recall that $P \to Q$ can be read P only if Q, P is sufficient for Q, or$Q$ is necessary for P then (a) $Q \to P$ (b) $P \to Q$ (c) $Q \to P$.

Problem 23 Rewrite the following statements without using the conditional.

(a) If it is cold, he wears a hat
(b) If productivity increases, then wages rise.

Solution: $P \to Q \Leftrightarrow \neg P \vee Q$

(a) It is not cold oe he wears a hat
(b) productivity does not increase or wages rise.

Find the truth table of $(P \leftrightarrow \neg Q) \rightleftarrows (Q \to P)$

Solution: Last column entries are: F T F F

Problem 24 Show that $P \wedge Q$ logically implies $P \rightleftarrows Q$.

Solution:

P	Q	$P \wedge Q$	$P \rightleftarrows Q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	T

Now $P \wedge Q$ istrue only in line 1 and in this case the proposition $P \rightleftarrows Q$ is also true. Thus $P \wedge Q$ logically implies $P \rightleftarrows Q$.

Problem 25 The arguement $\begin{array}{c} P \to Q \\ \neg P \\ \hline \therefore \neg Q \end{array}$ is a fallacy

Since $[(P \to Q) \wedge \neg Q] \to \neg Q$ is not a tautology.

Similarly the argument $\dfrac{P \to Q, \neg Q}{\therefore \neg P}$ is valid

Since $((P \to Q) \wedge \neg Q) \to \neg P$ is a tautology

Now Test the validity of each arguement

(a)

$$\begin{array}{c} \text{If its rains, chandu will be sick,} \\ \text{It did not rain} \\ \hline \text{Chandu was not sick} \end{array}$$

(b)

If it rains, chandu will be sick,
Chandu was not sick
———————————
It did not rain

Solution:
If P: It rains and
Q: Chandu is sick then

(a) $P \to Q$ (b) $P \to Q$
$\neg P$ $\neg Q$
$\therefore \neg Q$ $\therefore \neg P$

clearly

(a) is fallacy
(b) is valid

Problem 26 (a) $\{\neg, \wedge, \vee\}$ is an complete set of connectives
(b) Each of $\{\neg, \wedge, \vee, \to\}$ and $\{\neg, \to\}$ is an complete set of connectives

Solution: $\{\neg, \wedge, \vee\}$ is complete,`since DNF involves only these connectives. It is immediate that any bigger set is complete, and $\{\neg, \to\}$ is complete since $\wedge$ and $\vee$ can each be expressed in terms of $\neg$ and $\to$

$$P \wedge Q \Leftrightarrow \neg(P \to \neg Q)$$

$$P \vee Q \Leftrightarrow \neg P \to Q$$

Note Each of the sets $\{\neg, \wedge\}$, $\{\neg, \vee\}$, $\{\uparrow\}$, $\{\downarrow\}$ is complete.

Problem 27 Express the statement
'Every student in this class has studied calculus' as a Universal Qualification

Solution: Let $P(x)$: x has studied calculus
Then the statement 'Every student in this class has studied calculus' can be written as $\forall x P(x)$, where the universe of discourse consists of the students in the class
This statement can also be expressed as

$$\forall x(S(x) \to P(x)$$

where $S(x)$ is the statement "x is in this class" $P(x)$ is as before and the universe of discourse is the set of all students.

Problem 28 There are two restaurants next to each other. one has a sign that says "Good food is not cheap" and the other has a sign that says "cheap food is not good". Are the signs saying the same thing?

Solution:
Let P: Food is good
Q: Food is cheap
The first sign then be written as $P \to \neg Q$ and the second sign be written as $Q \to \neg P$

Now from the following truth table

P	Q	$\neg P$	$\neg Q$	$P \to \neg Q$	$Q \to \neg P$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

the two signs say the same thing.

Problem 29 Show that the truth value of the formula $((P \to Q) \wedge (Q \to R)) \to (P \to R)$ is independent of their components

Solution: Construct the truth table for the sbove statement formula, you wil find the all truth (T) in the last column, for any combination of truthvalues of its components the truth value of it is the same. Therefore the above assertion follows.

Problem 30 Show that $\neg(P \vee (\neg P \wedge Q))$ and $\neg P \wedge \neg Q$ are logically equivalent (without using truth table).

Solution:

$$\neg(P \vee (\neg P \wedge Q)) \Longleftrightarrow \neg P \wedge \neg(\neg P \wedge Q)$$
$$\Longleftrightarrow \neg P \wedge (\neg(\neg P) \wedge \neg Q)$$
$$\Longleftrightarrow \neg P \wedge (P \vee \neg Q)$$
$$\Longleftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q)$$
$$\Longleftrightarrow F \vee (\neg P \wedge \neg Q)$$
$$\Longleftrightarrow \neg P \wedge \neg Q$$

$\therefore$ $\neg(P \vee (\neg P \wedge Q))$ and $\neg P \wedge \neg Q$ are logically equivalent

Problem 31 Obtain a DNF of $P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$.

Solution:

$$P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$$
$$\Longleftrightarrow \neg P \vee ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$$
$$\Longleftrightarrow \neg P \vee ((\neg P \vee Q) \wedge \neg(\neg Q \vee \neg P))$$
$$\Longleftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P))$$
$$\Longleftrightarrow \neg P \vee ((\neg P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P)))$$
$$\Longleftrightarrow \neg P \vee F \vee (P \wedge Q)$$
$$\Longleftrightarrow \neg P \vee (P \wedge Q)$$

which is the required DNF.

Problem 32 Show $I_1 : \neg Q, P \to Q \Rightarrow \neg P$

Solution:

{1}	(1)	$P \to Q$	Rule P
{2}	(2)	$\neg Q \to \neg P$	Rule T, (1) and E_{18}
{3}	(2)	$\neg Q$	Rule P
{1, 3}	(4)	$\neg P$	Rule T, (2), (3) and I_{11}

Problem 33 Show that the arguement

$$P \to Q$$
$$R \to \neg Q$$
$$R$$

is valid

$$\therefore \neg P$$

Solution:

R	$\neg P$	$\neg Q$	$P \to Q$	$R \to \neg Q$
T	F	F	T	F
F	F	F	T	T
T	F	T	F	T
F	F	T	F	T
T	T	F	T	F
F	T	T	T	T
T	T	T	T	T
F	T	T	T	T

From the Row 7, it is clear that $\neg P$ follows and the arguement is valid

Problem 34 Show that the following premises are inconsistent

1. If Jill misses many classes through illness then he fails high school
2. If Jill fails high school, then he is uneducated
3. If Jill reads a lot of books, then he is not uneducated
4. Jill misses many classes through illness and reads a lot of books

Solution:

M: Jill misses many classes

F: Jill fail high schools

R: Jill reads a lot of books

U: Jill is uneducated.

The premises are $M \to F$, $F \to U$, $R \to \neg U$ and $M \wedge R$.

{1}	(1)	$M \to F$	Rule P
{2}	(2)	$F \to U$	Rule P
{1, 2}	(3)	$M \to U$	Rule T, (1), (2), and I_{13}
{4}	(4)	$R \to \neg U$	Rule P
{4}	(5)	$U \to \neg R$	Rule T, (4), E_{18}
{1, 2, 4}	(6)	$M \to \neg R$	Rule T, (3), (5), I_{13}
{1, 2, 4}	(7)	$\neg M \vee \neg R$	Rule (6), I_{16}
{1, 2, 4}	(8)	$\neg(M \wedge R)$	Rule T, (7), E_8
{9}	(9)	$M \wedge R$	Rule P
{1, 2, 4, 9}	(10)	$(M \wedge R) \wedge \neg(M \wedge R)$	Rule T, (8), (9), I_9

Problem 35 Show the following using the automatic theorem

(a) $P \Rightarrow (\neg P \to Q)$
(b) $P \wedge \neg P \wedge Q \Rightarrow R$
(c) $R \Rightarrow P \vee \neg P \vee Q$
(d) $P, \neg P \vee (P \vee Q) \Rightarrow Q$

Solution:

(a) $P \Rightarrow \neg P \to Q$

$P \Rightarrow \neg P, Q \qquad \Rightarrow \to$

$P, P \Rightarrow Q \qquad \Rightarrow \neg$

(b) $P \wedge \neg P \wedge Q \Rightarrow R$

$P, \neg P \wedge Q \Rightarrow R \qquad \wedge \Rightarrow$

$P, \neg P, Q \Rightarrow R \qquad \wedge \Rightarrow$

$P, Q \Rightarrow P, R \qquad \neg \Rightarrow$

(c) $R \Rightarrow P \vee \neg P \vee Q$

$R \Rightarrow P, \neg P \vee Q \qquad \Rightarrow \vee$

$R \Rightarrow P, \neg P, Q \qquad \Rightarrow \vee$

$P, R \Rightarrow P, Q \qquad \Rightarrow \neg$

(d) $P, \neg P \vee (P \wedge Q) \Rightarrow Q$

$P, \neg P \Rightarrow Q$ and $(P \wedge Q) \Rightarrow Q \qquad \vee \Rightarrow$

$P, \Rightarrow P, Q$ and $(P \wedge Q) \Rightarrow Q \qquad \neg \Rightarrow$

$P, \Rightarrow P, Q$ and $(P, Q) \Rightarrow Q \qquad \wedge \Rightarrow$

Problem 36 If P and R are true and Q, S are false then find the truth values of

(i) $[(P \wedge \neg Q) \vee R] \vee \neg[R \wedge (\neg P \Rightarrow Q)]$
(ii) $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow S)$

Solution:

(i) For the given assignments P, $\neg Q$, R are true and $(P \wedge \neg Q) \vee R$ is true, $\neg P$ is false and so $\neg P \Rightarrow Q$ is true
$\therefore$ $R \wedge (\neg P \Rightarrow Q)$ is true and this $\neg[R \wedge (\neg P \Rightarrow Q)]$ is false. But the full statement is of the form $T \vee F$ and hence is true.
(ii) Try yourself.

Problem 37 Express $P \rightleftarrows Q$ in terms of $\{\neg, \vee\}$ only

Solution:

$$\begin{aligned}(P \rightleftarrows Q) &\Longleftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)\\ &\Longleftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)\\ &\Longleftrightarrow \neg(\neg(\neg P \vee Q) \vee \neg(\neg Q \vee P))\end{aligned}$$

Problem 38 Establish the analgoues of Demorgan's laws involving $\uparrow$ and $\downarrow$

(i) $\neg(P \uparrow Q) \Longleftrightarrow \neg P \downarrow \neg Q$
(ii) $\neg(P \downarrow Q) \Longleftrightarrow \neg P \uparrow \neg Q$

Solution:

(i)

$$\begin{aligned}\text{RHS} : \neg P \downarrow \neg Q &\Longleftrightarrow \neg(\neg P \vee \neg Q) \Longleftrightarrow P \wedge Q\\ &\Longleftrightarrow \neg(\neg(\neg(P \wedge Q))\\ &\Longleftrightarrow \neg(P \downarrow Q) = \text{LHS}\end{aligned}$$

(ii) follows similarly

Problem 39 obtain equivalent DNF and CNF for $\neg(\neg(P \rightleftarrows Q) \wedge R)$

Solution:

$$\begin{aligned}\neg(\neg p \rightleftarrows Q) \wedge R) &\Longleftrightarrow \neg\{[\neg(P \rightarrow Q) \wedge (Q \rightarrow P)] \wedge R\}\\ &\Longleftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P) \vee \neg R\\ &\Longleftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P) \vee \neg R\\ &\Longleftrightarrow (\neg P \vee Q \vee \neg R) \vee (P \vee \neg Q \vee \neg R) \quad \text{(CNF)}\end{aligned}$$

$$\Longleftrightarrow [(\neg P \vee Q \vee \neg R) \wedge \neg P] \vee [\neg P \vee Q \vee \neg R) \wedge \neg Q] \vee [(\neg P \vee Q \vee \neg R) \wedge \neg R]$$

$$\Longleftrightarrow (Q \wedge P) \vee (\neg R \wedge P) \vee (\neg P \wedge \neg Q) \vee (\neg R \wedge \neg Q) \vee (\neg P \wedge \neg R) \vee (Q \wedge \neg R) \vee \neg R$$

$$\Longleftrightarrow (P \wedge Q) \vee (P \wedge \neg R) \vee (\neg P \wedge \neg Q) \vee (\neg Q \wedge \neg R) \vee (\neg P \wedge \neg R) \vee (Q \wedge \neg R) \vee \neg R$$

which is (DNF).

Problem 40 Show that $\neg P(a, b)$ follows logically from $(x)(y)(P(x, y) \rightarrow w(x, y))$ and $\neg w(a, b)$

Solution:

1.	$(x)(y)(P(x, y) \rightarrow w(x, y))$	Rule P
2.	$(y)(P(a, y) \rightarrow w(a, y))$	US, (1)
3.	$P(a, b) \rightarrow w(a, b)$	US, (2)
4.	$\neg W(a, b)$	Rule P
5.	$\neg P(a, b)$	T, (3), (4).

Problem 41 Find the truth value of

$[P \rightarrow ((Q \wedge (7R)) \vee S)] \wedge [(7T) \longleftrightarrow (S \wedge R)]$

(treat A $\wedge$ B)

where P, Q, R and S are all true while T is false

Solution: We evaluate the expression step by step, showing just the relevant row of the truth table

P	Q	R	S	T	TR	$Q \wedge (7R)$	$(Q \wedge (7R)) \vee S$	A	B	$A \wedge B$
T	T	T	T	F	F	F	T	T	T	T

Problem 42 Show that $(7P) \rightarrow (P \rightarrow Q)$ is a tautology

Solution:

$$\begin{aligned} [(7P) \rightarrow (P \rightarrow Q)] &\Longleftrightarrow [(7P) \rightarrow ((7P) \vee Q)] \\ &\Longleftrightarrow [(7(7P)) \vee ((7P) \vee Q)] \\ &\Longleftrightarrow P \vee [(7P) \vee Q] \\ &\Longleftrightarrow [P \vee (7P)] \vee Q \\ &\Longleftrightarrow T \vee Q \\ &\Longleftrightarrow T \end{aligned}$$

QUIZ Questions

1. $P \wedge Q$ has the truth value T whenever both P and Q have the truth value ________ (Ans: T)
2. $P \vee Q$ has the truth value F only when both P and Q have the truth value ________ (Ans: F)
3. $P \vee Q$ is true if either P is ____ or Q is __ or both P and Q are ________ (Ans: True)
4. $P \rightarrow Q$ has a truth value F when Q has the truth value ________ and P the truth value ________; otherwise it has the truth value ________ (Ans: F, T, T)
5. The converse of $P \rightarrow Q$ is ________ (Ans: $Q \rightarrow P$)
6. The contrapositive of $P \rightarrow Q$ is ______ (Ans: $\neg Q \rightarrow \neg P$)
7. The inverse of $P \rightarrow Q$ is ________ (Ans: $7P \rightarrow 7Q$)
8. $P \rightarrow Q$ and $\neg Q \rightarrow \neg P$ have the ________ truth values (Ans: same)
9. $p \rightleftarrows Q$ has the truth value T whenever both P and Q have ______ truth values (Ans: identical)
10. $P \rightleftarrows Q$ and $(P \rightarrow Q) \wedge (Q \rightarrow P)$ are ________ (Ans: equivalent)
11. $P \wedge \neg P$ is always ______ (Ans: F)
12. $P \vee \neg P$ is always ______ (Ans: T)
13. $P \vee \neg P$ is a ______ (Ans: tautology)
14. $P \wedge \neg P$ is a ______ (Ans: Contradiction)
15. $(P \vee Q) \rightarrow P$ is not a ______ (Ans: tautology)
16. $(P \wedge (P \leftrightarrows Q)) \rightarrow Q$ is a ________ (Ans: tautology)
17. Any substitution instance of a tautology is a ________ (Ans: tautology)
18. If $A \leftrightarrows B$ is a tautology, then ________

    (Ans: $A \Longleftrightarrow B$)
19. If $A \rightarrow B$ is a tautology, then we write ________

    (Ans: $A \Longrightarrow B$ or A tautologically imply B)
20. $A \Longrightarrow B$ guarantees that B has the truth value T whenever A has the ____ (Ans: T)
21. Both Implication and equivalence are ________

    (Ans: transitive)

22. {↑}, {↓} are ________

(Ans: functionally complete)

23. A product (Conjunction) of variables and their negations is called an ________

(Ans: elementary product)

24. A sum (disjunction) of variables and their negations is called an ________

(Ans: elementary sum)

25. If A has the truth value T for atleast one combination of truth values assigned to $P_1, P_2 \ldots P_n$ then A is said to be ________

(Ans: satisfiable)

26. DNF is the ________

(Ans: Sum of elementary sums)

27. CNF is the ________

(Ans: product of elementary sums)

28. PDNF:__ (Disjunctions of minterms only)

29. PCNF: ________ (Ans: Conjunctions of maxterms only)

30. PCNF, PDNF are ____ except for the rearrangements of the terms (Unique)

31. Rules of inference of statement calculus: ______

(Ans: Rule P, Rule T, Rule CP)

32. Rules of predicate calculus are: ________

(Ans: Rule US, Rule UG, Rule ES, Rule EG).

33. If $H_1, H_2, \ldots, H_m$ and P imply Q

then $H_1, H_2, \ldots, H_m$ imply ________

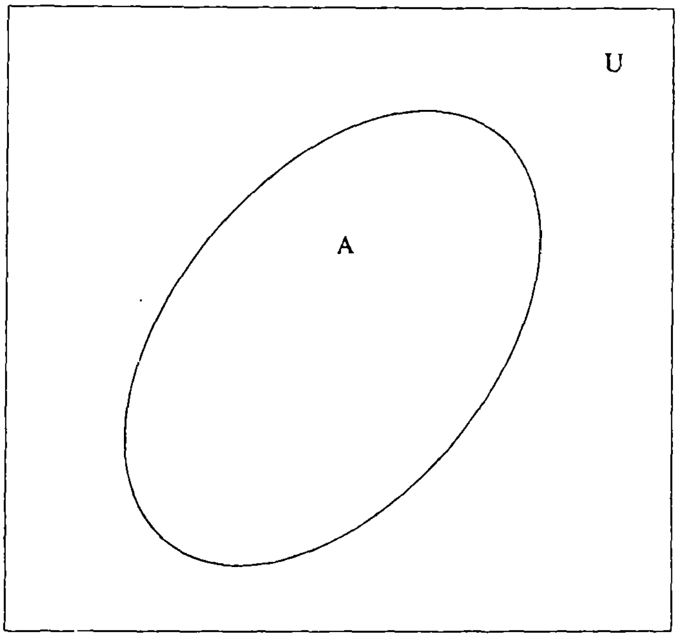
(Ans: $P \rightarrow Q$)

34. The connective ↑ is not ________ (Ans: associative)

35.
$$\begin{array}{r} P \rightarrow Q \\ P \\ \hline \therefore ? \\ \hline \end{array}$$

(Ans : Q)

2. Set Theory



1. Introduction

One of the most important tools in mathematics is the theory of sets. The notation, terminology and concepts of set theory are helpful in studying any branch of mathematics. Every branch of mathematics can be considered as a study of sets of objects of one kind or another. Further, sets and mathematical logic are now basic to the design of computers and electrical circuits.

2. Sets

Definition 2.1 A set is a collection of objects, called elements which share some common property.

A fundamental concept of set theory is that of membership of belonging to a set. Any object belonging to a set is called a member or an element of that set. The elements in a set can be anything: numbers, people, buildings, cars, letters etc.,

A set is said to be well defined if it is possible to determine, bymeans of certain rules, whether any given object is a member of the set.

The elements of the set are said to belong to the set.

Sets will be denoted by capital letters $A, B, C, \ldots, X, Y, Z$. Elements will be denoted by lower case letters $a, b, c, \ldots, x, y, z$. The phrase "is an element of" will be denoted by the symbol ϵ. Thus we write $X \in A$ for "x is an element of A". In analogous situations we write $x \notin A$ for "x is not an element of A".

Example 2.1 (i) The set of numbers: 1, 3, 5, 7, 9 (or) $A = \{1, 3, 5, 7, 9\}$

(ii) The set of rivers in Andhra pradesh.

We have five ways used to describe a set:

Describe a set

(i) by describing the properties of the elements of the set

(ii) by listing all its elements

(iii) by its characteristic function defined as $\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

(iv) by a recursive formula

(v) by an operation on some other sets.

Example 2.2 Describe the set containing all the nonnegative integers lessthan or equal to 6.

Let A denote the set. Then the set A can be described in the following ways:

(i) $A = \{x/x$ is a nonegative integer less than or equal to $6\}$

(ii) $A = \{0, 1, 2, 3, 4, 5, 6\}$

(iii) $\mu_A(x) = \begin{cases} 1 & \text{for } x = 0, 1, \ldots, 6 \\ 0 & \text{otherwise} \end{cases}$

(iv) $A = \{x_{i+1} = x_i + 1, i = 0, 1, \ldots, 5, \text{ where } x_0 = 0\}$

(v) Will be discussed later after operations on sets introduced.

Note 2.1 For a given set, not all the five ways of describing it are always possible.

A set is finite if it contains a finite number of distinguishable elements; otherwise, the set is infinite.

Definition 2.2 Let A and B be two sets. Then A is said to be a subset of B if every element of A is an element of B; A is said to be a proper subset of B if A is a subset of B and there is atleast one element of B which is not in A.

If A is a subset of B, We say A is contained in B. Symbolically we write $A \subseteq B$. If A is proper subset of B, then we say A is strictly contained in B, denoted by $A \subset B$.

The inclusion or containment of sets has the following properties:

Let A, B and C be sets

1. $A \subseteq A$
2. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
3. If $A \subseteq B$ and $B \subset C$, then $A \subset C$.
4. If $A \subseteq B$ and $A \nsubseteq C$, then $B \nsubseteq C$, where $\nsubseteq$ means "is not contained in".

Definition 2.3 Two sets A and B are equal $\Longleftrightarrow A \subseteq B$ and $B \subseteq A$. wewrite $A = B$.

Note 2.2 To show that two sets A and B are equal. We must show that each element of A is also an element of B and conversely.

Definition 2.4 A set is called universal set if it includes every set under discussion. A universal set will be denoted by U.

Definition 2.5 A set which does not contain any element is called an empty set or a null set, denoted by ϕ.

A set containing a single element is called a singleton.

Given any set A, the null set ϕ and the set A are both subsets of A.

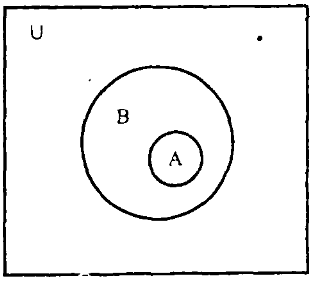
Definition 2.6 For a set A, a collection or family of all subsets of A is called the power set of A, denoted by P(A).

3. Venn Diagrams

A simple and instructive way of representing a set is with the help of diagrams known as Venn diagrams. In the venn diagram, a rectangle represents the universal set and any other set A is represented by the interior of a simple closed curve inside the rectangle. The form of the curve is immaterial, usually it is a circle.

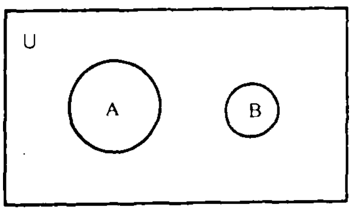
Example 3.1 A relationship between two sets can be conveniently denoted by the venn diagram.

Suppose A is a subset of B ie., $A \subset B$, this can be denoted as



Definition 3.1 If A and B are sets with no elements in common i.e., no element of A is in B and no element of B is in A, then the sets are said to be disjoint.

By venn diagram:



4. Operations On Sets:

We introduce certain basic operations on sets. Using these operations one can construct new sets by combining the elements of given sets

Definition 4.1 The intersection of any two sets A and B, written as $A \cap B$ is the set consisting of all the elements which belong to both A and B. Symbolically

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

From the definition

i) $A \cap B = B \cap A$

ii) $A \cap A = A$

iii) $A \cap \phi = \phi$

iv) $(A \cap B) \cap C = A \cap (B \cap C)$

v) $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n = \{x \mid x \in A_i \text{ for all } i\}$

Definition 4.2 Two sets A and B are called disjoint $\iff A \cap B = \phi$ i.e., A and B have no element in common.

Definition 4.3 A collection of sets is called a disjoint collection if, for every pair of sets in the collection, the two sets are disjoint. The elements of a disjoint collection are said to be mutually disjoint.

i.e., $A_i \cap A_j = \phi$ for all i, j and $i \neq j$.

Definition 4.4 For any two sets A and B, the *Union* of A and B, written as $A \cup B$ is the set of all elements which are members of the set A or the set B or both.

Symbolically, it is written as $A \cup B = \{x \mid x \in A \vee x \in B\}$

From the definition

(i) $A \cup A = A$

(ii) $A \cup \phi = A$

(iii) $A \cup B = B \cup A$

(iv) $(A \cup B) \cup C = A \cup (B \cup C)$

(v) $\bigcup_{i=1}^{n} = A_1 \cup A_2 \cup \ldots \cup A_n = \{x \mid x \in A_i \text{ for atleast one } i\}$

Definition 4.5 Let A and B be any two sets. The relative complement of B in A (or of B w.r.t. to A). Written as A–B, is the set consisting of all elements of A which are not elements of B, that is

$$A-B = \{x | x \in A \wedge x \notin B\}$$

The relative complement of B in A is also called the difference of A and B

Definition 4.6 Let U be the Universal set. For any set A, the relative complement of A with respect to U, that is U-A is called the (absolute) complement of A and is denoted by A' (or A^c or $\bar{A}$). Symbolically, $A' = U - A = \{x | x \in U \wedge x \notin A\} = \{x | x \notin A\}$

From the definition (i) $(A')' = A$ (ii) $\phi' = U$ (iii) $U' = \phi$ (iv) $A \cup A' = U$ (v) $A \cap A' = \phi$.

Definition 4.7 Let A and B be any two sets. The symmetric difference of A and B is the set consisting of all the elements that belong to A or B, but not to both A and B. It is denoted by $A \Delta B$.

Thus

$$\begin{aligned} A \Delta B &= \{x | (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\} \\ &= (A - B) \cup (B - A). \end{aligned}$$

From the definition

(i) $A \Delta B = B \Delta A$

(ii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

(iii) $A \Delta \phi = A$

(iv) $A \Delta A = \phi$

(v) $A \Delta B = (A \cap B') \cup (B \cap A')$

5. Properties of set operations:

All the operations on sets so far defined satisfy many algebraic properties. Many of the important properties listed in the subsequent paragraphs can be proved directly from the definitions. They can also be verified by Venn diagrams.

(*I*) *Properties of Union operation:*

Let A, B, C be subsets of U. Then

(1) $A \cup A = A$ (I dempotent property)

(2) $A \cup \phi = A$

(3) $A \cup U = U$, $A \cup A' = U$

(4) $A \subseteq A \cup B$ and $B \subseteq A \cup B$

(5) $A \cup B = B \cup A$ (commutative property)

(6) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)

Let us prove associative law

First, show that $A \cup (B \cup C) \subset (A \cup B) \cup C$

$$\begin{aligned} x \in A \cup (B \cup C) &\Rightarrow x \in A \text{ or } x \in (B \cup C) \\ &\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C) \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C \\ &\Rightarrow x \in (A \cup B) \text{ or } x \in C \\ &\Rightarrow x \in (A \cup B) \cup C \end{aligned}$$

$$\therefore A \cup (B \cup C) \subset (A \cup B) \cup C \tag{1}$$

Similarly we can prove

$$(A \cup B) \cup C \subset A \cup (B \cup C) \tag{2}$$

From (1) and (2), we have $A \cup (B \cup C) = (A \cup B) \cup C$.

(*II*) *Properties of Intersection operation:*

Let A, B, C be subsets of U. Then

(i) $A \cap A = A$

(ii) $A \cap B = A$ wherever $A \subseteq B$. Inparticular $A \cap U = A$

(iii) $A \cap B = B$ wherever $B \subseteq A$. Inparticular $A \cap \phi = \phi$

(iv) $A \cap B \subseteq A$ and $A \cap B \subseteq B$

(v) $A \cap B = B \cap A$

(vi) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)

(*III*) *Properties of the Complement:*

(i) $A \cup A' = U$

(ii) $A \cap A' = \phi$

(iii) $U' = \phi$

(iv) $\phi' = U$

(v) $(A')' = A$

(vi) $(A \cup B)' = A' \cap B'$

(vii) $(A \cap B)' = A' \cup B'$

(*IV*) *Distributive Law:*

For any three sets, we have

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(*V*) *Demorgan's Laws:*

For any two sets A and B

i) $(A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

Note 5.1 De Morgan's laws can be extended as follows:

If $A_1, A_2, \ldots, A_n$ are sets, then

(i) $(A_1 \cup A_2 \cup \ldots \cup A_n)' = A_1' \cap A_2' \cap \ldots \cap A_n'$

(ii) $(A_1 \cap A_2 \cap \ldots \cap A_n)' = A_1' \cap A_2' \cap \ldots \cap A_n'$

or simply

$$\left(\bigcup_i A_i\right)' = \bigcap_i (A_i)'$$

$$\left(\bigcap_i A_i\right)' = \bigcup_i (A_i)'$$

(*VI*) *Properties of the Difference operation:*

Let A, B, C are sets and U be the Universal set

Then

(i) $A' = U - A$

(ii) $A - B = A \cap B'$

(iii) $A - A = \phi$

(iv) $A - \phi = A$

(v) $A - B = B - A \iff A = B$

(vi) $A - B = A \iff A \cap B = \phi$

(vii) $A - B = \phi \iff A \subseteq B$

(*VII*) *Properties of Symmetric Difference:*

If A and B are sets, then

(i) $A \Delta A = \phi$

(ii) $A \Delta \phi = A$

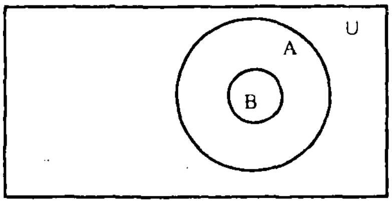
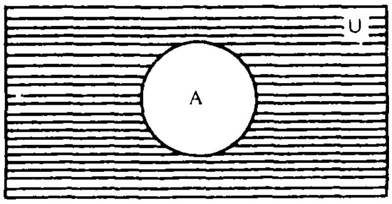
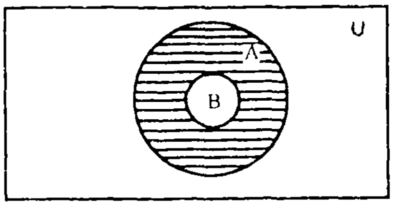
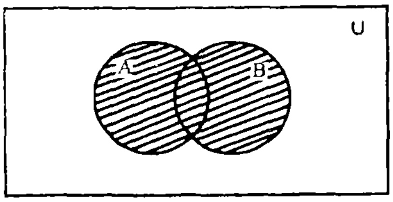
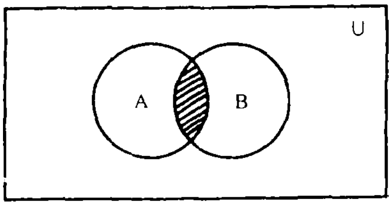
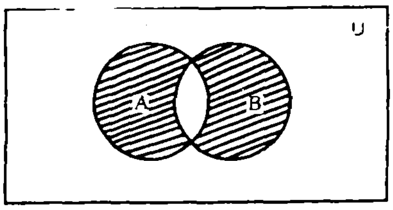
(iii) $A \Delta B = B \Delta A$

(iv) $A \Delta B = (A \cup B) - (A \cap B)$

6. Basic Laws of set Algebra

Idempotent Laws:	$A \cup A = A$ $A \cap A = A$
Associative Laws:	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws:	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Distributive Laws:	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Absorption Laws:	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Complement and Demorgan's Laws:	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$ $\phi' = U$ $U' = \phi$ $(A')' = A$ $A \cup \phi = A$ $A \cap U = A$ $A \cup U = A$ $A \cap \phi = \phi$ $A \cup A' = U$ $A \cap A' = \phi$

7. Venn Diagrams of set operations

Set operation	Symbol	Venn diagram
Set B is Contained in Set A	$B \subset A$	
Complement of A [Shaded region]	$A^C (A' or \bar{A})$	
The relative complement of set B w.r.t. A [Shaded region]	$A - B$	
The Union of two sets A and B [Shaded portion]	$A \cup B$	
The intersection of sets A and B [Shaded portion]	$A \cap B$	
The Symmetrical difference of sets A and B [Shaded portion]	$A \Delta B$	

8. Relations

In this discussion, our main concern is sets whose elements are ordered pairs. By an *ordered pair* we mean that each set is specified by two objects in a given fixed order. Note that an ordered pair is not a set consisting of two objects. The ordering of the two objects is important. The two objects need not be distinct. We denote an ordered pair by (a,b). We also define that $(a, b) = (c, d) \iff a = c$ and $b = d$.

For Example: A familiar example of an ordered pair is the representation of a point in a two-dimensional plane in cartesian coordinates.

Definition 8.1 Let A and B be two sets. The cartesian product of A and B is defined as $A \times B = \{(a, b) | a \in A \text{ and } B \in B\}$.

More generally, the cartesian product of n sets $A_1, A_2, \ldots, A_n$ is defined as $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i, i = 1 \text{ to } n\}$

The expression $(a_1, a_2, \ldots, a_n)$ is called an ordered n- tuple.

Example 8.1 Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 7\}$
Then

$$A \times B = \left\{ \begin{array}{l} (1, 2), (1, 3), (1, 7), (2, 2), (2, 3), (2, 7) \\ (3, 2), (3, 3), (3, 7), (4, 2), (4, 3), (4, 7) \end{array} \right\}$$

$$B \times A = \left\{ \begin{array}{l} (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2) \\ (3, 3), (3, 4), (7, 1), (7, 2), (7, 3), (7, 4) \end{array} \right\}$$

It is to be noted that $A \times B \neq B \times A$, if the sets A and B are different.

Note 8.1 (i) If A has m elements and B has n elements, then $A \times B$ and $B \times A$ will have mn elements.

(ii) $(A \times B) \times C \neq A \times (B \times C)$

(iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

for any three sets A, B and C.

Definition 8.2 A (binary) relation R from a set A to a set B is a subset of $A \times B$. If $A = B$, we say R is a (binary) relation on A.

We shall call a binary relation simply a relation.

Example 8.2 Let $\mathbb{R}$ denote the set of real numbers. Then $R = \{(x, x^2) | x \in \mathbb{R}\}$ defines a relation of the square of a real number.

Example 8.0.1 Example: The relation $R = \{(x, y) | x$ and y are real numbers and $x^2 + y^2 < 1\}$ consists of all points inside a unit circle whose center is at the origin.

Definition 8.3 Let R be a relation from A to B. The domain of R denoted by domR is defined:

$$\text{domR} = \{x \mid x \in A \text{ and } (x, y) \in R \text{ for some } y \in B\}$$

The range of R, denoted by ranR is defined

$$\text{ranR} = \{y \mid y \in B \text{ and } (x, y) \in R \text{ for some } x \in A\}$$

Clearly domR $\subseteq A$ and ranR $\subseteq B$.

We sometimes write $(x, y) \in R$ as xRy which reads "x relates to y".

Example 8.3 Let $A = \{2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$

Define a relation R from A to B by $(a, b) \in R$ if a divides b (with zero remainder).

We obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

The domain of R is the set $\{2, 3, 4\}$ and the range of R is the set $\{3, 4, 6\}$

9. Properties of relations:

Definition 9.1 A relation R on a set A is said to be

(i) Reflexive if xRx or $(x, x) \in R\ \forall x \in A$.

(ii) Irreflexive if $x \not R x$ or $(x, x) \notin R\ \forall x \in A$

(iii) Symmetric if $xRy \Rightarrow yRx$, for all $x, y \in A$.

(iv) Antisymmetric if $x \neq y$ and $xRy \Rightarrow y \not R x$ or $(y, x) \notin R$, for all $x, y \in A$

(OR)

whenever xRy and yRx, then $x = y$.

(v) transitive if xRy and, $yRz \Rightarrow xRz$, for all, $x, y, z \in A$

(vi) assymmetric if $xRy \Rightarrow y \not R x$ or $(y, x) \notin R$.

Example 9.1 (i) The relation $\leq$ is reflexive in the set of real numbers.

(ii) But the relation $<$ is not reflexive in the set of real numbers.

(iii) The relation $<$ is irreflexive in the set of real numbers

(iv) The relation "Similarity" in the set of triangles is both reflexive and symmetric.

(v) The relation "inclusion" in the collection of the subsets of a universal set is antisymmetric.

(vi) The relations $\leq$, $<$ and $=$ are transitive in the set of real numbers.

(vii) The relation "divides" is assymmetric in the set of real numbers.

Note 9.1 (i) If $(x, x) \notin R$ for atleast one $x \notin A$, then the relation R is not reflexive

(ii) If $(x, x) \in R$ for atleast one $x \in A$, then R is not irreflexive

(iii) If $(x, y) \in R$, but $(y, x) \notin R$ for atleast one pair $(x, y) \in A \times A$. then R is not symmetric.

(iv) Any relation which is not reflexive is not necessarily irreflexive and vice versa. The relation $s = \{(1, 1), (1, 2), (3, 2), (2, 3), (3, 3)\}$ in the set $\{1, 2, 3\}$ is not reflexive and not irreflexive.

(v) It is possible to have a relation which is both symmetric and antisymmetric Antisymmetric is not the same as not symmetric.

(vi) Any relation which is irreflexive and symmetric cannot be transitive because xRy and $yRx \Rightarrow xRx$, which is not true.

Example 9.2 If $A = \{1, 2, 3, 4\}$ then

(i) The relation $\{(1, 2), (2, 4)\}$ is not reflexive, not symmetric and not transitive.

(ii) The relation $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 2)\}$ is reflexive but neither symmetric nor transitive

(iii) The relation $\{(1, 1), (1, 3), (3, 1), (3, 4), (4, 3)\}$ is symmetric but neither reflexive nor transitive

(iv) The relation $\{(1, 1), (1, 3)\}$ is transitive but neither reflexive nor symmetric

(v) The relation $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 1), (3, 4), (4, 3)\}$ is reflexive, symmetric but not transitive.

(vi) The relation $\{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$ is symmetric transitive but not reflexive

(vii) The relation $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3)\}$ is reflexive transitive but not symmetric.

10. Operations on Relations:

Since relations from A to B are subsets of $A \times B$, then the Usual operations on sets such as complementation. Union and intersection can be applied to relations also.

Definition 10.1 Let R and S be relations from A to B

(i) $R \cap S = \{(a, b) \in A \times B/(a, b) \in R$ and $(a, b) \in S\}$ is the intersection of the relations R and S

$$a(R \cap S)b \Longleftrightarrow aRb \wedge aSb$$

(ii) $R \cup S = \{(a, b) \in A \times B|(a, b) \in R$ or $(a, b) \in S\}$ is the Union of the relations R and S

$$a(R \cup S)b \Longleftrightarrow aRb \vee aSb$$

(iii) $R - S = \{(a, b) \in A \times B|(a, b) \in R$ and $(a, b) \notin S\}$ is the difference of the relations R and S

$$a(R - S)b \Longleftrightarrow aRb \wedge a\$b$$

(iv) $R^c = \{(a,b) \in A \times B | (a,b) \in R\}$ is the complement of the relation R

$$a(R^c)b \Longleftrightarrow aRb$$

Example 10.1 Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R = \{(1,1),(2,2),(3,3)\}$ and $S = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined to obtain

$$R \cup S = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$
$$R \cap S = \{(1,1)\}$$
$$R - S = \{(2,2),(3,3)\}$$
$$S - R = \{(1,2),(1,3),(1,4)\}$$
$$S^c = \{(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4)\}$$
$$R^c = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4)\}$$

(v) Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a.c) where $a \in A$, $a \in c$ and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$.

We denote the composite of R and S by SoR.

Example 10.2 what is the composite of the relations R and S where R is the relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R = \{(1,1),(1,4),(2,3),(3,1),(3.4)\}$ and S is the relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ with $S = \{(1,0),(2,0),(3\ 1),(3,2),(4,1)\}$?

Solution: SoR is constructed Using all ordered pairs in R and ordered pairs in S where the second element of the ordered pair in R agrees with the first element of the ordered pair in S.

For example, the ordered pairs (2,3) in R and (3,1) in S produce the ordered pair (2,1) in SoR. Computing all the ordered pairs in the composite, we find

$$SoR = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$$

(vi) Given a relation R from X to Y, a relation $\tilde{R}$ from Y to X is called the converse of R, where the ordered pairs of $\tilde{R}$ are obtained by interchaning the members in each of the ordered pairs of R. This means, for $x \in X$ and $y \in Y$, that $xRy \Longleftrightarrow y\tilde{R}x$.

11. Equivalence relations:

Definition 11.1 A relation R in a set X is called an equivalence relation if it is reflexive, symmetric and transitive.

Example 11.1

(i) Equality of numbers on a set of real numbers

(ii) Equality of subsets of a Universal set.

(iii) Similarity of triangles on the set of triangles
(iv) Relation of statements being equivalent in the set of statements.

Example 11.2 Let $X = \{1, 2, \ldots, 7\}$ and $R = \{(x, y) | x - y$ is divisible by 3$\}$ in X. Show that R is an equivalence relation.

Solution: Showing R is an equivalence reation

(i) For any $a \in X$, a-a is divisible by 3 hence aRa
$\therefore$ R is reflexive
(ii) For any $a, b \in X$ if a-b is divisible by 3, then b-a is also divisible by 3, that is $aRb \Rightarrow bRa$. Thus R is symmetric
(iii) For $a, b, c \in X$, if aRb and bRc, then both a-b and b-c are divisible by 3, so that $a - c = (a - b) + (b - c)$ is also divisible by 3, and hence aRc.
Thus R is transitive
$\therefore$ R is an equivalence relation.

In general, Let I denote the set of all the integers, and let m be a positive integer. For $x \in I$ and $y \in I$ define R as

$$R = \{(x, y) | x - y \text{ is divisible by m}\}$$

is an equivalence relation.

Example 11.3 If $A = \{1, 2, 3, 4\}$, the relation $\{(1, 1), (2, 2), (3, 3), (4, 4)(1, 2), (2, 1)\}$ is an equivalence relation.

Definition 11.2 Let R be an equivalence relation on a set X.
For any $x \in X$, the set $[x]_R \subseteq X$ given by

$$[x]_R = \{y | y \in X \wedge xRy\}$$

s called an R-equivalence class generated by $x \in X$.

Definition 11.3 Let S be a given set and $A = \{A_1, A_2, \ldots, A_m\}$ where each $A_i, i = 1, \ldots, m$ is a subset of S and

$$\bigcup_{i=1}^{m} A_i = S$$

Then the set A is called a covering of S, and the sets $A_1, A_2, \ldots, A_m$ are said to cover S. f, in addition the elements of A, which are subsets of S, are mutually disjoint, then A is called a partition of S, and the sets $A_1, A_2, \ldots A_m$ are called the blocks of the partition.

(OR)

Definition (Partition set) 11.4

Let A be a non empty set and $A_1, A_2, \ldots A_n$ are the subsets of A. A set denoted by Π is called a partition set of A if

(i) $A_i \cap A_j = \phi, i \neq j$ and

(ii) $\bigcup_{i=1}^{n} A_i = A$

Example 11.4 (i) Let $A = \{a, b, c, d, e, f, g, h, i, j\}$

Let the subsets of A are

$$A_1 = \{a, b, c, d, e\}$$
$$A_2 = \{f, g, h\}$$
$$A_3 = \{i, j\}$$
$$A_4 = \{a, b, c, d\}$$
$$A_5 = \{c\}$$

Now $\Pi_1 = \{A_1, A_2, A_3\}$ is a partition because

$$A_1 \cap A_2 = \phi, \quad A_1 \cap A_3 = \phi, \quad A_2 \cap A_3 = \phi$$

Also $A_1 \cup A_2 \cup A_3 = A$.

But

$\Pi_2 = \{A_1, A_4, A_5\}$ is not a partition because $A_1 \cap A_4 \neq \phi$

$\Pi_3 = \{A_2, A_4, A_5\}$ is not a partition because

$A_4 \cap A_5 \neq \phi$ and $A_2 \cup A_4 \cup A_5 \neq A$.

(ii)

Z^+ = Set of all positive integers

P_1 = Set of all positive odd integgers

P_2 = Set of all positive even integers, clearly

$\Pi(Z^+) = \{P_1, P_2\}$ is a partition of Z^+

(iii) List all partitions of $A = \{1, 2, 3\}$

Consider $A_1 = \{1\}$ $A_4 = \{1, 2\}$

$A_2 = \{2\}$ $A_5 = \{2, 3\}$

$A_3 = \{3\}$ $A_6 = \{1, 3\}$

Now partition sets of A are

$$\Pi_1 = \{A_1, A_2, A_3\}$$
$$\Pi_2 = \{A_1, A_5\}$$
$$\Pi_3 = \{A_2, A_6\}$$
$$\Pi_4 = \{A_3, A_4\}$$

(iv) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$A_1 = \{1, 2, 3, 4\}, \quad A_2 = \{5, 6, 7\}, \quad A_3 = \{4, 5, 7, 9\}$$
$$A_4 = \{4, 8, 10\}, \quad A_5 = \{8, 9, 10\}, \quad A_6 = \{1, 2, 3, 6, 8, 10\}$$

Which of the following are partition of A?

(a) $\{A_1, A_2, A_3\}$ (b) $\{A_1, A_3, A_5\}$

(c) $\{A_3, A_6\}$ (d) $\{A_2, A_3, A_4\}$

Clearly

(a) & (c) are partition set of A

(b) & (d) are not partition set of A.

Example 11.5 Let $S = \{a, b, c\}$ and consider the following collections of subsets of S

$$A = \{\{a, b\}, \{b, c\}\} \qquad B = \{\{a\}, \{a, c\}\} \qquad c = \{\{a\}, \{b, c\}\}$$
$$D = \{\{a, b, c\}\} \qquad E = \{\{a\}, \{b\}, \{c\}\} \qquad F = \{\{a\}, \{a, b\}, \{a, c\}\}$$

The sets A and F are coverings of S

The sets C, D and E are partitious of S. Also every partition is a covering.

The set B is neither a partition nor a covering of S.

The partition D has only one block while E has three.

For any finite set, the smallest partition consists of the set itself as a block while the largest partition consists of blocks containing only single elements.

Two partitions are said to be equal if they are equal as sets. For a finite set, every partition is a finite partition i.e., every partition contains only a finite number of blocks.

Now, coming to the R-equivalence class, the set $[x]_R$ consists of all the R-relatives of x in the set X. Sometimes $[x]_R$ is also written as $\frac{x}{R}$.

Now, by the definition, the R-equivalence class generated by any element $y \in X$ is equal to the R-equivalence class generated by $x \in X$ provided that $y \in [x]_R$. Otherwise the R-equivalence classes generated by x and y are disjoint.

Further, each element of X generates an R-equivalence class which is non empty. Therefore the R-equivalence classes generated by the elements of X cover X, i.e., their union in the set X. Since the R-equivalence classes generated by any two elements are either equal or disjoint, we can say that the family of R-equivalence classes generated by the elements of X defines a partition of X. Such a partition is unique because an R-equivalence class of any element of X is unique.

Now, formulating the above idea:

Every equivalence relation on a set generates a unique partition of the set. The blocks of this partition correspond to the R-equivalence classes.

Theorem 11.1 Let P be a partition of the set A. Define a relation R on A as $aRb \Leftrightarrow a$ and b are members of the same block. Then R is an equivalence relation on A.

Proof: From the statement we infer the following

1. If $a \in A$ then a and a are in the same block : So aRa
2. $aRb \Rightarrow a$ and b are in the same block $\Rightarrow bRa$
3. aRb and $bRc \Rightarrow a, b, c$ are in the same block $\Rightarrow aRc$

Hence R is reflexive, symmetric and transitive and therefore an equivalence relation. □

Theorem 11.2 Let R be an equivalence relation defined on A. Then R induces a partition on A.

Proof: Let $a \in A$. Then $[a] = \{x \in A | aRx\}$

Since R is reflexive $a \in [a]$ and hence $[a] \neq \phi$. Thus every element of A is in some $[x]$. Further

$$\begin{aligned}
[a] \cap [b] \neq \phi &\Rightarrow c \in ([a] \cap [b]) \quad \text{for some } c \in A \\
&\Rightarrow c \in [a] \text{ and } c \in [b] \\
&\Rightarrow aRc \text{ and } bRc \\
&\Rightarrow aRc \text{ and } cRb \quad \text{Since } R \text{ is symmetric} \\
&\Rightarrow aRb \quad \text{Since } R \text{ is transitive} \\
&\Rightarrow b \in [a] \\
&\Rightarrow a \in [b] \quad \text{Since } R \text{ is symmetric}
\end{aligned}$$

Now

$$\begin{aligned}
x \in [b] &\Rightarrow bRx \\
&\Rightarrow xRb \\
&\Rightarrow xRa \text{ and } bRa \\
&\Rightarrow aRx \\
&\Rightarrow x \in [a]
\end{aligned}$$

Therefore $[b] \subseteq [a]$. Similarly $[a] \subseteq [b]$. Thus $[a] = [b]$. Hence any $[a]$ and $[b]$ are either identical or disjoint.

Thus R induces a partition P of A by the subsets $[a]$ as

1. every element of A is one of the elements of P
2. $[a] \cap [b] \neq \phi \Rightarrow [a] = [b]$

The sets $[a]$ are called equivalence classes of R □

Note 11.1 The partition P will be denoted by $\frac{A}{R}$. The elements of $\frac{A}{R}$ are called quotient sets of A w.r. to R

Example 11.6 Let $A = \{1, 2, 3, 4\}$ and $P = \{\{1, 2, 3\}, \{4\}\}$ be a partition of A. Find the equivalence relation determined by P.

Solution: Each element in the block is related to every other element in the same block and only to those elements. Hence

$$R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3) \\ (4, 4)\}$$

Example 11.7 Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 1), (1, 2), (2, 1) (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$ be an equivalence relation on R. Determine $\frac{A}{R}$

Solution: Here $[1] = \{1, 2\} = [2], [3] = \{3, 4\} = [4]$
The partition of A is $\{(1, 2), (3, 4)\}$
Hence $\frac{A}{R} = \{[1], [3]\}$

General procedure for construction of $\frac{A}{R}$

1. Choose $a \in A$ and find $[a]$
2. If $[a] \neq A$ choose $b \in A$ and $b \notin [a]$ and find $[b]$
3. If $[a] \cup [b] \neq A$ choose $c \in A$ and $c \notin [a] \cup [b]$ and find $[c]$
4. Repeat steps until all elements of A are included in the computed equivalence classes.

Example 11.8 Let $\mathbb{Z}$ be the set of integers and let R be the relation called "congruence modulo 3" defined by

$$R = \{(x, y) | x \in \mathbb{Z} \wedge y \in \mathbb{Z} \wedge (x - y) \text{ is divisible by } 3\}$$

Determine the equivalence classes generated by the elements of $\mathbb{Z}$.

Solution: The equivalence classes are

$$[0]_R = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$
$$[1]_R = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$$
$$[2]_R = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$$
$$\frac{\mathbb{Z}}{R} = \{[0]_R, [1]_R, [2]_R\}$$

In a similar manner one can find the equivalence classes generated by a relation "congruence modulo m" for any integer m.

Example 11.9 Let s be the set of all statement functions in n variables and let R be the relation given by

$$R = \{(x, y) | x \in s \wedge y \in s \wedge x \Leftrightarrow y\}$$

Discuss the equivalence classes generated by the elements of S.

Solution: The number of possible distinct truth tables for statement functions which depend upon n statement variables is 2^{2^n}

Thus there are 2^{2^n} R-equivalence classes generated by the elements of S.

Example 11.10 Let $X = \{a, b, c, d, e\}$ and let $C = \{\{a, b\}, \{c\}, \{d, e\}\}$ show that the partition C defines an equivalence relation on X.

Solution:

$$R = \{(a, a), (b, b), (a, b), (b, a), (c, c), (d, d), (e, e), (d, e), (e, d)\}.$$

Example 11.11 Let R be the following equivalence relation on the set

$A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$

Find the partition of A induced by R i.e., Find the equivalence class of R

Solution: The elements related to 1 are 1 and 5 hence

$$[1] = \{1, 5\}$$

Now pick that element which does not belong to [1] i.e., 2 which is related to 2, 3 and 6 hence

$$[2] = \{2, 3, 6\}$$

The only element which does not belong to [1] or [2] is 4, which is the only element related to itself 4.

Thus $[4] = \{4\}$

Hence partition of A induced by R i.e;

$$\frac{A}{R} = \{\{1, 5\}, \{2, 3, 6\}, \{4\}\}$$

hence there are three distinct classes.

Example 11.12 Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Determine a relation R on A b $aRb \Longleftrightarrow 3$ divides $(a - b)$. Show that R is an equivalence relation. Also determine th partition generated by R.

Solution:

$$A = \{1, 2, 3, 4, 5, 6, 7\}$$
$$R = \{(a, b) | 3 \text{ divides}(a - b)\}$$
$$R = \{(1, 1), (1, 4), (1, 7), (2, 2), (2, 5), (3, 3), (3, 6), (4, 1)$$
$$(4, 4), (4, 7), (5, 2), (5, 5), (6, 3), (6, 6), (7, 1), (7, 4), (7, 7)\}$$

For $a \in A, 3|(a - a)$ $\therefore$ $(a, a) \in R$, **R is reflexive**

For $(a, b) \in R, 3|(a - b) \Rightarrow 3|(b - a) \Rightarrow (b, a) \in R$ $\therefore$ R is symmetric

For $(a, b) \in R, (b, c) \in R \Rightarrow 3|(a, b)$ and $3|(b - c) \Rightarrow 3|(a - c)$

$\therefore$ R is transitive.

$\therefore$ R is an equivalence relation on A

The partition generated by $R = \{\{1, 4, 7\}, \{2, 5\}, \{3, 6\}\}$ □

Example 11.13 Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3),$ $(3, 1), (3, 2), (3, 3), (4, 4)\}$. Is R an equivalence relation? If yes, Find the partition of A induced by R

Solution:

$(1, 1), (2, 2), (3, 3), (4, 4) \in R \rightarrow$ reflexive

$$\left.\begin{matrix}(1, 2) \in R, (2, 1) \in R \\ (2, 3) \in R, (3, 2) \in R\end{matrix}\right\} \rightarrow \text{symmetric}$$

$$\left.\begin{matrix}(2, 1) \in R \text{ and } (1, 2) \in R \text{ then } (2, 2) \in R \\ (3, 2) \in R \text{ and } (2, 1) \in R \text{ then } (3, 1) \in R\end{matrix}\right\} \rightarrow \text{transitive}$$

R is an equivalence relation on A

Now

Partition of A induced by $R = \{\{1, 2, 3\}, \{4\}\}$. □

Example 11.14 If $\{\{1, 3, 5\}, \{2, 4\}\}$ is a partition set of the set $A = \{1, 2, 3, 4, 5\}$. Determine the corresponding equivalence relation?

Solution: The equivalence relation corresponding to the partition set is

$$R = \{\{1, 3, 5\} \times \{1, 3, 5\}, \{2, 4\} \times \{2, 4\}\}$$
$$= \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5), (2, 2),$$
$$(2, 4), (4, 2), (4, 4)\}$$

Example 11.15 List all partitions of $A = \{1, 2, 3\}$. Show that any partition of the set induces an equivalence relation

Solution: Writing the subsets of A

$$P_1 = \{1\},\ P_2 = \{2\},\ P_3 = \{3\},\ P_4 = \{1, 2\} P_5 = \{2, 3\},\ P_6 = \{1, 3\}$$

The partition sets of P are

(i)

$$\Pi_1 = \{\{1\}, \{2\}, \{3\}\}$$

Π_1 doesnot induce an equivalence relation

(ii)

$$\Pi_2 = \{\{1\}, \{2, 3\}\}$$

$\Rightarrow R = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$ is an equivalence relation.

Thus Π_2 induces an equivalence relation

(iii)

$$\Pi_3 = \{\{2\}, \{1, 3\}\}$$

$R = \{(2, 2), (1, 1), (1, 3), (3, 1), (3, 3)\}$ is an equivalence relation.

$\therefore\ \Pi_3$ induces an equivalence relation

(iv)

$$\Pi_4 = \{\{3\}, \{1, 2\}\}$$

$R = \{(1, 1), (3, 3), (1, 2), (2, 1), (2, 2)\}$ is an equivalence relation

$\therefore\ \Pi_4$ induces an equivalence relation.

Example 11.16 Let $A = \{a, b, c, d, e, f\}$ and $P = \{\{a, b, d\}, \{c, e, f\}\}$ is a partition of A. Determine the corresponding equivalence relation R.

Solution: Let $A_1 = \{a, b, d\}$ and $A_2 = \{c, e, f\}$

$$R = \{\{A_1 \times A_1\}, \{A_2 \times A_2\}\} \text{ is the equivalence relation}$$

$$\Rightarrow \quad R = \{(a, a), (a, b), (a, d), (b, a), (b, b), (b, d), (d, a), (d, b), (d, d), (c, c), (c, e), (c, f), (e, c), (e, e), (e, f), (f, c), (f, e), (f, f)\}$$

Example 11.17 Let $A = \{1, 2, 3, 4, 5\}$ and $P = \{\{1, 2, 4\}, \{3, 5\}\}$ is a partition of A. Determine the corresponding equivalence relation R

Solution: Let $A_1 = \{1, 2, 4\}$ and $A_2 = \{3, 5\}$

$$R = \{\{A_1 \times A_1\}, \{A_2 \times A_2\}\}$$

$$\Rightarrow \quad R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (4, 1), (4, 2), (4, 4), (3, 3), (3, 5), (5, 3), (5, 5)\}.$$

Example 11.18 Let $S = \{1, 2, 3, 4\}$ and $A = S \times S$. Define a relation R on A by $(a, b)R(a', b') \Longleftrightarrow a + b = a' + b'$

(i) Show that R is an equivalence relation

(ii) Compute $\frac{A}{R}$

Solution:

$$S = \{1, 2, 3, 4\}, A = S \times S, R = \{(a, b)R(a', b')|a + b = a' + b'\}$$

Clearly by $a + b = a + b, a + b = a' + b' \Rightarrow a' + b' = a + b$ and

$$a + b = a' + b' \text{ and } a' + b' = a'' + b'' \Rightarrow a + b = a'' + b''$$

R is reflexive, symmetric, transitive $\Rightarrow$ R is an equivalence relation.

$$A = \{\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}\}$$

$$\frac{A}{R} = \{\{(1, 1), (2, 2), (3, 3), (4, 4)\}, \{(1, 2), (2, 1), \}$$
$$\{(1, 3), (3, 1)\}, \{(1, 4), (4, 1)\}\{(2, 3), (3, 2)\}$$
$$\{(2, 4), (4, 2)\}, \{(3, 4), (4, 3)\}\}$$

Example 11.19 Let $A = \{1, 2, 3, 4, 5\}$ and $P = \{A \times A\}$. R is a relational set such that $(x, y)R(x', y') \Longleftrightarrow xy' = x'y$. Show that R is an equivalence relation. Compute $\frac{A}{R}$

Solution:

$$A = \{1, 2, 3, 4, 5\} \text{ and } P = \{A \times A\}$$
$$R = \{(x, y), (x', y')|xy' = x'y\}$$

1. $(x, y)R(x, y) \Longleftrightarrow xy = xy$
2.

$$\begin{aligned}(x, y), R(x', y') &\Rightarrow xy' = x'y \\ &\Rightarrow y'x = yx' \\ &\Rightarrow ((x', y'), (x, y)) \in R\end{aligned}$$

3. $((x, y), (x', y')) \in R$ and $((x', y'), (x'', y'')) \in R$ then $xy' = x'y$ and $x'y'' = x''y'$

Now

$$xy'' = x\left(\frac{x''y'}{x'}\right) = (xy')\left(\frac{x''}{x'}\right)$$
$$= x''y$$
$$\Rightarrow \quad ((x, y), (x'', y'')) \in R$$

$\therefore$ R is an equivalence relation

$$\frac{A}{R} = \{\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), \}, \{(1, 2), (2, 4)\},$$
$$\{(2, 1), (4, 2)\}, \{(1, 3)\}, \{(1, 4)\}, \{(2, 3)\}, \{(2, 5)\},$$
$$\{(3, 1)\}, \{(3, 2)\}, \{(3, 4)\}, \{(1, 5)\}, \{(3, 5)\}, \{(4, 1)\},$$
$$\{(4, 3)\}, \{(4, 5)\}, \{(5, 1)\}, \{(5, 2)\}, \{(5, 3)\}, \{(5, 4)\}\}.$$

Example 11.20 Let R denote a relation on the set of all ordered pairs of positive integers by

$$(x, y)R(u, v) \quad \text{if and only if} \quad xv = yu$$

Show that R is an equivalence relation

Solution:

(i) As $xy = yx$ is true for all positive integers x and y we have $(x, y)R(x, y)$ is fo all ordered pair (x, y) of positive integers. So the relation R is reflexive.

(ii)

$$(x, y)R(u, v) \Longrightarrow xv = yu$$
$$\Longrightarrow yu = xv$$
$$\Longrightarrow uy = vx$$
$$\Longrightarrow (u, v)R(x, y)$$

So R is symmetric

(iii) Let x, y, u, v, m, n be positive integers

$$(x, y)R(u, v) \quad \text{and} \quad (u, v)R(m, n)$$

$$\Longrightarrow xv = yu \text{ and } un = vm$$
$$\Longrightarrow xvun = yuvm$$
$$\Longrightarrow xn = ym \quad \text{by Cancelling } vu(\neq 0)$$
$$\Longrightarrow (x, y)R(m, n)$$

So R is transitive.

$\therefore$ Thus R is an equivalence relation.

Example 11.21 Let R and S be relations on A. Then

1) If R and S are reflexive, then $R \cup S$ is reflexive
2) If R and S are reflexive, then $R \cap S$ is reflexive
3) If R and S are Symmetric, then $R \cap S$ and $R \cup S$ are symmetric
4) If R and S are transitive, then $R \cap S$ is transitive
5) If R and S are equivalence relations, then So is $R \cap S$
6) If R is reflexive, the $\tilde{R}$ is also reflexive
7) If R is transitive, then $\tilde{R}$ is also transitive
8) If R is an equivalence relation, then $\tilde{R}$ is also an equivalence relation

Solution: Proving (3): Assume that both R and S to be Symmetric. Then if $(a, b) \in R \cap S$, then $(a, b) \in R$ and $(a, b) \in S$

As $(a, b) \in R$ and as R is symmetric $(b, a) \in R$

As $(a, b) \in S$ and as S is symmetric $(b, a) \in S$

Then $(a, b) \in R \cap S \Longrightarrow (b, a) \in R \cap S$

So $R \cap S$ is also symmetric

proving (7): If R is transitive, then $(a, b), (b, c) \in R \Longrightarrow (b, a), (c, b) \in \tilde{R}$

$$(a, c) \in R \Longrightarrow (c, a) \in \tilde{R}$$

$\therefore$ $\tilde{R}$ is transitive

Definition 11.4 Let R be a relation from a set A to a set B . The inverse relation from B to A, denoted by R^{-1} is the set of ordered pairs $\{(b, a) : (a, b) \in R\}$.

We can prove the remaining results in similar manner.

Theorem 11.3 A relation R on A is symmetric if and only if $R = R^{-1}$

Proof: Suppose R is symmetric and $(a, b) \in R$

$$(a, b) \in R \Rightarrow (b, a) \in R$$
$$\Rightarrow (a, b) \in R^{-1}$$

$\therefore$ $R \subseteq R^{-1}$

$$(a, b) \in R^{-1} \Rightarrow (b, a) \in R$$
$$\Rightarrow (a, b) \in R^{-1} \quad (\because \quad R \text{ is symmetric})$$

$R^{-1} \subseteq R$

Hence $R = R^{-1}$

Conversely suppose $R = R^{-1}$ and $(a, b) \in R$

Then $(a, b) \in R^{-1} \Rightarrow (b, a) \in R$

$\therefore$ R is symmetric □

Theorem 11.4 A relation R on a set A is reflexive $\Longleftrightarrow$ the inverse relation R^{-1} is reflexive

Proof:

$$\begin{aligned} R \text{ is reflexive} &\Longleftrightarrow (a, b) \in R \\ &\Longleftrightarrow (a, a) \in R^{-1} \\ &\Longleftrightarrow R^{-1} \text{ is reflexive} \end{aligned}$$

□

Theorem 11.5 The relation R on a set A is transitive $\Longleftrightarrow$ $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Proof: Suppose $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$ in particular $R^2 \subseteq R$.

Suppose $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R^2$. Since $R^2 \subseteq R$ we have $(a, c) \in R$. Therefore R is transitive conversely suppose that R is transitive. Now we prove $R^n \subseteq R$ by induction. When $n = 1$ then the result is true

Suppose $R^n \subseteq R$ we prove $R^{n+1} \subseteq R$

Let $(a, b) \in R^{n+1}$

Since $R^{n+1} = R^n o R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$

By inductive hypothesis $(x, b) \in R$ ($\because$ $R^n \subseteq R$)

Since R is transitive $(a, x) \in R$ and $(x, b) \in R \Rightarrow (a, b) \in R$

Therefore $R^{n+1} \subseteq R$

$\therefore$ Hence the result by induction on n. □

Result 11.1 Let A be any set and R be the reflexive relation on A. Then R^{-1} and $R \cup S$ are reflexive for any relation S on A.

Proof: Let $a \in A$

$$\begin{aligned} \text{Since } R \text{ is reflexive} &\Rightarrow aRa \\ &\Rightarrow aR^{-1}a \\ &\Rightarrow R \text{ is reflexive} \end{aligned}$$

Now $(a, a) \in R \subseteq R \cup S$ for every $a \in A$

$\therefore$ $R \cup S$ reflexive □

Result 11.2 Let A be any set and R be an anti symmetric relation on A. Show that R^{-1} is anti symmetric and $R \cap S$ is anti symmetric for any relation S on A.

Proof: Suppose $(a, b), (b, a) \in R^{-1}$

$$\Rightarrow (b, a), (a, b), \in R$$
$$\Rightarrow a = b \quad \text{(Since } R \text{ is antisymmetric)}$$

$\therefore$ R^{-1} is antisymmetic.

(ii) Suppose (a, b) and $(b, a) \in R \cap S$

$$\Rightarrow (a, b), (b, a), \in R$$

Since R is antisymmetric $\Rightarrow a = b$

Hence $R \cap S$ is anti symmetric □

Result 11.3 If R is an equivalence relation on a set A then R^{-1} is also an equivalence relation on A.

12. Representation of Relations

In this section we will discuss two alternative methods for representing relations. One method uses (zero-one) matrices, the other method uses directed graphs

Relation Matrix 12.1 If $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are finite sets containing m and n elements respectively and R is a relation from A to B, the we can represent the relation R by an $m \times n$ matrix, called Relation matrix, denoted by $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if} \quad (a_i, b_j) \in R \\ 0 & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

In otherwords, the matrix representing R has a 1 as its i-jentry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j. (Such a representation depends on the orderings used for A and B)

Example 12.1 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$, the relation R from A to B is given by

$$R = \{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3)\}$$

Now the relation matrix M_R is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Not only giving the relation matrix when a relation R is given, but also obtain the relation if the relation matrix is given.

Example 12.2 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$, which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution: Since R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$$

A relation matrix reflects some of the properties of a relation. In otherwords, the matrix of a relation on a set, which is square matrix, can be used to determine whether the relation has certain properties.

i) R is reflexive if all the elements on the main disagonal of M_R are equal to 1.

ii) R is Symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently R is symmetric if and only if $m_{ij} = m_{ji}$ for all pairs of integers i and j with $i = 1$ to n and $j = 1$ to n.
R is Symmetric if and only if $M_R = (M_R)^T$

iii) R is antisymmetric if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. In otherwords, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$

Example 12.3 Suppose that the relation R on a set is represented by the matrix

$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ Is R reflexive, Symmetric and/or antisymmetric?

Solution: Since all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, since M_R is symmetric it follows that R is symmetric. It is also easy to see that R is not antisymmetric.

Digraph of a relation 12.2 A relation can be represented pictorially by drawing its digraph as follows:

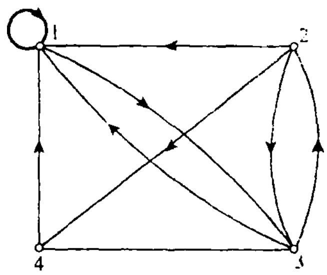
A small circle is drawn for each element of A and marked with the corresponding element. These circles are called vertices.

An arrow is drawn from the vertex a_i to the vertex a_j if and only if $a_i R a_j$. This is called an edge (directed). Note that an element of the form (a, a) in a relation corresponds to a directed edge from a to a. Such an edge is called a loop.

This pictorial representation of R is called a directed graph or digraph of R.

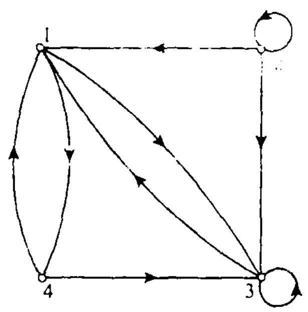
Example 12.4 The directed graph of the relation $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$ is:

Solution:



Example 12.5 What are the ordered pairs in the relation R represented by the directed graph shown below?

Solution:



The ordered pairs (x, y) in the relation are

$$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}.$$

Each of these pairs corresponds to an edge of the directed graph with (2, 2) and (3, 3) corresponding to loops

The directed graph representing a relation can be used to determine whether the relation has various properties.

(i) A relation is reflexive if and only if there is a loop at every vertex of the directed graph, so that every ordered pair of the form (x, x) occurs in the relation.

(ii) A relation is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so tht (y, x) in the relation whenever (x, y) is in the relation.

(iii) A relation is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices

(iv) A relation is transitive if and only if whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from x to z (completing a triangle where each side is a directed edge with the Correct direction).

Theorem: Prove that the transitive closure R^+ of a relation R on a set A is the smallest transitive relation on A containing R.

Proof: To prove (*i*) R^+ is a transitive relation on A

Let $a, b, c \in R^+$ so that aR^+b, bR^+c

$$\Rightarrow \quad (a, b) \in R^+, (b, c) \in R^+$$

We know that $R^+ = R \cup R^2 \cup R^3 \cup \ldots$

$\therefore$ $(a, b) \in R^k$, $(b, c) \in R^l$ for some positive integers $k\&l$, using $(a, b) \in R^k$, there are elements $c_1, c_2, \ldots c_k \in A$ such that $aRc_1, c_1Rc_2, c_2Rc_3, \ldots c_{k-1}Rc_k = b$ using $(b, c) \in R^l$ there exist elements $d_1, d_2, \ldots, d_l \in A$ such that $bRd_1, d_1Rd_2, \ldots, d_{l-1}Rd_l = c$

$$\therefore \quad aRc_1, c_1Rc_2 \ldots c_{k-1}Rc_k = b, bRd_1, d_1Rd_2, \ldots d_{l-1}Rd_l = c$$

$\therefore$ By definition $aR^{k+l}c \Rightarrow (a, c) \in R^{k+l}$

$$\Rightarrow \quad (a, c) \in R^+ \Rightarrow aR^+c$$

$\therefore$ R^+ is transitive relation on the set A.

To prove (ii) R^+ is the smallest transitive relation containing R

Clearly $R \subseteq R^+$

Let T be a transitive relation containing R

Then $R \subseteq T$

To prove $R^+ \subseteq T$

Let $(x, y) \in R^+ \Rightarrow xR^+y$ and then $(x, y) \in R^m$ for some m

There exists elements $y_1, y_2, \ldots y_m \in A$ satisfying $xRy_1, y_1Ry_2, y_2Ry_3 \ldots, y_{m-1}, Ry_m = y$

$\Rightarrow$ $(x, y_1) \in R, (y_1, y_2) \in R, (y_2, y_3) \in R \ldots, (y_{m-1}, y_m) \in R$

$\Rightarrow$ $(x, y_1) \in T, (y_1, y_2) \in T, (y_2, y_3) \in T \ldots, (y_{m-1}, y_m) \in T$ $\quad (\because \; R \subseteq T)$

$\Rightarrow$ $xTy_1, y_1Ty_2, \ldots y_{m-1}Ty_m = y$

$\Rightarrow$ xTy $\quad (\because \; T$ is transitive)

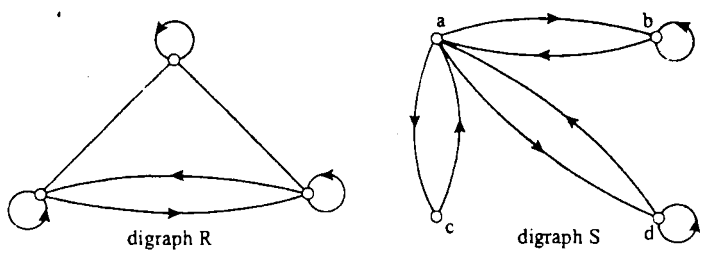
$\therefore$ $(x, y) \in T$

$\therefore$ $(x, y) \in R^+ \Rightarrow (x, y) \in T$

$R^+ \subseteq T$

Hence the proof. □

Example 12.6 Determine whether the relations for the directed graphs shown below are reflexive, symmetric, antisymmetric and/or transitive.



Since there are loops at every vertex of the directed graph of R, it is reflexive. R is neither Symmetric nor antisymmetric. Since there is an edge from a to b but not one from b to a but there are edges in both directions connecting b and c. Finally, R is not transitive since there is an edge from a to b and an edge from b to c, but no edge from a to c.

Since loops are not present at all the vertices of the directed graph of S, this relation is not reflexive. It is symmetric and not antisymmetric. Since every edge between distinct vertices is accompanied by an edge in the opposite direction. It is also not hard to see from the directed graph that S is not transitive. Since (c, a) and (a, b) belong to S, but (c, b) does not belong to S.

We obtain the matrices of Converse relation of R and composition of two relations as follows:

1. The relation matrix $M_{\tilde{R}}$ of $\tilde{R}$ can be obtained by simply interchanging the rows and columns of M_R.
   Such a matrix is called the transpose of M_R. Therefore

$$M_{\tilde{R}} = (M_R)^T$$

   The graph of $\tilde{R}$ is also obtained from that of R by simply reversing the arrows on each arc.

   Example 12.7 Given the relation matrix M_R Find $M_{\tilde{R}}$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

   Solution: $M_{\tilde{R}} =$ transpose of $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

2. *Boolean arithmetic with zero-one matrices:* This arithmetic is based on the Boolean operations $\vee$ and $\wedge$, which operate on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } \; b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } \; b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Definition 12.3 Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ (zero-one) matrices. Then the join of A and B is the zero-one matrix with $(i, j)^{\text{th}}$ entry $a_{ij} \vee b_{ij}$. The join of A and B is denoted dby $A \vee B$. The meet of A and B is the zero-matrix with $(i, j)^{\text{th}}$ entry $a_{ij} \wedge b_{ij}$. The meet of A and B is denoted by $A \wedge B$.

Example 12.8 Find the join the meet of the zero-one matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution: Join of A and B $(A \vee B) = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} =$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

meet of A and B $(A \wedge B) = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Definition 12.4 Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be an $k \times n$ zero-one matrix. Then the Boolean product of A and B, denoted by $A \odot B$, is the $m \times n$ matrix with $(i, j)^{\text{th}}$ entry $[c_{ij}]$ where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$$

Example 12.9 Find the Boolean product of A and B

Where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Solution: The Boolean product $A \odot B$ is given by

$$A \odot B = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The Boolean operations join and meet can be used to find the matrices representing the Union and the intersection of two relations. Suppose that R and S are relations on a set A represented by the matrices M_R and M_S respectively. The matrix representing the Union of these relations has a 1 in the positions where either M_R or M_S has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both M_R and M_S have a 1. Thus the matrices representing the Union and intersection of these relations are

$$M_{R\cup S} = M_R \vee M_S$$

and

$$M_{R\cap S} = M_R \wedge M_S$$

Example 12.10 Suppose that the relations R and S on a set A are represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

what are the matrices representing $R \cup S$ and $R \cap S$

Solution: The matrices of these relations are

$$M_{R\cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R\cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We now turn our attention to determining the matrix for the composite of relations. This matrix can be found using the Boolean product of the matrices for these relations.

In particular, Suppose that R is a relation from A to B and S is a relation from B to C. Suppose that A, B and C have m, p and n elements respectively. Let the zero-one matrices for SoR, R and S be $M_{SoR} = [t_{ij}]$, $M_R = [r_{ij}]$ and $M_S = [S_{ij}]$ respectively (these matrices have sizes $m \times p$, $m \times n$ and $n \times p$ respectively). The ordered pair (a_i, c_j) belongs to SoR if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S. It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k.

From the definition of the Boolean product, this means that

$$M_{SoR} = M_R \odot M_S$$

Example 12.11 Find the matrix representing the relations SoR where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution: The matrix for SoR is

$$M_{SoR} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

3. We now consider the Converse of a composite relation. For this purpose, Let R be a relation from A to B and S be a relation from B to C. Obviously $\tilde{R}$ is a relation from B to A, $\tilde{S}$ from C to B, SoR is a relation from A to C, and $\tilde{S}\tilde{o}R$ is a relation from C to A. Also the relation $\tilde{R}o\tilde{S}$ is from C to A.
   We now show that $\tilde{S}\tilde{o}R = \tilde{R}o\tilde{S}$
   If aRb and bSc, then $a(SoR)c$ and $c(\tilde{S}\tilde{o}R)a$. But $c\tilde{S}b$ and $b\tilde{R}a$ so that $c(\tilde{R}o\tilde{S})a$. This is true for any $a \in A$ and $c \in C$, hence the required result.

The same rule can be expressed in terms of the relation matrices by saying that the transpose of M_{SoR} is the same as the matrix $M_{\tilde{R}o\tilde{S}}$. The matrix $M_{\tilde{R}o\tilde{S}}$ can be obtained from the matrices $M_{\tilde{R}}$ and $M_{\tilde{S}}$, which inturn can be obtained from the matrices M_R and M_S.

Example 12.12 Given the relation matrices M_R and M_S
Find M_{SoR}. $M_{\tilde{R}}$, $M_{\tilde{S}}$, $M_{S\tilde{o}R}$ and show that $M_{S\tilde{o}R} = M_{\tilde{R}o\tilde{S}}$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Solution: $M_{\tilde{R}} =$ transpose of $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$M_{\tilde{S}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \text{transpose of} M_S$$

$$M_{SoR} = M_R \odot M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{S\bar{o}R} = \text{ transpose of } M_{S\bar{o}R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_{\bar{R}o\bar{S}} = M_{\bar{S}} \odot M_{\bar{R}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = M_{S\bar{o}R}$$

Result 12.1 The following hold for any relation R and S

(1) $(\bar{\bar{R}}) = R$
(2) $R = S \Longleftrightarrow \bar{R} = \bar{S}$
(3) $R \subseteq S \Longleftrightarrow \bar{R} \subseteq \bar{S}$
(4) $R\bar{\cup}S = \bar{R} \cup \bar{S}$
(5) $R\bar{\cap}S = \bar{R} \cap \bar{S}$

Definition 12.5 The transitive closure of a relation R is the smallest transitive relation containing R. We denote transitive closure of R by R^+. Let X be any finite set containing n elements and R be a relation in X. The relation $R^+ = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$ in X is called the transitive closure of R in X.

Example 12.13 Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on X. Find R^+

Solution: Given

$$R = \{(1, 2), (2, 3), (3, 4)\}$$

$$R^2 = \{(1,3),(2,4)\}$$
$$R^3 = \{(1,4)\}$$
$$R^4 = \phi$$
$$\therefore R^+ = R \cup R^2 \cup R^3 \cup R^4 \qquad (\because R^4 = \phi)$$
$$= \{(1,2),(2,3),(3,4),(1,3),(2,4),(1,4)\}$$

Definition 12.6 Let M be a square [zero-one] matrix and let r be a positive integer. The r^{th} Boolean power of M is the Boolean product of r factors of M. The r^{th} Boolean product of M is denoted by $M^{[r]}$. Hence

$$M^{[r]} = \underbrace{M \odot M \odot M \odot \ldots \odot M}_{\text{r times}}$$

This is well defined since the Boolean Product of matrices is associative

Example 12.14 Let $M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Find $M^{[n]}$ for all + ve integers n.

Solution: We find that $M^{[2]} = M \odot M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

We also find that $M^{[3]} = M^{[2]} \odot M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$M^{[4]} = M^{[3]} \odot M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

We can set that $M^{[n]} = M^{[5]}$ for all positive integers n with $n \geq 5$.

Definition 12.7 Let M_R be the (zero-one) matrix of the relation R on a set X with n elements. Then the matrix of the transitive closure R^+ is

$$M_{R^+} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}$$

Also $M_{R^n} = M_R^{[n]}$

Example 12.15 Find the matrix of the transitive closure of the relation R, where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution: Matrix of the transitive closure

$$M_{R^+} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$$

$$M_R^{[2]} = M_R \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_R^{[3]} = M_R^{[2]} \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Then

$$M_{R^+} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Note 12.1 Transitive closures of relations have important applications in certain areas such as networks, Syntactic analysis, fault detection and diagnosis in switching circuits.

13. Warshall's Algorithm:

Warshall's algorithm, is an efficient method for computing the (matrix of) transitive closure of a relation.

Algorithm: Given the relation matrix M_R of a simple digraph, then the following steps produce that matrix of transitive closure of the relation R

Step 1 : $P^{[0]} = M_R$

Step 2 : $k = 1$

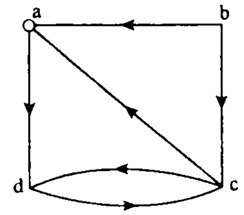
Step 3 : $i = 1$

Step 4 : $P_{ij}^{[k]} = P_{ij}^{[k-1]} \vee \left(P_{ik}^{[k-1]} \wedge P_{kj}^{[k-1]}\right) \forall j = 1$ to n

Step 5 : $i = i + 1$ If $i \leq n$, goto step 4

Step 6 : $k = k + 1$ If $k \leq n$, goto step3; otherwise stop.

Example 13.1 Let R be the relation with directed graph. Find the matrices M_R and



M_R+ (transitive closure of R) by Warshall's algorithm.

Solution:

$$M_R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = P^{[0]}(say)$$

$$P^{[1]} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = P^{[2]}$$

$$P^{[3]} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

and $P^{[4]} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ is the matrix of the transitive closure

Note 13.2 Warshall's algorithm computes M_R+ by efficiently computing $P^{[0]} = M_R$

$$P^{[1]}, P^{[2]}, \ldots P^{[n]} = M_R+.$$

14. Compatibility Relations

Definition 14.1 A relation R in X is said to be a compatibility relation if it is reflexive and Symmetric.

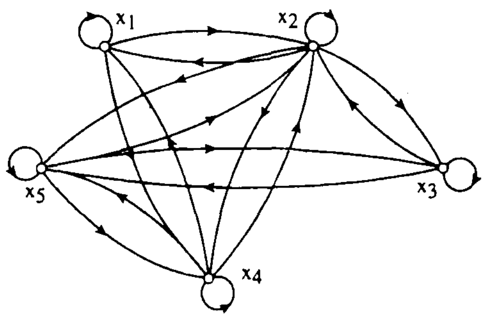
Clearly, all equivalence relations are compatibility relations. But we will consider those compatibility relations which are not equivalence relations.

Example 14.1 Let $X = \{$, ball, bed, dog, let, egg$\}$ and let the relation R be given by

$$R = \{(x, y) | x, y \epsilon X \wedge xRy \text{ if } x \text{ and } y \text{ contain some common letter}\}$$

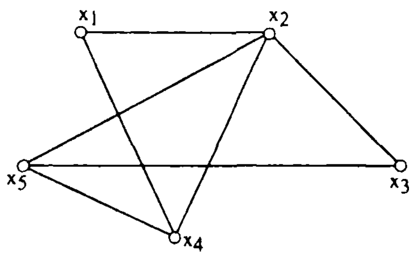
Then R is a compatibility relation, and x, y are called compatible if xRy. A compatibility relation is sometimes denoted by $\approx$ Note that ball $\approx$ red, bed $\approx$ egg, but ball $\approx$ egg. Thus $\not\approx$ is not transitive.

Denoting ball by x_1, bed by x_2, dog by x_3, Let by x_4 and egg by x_5, the graph of $\approx$ is as follows:



Since $\approx$ is a compatibility relation, it is not necessary to draw the loops at each element nor is it necessary to draw both xRy and yRx.

Thus, we can simplify the graph of $\approx$:



The elements in each of the sets $\{x_1, x_2, x_4\}$ and $\{x_2, x_3, x_5\}$ are related to each other i.e., the elements are mutually compatible further, these two sets define a covering of X. The set $\{x_2, x_4, x_5\}$ also has elements compatible to each other.

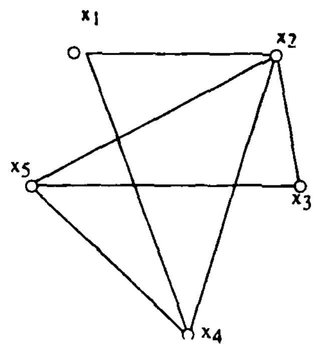
An equivalence relation on a set defines a partition of the set into equivalence classes, a compatibility relation does not necessarily define a partition. However, a compatibility relation does define a covering of the set.

The relation matrix of a compatibility relation is symmetric and has its diagonal elements unity. Therefore, it is sufficient to give only the elements of the lower triangular part of the relation matrix.

Now, The relation matrix for the above compatibility relation is

x_1	1			
x_2	0	1		
x_3	1	1	0	
x_4	0	1	1	1
	x_1	x_2	x_3	x_4

Definition 14.2 Let X be a set and $\approx$ a compatibility relation on X. A subset $A \subseteq X$ is called a maximal compatibility block if any element of A is compatible to every other element of A and no element of X-A is compatible to all the elements of A.



Example 14.2 The subsets $\{x_1, x_2, x_4\}$, $\{x_2, x_3, x_5\}$ and $\{x_2, x_4, x_5\}$ are maximal compatibility blocks. These sets are not mutually disjoint, and therefore they only define a covering of X.

Procedures to find the maximal compatibility blocks corresponding to a compatibility relation:

(I) (i) *First draw a simplified graph of the compatibility relation and*

(ii) pick from this graph the largest complete polygons. By a Largest complete polygon, we mean a polygon in which any vertex is connected to every other vertex.

For example, a triangle is always a complete polygon. In addition to these examples, any element of the set which is related only to itself forms a maximal compatibility block.

Similarly, any two elements which are compatible to one another but to no other elements also form a maximal compatibility block.

(II) Another procedure for finding the maximal compatibility blocks from the table of the relation matrix can be described in the following manner.

It is assumed that first a simplified table is obtained in which those elements which are only compatible to themselves are deleted, because they are in a maximal compatibility block by themselves and are in no other compatibility block. Such blocks are included in the list at the end.

(i) Start in the rightmost column of the table and proceed to the left until a column containing at least one nonzero entry is encountered. List all the compatible pairs represented by the entries in that column

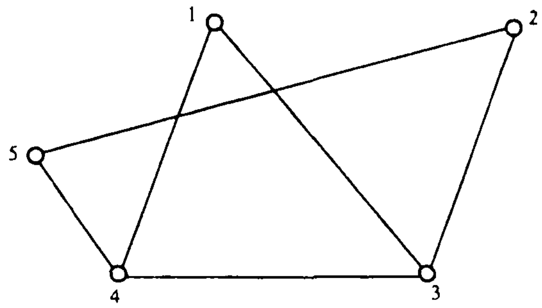
(ii) Proceed left to the next column that contains at least one nonzero entry. If any element is compatible to all the members of some previously defined compatibility class, then add this element to that class. If a member is compatible to only some members of a previously defined class, then form a new class which includes all the members that are compatible. Next, list all the compatible pairs not included in any previously defined class.

(iii) Repeat Step (ii) until all the columns are considered

The final sets of compatibility classes including those which are isolated elements constitute the maximal compatibility classes.

Example 14.3

(i)



2	0			
3	1	1		
4	1	0	1	
5	0	1	0	1
	1	2	3	4

The maximal compatibility blocks are

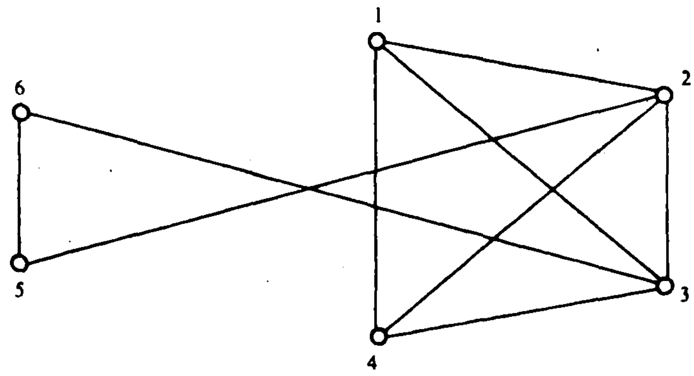
$$\{1, 3, 4\}, \{2, 3\}, \{4, 5\}, \{2, 5\}$$

Since 1,3,4 are compatible to each other, i.e., connected to each other, 2 and 3 connected each other, 4 and 5 connected each other, 2 and 5 connected each other, but not 5,3; 2,4;

(ii)

2	1				
3	1	1			
4	1	1	1		
5	0	1	0	0	
6	0	0	1	0	1
	1	2	3	4	5

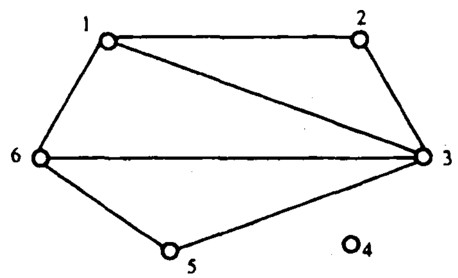
In graph



maximal compatibility blocks are

$$\{2, 1, 3, 4\}, \{2, 5\}, \{3, 6\}, \{5, 6\}$$

(iii)



2	0			
3	1	1		
5	0	0	1	
6	1	0	1	1
	1	2	3	5

maximal compatibility blocks are:

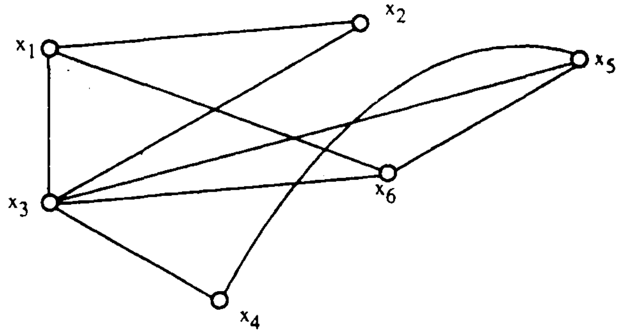
$$\{1, 2, 3\}, \{1, 3, 6\}, \{3, 5, 6\}, \{4\}$$

Example 14.4 Let the compatibility relation on a set $\{x_1 x_2 \ldots x_6\}$ be given by the matrix

x_2	1				
x_3	1	1			
x_4	0	0	1		
x_5	0	0	1	1	
x_6	1	0	1	0	1
	x_1	x_2	x_3	x_4	x_5

Draw the graph and find the maximal compatibility blocks of the relation.

Solution: Graph:



Maximal compatibility blocks are:

$$\{x_1, x_2, x_3\}, \{x_1, x_3 x_6\}, \{x_3, x_5 x_6\}, \{x_3, x_4, x_5\}.$$

15. Partial Order Relations

Definition 15.1 A binary relation R in a set P is called a partial order relation or a partial ordering in P iff R is reflexive, antisymmetric and transitive. We denote a partial ordering by the symbol $\leq$. If $\leq$ is a partial ordering on P, then the ordered pair $(P, \leq)$ is called a partially ordered set or a poset.

Definition 15.2 Let $(P, \leq)$ be a poset. If for every $x, y \in P$ we have either $x \leq y \vee y \leq x$, then $\leq$ is called a simple ordering or linear ordering on P, and $(P, \leq)$ is called a totally ordered or simply ordered set or a chain.

Note 15.1 It is not necessary to have $x \leq y$ or $y \leq x$ for every x and y in a poset P. Infact, x may not be related to y, in which case we say that x and y are incomparable.

If R is a partial ordering on P, then it is easy to see that the converse of R, namely $\tilde{R}$ is also a partial ordering on P. If R is denoted by $\leq$, then $\tilde{R}$ is denoted by $\geq$. This means that if $(P, \leq)$ is a poset, then $(P, \geq)$ is also a poset. $(P, \geq)$ is called the dual of $(P, \leq)$.

Definition 15.3 Another relationship which is associated with every partial ordering $\leq$ on P and which is denoted by $<$. This relation $<$ is defined, for every $x, y \in P$ as

$$x < y \Longleftrightarrow x \leq y \wedge x \neq y$$

Similarly, corresponding to the converse partial ordering $\geq$, there is a relation $>$ such that

$$x > y \Longleftrightarrow x \geq y \wedge x \neq y$$

Note 15.2 The relations $<$ and $>$ are irreflexive, antisymmetrix and transitive.

Example 15.1 Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Solution: Since $a \geq a$ for every integer a, $\geq$ is reflexive.
If $a \geq b$ and $b \geq a$, then $a = b$. Hence $\geq$ is antisymmetric finally $\geq$ is transitive since $a \geq b$ and $b \geq c$ imply that $a \geq c$ It follows that $\geq$ is a partial ordering on the set of integers and $(Z, \geq)$ is a poset.

Example 15.2 Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution: Since $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive.
It is antisymmetric since $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally $\subseteq$ is transitive, since $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence $\subseteq$ is a partial ordering on $P(S)$ and $(P(S), \subseteq)$ is a poset.

Example 15.3 The poset $(\mathbb{Z} \leq)$ is totally ordered, since $a \leq b$ or $b \leq a$ whenever a and b are integers.

Example 15.4 The divisibility relation $|$ is a partial ordering on the set of positive integers. Therefore $(Z^+, |)$ is a poset and it is not totally ordered since it contains elements that are incomparable, such as 5 and 7.

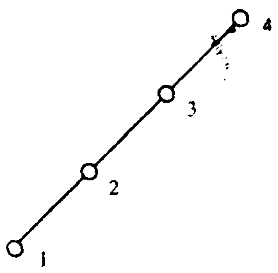
Definition 15.4 In a poset $(P, \leq)$, an element $y \in P$ is said to *cover* an element $x \in P$ if $x < y$ and if there does not exist any element $z \in P$ such that $x \leq z$ and $z \leq y$: that is y covers $x \iff (x < y \wedge (x \leq z \leq y \Rightarrow x = z \vee z = y))$.

16. Hasse Diagram

A partial ordering $\leq$ on a set P can be represented by means of a diagram known as Hasse diagram of $(P, \leq)$. In such a diagram, each element is represented by a small circle. The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$ and a line is drawn between x and y if y covers x. If $x < y$ but y does not cover x, then x and y are not connected directly by a single line. However, they are connected through one or more elements of P. It is possible to obtain the set of ordered pairs in $\leq$ from such a diagram.

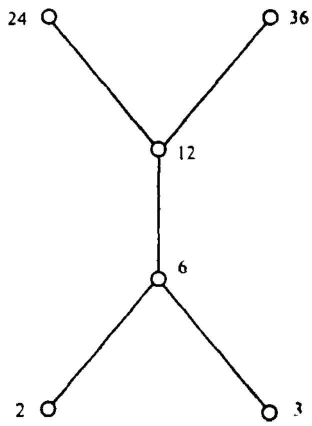
For a totally ordered set $(P, \leq)$, the Hasse diagram consists of circles one below the other. Thus a toset is called a chain

Example 16.1 Let $P = \{1, 2, 3, 4\}$ and $\leq$ be the relation "less than or equal to" then the Hasse diagram is:



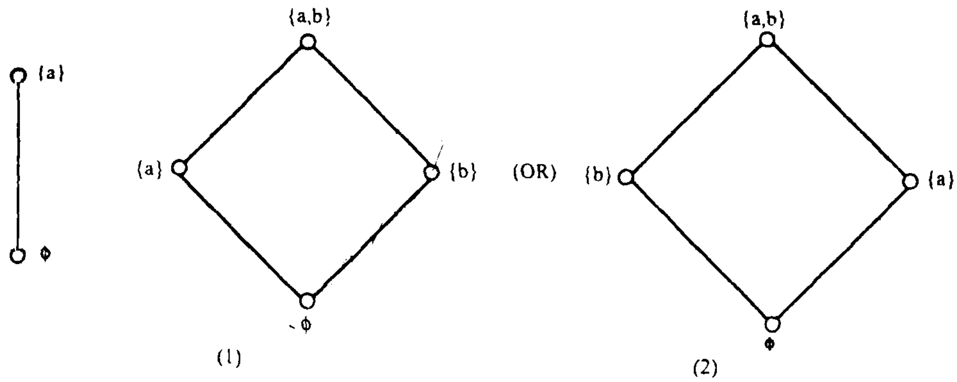
Example 16.2 Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if x divides y.

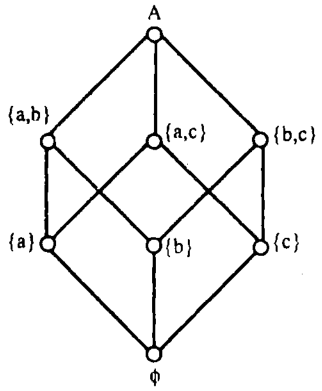
Hasse diagram is



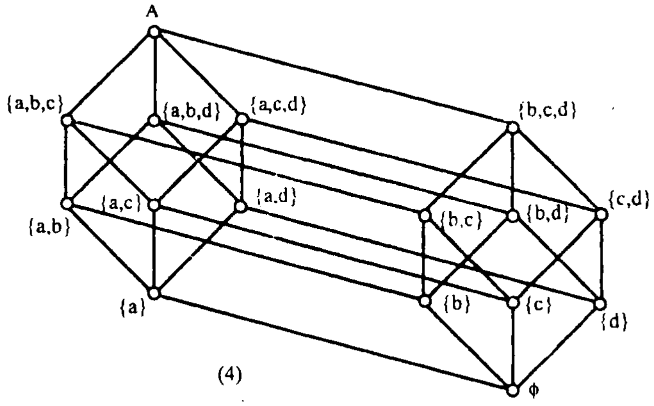
Example 16.3 Let A be a given finite set and $P(A)$ its power set. Let $\subseteq$ be the inclusion relation on the elements of $P(A)$. Draw Hasse diagrams of $(P(A), \subseteq)$ for $a) A = \{a\}$: $(b) A = \{a, b\} (c) A = \{a, b, c\} (d) A = \{a, b, c, d\}$

Solution: The required Hasse diagrams are:





(3)



(4)

Hasse diagrams of $(P(A), \subseteq)$

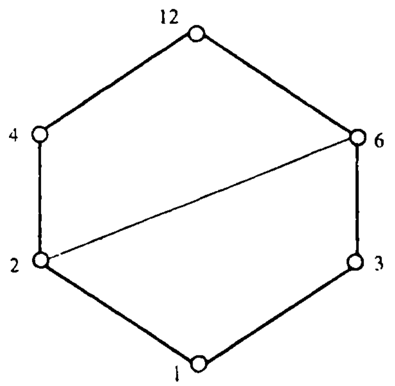
Example 16.4 Let A be the set of factors of a particular positive integer m and let $\leq$ be the relation divides i.e.,

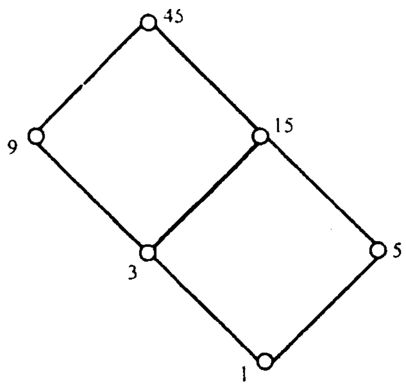
$$\leq = \{(x, y) | x \in A \wedge y \in A \wedge (x \text{ divides } y)\}$$

Draw Hasse diagrams for (a) $m = 2$ (b) $m = 6$ (c) $m = 30$ (d) $m = 210$ (e) $m = 12$ and (f) $m = 45$.

Solution: The required Hasse diagrams for (a) to (d) are the same as previous Hasse diagrams 1,2,3 and 4.

Hasse diagrams of (e) and (f) are





Note 16.1 (i) For a given poset, a Hasse diagram is not unique, as can be seen from Fig 2.

(ii) From a Hasse diagram of $(P, \leq)$, the Hasse diagram of $(P, \geq)$ which is the dual of $(P, \leq)$ can be obtained by rotating the diagram through 180° so that the points at the top become the points at the bottom.

(iii) Some Hasse diagrams have a unique point which is below all other points.

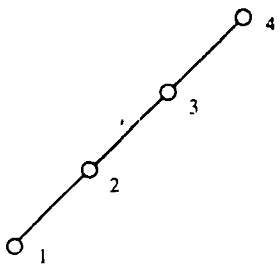
(iv) The Hasse diagrams become more complicated when the number of elements in the poset is large.

Definition 16.1 If there exists an element $y \in P$ such that $y \leq x$ for all $x \in P$, then y is called the least member in P relative to the partial ordering $\leq$. Similarly, if there exists an element $y \in P$ such that $x \leq y$ for all $x \in P$, then y is called the greatest member in P relative to $\leq$.

Note 16.2 From the definition it is clear that the least member if it exists, is unique, so also is the greatest member. It may happen that the least or the greatest member does not exist. The least member is usually denoted by 0 and the greatest by 1.

If the Hasse diagram of a poset is available, then it is easy to see whethere the least or the greatest member exists.

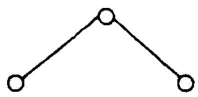
Example 16.5

From 

it is clear that the least member is 1 and the greatest is 4.

In Example 16.2, there is no least or greatest member.

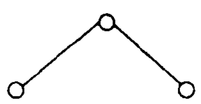
In Example 16.3, the least member is ϕ and the greatest member is A in all cases.



The above diagram shows that the greatest member exists but there is no least member.

Observations 16.1 In every simple ordering or chain, the least and the greatest members always exists.

Definition 16.2 An element $y \in P$ is called a minimal member of P relative to a partial ordering $\leq$ if for no $x \in P$ is $x < y$. Similarly, an element $y \in P$ is called a maximal member of P relative to a partial ordering $\leq$ if for no $x \in P$ is $y < x$.

Example 16.6 In the Hasse diagram,  there are two minimal members, and one maximal member.

Note 16.3 A minimal member need not be unique. All those members which appear at the lowest level of a Hasse diagram of a poset are minimal members.

Distinct minimal members are incomparable and distinct maximal members are also incomparable.

Note 16.4 It is not always necessary to draw the Hasse diagrams of a poset in order to determine the least, greatest, maximal and minimal members. However, their determination becomes simple when such a diagram is available.

Definition 16.3 Let $(P, \leq)$ be a poset and let $A \subseteq P$. Any element $x \in P$ is an upper bound for A if for all $a \in A, a \leq x$. Similarly, any element $x \in P$ is a lower bound for A if for $a \in A, x \leq a$.

Example 16.7 Let us consider the poset $(P(A), \subseteq)$. We choose a subset B of $P(A)$ given by $\{\{b, c\}, \{b\}, \{c\}\}$. Then $\{b, c\}$ and A are upper bounds for B, while ϕ is its lower bound. For the subset $C = \{\{a, c\}, \{c\}\}$, the upper bounds are $\{a, c\}$ and A while the lower bounds are $\{c\}$ and ϕ.

In the Example 16.2 if $A = \{2, 3, 6\}$, then 6, 12, 24 and 36 are upper bounds of A and there is no lower bound.

Note 16.5 Upper and lower bounds of a subset are not necessarily unique.

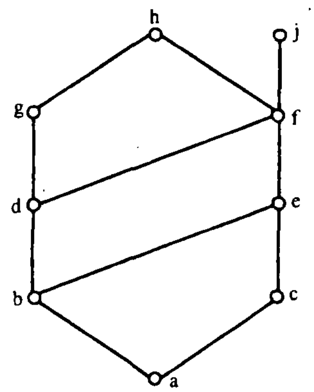
Definition 16.4 Let $(P, \leq)$ be a poset and let $A \subseteq P$. An element $x \in P$ is a least upper bound or supermum, for A if x is an upper bound for A and $x \leq y$ where y is any upper bound for A.

Similarly, the greatest lower bound or infimum, for A is an element $x \in P$ shuch that x is a lower bound and $y \leq x$ for all lower bounds y.

Observations 16.2 (i) A least upper bound, if it exists is unique and the same is true for a greatest lower bound. The least upper bound is abbreviated as 'LUB' or 'sup' and the greatest lower bound is abbreviated as 'GLB' or 'Inf'.

(ii) For a simply ordered chain, every subset has a supremum and as infimum.

Example 16.8 Find the greatest lower bound and the least upper bound of $\{b, d, g\}$ if they exist, in the poset show below:



Solution: The upper bounds of $\{b, d, g\}$ are g and h since $g < h$, g is the least upperbound. The lower bounds of $\{b, d, g\}$ are a and b. Since $a < b$, b is the greatest lower bound.

Example 16.9 The posets in Example 16.1 and Example 16.3 are such that every subset has a supremum and an infimum.

Example 16.10 In example 16.2, the set $A = \{2, 3, 6\}$ has the $LUBA = 6$, while the GLBA does not exist.

Similarly, for the subset $\{2, 3\}$, the supremum is again 6, but there is no infimum.

For the subset $\{12, 6\}$, the supremum is 12 and the infimum is 6.

Observations 16.3 For a poset $(P, \leq)$, we know that its dual $(P, \geq)$ is also a poset. The least member of P relative to the ordering $\leq$ is the greatest member in P relative tot the ordering $\geq$ and vice versa. Similarly the maximal and minimal elements are interchanged. For any subset $A \subseteq P$, the GLBA in $(P, \leq)$ is the same as the LUBA in $(P, \geq)$.

Definition 16.5 A poset is called well-ordered if every non empty subset of it has a least member.

Note 16.6 By the above definition, every well-ordered set is totally ordered, because for any subset say $\{a, b\}$ we must have either a or b as its least member. But, every totally ordered set need not be well- ordered A finite totally ordered set is also well-ordered.

Example 16.11 $I_n = \{1, 2, \ldots, n\}$ is a well-ordered set.

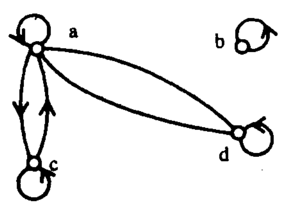
Exercise 1

(1) Determine whether the relation R on the set of all integers is reflexive, symmetric antisymmetric and/or transitive where $(x, y) \in R$ if and only if

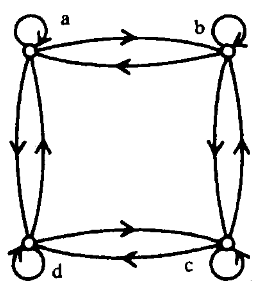
(a) $x \neq y$
(b) $xy \geq 1$
(c) $x = y + 1$ or $y - 1$
(d) $x \equiv y \pmod 7$
(e) x is a multiple of y
(f) x and y are both negative or both non negative
(g) $x = y^2$
(h) $x \geq y^2$

(2) Show that propositional equivalence is an equivalence relation on the set of a compound propositions.

(3) Determine whether the relation with the directed graphs shown is an equivalenc relation.



(a)

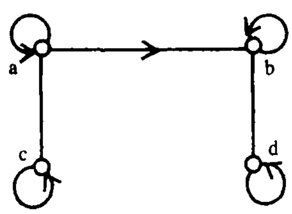


(b)

(4) Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}|) are maximal and whic are minimal?

(5) Which of the following are posets
(a) $(Z, =)$ (b) $(Z, \neq)$

(6) Determine whether the relation with directed graph shown is a partial order



(7) A relation R on set $A = \{1, 2, 3\}$ is given by $R = \{(1, 1), (1, 2), (2, 1), (2, 2)$ $(2, 3), (3, 2), (3, 3)\}$ Show that it is compatibility but not equivalence relation?

Answer (1)

(a) Symmetric
(b) Symmetric, transitive
(c) Symmetric
(d) Reflexive, Symmetric, transitive
(e) Reflexive, transitive
(f) Reflexive, Symmetric, transitive
(g) Antisymmetric
(h) Antisymmetric, transitive

(3) (a) No (b) No
(4) Maximal : 12,20,25
Minimal : 2 and 5
(5) (a) Yes, (b) No
(6) No

---

17. Functions

Introduction

Functions, a particular class of relations. A relation is a correspondence between the elements of two sets, associating elements of the first set with those of the second. It is possible that a given relation associates to any element of the first set several different elements of the second set. It is also possible that some element of the first set is not associated with any from the second.

A special type of relation is that which associates to each member of the first set only one member of the second. Such a relation or correspondence is called a function from one set into the other. Thus function is only a special type of a relation or correspondence. We will consider only discrete functions which associate a finite set to another finite set.

Example 17.1 (i) The relation R "is less than" in the set of all real numbers is not a function in the sense that if we consider any one number of the set, say 3, we have $3R4$, $3R5$, $3R6$, etc., so that the element 3 is related to several elements.

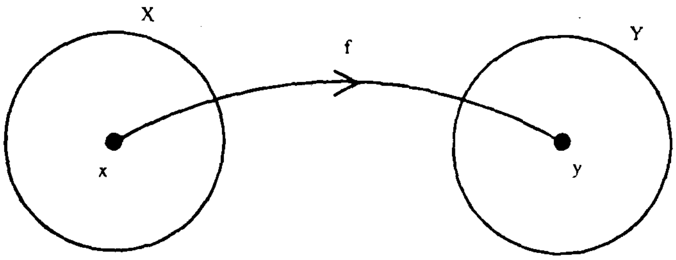
(ii) The relation "is a square of" in the set of real numbers is a function since there is only one real number y which is the square of a given real number x.

Definition 17.1 Let X and Y be any two sets. A relation f from X to Y is called a function if for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$.

We write $y = f(x)$. A function from X to Y is also written as $f : X \to Y$ which is read as "a function of X into Y". The set X is called the domain of the function f and

Y is called the co-domain of f. If x is an element of the set X, then the element y in Y which is assigned to x is called the image of x so that we have $(x, y) \in f$.

Functions are also called mappings or transformations.



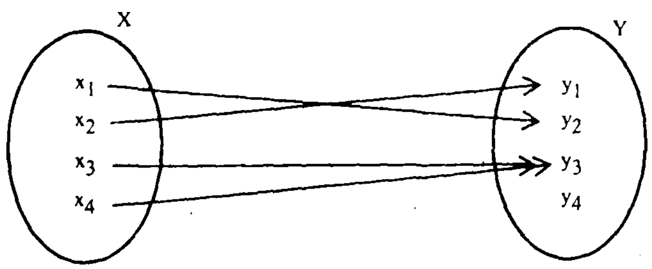
If the domain and co-domain of a function f are both the same set, say $f : A \to A$ then f is called an operator or transformation on A. Two functions $f : X \to Y$ an $g : X \to Y$ are said to be same, and we write $f = g$, if $f(x) = g(x)$, for ever $x \in X$.

Example 17.2 Let A be the set of all real numbers. We define a rule f which assign to each real number x. Its square x^2 that is $f(x) = x^2.x^2$is non-negative for all rea numbers then f given by $f(x) = x^2$ is a function from A to B.

Definition 17.2 Let f be a mapping of X into Y i.e., let $f : X \to Y$. Each element i Y need not appear as an image of some element in X. The range of f consists of exactl those elements in Y which appear as the image of atleast one element in X. We denote th range of $f : X \to Y$ by $f(X)$. It is also denoted by $R(f) = f(X) = \{f(x) \in Y | x \in X$ clearly the range of f is a subset of Y.

Note 17.1 Domain of f is denoted by $D(f)$.

Note 17.2 The rule of correspondence which defines a function may be indicated t a diagram as shown below

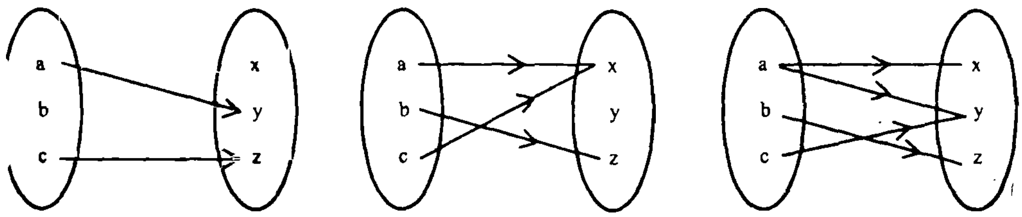


Function f from X to Y

Note 17.3 A function may take the same value at two different elements to X.

Example 17.3 A program written in a high level language is transformed (or mapped) nto a machine language by a compiler. Similarly, the output from a computer is a function of its input.

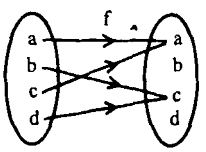
Example 17.4 State whether or not each of the diagrams given below defines a unction of $A = \{a, b, c\}$ into $B = \{x, y, z\}$



Solution:

(i) is not *a* function since nothing is assigned to *b*
(ii) is *a* function
(iii) is not *a* function since the element *a* in the domain is assigned to two elements *x* and *y* in the codomain. But in a function, each element in the domain can be assigned to only one element in the co-domain.

xample 17.5 The following figure defines a function f which maps the set $\iota, b, c, d\}$ to itself. Find the range of f.



Solution: The range consists of all the image points. The elements *a* and *c* only ppear as images. So range of f is the set $\{a, c\}$.

xample 17.6 If the function f is defined by $f(x) = x^2 + 1$ on the set $-2, -1, 0, 1, 2\}$. Find the range of f.

Solution: We compute the image of each element

$$f(-2) = 5, \quad f(-1) = 2, \quad f(0) = 1, \quad f(1) = 2, \quad f(2) = 5$$

ence the range of f is the set $\{1, 2, 5\}$.

finition 17.3 If $f : X \to Y$ and $A \subseteq X$, then $f \cap (A \times Y)$ is a function from $\to Y$ called the restriction of f to A and is sometimes written as $\frac{f}{A}$. If g is a restriction f, then f is called the extension of g.

Note 17.4 $(\frac{f}{A}) : A \to Y$ is s.t for any $a \in A$, $(\frac{f}{A})(a) = f(a)$. The domain of $(\frac{f}{A})$ is A, while that of f is X. Obviously, if g is a restriction of f, then $D(g) \subseteq D(f)$ and $g(x) = f(x)$ for $x \in D(g)$ and $g \subseteq f$.

18. Types of functions

Definition 18.1 A mapping $f : X \to Y$ is called onto (Surjective, a surjection) if the range $R(f) = Y$; otherwise it is called into.

Example 18.1 Let $X = Y = R$ and $f(x) = x^2 + 2$, $D(f) = R$ and $R(f) \subseteq R$. The values of f for different values of $x \in R$ all lie on a parabola, clearly f is an into mapping

Example 18.2 Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$

Solution: Since all three elements of the codomain are images of elements in the domain. $\therefore$ f is onto.

Definition 18.2 A mapping $f : X \to Y$ is called one-to-one (injective or $1-1$) if distinct elements of X are mapped into distinct elements of Y.

In other words, f is one-to-one if

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

or equivalently $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Example 18.3 The function $f : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$ and $f(d) = 3$ is one to one since f takes on different values at the four elements of its domain

Example 18.4 The function $f(x) = x^2$ from the set of integers to the set of integers is not one-to-one because, for instance $f(1) = f(-1) = 1$ but $1 \neq -1$.

Note 18.1 When X and Y are finite sets, a mapping $f : X \to Y$ can be one-to-one if the number of elements in X is less than or equal to the number of elements in Y.

Definition 18.3 A mapping $f : X \to Y$ is called one-to-one onto (objective) if it is both one-to-one and onto. Such a mapping is also called a one to one correspondence between X and Y.

Observations 18.1 For $f : X \to Y$ to be bijective when X and Y are finite requires that the number of elements in X be the same as the number of elements in Y.

Example 18.5 The function $f : \{a, b, c, d\} \to \{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$ is one-to-one and onto. It is one-to-one since all four elements of the domain are images of elements in the domain. Hence f is *a* bijection.

Example 18.6 The function $f(x) = 2x + 1$ on R is a bjection from R to R.

Definition 18.4 A mapping in which many elements in the domain have the same image element in the co-domain is called a many-to-one mapping

Note 18.2 A function $f : X \to Y$ is many-to-one if it is not one to one.

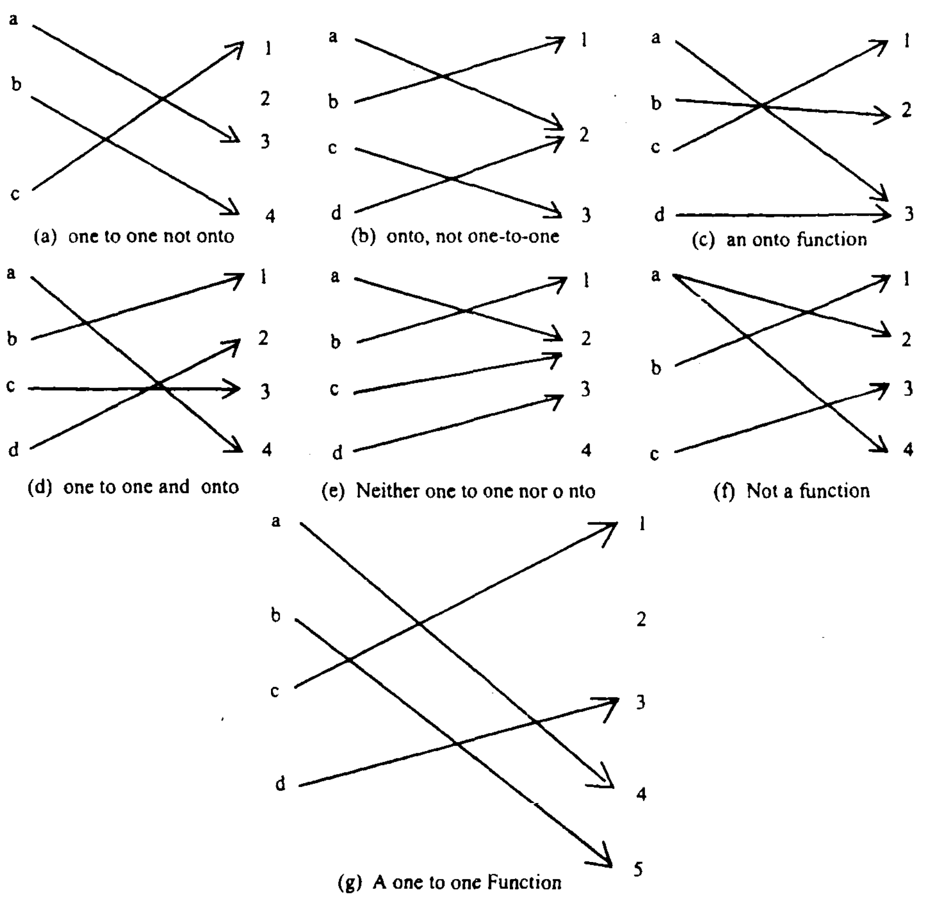
Example 18.7 Consider $f : Z \to Z$ given by $f(x) = x^2$. This is a many to one mapping since $f(-1) = f(1) = 1$, $f(-2) = f(2) = 4$ etc.,

Definition 18.5 Let X be any set. Let the function $I_X : X \to X$ be defined by $I_X(x) = x \; \forall \, x \in X$. Then I_X is called the identity function or the identity transformation on X.

Note 18.3 The identity function assigns each element to itself.

$\therefore$ It is one-to-one and onto, so that it is a bijection.

Examples of different types of correspondences



(a) one to one not onto

(b) onto, not one-to-one

(c) an onto function

(d) one to one and onto

(e) Neither one to one nor o nto

(f) Not a function

(g) A one to one Function

19. Inverse Function

Let $f : X \to Y$ be a one to one correspondence from the set X to the set Y. The inverse function of f is the function that assigns to an element $y \in Y$ the unique element $x \in X$ such that $f(x) = y$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(y) = x$ when $f(x) = y$

Definition 19.1 A function $f : X \to Y$ is said to be invertible if the inverse relation $f^{-1} : Y \to X$ is also a function. Then f^{-1} is called the inverse of f.

Theorem 19.1 A function $f : X \to Y$ is invertible $\Longleftrightarrow$ f is one one and onto.

Proof: f is invertible means f^{-1} is a function from $Y \to X$, f^{-1} will be a function from $Y \to X$ if and only if

1. For every $y \in Y$, there must be an $x \in X$ such that $f^{-1}(y) = x$ That is $f(x) = y, \Rightarrow f$ must be onto
2. For each $y \in Y$, $f^{-1}(y)$ must be unique. This is possible if and only if f is one to one

Hence f is invertible if and only if f is one to one and onto □

Note 19.1
1. If f is a bijection from A to B, then the relation f^{-1} from B to A is a function from B to A
2. If $f : A \to B$ is a one one onto mapping then $f^{-1} : B \to A$ is unique
3. Let A and B be two non empty sets. If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.

If a function f is not a one to one correspondence, we cannot define an inverse function of f.

A one to one correspondence is called invertible, since we can define an inverse of this function. A function is not invertible if it is not a one to one correspondence, since the inverse of such a function does not exist.

Example 19.1 The function of from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$ is invertible since it is a one to one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so that $f^{-1}(1) = c$, $f^{-1}(2) = a$ and $f^{-1}(3) = b$.

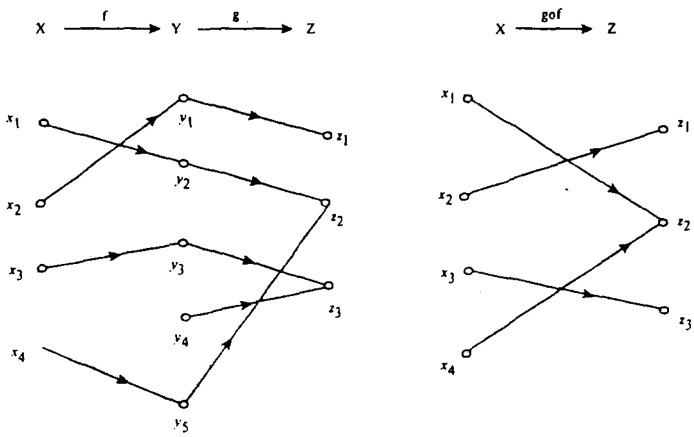
Example 19.2 The function $f(x) = x^2$ from Z to Z is not invertible, since $f(-1) = f(1) = 1$, f is not one to one.

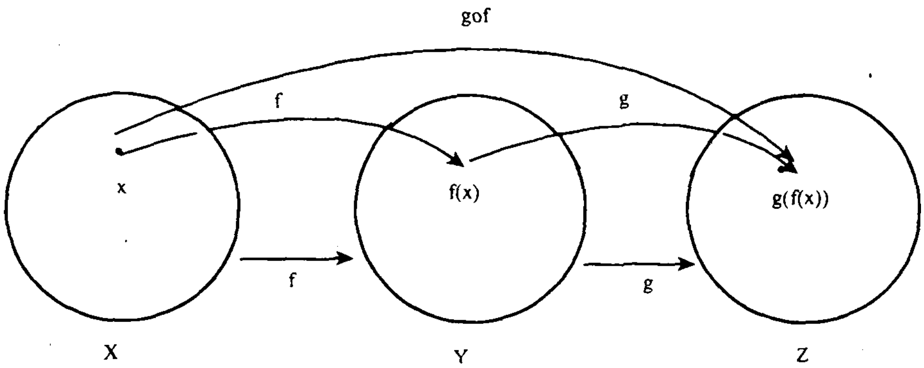
20. Composition of Functions

The operation of composition of relations can be extended to functions in the following manner.

Definition 20.1 Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions. The composite function $(gof) : X \rightarrow Z$ defined by $(gof)(x) = g(f(x))$ is called the *composition* of functions.

The composition of functions is shown below:





Theorem 20.1 If $f : A \rightarrow B$ and $g : B \rightarrow C$ are two one-one (injective) functions then the mapping gof: $A \rightarrow C$ is one-one

Proof: $f : A \rightarrow B$ and $g : B \rightarrow C$ are one-one

$$\therefore \quad gof : A \rightarrow C$$

To prove that gof is one-one

Let $a_1, a_2 \in A$ $\therefore$ $f(a_1), f(a_2) \in B$ and $g(f(a_1)), g(f(a_2)) \in C$ i.e $(gof)(a_1)$, $(gof)(a_2) \in C$

Now $(gof)(a_1) = (gof)(a_2)$

$$\Rightarrow \quad g(f(a_1)) = g(f(a_2)) \quad \Rightarrow \quad f(a_1) = f(a_2) \quad (\because \quad g \text{ is one one})$$

$$\Rightarrow \quad a_1 = a_2 \quad (\because \quad f \text{ is one one})$$

Hence gof: $A \to C$ is a one-one function □

Note 20.1 The converse of the above theorem is not true

If $f : A \to B$, $g : B \to C$ and gof is one one then both f and g need not be one one

Theorem 20.2 If $f : A \to B$, $g : B \to C$ are mappings and gof is one one then f is necessarily an injection.

Proof: Let $x, y \in A$, since $f : A \to B$, $f(x), f(y) \in B$

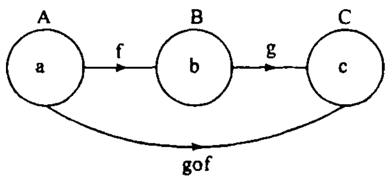
$$\text{Now } f(x) = f(y) \Rightarrow g(f(x)) = g(f(y))$$
$$\Rightarrow \quad (gof)(x) = (gof)(y)$$
$$\Rightarrow \quad x = y \quad (\because \quad gof \text{ is an injection})$$

Hence f is an injection □

Theorem 20.3 If $f : A \to B$ and $g : B \to C$ are two onto onto (surjective) functions then the mapping gof: $A \to C$ is onto.

Proof: $f : A \to B$ and $g : B \to C$ are onto $\therefore$ $gof : A \to C$

To prove that gof is onto



Let c be any element of C

Since $g : B \to C$ is an onto function, there exists an element $b \in B$ such that $g(b) = c$

Since $f : A \to B$ is an onto function, there exists an element $a \in A$ such that $f(a) = b$

Now $g(b) = c \Rightarrow \quad g(f(a)) = c \Rightarrow \quad (gof)(a) = c$

Thus for any element $c \in C$ there is an element $a \in A$ such that (gof) $(a) = c$.

$\therefore$ gof: $A \to C$ is an onto function □

Note 20.2 The converse of the above theorem is not true If $f : A \to B$, $g : B \to C$ and gof is a surjection then both f and g need not be surjections.

Theorem 20.4 If $f : A \to B$ and $g : B \to C$ such that gof is a surjection, then g is necessarily a surjection

Proof: Let $c \in C$, since gof is a surjection from A to C there exists an element $a \in A$ such that (gof)$(a) = c$ i.e $g(f(a)) = c$ since $g : B \to C$ and $f(a) \in B \forall c \in C$ there exists an element belonging to B. Hence g is a surjection. □

Theorem 20.5 If $f : A \to B$ and $g : B \to C$ are two bijective functions then the mapping gof: $A \to C$ is a bijection

Proof: f and g are injections $\Rightarrow$ gof: $A \to C$ is an injection
f and g are surjections $\Rightarrow$ gof: $A \to C$ is an surjection
Hence it follows that if f and g are bijections. gof is also a bijection. □

Note 20.3 The converse of the above theorem is not true.

Theorem 20.6 If $f : A \to B, g : B \to C$ and $h : C \to D$ are functions then $ho(gof) = (hog)of$

Proof:

$$f : A \to B \quad \text{and} \quad g : B \to C \quad \Rightarrow \quad gof : A \to C$$

$$\text{Now} \quad gof : A \to C \quad \text{and} \quad h : C \to D \quad \Rightarrow \quad ho(gof) : A \to D$$

Similarly (hog) of : $A \to D$
Thus ho(gof) and (hog)of both exist and have the same domain A and codomain D
Let a be an element of A

$$\begin{aligned} \text{Now} \quad [ho(gof)](a) &= h[(gof)(a)] = h[(g(f(a))] = (hog)[f(a)] \\ &= [(hog)of](a) \\ \therefore \quad ho(gof) &= (hog)of \end{aligned}$$

□

Note 20.4 Composition of mappings is said to be associative

Theorem 20.7 The composite of any function with the identity function is the function it self

Proof: Let $f : A \to B$, $I_A : A \to A$ and $I_B : B \to B$
To prove that (i) $I_B of = f$ and (ii)$foI_A = f$
By definition $I_B of : A \to B$
Let $a \in A$, then $f(A) \subset B$

(i)

$$(I_B of)(a) = I_B(f(a)) = f(a) \quad (\because \quad I_B \text{is the identity mapping on } B)$$

$$\therefore \quad I_B of = f$$

(ii) Also by definition $foI_A : A \to B$

$$(foI_A)(a) = f(I_A(a)) = f(a) \quad (\because \quad I_A \text{is the identity mapping on } A)$$

$$\therefore \quad foI_A = f$$

□

Note 20.5

$$f : A \to A, I : A \to A \Rightarrow foI = f.$$

Since $a \in A$, $(foI)(a) = f(I(a)) = f(a)$, similarly $Iof = f$

$$\therefore \quad foI = Iof = f$$

Theorem 20.8 If $f : A \to B$ is bijection, then $f^{-1}of = I_A$ and $fof^{-1} = I_B$

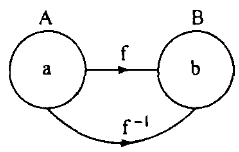
Proof: $f : A \to B$ is a bijection $\Rightarrow f^{-1} : B \to A$ is a bijection

By definition $fof^{-1} : B \to B$ and $fof : A \to A$

To prove that $f^{-1}of = I_A$

Let $a \in A$

Since $f : A \to B$ there exists a unique element $b \in B$ such that $f(a) = b$



$\therefore \quad a = f^{-1}(b)$ Since f is a bijection

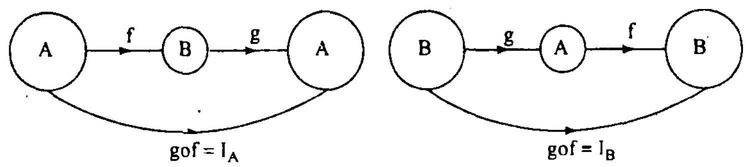
$$\therefore \quad (f^{-1}of)(a) = f^{-1}[f(a)] = f^{-1}(b) = a = I_A(a)$$

$$\therefore \quad f^{-1}of = I_A$$

Similarly it can be shown that $fof^{-1} = I_B$ □

Theorem 20.9 If $f : A \to B$ and $g : B \to C$ are two function such that $gof = I_A$ and $fog = I_B$ then $g = f^{-1}$

Proof: (i) To prove that f is one-one



Let $a_1, a_2 \in A$ and since $f : A \to B$, $f(a_1), f(a_2) \in B$

$$\text{Now} \quad f(a_1) = f(a_2) \Rightarrow \quad g[f(a_1)] = g[f(a_2)]$$

$$\Rightarrow (gof)(a_1) = (gof)(a_2) \Rightarrow I_A(a_1) \quad \therefore a_1 = a_2$$

$\therefore \quad f$ is one-one

(ii) To prove that f is onto
Let b be an element of B

$$I_B(b) = (fog)(b) \Rightarrow b = f(g(b)) \Rightarrow f(g(b)) = b$$

i.e., there exists a pre-image $g(b) \in A$ for b under the mapping f, $\therefore$ f is onto
Thus f is one one onto and hence $f^{-1} : B \rightarrow A$ exists and is also one-one onto

(iii) To prove $g = f^{-1}$
Now $g : B \rightarrow A$ and $f^{-1} : B \rightarrow A$
Let $a \in A$ and b be the f-image of a where $b \in B$

$$\therefore \quad f(a) = b \Rightarrow a = f^{-1}(b)$$

Now $g(b) = g(f(a)) = (gof)(a) = I_A(a) = a$

$$\Rightarrow \quad a = f^{-1}(b) \quad \therefore \quad g = f^{-1}$$

□

Theorem 20.10 If $f : A \rightarrow B$ is invertible then $(f^{-1})^{-1} = f$

Proof: f invertible means f is one one and onto
Now

$$f^{-1}(a_1) = f^{-1}(a_2) \qquad \text{where } f(x_1) = a_1 \text{ and } f(x_2) = a_2$$
$$\Rightarrow \quad x_1 = x_2$$
$$\Rightarrow \quad f(x_1) = f(x_2) \qquad \text{since } f \text{ is onetoone}$$
$$\Rightarrow \quad a_1 = a_2$$

Hence f^{-1} is one to one

Further let $a_0 \in A$. Then there exists a unique $b_0 \in B$ such that $f(a_0) = b_0$. That is there exists $b_0 \in B$ such that $f^{-1}(b_0) = a_0$.

Hence f^{-1} is onto

Since f^{-1} is the inverse relation of f and vice versa $(f^{-1})^{-1} = f$. □

Theorem 20.11 If $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijective functions, then $(gof)^{-1} = f^{-1}og^{-1}$

Proof: $f : A \rightarrow B$, $g : B \rightarrow C$ are bijections

$$\Rightarrow \quad gof : A \rightarrow C \text{ is a bijection}$$
$$\Rightarrow \quad (gof)^{-1} : C \rightarrow A \text{ is a bijection}$$

Also $g^{-1} : C \rightarrow B$ and $f^{-1} : B \rightarrow A$ are bijections

$$f^{-1}og^{-1} : C \rightarrow A \text{ is a bijection}$$

Let c be any element of C. Then $\exists$ an element $b \in B$ such that $g(b) = c \Rightarrow b = g^{-1}(c)$
x Also $\exists$ an element $a \in A$ such that $f(a) = b \Rightarrow a = f^{-1}(b)$

$$\text{Now } (gof)(a) = g(f(a)) = g(b) = c$$

$$\Rightarrow \quad a = (gof)^{-1}(c) \Rightarrow (gof)^{-1}(c) = a \tag{1}$$

$$\text{Also } (f^{-1}og^{-1}) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a \tag{2}$$

$\therefore$ From (1) and (2)

$$(gof)^{-1}(c) = (f^{-1}og^{-1})(c) \quad \Rightarrow \quad (gof)^{-1} = f^{-1}og^{-1}$$

□

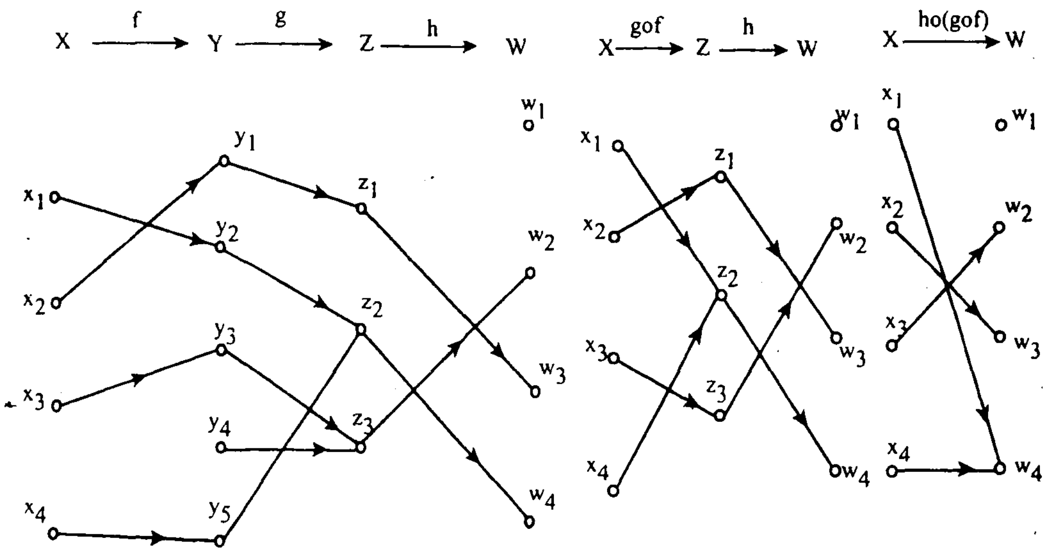
Note 20.6

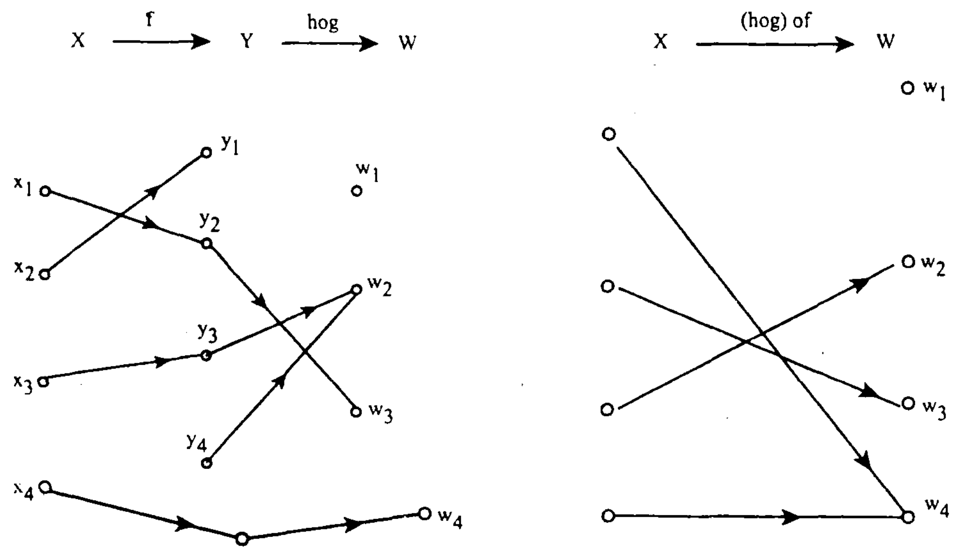
(i) It is assumed that the range $R(f)$ is a subset of the $D(g)$, which is Y, that is $R(f) \subseteq D(g)$; otherwise, gof empty.

(ii) Given $f : X \to Y$ and $g : Y \to Z$, we have gof. However, the composite function fog may or may not exist. For the existence of fog, it is necessary that $R(g) \subseteq D(f)$.

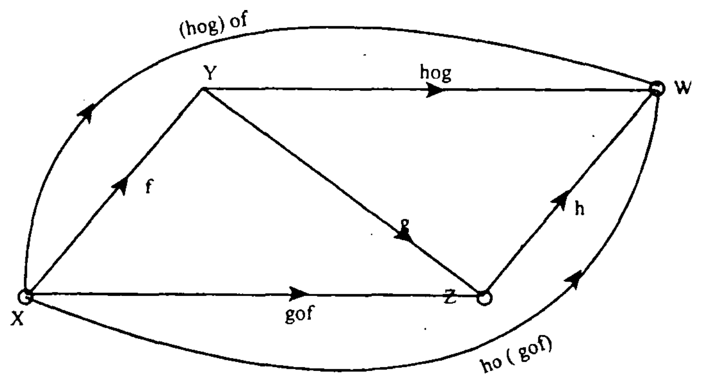
(iii) For functions $f : X \to X$ and $g : X \to X$, the composite functions such as fog, gof, fof gog etc., can be formed.

(iv) Consider three functions $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$. The composite functions $gof : X \to Z$ and hog: $Y \to W$ can be formed. Other composite functions such as ho(gof) and (hog)of can also be formed. Both of these functions are from X to W.





Note 20.7 hogof = ho(gof) = (hog) of



Example 20.1 Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for $x \in \mathbb{R}$, the set of real numbers. Find gof, fog, fof, gog, foh, hog, hof and fohog

Solution:

$$gof = \{(x, x) | x \in R\}$$

$$fog = \{(x, x) | x \in R\} = gof$$

$$fof = \{(x, x + 4) | x \in R\}$$

$$gog = \{(x, x - 4) | x \in R\}$$

$$foh = \{x, 3x + 2)|x \in R\}$$
$$gog = \{(x, x - 4)|x \in R\}$$
$$foh = \{x, 3x + 2)|x \in R\}$$
$$hog = \{(x, 3x - 6)|x \in R\}$$
$$hof = \{(x, 3x + 6)|x \in R\}.$$
$$(foh)og = \{(x, 3x - 4)|x \in R\} = fo(hog) = fohog$$

Example 20.2 Let $f(x) = 2x + 3,\ g(x) = 3x + 2$ for $x \in \mathbb{Z}$ Is gof = fog ?

Solution: $(fog)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$
Similarly $(gof)(x) = 6x + 11$

Note 20.8 Even though fog and gof are defined for the functions f and g, fog and gof are not equal. In otherwords, the commutative law does not hold for the composition of functions always.

Example 20.3 Let $f : R \to R$ be given by $f(x) = -x^2$ and $g : R_+ \to R_+$ be given by $g(x) = \sqrt{x}$ where R_+ is the set of non negative real numbers and R is the set of real numbers. Find fog, Is gof defined?

Solution: $(fog)(x) = -x \forall x \in R_+$. The function $fog : R_+ \to R$ is defined because the range of g is $R_+ \subseteq R$ and R is the domain of f.
On the other hand, the Range of f is not included in the domain of g; therefore gof is not defined. The only element common to $R(f)$ and $D(g)$ is 0.

Observations 20.1 (i) If $f : X \to Y$ is invertible then f^{-1} of $= I_X$ and $fof^{-1} = I_y$

(ii) Let $f : X \to Y$ and $g : Y \to X$. The function g is equal to f^{-1} only if $gof = I_x$ and $fog = I_y$

(iii) $(f^{-1})^{-1} = f$

(iv) Let $f : X \to Y$ and $g : Y \to X$ be such that $gof : X \to Z$ can be constructed. If f and g are both one to one and onto, then gof will also be one to one and onto and the inverses f^{-1}, g^{-1} and $(gof)^{-1}$ exist and are one to one and onto. Then $(gof)^{-1} = f^{-1}og^{-1}$
ie., the inverse of a composite function can be expressed interms of the composition of the inverses in the reverse order.

Example 20.4 Show that the functions $f(x) = x^3$ and $g(x) = x^{1/3}$ for $x \in \mathbb{R}$ are inverses of one another.

Solution: Since $(fog)(x) = f(x^{\frac{1}{3}}) = x = I_x$ and $(gof)(x) = g(x^3) = x = I_x$ then $f = g^{-1}$ or $g = f^{-1}$.

Example 20.5 Let f be the function from the set of integers to the set of integers such that $f(x) = x + 1$. Is f invertible and if it is, what is its inverse

Solution: The functions f has an inverse since it is a one to one correspondence. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbb{Z}$ that is sent to y by f. Consequently, $f^{1}(y) = y - 1$ (or) $f^{-1}(x) = x - 1$.

Example 20.6 If A has m elements and B has n elements, how many functions are there from A to B?

Solution: To define a function $f : A \rightarrow B$, for each $a \in A$, we have to select one element from B as the image of a. For a given $a \in A$, we have n choices viz., n elements of B. As there are m elements in A and for each element in A there are n choices, the number of such choices is n^m. Hence the number of distinct functions from A to B is n^m.

Exercise 2

1. Why is f not a function from R to R in the following equations
(a) $f(x) = \frac{1}{x}$ (b) $f(x) = \sqrt{x}$, (c) $f(x) = \pm\sqrt{x^2 + 1}$

2. Let f be a function from X to Y. Find whether f is one-one and whether it is onto in the following cases

   (i) $X = Y = \mathbb{Z},\ f(x) = x - 1$

   (ii) $X = \mathbb{R}, Y = \{x/x$ is real and $x \geq 0\};\ f(x) = |x|$

   (iii) $X = \mathbb{R}, Y = \{x/x$ is real and $x \geq 0\};\ f(x) = x^2$

3. Determine whether each of the following functions from $\mathbb{Z}$ to $\mathbb{Z}$ is onto
(a) $f(x) = x - 1$ (b) $f(x) = x^2 + 1$ (c) $f(x) = x^3$.
4. Determine whether each of the following functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$
(a) $f(x) = 2x + 1$ (b) $f(x) = \frac{x^2+1}{x^2+2}$
5. If the mappings f and g are given by $f = \{(1, 2), (3, 5), (4, 1)\}$ $g = \{(2, 3), (5, 1), (6, 3)\}$, then write down the pairs in the mapping fog and gof
6. Let $A = \{1, 2, 3\}$. Define $f : A \to A$ by $f(1) = 2,\ f(2) = 1$ and $f(3) = 3$ Find f^{-1}.
7. Let $f : R \to R$ and $g : R \to R$, where R is the set of real numbers be given by $f(x) = x^2 - 2$ and $g(x) = x + 4$ Find fog and gof. State whether these functions are injective, surjective and bijective

Answers (2)

1. (a) $f(0)$ is not defined
   (b) $f(x)$ is not defined for $x < 0$
   (c) $f(x)$ is not well defined since there are two distinct values assigned tc each x.
2. (i) one to one and onto
   (ii) onto but not one to one
   (iii) onto but not one to one
3. (a) onto
   (b) not onto
   (c) not onto
4. (a) Yes, bijection
   (b) No, Not a bijection
5. $fog = \{(2, 5), (5, 2), (6, 5)\}$ $gof = \{(1, 3), (3, 1)\}$
6. $f^{-1} = \{(2, 1), (1, 2), (3, 3)\}$
7. $fog = x^2 + 8x + 14$, not onto, not one to one $gof = x^2 + 2$, not onto, not on to one.

21. Binary Operations

Definition 21.1 Let X be a set and f be a mapping $f : X \times X \to X$. Then f is called a binary operation on X.

In general, a mapping $f : X^n \to X$ is called on n - ary operation and n is called the order of the operation.

For $n = 1$, $f : X \to X$ is called a Unary operation

Observations 21.1 If an operation on the members of *a set produces images which are also members of the same set*, then the set is said to be closed under that operation, and this property is called the closure property. The definition of binary or n-ary operations implies that the sets on which such operations are defined are closed under these operations.

Example 21.1 (i) The operations of addition, multiplication and subtraction are binary operations on the set of integers and also on the set of real numbers. (ii) Operations of set union and intersection are binary operations on the set of subsets of a universal set.

Definition 21.2 A binary operation $f : X \times X \to X$ is said to be commutative if for every $x, y \in X$, $f(x, y) = f(y, x)$.

Definition 21.3 A binary operation $f : X \times X \to X$ is said to be associative if for every $x, y, z \in X$, $f(f(x, y), z) = f(x, f(y, z))$

The above definitions can be rewritten using * to denote the binary relation on X. That is * is communtative if for any $x, y \in X$, $x * y = y * x$.

Similarly * is associative on X if for any $x, y, z \in X$,

$$(x * y) * z = x * (y * z)$$

Definition 21.4 A binary operation $f : X * X \to X$, denoted by * is said to be distributive over the operation $g : X * X \to X$, denoted by 'o', if for every $x, y, z \in X$

$$x * (yoz) = (x * y)o(x * z)$$

Example 21.2 The operations of addition and multiplication over the set of real numbers are commutative and associative.

Union and intersection over the power set of any set are other examples of commutative and associative operations.

Definition 21.5 Let * be a binary operation on X. If there exists an element $e_l \in X$ s.t $e_l * x = x$ for every $x \in X$, then e_l is called a left identity with respect to *. Similarly, if there exits an element $e_r \in X$ such that $x * e_r = x$ for every $x \in X$ then e_r is called a right identity with respect to *.

Result 21.1 Let * be a binary operation and let e_l and e_r be left and right identities with respect to *. Then $e_l = e_r = e(say)$, such that $e * x = x * e = x$ for every $x \in X$ and in such a case $e \in X$ is unique and is called the identity with respect to *.

Proof: Since e_l and e_r are left and right identities

$$e_l * e_r = e_l = e_r$$

Next, let us assume e_1 and e_2 are two distinct identities. Then $e_1 * e_2 = e_1 = e_2$. Which is a contradiction, hence an identity, if it exists, is unique □

Example 21.3 The element 0 is the identity for addition, and 1 is the identity for multiplication over the set of real numbers. The empty set $\emptyset$ is the identity for the operation of union, and the universal set E is the identity for the operation of intersection over the subsets of a universal set.

Definition 21.6 Let * be a binary operation on X. If there exists an element $0_l \in X$ s.t $0_l * x = 0_l$ for every $x \in X$, then 0_l is called a left zero with respect to *. Similarly, if there exists an element $0_r \in X$ such that $x * 0_r = 0_r$ for every $x \in X$, then 0_r is called a right zero with respect to *.

Result 21.2 Let * be a binary operation, and 0_l and 0_r be left and right zeros with respect to *. Then $0_l = 0_r = 0$ such that

$$0 * x = x * 0 = 0 \text{ for all } x \in X.$$

$0 \in X$ is unique and is called the zero with respect to *.

Example 21.4 The element 0 is the zero for multiplication on a set of real numbers. The empty set $\emptyset$ is the zero for intersection and universal set E is the zero for the union of subsets of a universal set.

Definition 21.7 Let * be a binary operation on X. An element $a \in X$ is called idempotent with respect to * if $a * a = a$.

Example 21.5 The identity and Zero elements with respect to a binary operation are idempotent. But there may be other idempotent elements besides the identity and zero elements. For example, every set is idempotent with respect to the operations of union and intersection.

Definition 21.8 Let * be a binary operation on X with the identity e. An element $a \in X$ is said to be left-invertible if there exists an element $x_l \in X$ such that $x_l * a = e$ x_l is called a left inverse of a. Similarly $a \in X$ is said to be right invertible if there exists an element $x_r \in X$ such that $a * x_r = e$. x_r is called a right inverse of a. If an element $a \in X$ is both left-invertible, then a is called invertible.

Clearly, if a binary operation * on X with the identity e is commutative, then any element that is left or right invertible is invertible.

Result 21.3 Let * be a binary operation on X which is associative and which has the identity $e \in X$. If an element $a \in X$ is invertible, then both its left and right inverses are equal. Such an element is called the inverse of a because it is unique.

Proof: Let x_ℓ and x_r be any left and right inverses of a respectively we show that $x_\ell = x_r$ as follows

$$x_\ell * a = a * x_r = e$$

hence

$$\begin{aligned} x_\ell * a * x_r = (x_\ell * a) * x_r = x_\ell * (a * x_r) = e * x_r &= x_r \\ &= x_\ell * e \\ &= x_\ell \end{aligned}$$

To show uniqueness, let us assume that x and y are two distinct inverses of a.

Thus $y = y * e = y * (a * x) = (y * a) * x = e * x = x$

Which is a contradiction.

∴ Inverse of a is unique.

Hence the theorem. □

Note 21.1 The Unique inverse of an element $a \in X$, if it exists is denoted by a^{-1}, so $a^{-1} * a = a * a^{-1} = e$

Clearly it follows that $(a^{-1})^{-1} = a$.

Definition 21.9 An element $a \in X$ is called Cancellable with respect to a binary operation * on X, if for every $x, y \in X$,

$$(a * x = a * y) \vee (x * a = y * a) \Rightarrow (x = y)$$

Example 21.6 Determine whether usual multiplication on the set $A = \{1, -1\}$ is a binary operation.

Solution: Since $(-1) \cdot 1 = -1 \in A$, $1 \cdot 1 = 1 \in A$, $(-1)(-1) \in A$, $1 \cdot (-1) = -1 \in A$ usual multiplication on A is a binary operation

Example 21.7 Examine whether matrix multiplication on the set $M = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} : a, b \in R \right\}$ is a binary operation

Solution: Let $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \in M$

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & bc \end{bmatrix} \notin M$$ (∵ This is not in the form of the element of M).

Matrix multiplication is not a binary operation on the set M

Example 21.8 Determine whether * defined by $a * b = \frac{a+b}{ab}$ on a set N is a binary operation

Solution: * is not a binary operation in N, since some elements do not have the images in N.
For example, $1 * 2 = \frac{1+2}{1.2} = \frac{3}{2} \notin N$.

Example 21.9 Determine whether the binary * defined is commutative and whether it is associative on the set Z where $a * b = a - b$

Solution:

(i) * is not commutative since $a * b \neq b * a$ ie., $a - b \neq b - a$

(ii) * is not associative since $a * (b * c) \neq (a * b) * c$
i.e., $a - (b - c) \neq (a - b) - c$

Example 21.10 On Q, where $a * b$ is $\frac{ab}{2}$

Solution: Clearly $\frac{ab}{2} = \frac{ba}{2} \Rightarrow a * b = b * a$ in Q
Also $a * (b * c) = \frac{a(bc)}{4} = \left(\frac{ab}{2}\right)\frac{c}{2} = (a * b) * c$ in Q
$\therefore$ * is commutative and Associative in Q.

Example 21.11 Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $\mathbb{R}$ is the set of integers and $g(x, y) = x * y = x + y - xy$. Show that the binary operation * is commutative and associative. Find the identity element and indicate the inverse of each element (especially when $x \neq 1$).

Solution: Clearly * is a binary operation on $\mathbb{R}$.

(i) $x * y = x + y - xy = y + x - yx = y * x$
$\therefore$ * is commutative

(ii)
$$x * (y * z) = x * (y + z - yz) = x + y + z - y^2 - x(y + z - yz)$$
$$x + y + z - yz - xy - xz + xyz$$
$$(x * y) * z = (x + y - xy) * z = x + y - xy + z - (x + y - xy)z$$
$$= x + y + z - xy - xz - yz + xyz$$

$\therefore$ * is Associative

(iii) To find identity, Assume y is the identity $\therefore$ $x * y = x$
$\Rightarrow x + y - xy = x$
$\Rightarrow y(1 - x) = 0$
$\Rightarrow y = 0$ or $1 - x = 0$. Now $1 - x = 0$ only when $x = 1$.
Clearly, $y = 0$ is the identity element for any element $x \neq 1$

(iv) To find inverse for the elements $(x \neq 1)$

$$x * y = 0$$
$$\Rightarrow x + y - xy = 0$$
$$y = \frac{x}{x - 1} \ (x \neq 1),$$ Thus, the inverse of $x(\neq 1)$ is $\frac{x}{x-1}$

Every Element of R except 1 has an inverse.

Exercise 3

(I) Find whether * defined on the set is binary

(i) On $\mathbb{Z}$, where $a * b = \min\{a, b\}$
(ii) On $\mathbb{R}$, where $a * b = \frac{a}{b}$

(II) Determine whether the binary operation * defined is commutative and whether it is associative on the set

(i) On $\mathbb{Z}^+$, $a * b = a + b + 1$
(ii) On $\mathbb{R}$, $a * b = \max\{a, b\}$
(iii) On $\mathbb{R}$, $a * b = |a|^b$
(iv) On $\mathbb{N}$. $a * b = a^b$

(III) Find the identity element of the group of integers with the binary operation * defined by $a * b = a + b - 2, \forall a, b \in \mathbb{Z}$

(IV) What are the identity and inverse elements under * defined as $a*b = \frac{ab}{2} \forall a, b \in \mathbb{R}$

Note: Answer:

(I) (i) Binary (ii) Not binary
(II) (i) Commutative, associative
(ii) – do –
(iii) Not commutative, but associative
(iv) Not commutative, not associative
(III) identity $= -2$
(IV) 2 is the identity, $\frac{4}{a}$ is the inverse of a.

Definition 21.10 Characteristic Function of a set

Let U be a Universal set and A be a subset of U. Then function $\Psi_A : U \to \{0, 1\}$ defined by

$$\Psi_A(x) = \begin{cases} 1 \text{ if } & x \in A \\ 0 \text{ if } & x \notin A \end{cases}$$

is called the characteristic function of the set A.

Example 21.12 Let U be the set of all vegetables and A be the set of all potatoes Then Ψ_A associates the number 1 with each potatoe and 0 with any other vegetable.

We can use the characteristic functions of sets to determine set relations

Let A and B be any two subsets of a universal set U. Then the following hold for all $x \in E$

$$\Psi_A(x) = 0 \Leftrightarrow A = \emptyset \qquad \text{(i)}$$

$$\Psi_A(x) = 1 \Leftrightarrow A = U \qquad \text{(ii)}$$

$$\Psi_A(x) \leq \Psi_B(x) \Leftrightarrow A \leq B \qquad \text{(iii)}$$

$$\Psi_A(x) = \Psi_B(x) \Leftrightarrow A = B \qquad \text{(iv)}$$

$$\Psi_{A\cap B}(x) = \Psi_A(x) \cdot A\Psi_B(x) \qquad \text{(v)}$$

$$\Psi_{A\cup B}(x) = \Psi_A(x) + \Psi_B(x) - \Psi_{A\cap B}(x) \qquad \text{(vi)}$$

$$\Psi_{\sim A}(x) = 1 - \Psi_A(x) \qquad \text{(vii)}$$

$$\Psi_{A-B}(x) = \Psi_{A\cap\sim B}(x) = \Psi_A(x) - \Psi_{A\cap B}(x) \qquad \text{(viii)}$$

The above properties can easily be proved using the definition of characteristic functions

For example, (V) can be proved as follows:

$x \in A \cap B \Longleftrightarrow x \in A$ and $x \in B$, so that $\Psi_A(x) = 1$ and $\Psi_B(x) = 1$ and $\Psi_{A\cap B}(x) = 1 * 1 = 1$.

If $x \notin, A \cap B$, then $\Psi_{A\cap B}(x) = 0$ and $\Psi_A(x) = 0$ or $\Psi_B(x) = 0$ consequently $\Psi_A(x) * \Psi_B(x) = 0$.

Many set Identities and other relations can be proved by using characteristic functions and the usual arithmetic operations and relations.

Example 21.13 Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution:

$$\begin{aligned}
\Psi_{A\cap(B\cup C)}(X) &= \Psi_A(x) \cdot \Psi_{B\cup C}(x) \text{ Using (v)} \\
&= \Psi_A(x)(\Psi_B + \Psi_C - \Psi_{B\cap C}), \text{ Using (vi)} \\
&= \Psi_A\Psi_B + \Psi_A\Psi_C - \Psi_A\Psi_{B\cap C} \\
&= \Psi_{A\cap B} + \Psi_{A\cap C} - \Psi_{A\cap(B\cap C)}, \text{ Using (v)} \\
&= \Psi_{A\cap B} + \Psi_{A\cap C} - \Psi_{A\cap B\cap C} \\
&= \Psi_{A\cap B} + \Psi_{A\cap C} - \Psi_{(A\cap B)\cap(A\cap C)} \\
&= \Psi_{(A\cap B)\cup(A\cap C)}(x), \text{ Using (vi)}
\end{aligned}$$

Example 21.14 Show that $\sim\sim A = A$

Solution:

$$\begin{aligned}
\Psi_{\sim\sim A}(x) &= 1 - \Psi_{\sim A}(x) \\
&= 1 - (1 - \Psi_A(x)), \text{ Using (vii)} \\
&= \Psi_A(x)
\end{aligned}$$

22. Recursive Functions

Definition 22.1 Any function $f : N^n \to N$ is called total because it is defined for every n-tuple in N^n

Example 22.1 $f(x, y) = x + y$, which is defined for all $x, y \in N$ and hence is a total function.

Definition 22.2 If $f : D \to N$ where $D \subseteq N^n$, then f is called partial function

Example 22.2 $g(x, y) = x - y$, which is defined for only those $x, y \in N$ which satisfy $x \geq y$. Hence $g(x, y)$ is partial.

Note 22.1 A partial function can be made total function if we restrict the domain of the function only to those values for which function value is defined.

We now give a set of three functions called the Initial functions over N

Definition 22.3 The initial functions over N are

(i) Zero function Z : $Z(x) = 0$
(ii) Successor function S : $S(x) = x + 1$
(iii) Projection function $U_i^n : U_i^n(x_1, x_2, \ldots, x_n) = x_i$

Note 22.2 The projection function is also called the generalized identity function

Example 22.3 $U_1^2(x, y) = x$, $U_2^3(2, 4, 6) = 4$ etc.,

The above initial functions are used in defining other functions by induction.

* The *Composition* of functions can be extended to functions of more than one variable. For example let $f_1(x, y)$, $f_2(x, y)$ and $g(x, y)$ be any three functions. The composition of g with f_1 and f_2 is a function h given by

$$h(x, y) = g(f_1(x, y), f_2(x, y))$$

If f_1, f_2 and g are total, then h is also total. In general, let $f_1, f_2, \ldots f_n$ each be partial functions of m vairiables, and let g be a partial function of n variables. Then the composition of g with $f_1, f_2, \ldots f_n$ produces a partial function h given by

$$h(x_1, x_2, \ldots, x_m) = g(f_1(x_1, \ldots x_m) \ldots f_n(x_1, \ldots x_m))$$

Also h is total $\iff f_1, f_2, \ldots f_n$ and g are total.

* The following operation which defines a function $f(x_1, x_2 .. x_n, y)$ of $n + 1$ variables by using two other functions $g(x_1, x_2 \ldots x_n)$ and $h(x_1, x_2, \ldots, x_n, y, z)$ of n and $n + 2$ variables respectively is called *recursion*.

$$f(x_1, x_2, \ldots, x_n, 0) = g(x_1, x_2, \ldots, x_n)$$
$$f(x_1, x_2, \ldots, x_n, y + 1) = h(x_1, x_2, \ldots, x_n, y, f(x_1, x_2, \ldots, y))$$

variable.

In the above definition, the variable y is assumed to be the inductive variable in the sense that the value of f at $y + 1$ is expressed interms of the value of f at y. The variables $x_1, x_2, \ldots, x_n$ are treated as parameters and are assumed to remain fixed throughout the definition. Also it is assumed that both the functions g and h are know. Now, by the following definition, we impose restrictions on g and h which will guarantee that the function f which is defined recursively, Can actually be computed and is total.

Definition 22.4 A function f is called *primitive recursive* iff it can be obtained from the initial functions by a finite number of operations of composition and recursion.

Note 22.3 It is not always necessary to use only the initial functions in the construction of a particular primitive recursive function. We can use any of primitive recursive functions along with the initial functions to obtain another primitive recursive function, provided we restrict ourselves to the operations of composition and recursion only.

Example 22.4 Show that $b(x, y) = x + y, x, y \in N$ is primitive recursive

Solution: Consider

$$x + (y + 1) = (x + y) + 1 \tag{3}$$

L.H.S of (1) can be expressed in terms of f. R.H.S of (1) can be expressed in terms of the Successor function S. Now let us use recursion.
Define $f(x, 0) = x = U_1^1(x)$

$$f(x, y + 1) = S(U_3^3(x, y, f(x, y)) \left(\begin{array}{c} \because U_3^3(xyz) = z \\ S(x) = x + 1 \end{array} \right)$$

Now U_1^1, U_3^3, S are initial functions, SU_3^3 is by composition $\therefore$ f is got by applyin recursion for the functions U_1^1, U_3^3 and S.
Hence f is primitive recursive

Example 22.5 Show that $f(x, y) = x * y$ is a primitive recursive function

Solution:

$$f(x, 0) = x * 0 = 0 \tag{1}$$

$$f(x, y + 1) = x * (y + 1) = x * y + x \tag{2}$$

Comparing (1) and (2) with definitions of initial functions we can write

$$f(x, 0) = Z(x) \tag{3}$$

$$f(x, y + 1) = f_1(U_3^3(x, y, f(x, y)), U_1^3(x, y, f(x, y)) \tag{4}$$

Where $f_1(x, y) = x + y$, which is primitive recursive

Taking $g = Z$ and h defined by

$$h(x, y, z) = f_1(U_3^3(x, y, z), U_1^3(x, y, z))$$

Clearly (3), (4) defined f_1 by recursion.
As Z is an initial function $g = Z$ is primitive recursive. As h is defined using composition of f_1, which is primitive recursive, U_3^3, U_3^1, which are initial functions h is primitive recursive. Hence f obtained from g and h, using recursion is primitive recursive.

Example 22.6 Show that $f(x, y) = x^y$ is primitive recursive

Solution: Now $f(x, 0) = x^0 = 1$ for $x \neq 0$ and we put $x^0 = 0$ for $x = 0$

$$f(x, y + 1) = x^{y+1} = x^y * x = x * f(x, y)$$

Define

$$\begin{aligned} f(x, 0) &= 1 \\ f(x, y + 1) &= x * f(x, y) \\ &= U_3^1(x, y, f(x, y)) * U_3^3(x, y, f(x, y)) \end{aligned}$$

Now $f(x, 0) = S(Z(x))$ (SoZ is primitive recursive)

$f(x, y + 1) = h(x, y, f(x, y))$ where

$h(x, y, z)$ is defined by $U_3^1(x, y, z) * U_3^3(x, y, z)$

U_3^1, U_3^3 are initial functions and $f_2(x, y) = x * y$ is primitive recursive, we see that f is defined by applying recursion to primitive recursive functions $S(z(x))$ and h. Hence f is primitive recursive.

Example 22.7 Show that the following functions over N are primitive recursive

(i) Constant function over N
(ii) predecessor function
(iii) proper substraction function
(iv) zero test function
(v) odd and even parity function

Solution:

(i) Let $f(x) = K$ be a given constant function. Define $f(0) = K$, $f(n + 1) = U_2^2(n, f(n))$. As f is defined by using recursion on the initial function U_2^2, f is primitive recursive.
(ii) The predecessor function $P(x)$ defined by $P_1(x) = x - 1$, if $x \neq 0$ and $P(0) = 0$. Define P by $P(0) = Z(0)$, $P(x + 1) = U_2^1(x, P(x))$. Hence P is primitive recursive.

(iii) Proper subtraction function — is defined by $x \dot{-} y = x - y$ if $x \geq y$ and $x \dot{-} y = 0$ if $x < y$.
Define $x \dot{-} 0 = x$ and $x \dot{-} (y + 1) = p(x \dot{-} y)$. Here p is the predecessor function. As the function $\dot{-}$ is defined by using recursion on the primitive recursive function P it is primitive recursive.

(iv) The zero-test function $\overline{Sg}$ is defined by

$$\overline{Sg}(0) = 1$$
$$\overline{Sg}(x) = 0 \text{ if } x > 0$$

Define

$$\overline{Sg}(0) = S(z(0))$$
$$\overline{Sg}(x + 1) = Z(U_2^2(x, \overline{Sg}(x))).$$

clearly this function is primitive recursive

(v) Odd and even parity function (denoted by P_r) P_r is defined by $P_r(0) = P_r(2) = \ldots = 0$ and $P_r(1) = P_r(3) = \ldots 1$. Define P_r by $P_r(0) = Z(0)$, $P_r(x + 1) = \overline{Sg}(U_2^2(x, P_r(x)))$.
As P_r is defined using recursion on the primitive recursive function $\overline{Sg}$, P_r i primitive recursive.

Example 22.8 Show that if $f(x, y)$ defines the remainder upon division of y by x then it is a primitive recursive function

Solution: $f(x, 0) = 0$. Also $f(x, y)$ increases by 1 when y is increased by Until the value becomes equal to x, in which case it is equated to zero and the proces continues. Now let us define of using recursion on known primitive recursive and initia functions. f is defined by

$$f(x, 0) = 0,\ f(x, y + 1) = S(f(x, y) * Sg(x \dot{-} S(f(x, y)))$$

where Sg denotes the sign function defined by

$$Sg(0) = Z(0)$$
$$Sg(x + 1) = S(Z(U_2^2(x, Sg(x)))$$

Sign function is primitive recursive (check!)
Hence f is also primitive recursive, Since f is got by using recursion on known primitiv recursive functions and composition.

Exercise 4

1. Define a primitive recursive function and show that $f(x) = [x/2]$ is primitive recursive (where $[\frac{x}{2}]$ is the integral part of $x/2$).
2. Show that the function f defined by

$$f(x) = \begin{cases} \frac{x}{2} & \text{When x is even} \\ \frac{x-1}{2} & \text{When x is odd} \end{cases}$$

is primitive recursive
3. Show that the function $f(x_1, x_2, y)$ defined as

$$f(x_1, x_2, y) = \begin{cases} x_2 & x_1 > y \\ (x_1 * y) + x_2 & x_1 \leq y \end{cases}$$

is primitive recursive
4. Show that the following functions are primitive recursive
   (i) Absolute value function $f(x, y) = (x - y)$
   (ii) $min(x, y)$
   (iii) $max(x, y)$
   (iv) $f(x) = x^2$
5. Show that the factorial function is primitive recursive

Answers

1. Define f by $f(0) = 0$, $f(y + 1) = f(y) + Pr(y)$ where Pr is the odd even parity function. Then f is primitive recursive
2. Same as (1)
3. $f(x_1, x_2, y) = x_2 + (x_1 * y) * \overline{Sg}(x_1 \dot{-} y)$ clearly f is primitive recursive
4. (i) $|x - y| = (x \dot{-} y) + (y \dot{-} x)$
   (ii) $min(x, y) = x \dot{-} (x \dot{-} y)$
   (iii) $max(x, y) = y + (x \dot{-} y)$
   (iv) $f(y) = U_1^1(y) * U_1^1(y)$
   As $\dot{-}$, + (addition) and $*$ (multiplication) functions are primitive recursive, the given functions (i) - (iv) are primitive recursive.

efinition 22.5 Let $g(x_1, x_2, \ldots, x_n, y)$ be a total function over N. If there exists least one value of y, say $y_0 \in N$, s.t the function $g(x_1, x_2, \ldots, x_n, y_0) = 0$ for all tuples $(x_1, x_2, \ldots, x_n) \in N^n$, then g is called a regular function

:ample 22.9 $g(x, y) = min(x, y)$ is a regular function since $g(x, 0) = 0$ for all $\in N$.

Note 22.4 Not all total functions are regular

Example 22.10 If $g(x, y) = |y^2 - x|$, obviously $g(x, y)$ is total but $|y^2 - x| = 0$ for only those values of x which are perfect squares and not for all values of x. This shows that there is no value of $y \in N$ shch that $|y^2 - x| = 0$ for all x.

Example 22.11 The function $y \dot{-} x$ is regular because for $y = 0$, $y \dot{-} x$ is zero for all x.

Definition 22.6 A function $f(x_1, x_2, \ldots, x_n)$ is said to be defined from a total function $g(x_1, x_2, \ldots x_n, y)$ by minimization if

(a) $f(x_1, x_2, \ldots, x_n)$ is the least value of all $y's$ such that $g(x_1, x_2, \ldots, x_n, y) = 0$ if it exists. The least value is denotes by $\mu_r(g(x_1, x_2, \ldots, x_u, y) = 0)$

(b) $f(x_1, x_2, \ldots, x_n)$ is undefined if there is no y such that $g(x_1, x_2, \ldots, x_n, y) = 0$

Note 22.5 From the definition if follows that $f(x_1, x_2, \ldots x_n)$ is well defined and total if g is regular. If g is not regular, then the operation of minimization may produce a partial function.

Definition 22.7 A function is said to be *recursive* iff it can be obtained from the initial functions by a finite number of applications of the operations of composition, recursion and minimization over regular functions.

Definition 22.8 A function is said to be *partial recursive* iff it can be obtained from the initial functions by a finite number of applications of the operations of composition, recursion and minimization

Example 22.12 Show that the function $f(x) = \frac{x}{2}$ is a partial recursive function

Solution: Let $g(x, y) = |2y - x|$. The function g is not regular because $|2y - x| = 0$ only for even values of x. Now define

$$f(x) = \mu_r(|2y - x| = 0).$$

Then f is defined only for even values of x and is equal to $\frac{x}{2}$. When x is odd, $f(x)$ is not defined. So f is a partial recursive.

Example 22.13 Let $\lfloor \sqrt{x} \rfloor$ be the greatest integer $\leq \sqrt{x}$ Show that $\lfloor \sqrt{x} \rfloor$ is primitive recursive

Solution: Observe that $(y + 1)^2 \dot{-} x$ is zero for $(y + 1)^2 \leq x$ and nonzero for $(y + 1)^2 > x$.

Hence $\overline{Sg}\left((y + 1)^2 \dot{-} x\right)$ is 1 if $(y + 1)^2 \leq x$ and cannot be equal to zero.

The smallest value of y for which $(y + 1)^2 > x$ is the required number $\lfloor \sqrt{x} \rfloor$; hence

$$\lfloor \sqrt{x} \rfloor = \mu_r(\overline{Sg}[(y + 1)^2 \dot{-} x] = 0).$$

$\overline{Sg}[(y+1)^2 - x]$ is a regualr function of x.
As $\lfloor\sqrt{x}\rfloor$ is got by minimization of a regular function $\lfloor\sqrt{x}\rfloor$ is recursive.
We are considering only sets whose elements are natural numbers or sets of n-tuples of the natural numbers.

Definition 22.9 To each such set A we can define the *characteristic function* χ_A

$$\text{Where } \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Example 22.14 If $A = \{a, b, c, d\}$
Then $\chi_A(x) = 1$ if $x = a, b, c,$ or d
and $\chi_A(x) = 0$ for all $x \neq a, b, c, d$

Definition 22.10 A set A is called recursive (partial recursive) if its characteristic function χ_A is recursive (partial recursive)

Example 22.15 Show that the sets of even and odd natural numbers are both recursive

Solution: Recall that the parity function is the required characteristic function for the set A of even natural numbers Hence A is primitive recursive.
Also the set of odd natural numbers is A^c, hence A^c is also primitive recursive.

Example 22.16 Show that the set of divisors of a positive integer n is recursive

Solution: A set is recursive iff its characteristic function is recursive. Now a number $x \leq n$ is a divisor of n if and only if $|x * i - n| = 0$ for some fixed i, $1 \leq i \leq n$.

Also $|x * i - n|$ is non zero for all i, $1 \leq i \leq n$, if x is not a divisor of n.

Let χ_A denote the characteristic function of the set of all divisors of n. Then

$$\chi_A(x) = \sum_{i=1}^{n} \overline{Sg}|x * i - n|$$

Where A denotes the set of divisors of n.

Also, Note that i is a divisor of $n \iff |x * i - n| = 0 \iff \overline{Sg}|x * i - n| = 1$.

As χ_A is a finite sum of primitive recursive functions, it is recursive.

Definition 22.11 **Ackermann's Function** $A(x, y)$
The Ackermann's Function $A(x, y)$ is defined by

$$A(0, y) = y + 1$$
$$A(x + 1, 0) = A(x, 1)$$
$$A(x + 1, y + 1) = A(x, A(x + 1, y))$$

By the definition, we can construct the value of $A(x, y)$ for fixed values of x and y. Therefore $A(x, y)$ is well defined and total.

It is known that $A(x, y)$ is not primitive recursive, but recursive.

Example 22.17 If A denotes Ackermann's function evaluate

$(i) A(1, 1) (ii) A(1, 2) (iii) A(2, 1)$

Solution: From the definition of $A(x, y)$

(i)

$$\begin{aligned} A(1,1) &= A(0+1, 0+1) = A(0, A(1,0)) \\ &= A(0, A(0,1)) \\ &= A(0,2) \quad (\because A(0,1) = 1+1 = 2) \\ &= 3 \end{aligned}$$

(ii) $A(1,2) = 4$(check!)
(iii) $A(2,1) = 5$(check!)

Example 22.18 Let x, y be positive integers and suppose Q is defined recursively as follows:

$$Q(x,y) = \begin{cases} 0 & \text{if } x < y \\ Q(x-y, y) + 1 & \text{if } y \le x \end{cases}$$

Compute $Q(4, 7)$ and $Q(14, 6)$

Solution:

$$\begin{aligned} Q(4,7) &= 0 \quad \text{since} \quad 4 < 7 \\ Q(14,6) &= Q(14-6, 6) \\ &= Q(8,6) + 1 \\ &= Q(8-6, 6) + 1 + 1 \\ &= Q(2,6) + 2 \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

□

Exercise 5

1. Show that $f(x, y) = x - y$ is partial recursive
2. Using $A(3, 2) = 17$, evaluate $A(3, 3)$
3. Show that $A(1, y) = y + 2$

Answers (5)

2. 37

23. Lattices

Relation, is one of the most important concept in Mathematics. Equivalence relations, partial ordering relations, compatible relations, and Functions are special types of relations.

In this discussion, we introduce lattice as a partially ordered set, and we study some properties of lattice.

Definition 23.1 A Lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest Lower bound (G L B) and a least upper bound (L U B).

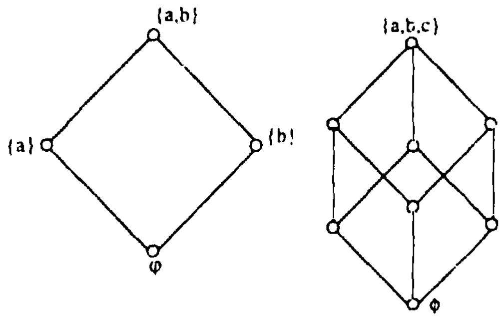
The GLB of a subset $\{a, b\} \subseteq L$ will be denoted by $a * b$ and the least upper bound (LUB) by $a \oplus b$

$$\text{i.e., GLB}\{a, b\} = a * b \text{ (meet or product of a and b)}$$
$$\text{LUB}\{a, b\} = a \oplus b \text{ (join or sum of a and b)}$$

From the definition of a lattice that both $*$ and $\oplus$ are binary operations on L because of the Uniqueness of the LUB and GLB of any subset of a poset.

It is obvious that, a totally ordered set is trivially a lattice, but not all partially ordered sets are lattices, can be concluded from Hasse diagrams of posets.

Example 23.1 Let S be any set any $P(S)$ be its power set. The poset $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively. In particular, when S has a single element, the corresponding lattice is a chain containing

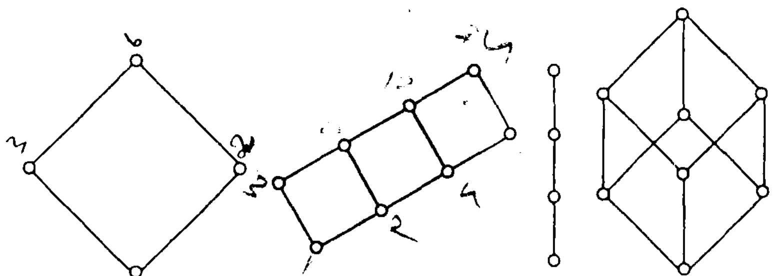


two elements. When S has two and three elements, the diagrams of the corresponding lattices are as shown below, respectively

Example 23.2 Let I_+ be the set of all positive integers and let D denote the relation of "devision" in I_+ such that for any $a, b \in I_+, aDb \iff a$ divides b. Then (I_+, D) is a lattice in which the join of a and b by least common multiple (LCM) of a and b

i.e., $a \oplus b$ = LCM of a and b, and the meet of a and b

i.e., $a * b$ = GCD of a and b (greatest common divisor of a and b)



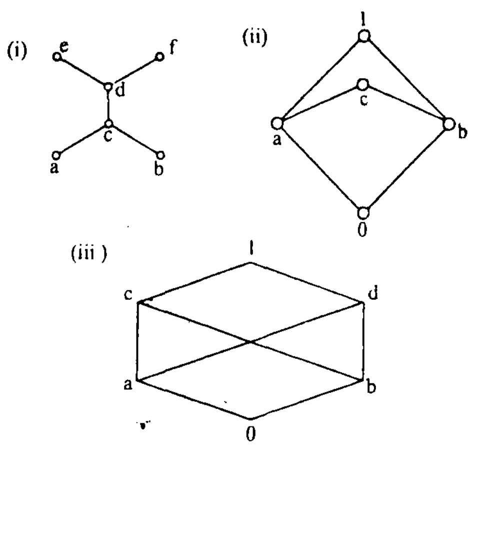
(S_6, D) (S_{24}, D) (S_8, D) (S_{30}, D)

Example 23.3 Let n be a positive integer and S_n be the set of all divisors of n; for example $n = 6, S_6 = \{1, 2, 3, 6\}$ and for $n = 24, S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$. Let D denote the relation of "division" as defined in above example 23.2. The lattices $(S_6, D), (S_{24}, D), (S_8, D)$, and (S_{30}, D) are given respectively

Note 23.1 Symbols such as $\wedge$ and $\vee$ (OR) and $+$ are also be used to denote the meet $(*)$ and join $(\oplus)$ of two elements respectively

Previous examples show that different lattices can be represented by the same diagram except that the nodes have different labels.

Example 23.4 Explain why the posests given below are not lattices



Solution: Consider (iii)
We know that a poset is a lattice iff sup(a, b) and inf(a, b) exist for each pair of a, b in the set.
In the diagram (iii), $\{a, b\}$ has three upper bounds c, d and 1 and no one of them preceeds the other two i.e., GLB and LUB of $\{a, b\}$ does not exist.
Similarly (ii) is not a lattice.

For (i) GLB $=?$

LUB $=?$ in general $\quad \therefore$ It is also not a lattice.

23.1 Principle of Duality

Any statement about lattices involving the operations $*$ and $\oplus$ and the relations $\leq$ and $\geq$ remains true if $*$ is replaced by $\oplus$, $\oplus$ by $*$, $\leq$ by $\geq$ and $\geq$ by $\leq$.

The operations $*$ and $\oplus$ are called duals of each other as are the relations $\leq$ and $\geq$. Similarly, the lattices $(L, \leq)$ and $(L, \geq)$ are called duals of each other.

Some properties of lattices:

Result 23.1 Let $(L, \leq)$ be a lattice. The L satisfies the following laws: For any $a, b, c \in L$. We have

(1) Idempotent laws

$$a * a = a \text{ and } a \oplus a = a$$

(2) Commutative laws

$$a * b = b * a \text{ and } a \oplus b = b \oplus a$$

(3) Associative laws

$$(a * b) * c = a * (b * c) \text{ and } (a \oplus b) \oplus c = a \oplus (b \oplus c)$$

(4) Absorption laws

$$a * (a \oplus b) = a \text{ and } a \oplus (a * b) = a$$

Proof: (1) Let $a, b, c \in L$, by the definition of GLB of a and b we have

$$a * b \leq a \qquad \text{(i)}$$

and if $a \leq a$ and $a \leq b$, then

$$a \leq a * b \qquad \text{(ii)}$$

As $a \leq a$, from (1) and (ii) we have
$a * b \leq a$ and $a \leq a * a$ respectively.
By the antisummetric property it follows that $a = a * a$, Similarly we can prove that $a \oplus a = a$

(2) Given $a, b, \in L$, both $a * b$ and $b * a$ are GLB of a and b. By the Uniqueness of GLB of a and b, we have $a * b = b * a$ Similarly $a \oplus b = b \oplus a$ holds good.

(3) Let $a, b, c \in L$. By the definition we have

$$(a * b) * c \leq a * b$$
$$\text{and } (a * b) * c \leq c$$

By the definition of GLB of a and b, We have $a * b \leq a$ and $a * b \leq b$, So by transitive property of $\leq$ we have

$$(a * b) * c \leq a$$
$$\text{and } (a * b) * c \leq b$$

As $(a * b) * c \leq b$ and $(a * b) * c \leq c$
We see that $(a * b) * c$ is a alower bound for b and c, From the definition of $b * c$ it follows that $(a * b) * c \leq b * c$
As $(a * b) * c \leq a$ and $(a * b) * c \leq b * c$,
From the definition of $a * (b * c)$, we have

$$(a * b) * c \leq a * (b * c) \qquad \text{(iii)}$$

Now $a * (b * c) \leq a$ and $a * (b * c) \leq b * c$
As $b * c \leq b$, by transitivity $a * (b * c) \leq b$
Since $a * (b * c) \leq a$ and $a * (b * c) \leq b$, we have $a * (b * c) \leq (a * b)$.

$$\text{As } a * (b * c) \leq b * c \leq c$$
$$a * (b * c) \leq (a * b) * c \qquad \text{(iv}$$

From (iii) and (iv), by antisymmetric property, it follows that

$$a * (b * c) = (a * b) * c$$

Similarly, we can prove that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(4) Let $a, b, \in L$. Then $a \leq a$ and $a \leq a \oplus b$. So $a \leq a * (a \oplus b)$. On the othe hand $a * (a \oplus b) \leq a$. By antisymmetric property of $\leq$ we have $a = a * (a \oplus b$
Similarly, we have $a = a \oplus (a * b) \forall a, b \in L$.

□

Theorem 23.1 Let $(L, \leq)$ be a lattice. For any $a, b \in L$ the following are equivalent.

(i) $a \leq b$

(ii) $a * b = a$
(iii) $a \oplus b = b$

Proof: At first, consider $(i) \Longleftrightarrow (ii)$

We have $a \leq a$, assume $a \leq b$ Therefore $a \leq a * b$. By the definition of GLB, we have

$$a * b \leq a$$

Hence by antisymmetric property, $a * b = a$

Assume that $a * b = a$, but is only possible if $a \leq b \Rightarrow a * b = a \Rightarrow a \leq b$.

Combining these two results, we have $a \leq b \Longleftrightarrow a * b = a$

Similarly, $a \leq b \Longleftrightarrow a \oplus b = b$

Alternatively, $(ii) \Longleftrightarrow (iii)$ as follows:

Assume $a * b = a$, we have $b \oplus (a * b) = b \oplus a = a \oplus b$, but by absorption $b \oplus (a * b) = b$ Hence $a \oplus b = b$.

By repeating similar steps, we can show that $a * b = a$ follows from $a \oplus b = b$ $\therefore$ $(ii) \Longleftrightarrow (iii)$

Hence the theorem. □

Theorem 23.2 Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$, the following properties called isotonicity hold.

$$b \leq c \Rightarrow \begin{cases} a * b \leq a * c \\ a \oplus b \leq a \oplus c \end{cases}$$

Proof: From the previous theorem

$$b' \leq c' \Longleftrightarrow b' * c' = b' \qquad \text{(i)}$$

Now, To prove $a * b \leq a * c$ take b' as $a * b$, c' as $a * c$

$$\begin{aligned} \text{i.e.,}(a * b) * (a * c) &= (a * a) * (b * c) \quad \text{by associative law} \\ &= a * (b * c) \\ &= a * b \quad [\because b \leq c \Longleftrightarrow b * c = b] \end{aligned}$$

Therefore $a * b \leq a * c$ by (i)

The second statement, is the dual of the first statement, can be proved in a similar manner. □

Corollary 23.2.1 For any a, b, c, d in a lattice $(L, \leq)$ if $a \leq b$ and $c \leq d$, then $a \oplus c \leq b \oplus d$ and $a * c \leq b * d$

Proof:

As $a \leq b$, We have $a \oplus c \leq b \oplus c$

As $c \leq d$, We have $b \oplus c \leq b \oplus d$

By transitivity of $\leq$, it follows that $a \oplus c \leq b \oplus d$

Similarly, We can obtain $a * c \leq b * d$. □

Theorem 23.3 Let $(L, \leq)$ be a lattice. For any $a, b, c, \in L$, the following inequalities, hold:

1. Distributive Inequalities

   (i) $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$
   (ii) $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

2. Modular Inequalities

   (i) $a \leq c \Longleftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$
   (ii) $a \geq c \Longleftrightarrow a * (b \oplus c) \geq (a * b) \oplus c$

Proof: As (ii) in 1, (ii) in 2 are duals of (i) in 1 and (i) in 2 respectively, it is enough to prove (i) in 1. and (i) in 2 only.

Consider (i) in 1:

let $a, b, c, \in L$. As $a \leq a \oplus b$ and $a \leq a \oplus c$ we have $a \leq [(a \oplus b) * (a \oplus c)]$

As $b * c \leq b \leq a \oplus b$ and $b * c \leq c \leq a \oplus c$, we have $(b * c) \leq (a \oplus b) * (a \oplus c)$

So $(a \oplus b) * (a \oplus c)$ is an upper bound for a and $b * c$ and hence $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

Thus (i) in 1 is proved.

The inequality (i) in 2 is a special case of (i) in 1.

If $a \leq c$, then $a \oplus c = c$ and from (i) in 1 we obtain

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c) = (a \oplus b) \oplus c$$

which is inequality (i) in 2.

Hence the theorem. □

24. Lattice as algebraic system

Definition 24.1 A Lattice is an algebraic system $(L, *, \oplus)$ with two binary operations $*$ and $\oplus$ on L which are both (1)commutative and (2) associative and (3) satisfy the absorption laws.

Note 24.1 The absence of idempotent laws is due to the absorption laws for any $a \in L$,

$$a * a = a * [a \oplus (a * a)] = a$$

Note 24.2 The above definition does not assume the existence of any partial ordering on L.

Now, we show that a partial ordering relation on L follows as a consequence of the properties of the operations $*$ and $\oplus$.

(I) Define a relation R or L s.t for $a, b, \in L$ $\quad aRb \Longrightarrow a * b = a$

(i) For any $a \in L, a * a = a$, So $aRa \Rightarrow R$ is reflexive

(ii) For $a, b \in L$, given aRb and bRa,

i.e., $a * b = a$ and $b * a = b$

But $a * b = b * a$ and So $a = b$

i.e., aRb and $bRa \Rightarrow a = b \Rightarrow R$ is antisymmetric.

(iii) For $a, b, c \in L$, given aRb and bRc

Then $a * b = a$ and $b * c = b$, Consider $a * c = (a * b) * c = a * (b * c) = a * b = a \Rightarrow aRc \Rightarrow R$is transitive

Thus R is a partial ordering relation

We know that $a * b = a \Rightarrow a \oplus b = b$.

$\therefore$ the relations R on L : $a \oplus b = b$ for any $a, b, \in L$ is also a partial ordering relation on L.

Now, clearly, withrespect to this partial orderings, $a * b$ and $a \oplus b$ are the GLB and LUB of $\{a, b\} \subseteq L$ respectively.

25. Some Special Lattices

Let $(L, *, \oplus)$ be a lattice and $S \subseteq L$ be a finite subset of L where $S = \{a_1, a_2, \ldots, a_n\}$. The GLB and the LUB of S can be expressed as

$$GLBS = \mathop{*}_{i=1}^{n} a_i \text{ and } LUBS = \bigoplus_{i=1}^{n} a_i$$

Where

$$\mathop{*}_{i=1}^{n} a_i = a_1 * a_2 * \ldots * a_n$$

and

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus a_2 \oplus \ldots \oplus a_n$$

Definition 25.1 A lattice is called complete if each of its non empty subsets has a LUB and a GLB.

Clearly, Every Finite lattice must be complete. Also Every complete lattice must have a least element and a greatest element.

The least and the greatest elements of a lattice, are called the bounds (Units, Universal bounds) of the lattice and are denoted by 0 and 1 respectively.

A lattice which has both elements 0 and 1 is called a bounded lattice.

For the lattice $(L, *, \oplus)$ with $L = \{a_1, \dots a_n\}$

$$\overset{n}{\underset{i=1}{*}} a_i = 0 \quad \text{and} \quad \overset{n}{\underset{i=1}{\oplus}} a_i = 1$$

The bounds 0 and 1 of a Lattice $(L, *, \oplus, 0, 1)$ satisfy the following identities.

For any $a \in L$, $\quad a \oplus 0 = a \quad a * 1 = a$

$$a \oplus 1 = 1 \quad a * 0 = 0$$

Clearly, 0 is the identity of the operation $\oplus$ and 1 is the identity of the operation $*$. Similarly, 0 and 1 are zeros with respect to the operations $*$ and $\oplus$ respectively.

In a bounded Lattice, 1 and 0 are duals of each other.

Definition 25.2 In a bounded Lattice $(L, *, \oplus, 0, 1)$ an element $b \in L$ is called a complement of an element $a \in L$ is

$$a * b = 0 \quad \text{and} \quad a \oplus b = 1$$

Note 25.1 (i) The definition of a complement is symmetric in a and b i.e., b is a complement of a if a is a complement of b.

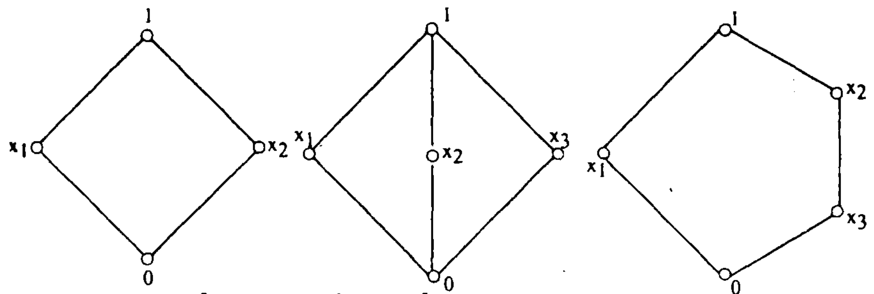
(ii) Any element $a \in L$ may or may not have a complement.

(iii) Furthermore, an element of L may have more than one complement in L,

(iv) We have $0 * 1 = 0$ and $0 \oplus 1 = 1$, $\therefore$ 0 and 1 are complements of each other. Als 1 is the only complement of 0, and 0 is the only complement of 1.

Definition 25.3 A Lattice $(L, *, \oplus, 0, 1)$ is said to be a complemented Lattice if every element of L has at least one complement.

Example 25.1

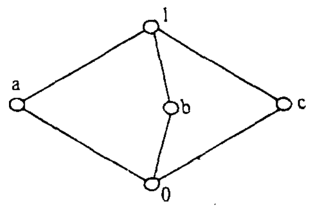


$x_1 * x_2 = 0$
$x_2 \oplus x_2 = 1$
complements of x_1 is x_2

complements of x_1 are x_2, x_3
complements of x_2 are x_1, x_3

complements of x_1 are x_2, x_3

Note 25.2 If a's complement is b, then b need not be the only complement of a.



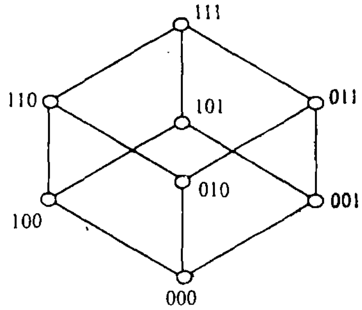
Example 25.2 As $a * b = 0 = a * c$ and $a \oplus b = 1 = a \oplus c$, both b and c are complements of a.

Also, a is a complement of both b and c.

Note 25.3 Not every Lattice with 0 and 1 is complemented.

Example 25.3 For example, if L is a finite chain with more than two elements then L is not complemented. Take $x \in L, x \neq 0, x \neq 1$. Then if $x \oplus y = 1$ then $y = 1$ and $x * y = x \neq 0$. So x has no complement.

Example 25.4 Let $(L^n, \leq_n)$ be the Lattice of n-tuples of 0 and 1



$(L_3, \leq_3)$

This is a complemented lattice in which every element has a Unique complement. The complement of an element of L^n can be obtained by interchanging 1 by 0 and 0 by 1 in the n-tuple representing the element. As a special case when $n = 3$, $(L^3, \leq_3)$ is shown. The bounds of $(L^3, \leq_3)$ are (0, 0, 0) and (1, 1, 1). The complement of (1, 0, 1) is (0, 1, 0).

Example 25.5 The Lattice $(\rho(s), \leq)$ of the power set of any set S is isomorphic to the Lattice $(L^n, \leq_n)$ provided S has n elements. The meet and join operations on $\rho(S)$ are $\cap$ and $\cup$ respectively, while the bounds are ϕ and S. The Lattice $(\rho(s), \leq)$ is a complemented Lattice in which the complement of any subset A of S is the set $S-A$.

Definition 25.4 A Lattice $(L, *, \oplus)$ is called a distributive Lattice if for any $a, b, c \in L$,

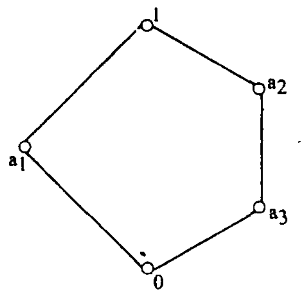
$$a * (b \oplus c) = (a * b) \oplus (a * c)$$
$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

In other words, in a distributive Lattice the operations $*$ and $\oplus$ distribute over each other.

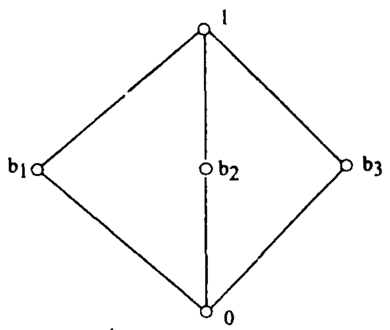
Note 25.4 If one of the distributive equality holds $\forall a, b, c \in L$, then by duality principle, the other distributive equality also holds for all $a, b, c \in L$.

Note 25.5 It is sufficient to verify any one of these two equalities for all possible combinations of the elements of a Lattice. Note that the distributive equalities may be satisfied by some elements of a Lattice, but this does not guarantee that the Lattice is distributive

Example 25.6 Show that the Lattices given by the diagrams are not distributive



(A)



(B)

Solution: In Lattice (A)

$$a * (a_1 \oplus a_2) = a_3 * 1 = a_3 = (a_3 * a_1) \oplus (a_3 * a_2)$$
$$a_1 * (a_2 \oplus a_3) = 0 = (a_1 * a_2) \oplus (a_1 * a_3)$$

but

$$a_2 * (a_1 \oplus a_3) = a_2 * 1 = a_2$$
$$(a_2 * a_1) \oplus (a_2 * a_3) = 0 \oplus a_3 = a_3$$

Hence the Lattice (A) is not distributive.
In (B), $b_1 * (b_2 \oplus b_3) = b_1$ while $(b_1 * b_2) \oplus (b_1 * b_3) = 0$ which shows that the Lattice is not distributive.

Theorem 25.1 Every chain is a distributive Lattice.

Proof: Let $(L, \leq)$ be a chain and $a, b, c \in L$. Consider the following possible cases:

(i) $a \leq b$ or $a \leq c$ and
(ii) $b \leq a$ or $c \leq a$

In case (i), $a \leq b \oplus c$. So $a * (b \oplus c) = a$ and $(a * b) \oplus (a * c) = a$

In case (ii), $b \oplus c \leq a$. So $a * (b \oplus c) = b \oplus c$ and $(a * b) \oplus (a * c) = b * c$. Thus for all $a, b, c \in L$, the distributive equation $a * (b \oplus c) = (a * b) \oplus (a * c)$ holds and hence L is distributive. □

Theorem 25.2 Let $(L, *, \oplus)$ be a distributive Lattice.
For any $a, b, c \in L$,

$$(a * b = a * c) \wedge (a \oplus b = a \oplus c) \Longrightarrow b = c$$

Proof:

$$(a * b) \oplus c = (a * c) \oplus c = c$$
$$(a * b) \oplus c = (a \oplus c) * (b \oplus c) = (a \oplus b) * (b \oplus c)$$
$$= b \oplus (a * c) = b \oplus (a * b) = b$$

□

Hence the theorem

Note 25.6 By the above theorem, if an element $a \in L$ has a complement, then it must be unique, in a distributive lattice.
Suppose that b and c are complements of a, then

$$a * b = a * c = 0 \quad \text{and} \quad a \oplus b = a \oplus c = 1.$$

By the above theorem, $b = c$.

Definition 25.5 A Lattice is said to be modular if

$$a \leq c \Longrightarrow a \oplus (b * c) = (a \oplus b) * c$$

Theorem 25.2 Every distributive Lattice is modular

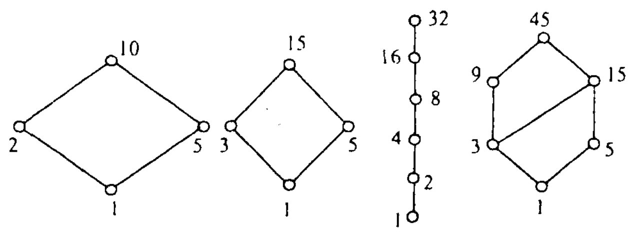
Proof: Let $(L, \leq)$ be a distributive Lattice.
For all $a, b, c \in L$, we have $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$
Thus if $a \leq c$, then $a \oplus c = c$ and
$a \oplus (b * c) = (a \oplus b) * c$. So if a $\leq c$, the modular equation is satisfied and L is modular. □

Note 25.7 Converse is not true always.

Note 25.8 Any chain is modular

Example 25.7 If $D(n)$ denotes the Lattice of all the divisors of the integer n, draw the Hasse diagrams of $D(10)$, $D(15)$, $D(32)$ and $D(45)$.

Solution:



Example 25.8 Let L be a complemented, distributive Lattice. For $a, b \in L$, the following are equivalent.

(i) $a \leq b$

(ii) $a * b' = 0$

(iii) $a' \oplus b = 1$

(iv) $b' \leq a'$ Where "′" denotes corresponding complement

Solution:

$$
\begin{aligned}
a \leq b &\Longrightarrow a \oplus b = b \\
&\Longrightarrow (a \oplus b) * b' = 0 \qquad \text{as } b * b' = 0 \\
&\Longrightarrow (a * b') \vee (b * b') = 0 \\
&\Longrightarrow a * b' = 0 \qquad \text{as } \ b * b' = 0
\end{aligned}
$$

Hence $(i) \Longrightarrow (ii)$

$$
\begin{aligned}
a * b' = 0 &\Longrightarrow (a * b')' = 1 \\
&\Longrightarrow a' \oplus (b')' = 1 \\
&\Longrightarrow a' \oplus b = 1
\end{aligned}
$$

Hence $(ii) \Longrightarrow (iii)$

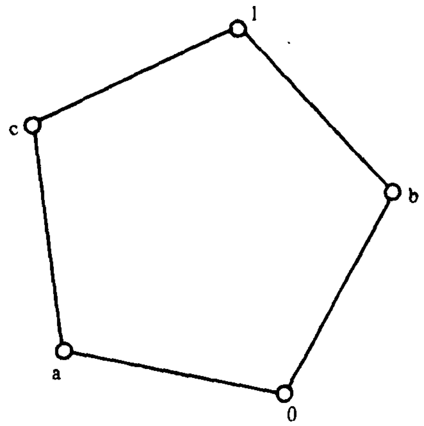
$$
\begin{aligned}
a' \oplus b = 1 &\Longrightarrow (a' \oplus b) * b' = b' \\
&\Longrightarrow (a' * b') \oplus (b * b') = b' \quad \text{(distributive law)} \\
&\Longrightarrow a' * b' = b' \qquad \text{as } b * b' = 0 \\
&\Longrightarrow b' \leq a'.
\end{aligned}
$$

Hence $(iii) \Longrightarrow (iv)$

$$
\begin{aligned}
b' \leq a' &\Longrightarrow a' * b' = b' \\
&\Longrightarrow a \oplus b = b \qquad \text{(taking complement on bothsides by Demorgan's law)} \\
&\Longrightarrow a \leq b
\end{aligned}
$$

Hence $(iv) \Longrightarrow (i)$
Thus $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (i)$

Example 25.9 Prove that the following Lattice is not modular.



Solution: For this Lattice when $a \leq c$

$$a \oplus (b * c) \neq (a \oplus b) * c$$

Since $a \oplus (b * c) = a \oplus 0 = a$

$$\text{but} \qquad (a \oplus b) * c = 1 * c = c.$$

∴ it is not a modular lattice.

Note 25.9 (i) A Lattice L is modular $\iff$ For all $a, b, c \in L$

$$a \oplus (b * (a \oplus c)) = (a \oplus b) * (a \oplus c)$$

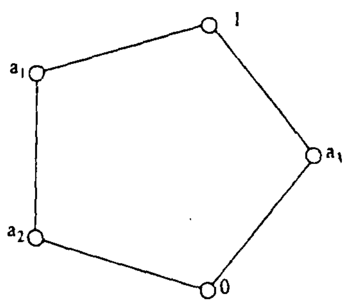
(ii) A Lattice L is modular $\iff$ For all a, b, c In L

$$(a \oplus (b * c)) * (b \oplus c) = (a * (b \oplus c)) \oplus (b * c) \quad \text{(Try!)}$$

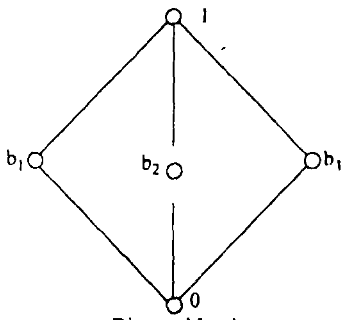
Note 25.10 A Lattice is distributive $\iff$ for all, $a, b, c \in L$

$$(a * b) \oplus (b * c) \oplus (c * a) = (a \oplus c) * (b \oplus c) * (c \oplus a) \quad \text{(Try!)}$$

Result: Already we saw that the pentagonal lattice, diamond lattice are not distributive.



Pentagonal Lattice



Diamond Lattice

Theorem: A lattice is non distributive if and only if it contains a sublattice isomorphic to one of the above two lattices (pentagonal or diamond lattice).

Every distributive lattice is modular.

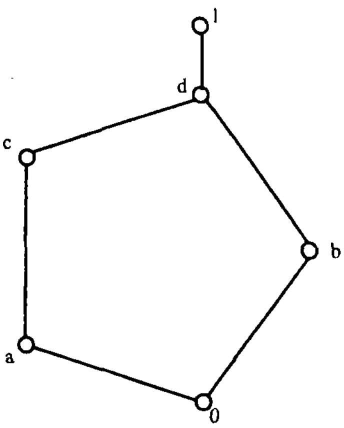
But there are modular lattices which are not distributive.

Example: The diamond lattice is a non distributive modular lattice.

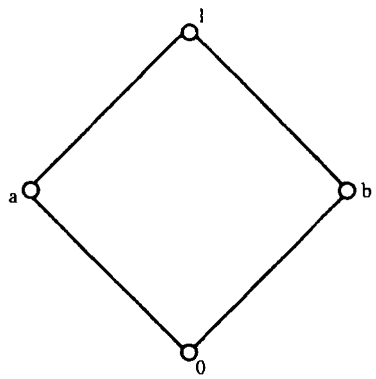
But pentagonal lattice is in fact not even modular.

Theorem: A lattice is non modular if and only if it contains a sublattice isomorphic to pentagonal lattice

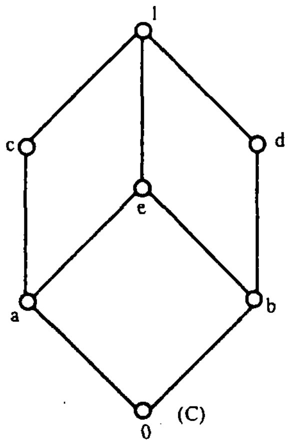
Example 25.10 Consider the lattices given by following Hasse diagrams which of them, are modular? distributive? complemented?



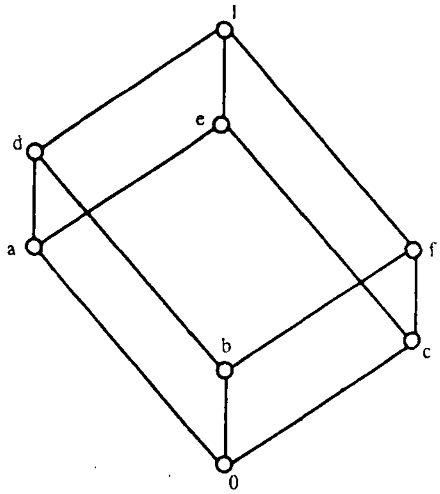
(A)



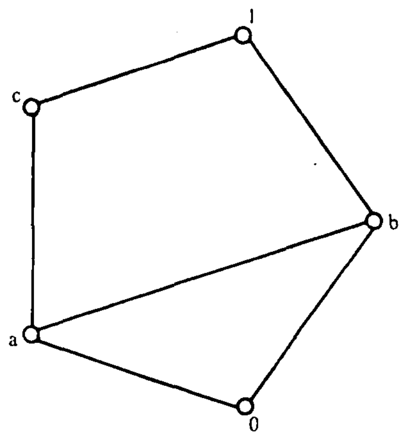
(B)



(C)



(D)



(E)

(F)

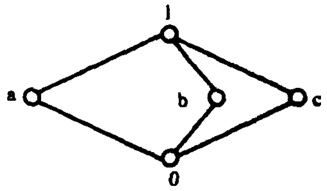
Solution:

(A) not modular, not distributive, As the element b has no complement, the lattice is not complemented.
(B) Modular, distributive, complemented
(C) Not Modular (as $0, a, c, d, 1$ forms pentagon), not distributive, not complemented (e has no complement)
(D) Modular, distributive and complemented
(E) Modular, distributive, not complemented
(F) Modular, distributive, but not complemented.

Exercise 6 1. In a Lattice L, show that, for all $a, b, c, d \in L$.

$$(a * b) \oplus (c * d) \leq (a \oplus c) * (b \oplus d)$$

2. Is the following Lattice



distributive? modular?

3. In a distributive Lattice, prove that the following are equivalent

(*i*) $a * b \leq A \leq a \oplus b$

(*ii*) $A = (a * A) \oplus (b * A) \oplus (a * b)$.

4. Show that a Lattice L is distributive if and only if for all $a, b, c, in L$,

$$(a \oplus b) * c \leq a \oplus (b * c)$$

5. Let L be a distributive Lattice with 0 and 1. If an element $a \in L$ has a complemen a', then show that $a \oplus (a' * b) = a \oplus b$.

6. Show that in a Lattice if $a \leq b \leq c$ then

(*i*) $a \oplus b = b * c$

and (*ii*) $(a * b) \oplus (b * c) = b = (a \oplus b) * (a \oplus c)$.

Solved Problems

Problem 1 Let $A = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$

and $A_1 = \{1, 2, 3, 4\}$, $A_2 = \{5, 6, 7\}$
$A_3 = \{4, 5, 7, 9\}$, $A_4 = \{4, 8, 10\}$
$A_5 = \{8, 9, 10\}$, $A_6 = \{1, 2, 3, 6, 8, 10\}$

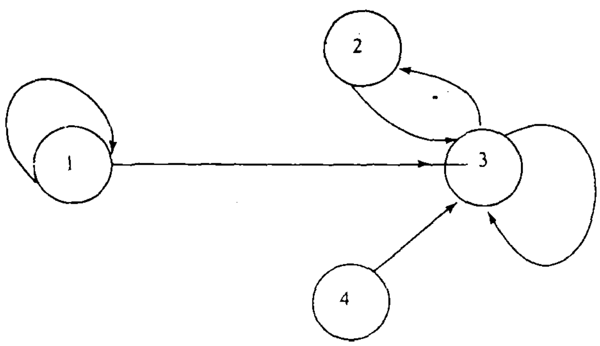
Which of the following are partitions of A?

(a) $\{A_1, A_2, A_5\}$
(b) $\{A_1, A_3, A_5\}$
(c) $\{A_3, A_6\}$
(d) $\{A_2, A_3, A_4\}$

Solution:

(a) $\{A_1, A_2, A_5\}$ is a partition of A, since $\{A_1, A_2, A_5\}$ are mutually disjoint
(b) $\{A_1, A_3, A_5\}$ is not a partition of A, since $A_1 \cap A_3 \neq \emptyset$
(c) $\{A_3, A_6\}$ is a partition of A, since $A_3 \cap A_6 = \emptyset$
(d) $\{A_2, A_3, A_4\}$ is not a partition of A, since A_2, A_3, A_4 are not mutually disjoint

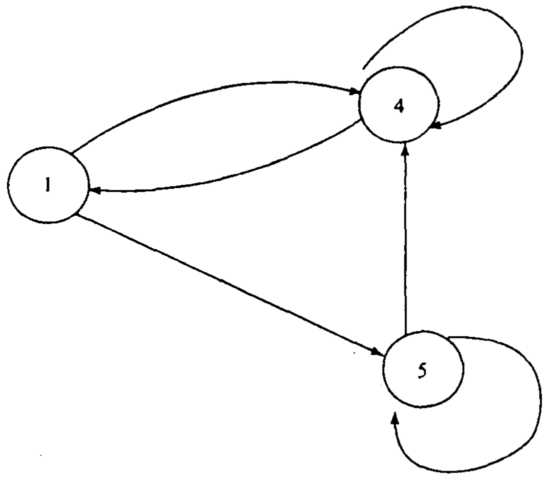
Problem 2 Find the relation determined by the following digraph



Solution: Since $a_i R a_j \iff$ there is an edge from a_i to a_j

$\therefore \quad R = \{(1, 1), (1, 3), (2, 3), (3, 2), (3, 3), (4, 3)\}$

Problem 3 Let $A = \{1, 4, 5\}$ and Let R be given by the following digraph. Find M_R and R



Solution:

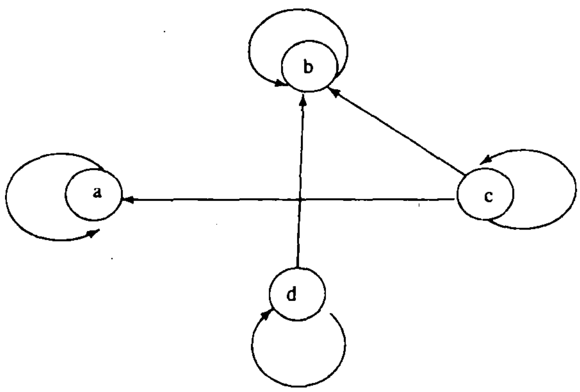
$$M_R = \begin{pmatrix} 011 \\ 110 \\ 011 \end{pmatrix}$$

$R = \{(1, 4), (1, 5), (4, 1), (4, 4), (5, 4), (5, 5)\}$

Problem 4 Let $A = \{a, b, c, d\}$ and Let R be the relation on A that has the matrix

$$M_R = \begin{pmatrix} 1000 \\ 0100 \\ 1110 \\ 0101 \end{pmatrix}, \quad \text{Construct the digraph of} R.$$

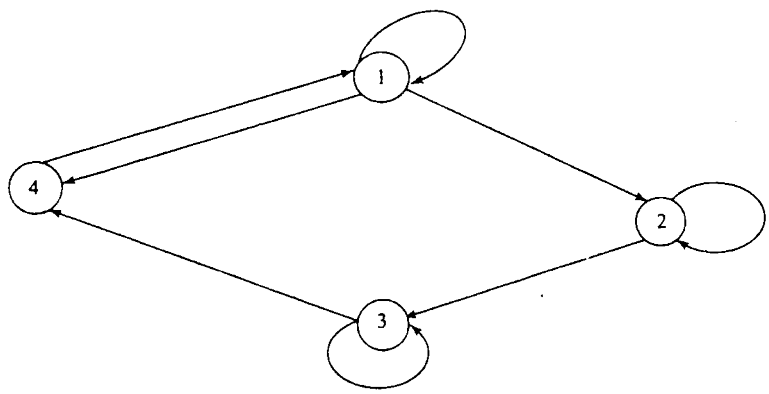
Solution:



Problem 5 Find the relation R defined on $A = \{1, 2, 3, 4\}$ and

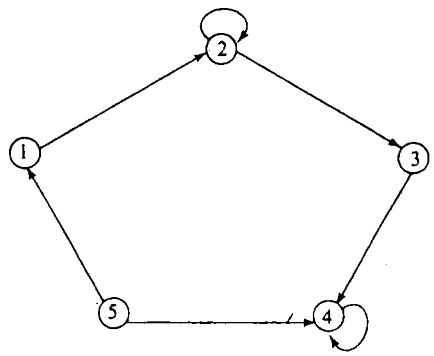
$$M_R = \begin{pmatrix} 1101 \\ 0110 \\ 0011 \\ 1000 \end{pmatrix}, \quad \text{and its digraph}$$

Solution: $R = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 3), (3, 4), (4, 1)\}$



Problem 6 Find the relation determined by the digraph and give its matrix

Solution:



Solution: $R = \{(1, 2), (2, 3), (2, 2), (3, 4), (4, 4), (5, 1), (5, 4)\}$

$$M_R = \begin{bmatrix} 10000 \\ 01100 \\ 00010 \\ 00010 \\ 10010 \end{bmatrix}$$

Problem 7 Let $A = Z$ the set of integers, and let
$R = \{(a, b) \in A \times A | a < b\}$
so that R is the relation less than.

Is R symmetric, asymmetric or antisymmetric?

Solution:

Symmetry: If $a < b$, then it is not true that $b < a$

$\therefore$ R is not symmetric

Asymmetry: If $a < b$, then $b \nless a$(b is not less than a)

So R is asymmetric

Antisymmetry: If $a \neq b$, then either $a \nless b$ or $b \nless a$

So that R is antisymmetric

Problem 8 Consider the $A = \{1, 2, 3\}$ and Let R be the relation on A Whose matrix is

$$M_R = \begin{pmatrix} 111 \\ 001 \\ 001 \end{pmatrix}$$

Show that R is transitive

Solution: By direct computation $(M_R)^2_{\odot} = M_R$

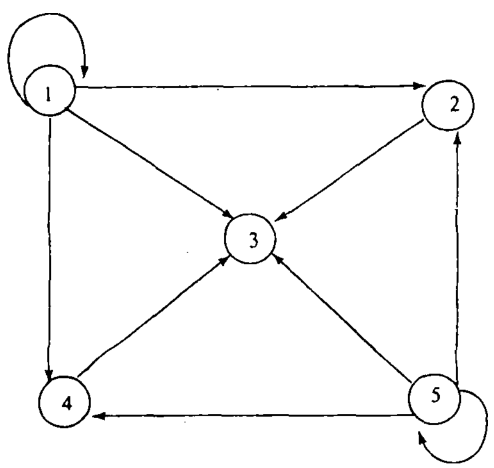
$\therefore$ R is transitive

Problem 9 Let $A = \{1, 2, 3, 4\}$ and Let $R = \{(1, 2), (1, 3), (4, 2)\}$ Is R transitive?

Solution: Since there are no elements a, b and c in A such that aRb and bRc but $a \not R c$ we conclude that R is transitive

Problem 10 Determine whether the relation, given is reflexive irreflexive, symmetric, asymmetric, antisymmetric or transitive.

(a)



(b)

$$M_R = \begin{bmatrix} 0101 \\ 1011 \\ 0100 \\ 1100 \end{bmatrix}$$

(c) $A = Z^+$; $aRb \iff a = b^k$ for some $k \in z^+$

Solution:

(a) Antisymmetric transitive
(b) Irreflexive, symmetric
(c) Reflexive, antisymmetric, transitive

Problem 11 Show that if a relation on a set A is transitive and irreflexive then it is asymmetric

Solution: Let R be transitive and irreflexive. Suppose aRb and bRa. Then aRa since R is transitive. But this contradicts the fact that R is irreflexive. Hence R is asymmetric

Problem 12 Let $A = \{1, 2, 3, 4\}$ and consider the partition $\{\{1, 2, 3\}, \{4\}\}$ of A. Find the equivalence relation R on A determined by partition.

Solution: The blocks of partition are $\{1, 2, 3\}$ and $\{4\}$. Each element in a block is related to every other element in the same block and only to those elements. Thus in his case
$R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)\}$.

roblem 13 Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2\}$, $D = d_1, d_2, d_3, d_4\}$ Consider the following four functions from A to B, A to D, B to C and D to B respectively

(a) $f_1 = \{(a_1, b_2), (a_2, b_3), (a_3, b_1)\}$
(b) $f_2 = \{(a_1, d_2), (a_2, d_1), (a_3, d_4)\}$
(c) $f_3 = \{(b_1, c_2), (b_2, c_2), (b_3, c_1)\}$
(d) $f_4 = \{(d_1, b_1), (d_2, b_2), (d_3, b_1)\}$

Determine whether or not each function is one to one, whether each function is onto.

Solution:

(a) f_1 is one to one and onto
(b) f_2 is one to one and but not onto
(c) f_3 is onto but is not one to one
(d) f_4 is not one to one and not onto.

roblem 14 Let R be the set of real numbers and Let $f : R \to R$ be defined by $(x) = x^2$. Is f invertible?

Solution: We must determine whether f is one to one since $f(2) = f(-2) = 4$
We conclude that f is not one to one. Hence, f is not invertible

Problem 15 Show that any constant function is primitive recursive

Solution:

$$\text{If} f(x) = 2 \forall x$$
$$\text{then } f(x) = SSZ(x)$$

Problem 16 Show that

$$\delta(x) = \begin{cases} 0 & \text{if } x = 1 \\ x - 1 & \text{if } x > 0 \end{cases} \text{ is primitive recursive}$$

Solution:

$$\text{Since } \delta(x) = \delta^1(Z(x), \cup_1^1(x))$$
$$\text{where } \delta^1(x, 0) = 0 = Z(x)$$
$$\delta^1(x, y+1) = y = \cup_2^3(x, y, \delta^1(x, y))$$

Problem 17 Show that $f(x, y) = |x - y|$ is primitive recursive

Solution: Since $g(x, y) = x \dot{-} y$ and $A(x, y) = x + y$ are primitive recursive

$$|x - y| = (x \dot{-} y) + (y \dot{-} x) = A(g(x, y), g(y, x))$$
$$= A(g(x, y), g(\cup_2^2(x, y), \cup_1^2(x, y))$$

Problem 18 Show that

$$f(x) = \begin{cases} x & \text{if } x \text{ is odd} \\ \frac{x}{2} & \text{if } x \text{ is even} \end{cases}$$

is primitive recursive

Solution: Note that $g(x) = \left[\frac{1}{2}x\right]$ is primitive recursive

$$\text{Since } g(0) = 0$$
$$\text{and } g(x+1) = g(x) + (x \bmod 2)$$

Now we obtain f as

$$f(x) = \left[\frac{1}{2}x\right](1 + (x \bmod 2)) + (x \bmod 2)$$

Problem 19 Show that

$$f(x, y) = \begin{cases} 1 & \text{if } y = 0 \\ x^{x^{x^{\cdot^{\cdot^{x}}}}} & y \text{ times if } y > 0 \end{cases}$$

is primitive recursive

Solution: clearly

$$f(x, 0) = 1$$
$$f(x, y + 1) = x^{f(x,y)}.$$

Problem 20 The function $\left[\frac{x}{2}\right]$ which is equal to the greatest integer which is $\leq \frac{x}{2}$ is primitive recursive

Solution:

$$\text{Clearly } \left[\frac{0}{2}\right] = 0, \left[\frac{1}{2}\right] = 0, \left[\frac{2}{2}\right] = 1, \left[\frac{3}{2}\right] = 1$$

$$\text{Also } \left[\frac{x}{2}\right] = \begin{cases} \frac{x}{2} & \text{when } x \text{ is even} \\ \frac{x-1}{2} & \text{when } x \text{ is odd} \end{cases}$$

Considering odd and even parity function

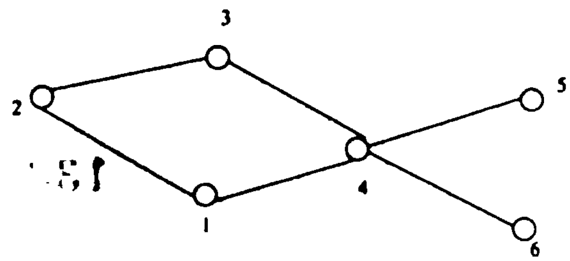
$$p_r(0) = 0$$
$$p_r(y + 1) = \overline{Sg}(U_2^2(y, p_r(y))$$
$$\text{where } \overline{Sg}(0) = 1$$
$$\overline{Sg}(y + 1) = 0$$

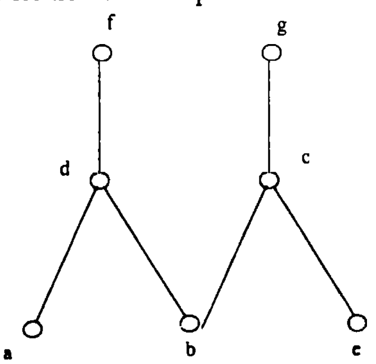
$\overline{Sg}$, p_r are primitive recursive functions

$$\left[\frac{0}{2}\right] = 0$$
$$\left[\frac{y+1}{2}\right] = \left[\frac{y}{2}\right] + p_r(y)$$

where $p_r(y)$ denoted the parity functions which is 1 when y is odd and which is zer when y is even

Problem 21 Determine all maximal and minimal elements of the poset



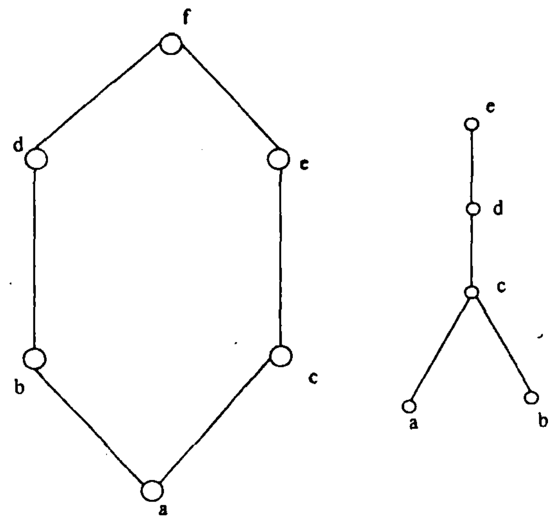


Solution:

(a) Maximal elements: 3, 5
Minimal elements: 1, 6

(b) Maximal elements: f, g
Minimal elements: a, b, c

Problem 22 Determine the greatest and least elements, if they exist of the poset



(a) (b)

Solution:

Ans (a) Greatest f; least a

Ans (b) greatest e; least none

Ans (c) $A = \{x/x$ is a real number and $0 < x < 1\}$ with usual partial order $\leq$

Ans (d) $A = \{x/x$ is a real number and $0 \leq x \leq 1\}$ with usual partial order $\leq$

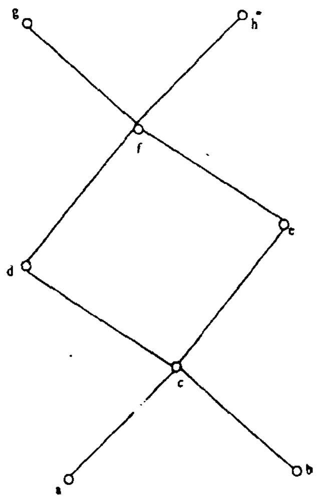
Ans (e) Greatest none; least none

Ans (f) greatest 1; least 0

Problem 23 Find if they exist

(a) all upper bounds of B
(b) all lower bounds of B
(c) the least upper bound of B
(d) the greatest lower bound of B

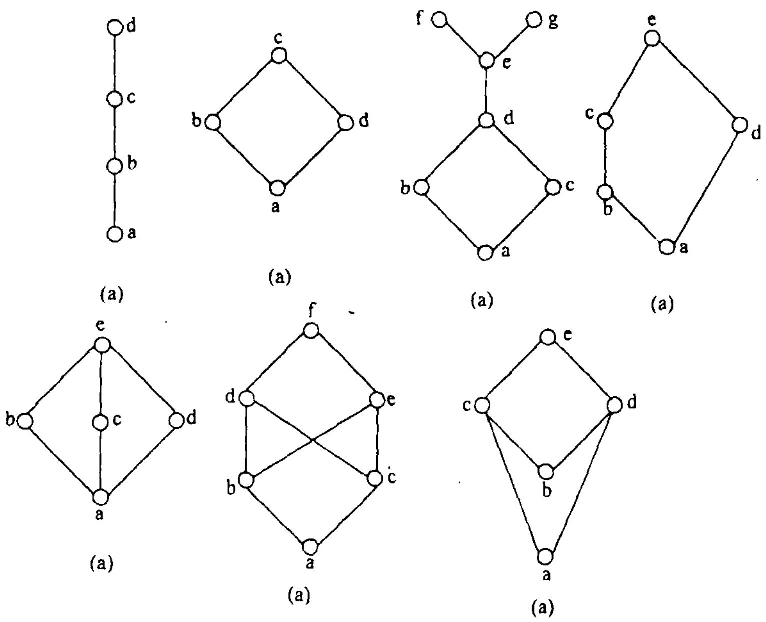
$$B = \{c, d, e\}$$



Solution:

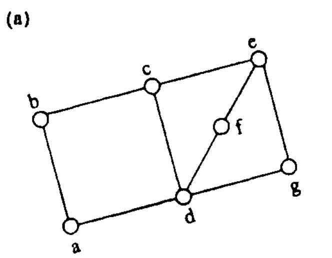
Ans (a) f, g, h
Ans (b) a, b, c
Ans (c) f
Ans (d) c

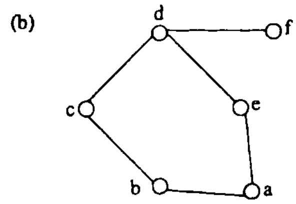
Problem 24 Which of the Hasse diagrams in following figures represent Lattices?



Solution: Hasse diarams (a), (b), (d) and (e) represent Lattices, Diagram (c) does not represent a Lattice, because $f \oplus g$ does not exist, Diagram (f) does not represent a Lattice be cause neither $d \oplus e$ nor $b * c$ exist. Diagram (g) doesnot represent a lattice because $c \oplus d$ does not exist

Problem 25 Determine whether each lattice is distributive complemented or both



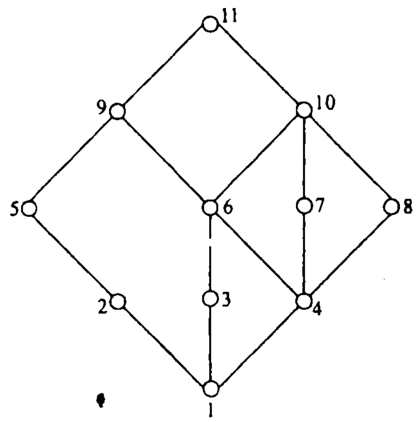


Solution:

(a) Neither

(b) Neither

Problem 26 Let $A = \{1, 2, 3, 4, 5, \ldots, 11\}$ be the poset whose Hasse diagram shown below. Find the LUB and GLB of $B = \{6, 7, 10\}$ if they exist



Solution: Exploring all upward paths from vertices 6 ,7 and 10 we find that $LUB(B) = 10$. Similarly by examining all downwards paths from 6, 7 and 10 we find that $GLB(B) = 4$.

Problem 27 In a distribution Lattice, if an element has a complement then this complement is unique

Solution: Suppose that an element a has two complements b and c, that is

$a \vee b = 1 \quad a \wedge b = 0$

$a \vee c = 1 \quad a \wedge c = 0$

We have

$$\begin{aligned} b &= b \wedge 1 \\ &= b \wedge (a \vee c) \\ &= (b \wedge a) \vee (b \wedge c) \\ &= 0 \vee (b \wedge c) \\ &= (a \wedge c) \vee (b \wedge c) \\ &= (a \vee b) \wedge c \\ &= 1 \wedge c \\ &= c \end{aligned}$$

Problem 28 Is the relation

$$R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 2), (3, 3), (4, 4), \}$$

reflexive on the set $A = \{1, 2, 3, 4\}$?

Solution: This relation is reflexive (all points on the main diagonal are present), but not symmetric (the X's are not symmetrically located with respect to the main diagonal)

	1	2	3	4
1	X	X		X
2	X	X		
3		X	X	
4				X

Problem 29 For (x, y) and (u, v) in $\mathbb{R}^2$ define $(x, y)R(u, v)$ if $x^2 + y^2 = u^2 + v^2$. Prove that R defines an equivalence relation on $\mathbb{R}^2$ and interpret the equivalence classes geometrically.

Solution: If $(x, y) \in \mathbb{R}^2$ then $x^2 + y^2 = x^2 + y^2$ so $(x, y)R(x, y)$

The Relation is reflexive.

If $(x, y)R(u, v)$, then $x^2 + y^2 = u^2 + v^2$ so $u^2 + v^2 = x^2 + y^2$ and $(u, v)R(x, y)$ The relation is symmetric

Finally $(x, y)R(u, v)$ and $(u, v)R(w, z)$ then $x^2+y^2 = u^2+v^2$ and $u^2+v^2 = w^2+z^2$. Thus $x^2 + y^2 = u^2 + v^2 = w^2 + z^2 \Rightarrow (x, y)R(w, z)$, So the relation is transitive.

The equivalence class of (a, b) is

$$[(a, b)] = \{(x, y)|(x, y)R(a, b)\} = \{(x, y)|x^2 + y^2 = a^2 + b^2\}$$

For example $[(1, 0)] = \{(x, y)|x^2 + y^2 = 1^2 + 0^2 = 1\}$, is the graph of a circle in the Cartesian plane with centre (0, 0) and radius 1.

In general (a, b), let $c = a^2 + b^2$, the $[(a, b)]$ is the set of points (x, y) satisfying $x^2 + y^2 = c$, is the circle with center (0, 0) and radius $\sqrt{c}$.

Problem 30 For $a, b \in R - \{0\}$, define $aRb \iff \frac{a}{b} \in Q$

(a) Prove that R is an equivalence relation
(b) Find the equivalence class of 1.
(c) Show that $[\sqrt{3}] = [\sqrt{12}]$

Solution: Reflexive: If $a \in R - \{0\}$, then $\frac{a}{a} = 1 \in Q \therefore \quad aRa$

symmetric: If aRb, then $\frac{a}{b} \in Q \quad (\because \quad 0 \notin R - \{0\})$

$$\frac{b}{a} = 1/\frac{a}{b} \in Q \quad \therefore \quad 1Ra$$

transitive: If aRb and bRc, then $\frac{a}{b} \in Q \& \frac{b}{c} \in Q$

Now $\frac{a}{c} = \frac{a}{b}\frac{b}{c} \in Q \quad \therefore \quad aRc$

$$[1] = \{a|aR1\} = \{a|\frac{a}{1} \in Q\} = \{a|a \in Q\} = Q - \{0\}$$

$$\text{Since} \quad \frac{\sqrt{12}}{\sqrt{3}} = 2 \in Q \quad \sqrt{3}R\sqrt{12} \quad \therefore \quad [\sqrt{3}] = [\sqrt{12}].$$

□

Problem 31 Is the function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{one one?}$$

Solution: The absolute value function is a function with domain R and range $[0, \infty) = \{y \in R | y \geq 0\}$

It is not one one since $|2| = |-2|$.

Problem 32 Show that the functions $f : R \to (1, \infty)$ and $g : (1, \infty) \to R$ defined by $f(x) = 3^{2x} + 1, g(x) = \frac{1}{2}\log_3(x - 1)$ are inverses.

Solution: For any $x \in R$

$$\begin{aligned}(gof)(x) = g(f(x)) &= g(3^{2x} + 1) \\ &= \frac{1}{2}(\log_3(3^{2x} + 1) - 1) \\ &= \frac{1}{2}\log_3 3^{2x} \\ &= \frac{1}{2}(2x) \\ &= x\end{aligned}$$

and for any $x \in (1, \infty)$

$$(fog)(x) = f(g(x)) = x.$$

$\therefore$ f and g are inverses.

Problem 33 Let a and b be natural numbers. Show that the number of positive integers less than or equal to a and divisible by b is $\left\lfloor \frac{a}{b} \right\rfloor$

Solution:

$$\text{As } a = qb + r \text{ with } 0 \le r < b$$

$$\text{Then } \left\lfloor \frac{a}{b} \right\rfloor = q.$$

Clearly the positive integers $b, 2b, \ldots, qb$ are all lessthen or equal to a and are all divisible by b.

On the other hand if $sb \le a$, then s must belong to the set $\{1, 2, \ldots q\}$.

Hence there are exactly q such natural numbers as required.

Problem 34 Show that the function

$$x \mod y = \begin{cases} \text{remainder on dividing } x \text{ by } y, & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases}$$

is primitive recursive

Solution:

$$x \mod y = \begin{cases} 0 \mod y = 0 \\ x + 1 \mod y = [(x \mod y) + 1][sg(y \dot{-} ((x \mod y) + 1))] \end{cases}$$

where the second clause tells us to increment $x \mod y$ by 1 except when this equals y(or $y = 0$), when we return to 0;

□

Quiz Questions

1. For a set A, $P(A)$ is ________
   (Ans: power set of A)
2. If $o(A) = n$ then $o(P(A)) =$ ____
   (Ans: 2^n)
3. $(A \cup B)^c =$ ____
   (Ans: $(A^c \cap B^c)$)
4. $(A \cap B)^c =$ ____
   Ans: $(A^c \cup B^c)$
5. $A \times B$ ____ $B \times A$ (Ans: $\neq$)
6. The relations 'divides' on set of +ve integers is ____
   (Ans: reflexive)
7. The function $f(x) = x^2$ from Z to Z is not ____
   (Ans: invertible)
8. Ackermann's function $A(x, y)$ is not ____ but ____
   (Ans: Primitive recursive, recursive)
9. x^y is ____
   (Ans: primitive recursive)
10. $x \dot{-} y$ is ____
    (Ans: Primitive recursive)
11. In a Lattice $(L, \leq)$, $a * (a \oplus b) =$ ________ and $a \oplus (a * b) =$ ________
    (Ans: a)
12. $a \leq b \Longleftrightarrow a * b = a \Longleftrightarrow a \oplus b = b$
13. Every chain is ____
    (Ans: distributive)
14. $a * (b \oplus c) =$ ________
    (Ans: $(a * b) \oplus (a * c)$)
15. $a \oplus (b * c) =$ ________
    (Ans: $(a \oplus b) * (a \oplus c)$)
16. A Lattice is an ________ with two binary operations $*$ and $\oplus$.
    (Ans: algebraic system)
17. What is the inverse function of $f(x) = ax + b$ from R to R: ________.
    (Ans: $f^{-1}(x) = \frac{x-b}{a}$)

18. The partition of Z:

    Ans: {Set of all even integers, set of all odd integers}.

19. An equivalence relation R in a non empty set A ____ and Conversely, a ____ of A defines an equivalence relation on A

    (Ans: Partitions, partition)

20. Let R be a relation on a finite set S with boolean matrix A. Then R is reflexive $\Longleftrightarrow$ all diagonal entries of A are ________

    (Ans: 1)

21. R is anti-reflexive $\Longleftrightarrow$ if all diagonal entries of A are ________

    (Ans: 0)

22. R is symmetric $\Longleftrightarrow$?________ (Ans: $A = A^T$)

23. How many functions are there from a set with m elements to one with n elements?

    (Ans: n^m)

24. How many one-to-one functions are there from a set with m elements to one with n elements?

    (Ans: $n(n-1)(n-2)\ldots(n-m+1)$)

25. Let $A_i = \{i, i+1, i+2, \ldots\}$

    Then $\bigcup_{i=1}^{n} A_i = \{$________$\}$

    (Ans: $\{1, 2, 3, \ldots\}$)

26. From Q 25, $\bigcap_{i=1}^{n} A_i =$ ________

    (Ans: $\{n, n+1, n+2, \ldots\}$

27. Is $f(x) = x^2$ from the set of integers to the set of integers is one to one?

    (Ans: No)

28. Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

    (Ans: No)

29. Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b)$ 2, $f(c) = 1$ and $f(d) = 3$. Is $f a$ bijection?

    (Ans: yes)

30. Let $f : R \to R$ be defined by $f(x) = 3x - 7$ Find a formula for the inverse function $f^{-1} : R \to R$

    (Ans: $f^{-1}(x) = \dfrac{x+7}{3}$)

26. Group theory

Introduction:

In this chapter, we introduce conceptual ideas related to algebraic structures. They are very important in the theory of sequential machines, formal languages, computes arithmetic, cryptography and many more other applications.

Algebraic system (structure) :

A set together with one or more n-any operations is called on algebraic system or simply an algebra.

Recall that, an n-any operation on a set X which is a mapping from $X^n \to X$. If $n = 1$ such an operation is called unary operation, if n = 2, it is called binary operation and so on.

Since the operations and relations on the set S define a structure on the elements of S, an algebraic system is called algebraic structure.

Binary operation:

When two real numbers are added together, the result is another real number. This is also the case when one real number is subtracted from another real number. Thus addition, subtraction and multiplication are rules of combination, Which, When applied to the elements of the set of real numbers, give results which also belong to this set. Thus we have

$$a + b \in R,\ a - b \in R,\ a.b \in R$$

Where a,b are two elements of R, the set of real numbers. Thus each of these operations is function which assigns a unique element to every pair of real numbers.

Definition:

Let S be a non-empty set. A binary operation on S is a rule that assigns to each ordered pair of elements of S, a unique element of S. This is,

A binary operation is a function from SXS into S (i.e.,) $f: S \times S \to S$ or f assigns an element $f(a,b)$ of S to each ordered pair (a,b) in $S \times S$.

Let S be any set and $*a$ brainy operation on S. Properties that apply to *are :

$a*b = b*a$ for all $a,b \in S$ (commutative).

$a*(b*c) = (a*b)*c$ for all $a,b,c \in S$ (Associatives).

For all $a \in S$, there exists an element $e \in S$ such that are $a*e$ = a = e *a (identity).

For each a $\in$ S, there exists an element b $\in$ S , such that a*b = e = b*a (Inverse).

a*b = c *b and b*a = b*c implies a = c (concellation).

Also, a * (bOc) = (a*b) O (a*c) for all a,b,c $\in$ S (Distributive)

Where *,o are two binary operations on S. The basic property is that if * is a binary operation on S and a and b are elements in S, then a*b $\in$ S, this property is expressed by saying that S is closed under the operation * . This means that by operating on elements of S with *, We can't get elements that are not in S.

A binary operation is so called because it combines two objects.

Example: Let Z be the set of integers with the two binary operations addition and multiplication + and *.Then (Z,+,*) is an algebraic system satisfying all the above properties.

Group theory is a well- developed branch of abstract algebra and is applied in various branches of physical sciences and in computer science.

Definition: It G is a non empty set and * is a binary operation in G, then (G,*) is called a 'Group' if the following conditions are satisfied:

(1) For all a, b ∈ G, a*b ∈ G (closure of G under *)

(2) For all a,b,c ∈ G, a* (b*c) = (a*b)*c (Associative)

(3) There exists e ∈ G with a*e = e*a = a, for all (The Existence of an identity)

(4) For each a ∈ G there is an element b ∈ G such that a *b = b*a = e (Existence of Inverses)

If in addition, a*b = b*a for all a,b, ∈ G, then G is called an abelian group.

Example: Under ordinary addition, each of Z,Q,R,C is any abelian group.

Semigroup: A non empty set S together with an **associative** binary operation * on it is called a semi group.

The semi group is denoted by (S,*)

Example: Let N be the set of positive integers. Then (N, +) and (0N,*) are semi-groups.

Monoid: A semi group (M,*) with an identity element is called a monoid. Thus (M, *) is a monoid if

(1) For any a,b,c, ∈ S, (a*b)*c = a* (b*c)and

(2) There exists an element e ∈ M such that for any $x \in M$, $x*e = e*x = x$

Example: (Z,+) is a semi group having the number 0 as the identity element. Hence (Z,+) is a monoid.

FACT: For every Group G

(1) The identity of G is Unique

(2) the inverse of each element of G is Unique

(3) if a,b,c, ∈ G and ab = ac, then b = c (left- cancellation property)

(4) if a,b,c ∈ G and ba = ca then b = c (Right cancellation property)

Subgroup:

Let G be a group and Φ ≠ H ⊆ G. If H is a group under the binary operation of G, then we call H a subgroup of G.

Example: Every group G has {e} and G as subgroups. These are the trivial subgroups of G. All others are termed nontrivial or proper.

(Z,+) is a subgroup of (R,+)

(Q^+,*) is not a subgroup of (R,.+) even though $Q^+ \subseteq R$.

FACT : If H is a non empty subset of a group G then H is a sub group of G if and only if (a) for all a, b $\in$ H, a.b $\in$ H and (b) for all a $\in$ H, $a^{-1} \in$ H

FACT : If G is a group and $\Phi \neq H \subseteq G$ with H finite, then H is a sub group G $\Leftrightarrow$ if H is closed under the binary operation of G.

Example : Let B={0,1} and the operation + is defined as follows :

+	0	1
0	0	1
1	1	0

Then (B,+) is a group with 0 as identity and each element is its own inverse.

Example : Consider (Z,*) where * is defined by a*b = a + b –ab, check whether (Z,*) is a group or monoid.

From group properties, we have

a*b = a + b –ab $\in$ Z, for a,b $\in$ Z

a*(b*c) = a* (b + c –bc)

= a + (b+ c – bc) – a(b + c –bc)

= a + b + c –bc – ab – ac + abc

(a*b)*c = a + b – ab + c –ac – bc + abc

∴ (a*b)*c = (a*b)*c

'o' $\in$ Z is the identity element as a*0 = 0*a = 1.

For 3 $\in$ Z, there is no $x \in$ Z such that

$$3 + x - 3x = 0 \text{ as } 3 + x\text{-}3x = 0 \Rightarrow x = \frac{3}{2} \notin Z$$

Hence, (Z,*) is not a group.

However, it is only a Monoid.

Definition :

If $(G,^\circ)$ and $(H,*)$ are groups and $f : G \rightarrow H$ then f is called a (group) **homomorphism** if for all $a,b \in G$

$$f(a^\circ b) = f(a) * f(b)$$

FACT : Let (G, °), (H,*) be groups with respective identities e_G, e_H. If f: $G \to H$ is a homomorphism, then

$$f(e_G) = e_H$$

$$f(a^{-1}) = (f(a))^{-1} \text{ for all } a \in G$$

$$f(a^n) = [f(a)]^n \text{ for all } a \in G \text{ and all } n \in Z$$

$f(S)$ is a subgroup of H for each subgroup S of G.

Definition : If f: $(G, °), \to (H,*)$ is homomorphism. We call f is on **isomorphism** if it is one to one and onto.

In this case, G, H are said to be **isomorphic** groups

Example : Let If: $(R^+ .°) \to (R,+)$ where $f(x) = \log_{10}(x)$.

This function is both one-to-one and onto.

For all $a, b \in R^+$, $f(ab) = \log_{10}(ab) = \log_{10} a + \log_{10} b$

$$= f(a) + f(b)$$

Therefore, f is an isomorphism and the group of positive real numbers under multiplication is abstractly the same as the group of all real numbers under addition.

Example : Let E be the set of all even integers. Show that the semigroups $(Z,+)$ and (E,+) are isomorphic.

Define f: $Z \to E$ by $f(x) = 2x$

$$f(a_1) = f(a_2) \Rightarrow 2a_1 = 2a_2$$

$$\Rightarrow a_1 = a_2 \text{ for } a_1, a_2 \in S$$

Hence, f is one-one

Let $b \in E$, since b is even, there exists $a \in Z$ such that

$$a = \frac{b}{2} \text{ or } f(a) = 2a = b$$

Hence, f is onto

$$f(a + b) = 2(a + b) = 2a + 2b$$

Hence f is an isomorphism

$\therefore$ (Z,+) and $(E,+)$ are isomorphic

3. Elementary Combinatorics

> Say mathematician, how many are the combinations in one composition with ingredients of six different tastes - sweet, pungent, astringent, sour, salt and bitter - taking them by ones, twos or threes, etc.,?
>
> -From Lilavathi by Bhaskara

1. Introduction

Combinatorics is the branch of mathematics meant to solve counting problems without enumerating all possible cases. At an elementary level, Combinatorics is usually considered as a part of discrete mathematics in which the main problem is that of counting the number of ways of arranging or choosing objects from a finite set according to some simple specified rules. At an advanced level, combinatorics deals with the enumeration analysis and optimization of discrete structures.

Combinational mathematics has a variety of applications. It is used in several physical and social sciences viz., computer science, operations research, statistics, probability, chemistry.

The basic ideas, and concepts of combinatorics are necessary to make an assessment about the amount of storage in a computer. Also, they are useful to solve many problems of computer science.

2. Basics of counting

Two basic counting principles

Two elementary principles act as building blocks for all counting problems.

2.1 Disjunctive (or) Sum Rule

If an event can occur in m ways and another event can occur in n ways, and if these two events cannot occur simultaneously, then one of the two events can occur in $m + n$ ways.

More generally, If $E_1, E_2, \ldots, E_n$ are n events such that no two of them can occur at the same time, and E_1 can happen in n_1 ways, E_2 can happen in n_2 ways ..., E_n can happen in n_n ways, then one of the n events (E_1 or E_2 or E_n) can occur in $n_1+n_2+\ldots+n_n$ ways.

The sum rule can also be stated in terms of choices:

If an object can be selected from a collection in n_1 ways and an object can be selected from a seperate collection in n_2 ways, then the selection of one object from either one collection or the other can be made in $n_1 + n_2$ ways.

Example 2.1.1 If there are 9 boys and 10 girls in a class, there are $9 + 10 = 19$ ways of selecting one student (either a boy or a girl) as class representative.

Example 2.1.2 Suppose E is the event of selecting a prime number less than 10 and F is the event of selecting an even number less than 10, Then E can happey in 4 ways, and F can happen in 4 ways. But because 2 is an even prime, E or F can happen in only $4 + 4 - 1 = 7$ ways.

2.2 Sequential (or) product rule

If an event can occur in m ways and a second event can occur in n ways, and if the number of ways the second event occurs does not depend upon how the first event occurs, then the two events can occur simultaneously in mn ways.

Moregenerally, If events $E_1, E_2, \ldots, E_n$ can happen in $n_1, n_2, \ldots$, and n_n ways respectively, then the sequence of events E_1 first, followed by $E_2, \ldots$, followed by E_n can happen in $n_1 \cdot n_2 \cdot n_3 \ldots n_n$ ways.

The product rule can also be stated interms of choices:

If a first object can be chosen in n_1 ways, a second in n_2 ways, ..., and an n^{th} object can be chosen in e_n ways, then a choice of a first, second, ..., and an n^{th} object can be made in $n_1 \cdot n_2 \ldots n_n$ ways.

Example 2.2.1 A book shelf holds 5 different mathematics books, 6 different computer science books, and 10 different books of statistics. Therefore $(i) 5.6.10 = 300$ ways of selecting 3 books, 1 in each subject $(ii) 5 + 6 + 10 = 21$ ways of selecting 1 book in any one of the subject.

Example 2.2.2 From the previous example, A mathematics book and a computer science book can be selected in $(5)(6) = 30$ ways; A mathematics book and a statistics book can be selected in $5.10 = 50$ ways; a computer science book and a statistics book can be selected in $6.1 = 60$ ways. Thus there are $30 + 5 + 60 = 140$ ways of selecting 2 books in 2 subjects.

Example 2.2.3 If each of 12 Questions in a objective type examination has 4 answers (1 correct and 3 wrong), the number of ways of answering all Questions is 4^{12}.

We summarize the sum rule by saying that we add the numbers of elements in each subset when the elements being counted can be decomposed into disjoint subsets.

Also we summarize the product rule by saying that we multiply together the numbers of ways of doing each step when an activity is constructed in successive steps.

If we are counting objects that are constructed in successive steps, we use the product rule. If we have disjoint sets of objects and we want to know the total number of objects, we use the sum rule. It is important to recognize when to apply each principle. This will comes from practice and careful thinking about each problem. The first two above examples illustrate both counting principles.

Again, we consider some more examples that illustrate both counting rules.

Example 2.2.4 A computer password consists of a letter of the alphabet followed by 3 or 4 digits. Find (a) the total number of passwords that can be created and (b) the number of passwords in which no digit repeats.

(a) The number of 4-character passwords is $(26) \times 10^3$, and the number of 5-character passwords is $(26) \times 10^4$, by the product rule. So the total number of passwords is $(26) \cdot 10^3 + (26) \cdot 10^4 = 286,000$ by the sum rule.

(b) The number of 4-character passwords is $(26)(10)(9)(8) = 18,720$ and the number of 5-character passwords is $(26)(10)(9)(8)(7) = 131,040$ for a total of 149,760.

Example 2.2.5 A six person committee composed of Shiva, Brahma, Vishnu, Narada, Indra & Yama is to select a chairperson, secretary and treasurer.

(a) In how many ways can this be done?

(b) In how many ways can this be done if either Shiva or Brahma must be chairperson?

(c) In how many ways can this be done Indra must hold one of the offices?

(d) In how many ways can this be done if both Narada and Yama must hold office?

Solution:

(a) We use the product rule. The officers can be selected in three successive steps: Select the treasurer. The chairperson can be selected in sixways, once the chairperson has been selected, the secretary can be selected in five ways. After selection of the chair person and secretary, the treasuner can be selected in four ways. Therefore, the total number of possibilities is

$$6.5.4 = 120$$

(b) Arguing as in part(a), if shiva is chairperson, there are 5.4 = 20 ways to select the remaining officers. Similarly, if Brahma in chair person, there are 20 ways to select the remaining officers. Since these cases are disjonint, by the sum rule, there are

$$20 + 20 = 40 \text{ possibilities}$$

(c) Arguing as in purt (a), if Indra is chair person, there are 20 ways to select the remaining officers. Similarly, if Indra is secretary, there are 20 possibilities, and if Indra is treasurer, there are 20 possibilities. Since these three cases are pairwise disjoint, by the sum rule, there are

$$20 + 20 + 20 = 60 \text{ possibilities.}$$

[Alternate solution]: Let us consider the activity of assigning Indra and two others to offices to be made up of three successive steps: Assign Indra an office, fill the highest remaining office, fill the last office. Once Indra has been assigned, there are five ways to fill the highest remaining office. Once Indra has been assigned and the highest remaining office filled, there are four ways to fill the last office. By the product rule, there are 3.5.4 = 60 possibilities

(d) Let us consider the activity of Narada, Yama and one other person to offices to be made up of three successive steps Assign Narada, assign Yama, fill the remaining office. There are three ways to assign Narada. Once Narada has been assigned, there are two ways to assign Yama. Once Narada and Yama have been assigned, there are four ways to fill the remaining office. By the product rule, there are 3.2.4 = 24 possibilities.

3. Permutations and Combinations

Definition 3.1 A permutation of n distinct objects $x_1, \dots x_n$ is an *ordering* of the n elements $x_1, \dots x_n$.

Example 3.1 There are Six permutations of three objects A, B, and C. They are ABC, ACB, BAC, BCA, CAB, CBA

Definition 3.2 An r-permutation of n (distinct) elements $x_1, \dots, x_n$ is an ordering of an r-element subset of $\{x_1, \dots x_n\}$. The number of r-permutations of a set of n distinct elements is denoted by $P(n, r)$

Example 3.2 2-permutations of A, B, C are AB, BA, CA, AC, BC, CB.

Definition 3.3 A combination is a selection of objects *without regard to order*.

Example 3.3 ABC is the combination of three objects A, B and C.

Definition 3.4 An r-combination of n (distinct) elements $x_1, \ldots x_n$ is an *Unordered* selection of an r-element subset of $\{x_1, \ldots x_n\}$ The number of r-combinations of a set of n district elements is denoted by $C(n, r)$ or $\binom{n}{r}$.

Example 3.4 2-combinations of A, B, C are AB, AC, BC.

Note 3.1 In general, when order matters, we count the number of permutations, when order does not matter. We count the number of combinations.

4. Enumeration of permutations and combinations, (Enumeration without repetition)

4.1 Enumerating r-permutations without repetitions

$$P(n, r) = n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!} \quad (0! \equiv 1)$$

Example 4.1 There are $P(10, 4) = 5040$ 4-digit numbers that contain no repeated digits, since each such number is just an arrangement of four of the digits 0, 1, 2, 3, ... , 9 (leading zeroes are allowed).

4.2 Enumerating r-combinations without repetitions

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

Example 4.2 A committee of 5 be chosen from 9 people in $C(9, 5)$ ways.

Observations 4.1 (i) $P(n, n) = n!$ $\quad n! = n(n-1)\ldots 2.1$

(ii) There are $n!$ permutations of n distinct objects

(iii) $C(n, r) = C(n, n-r)$

(iv) There are $(n-1)!$ permutations of n distinct objects in a circle.

Example 4.3 How many words of three distinct letters can be formed from the letters of the word MAST?

Solution: The number is $P(4, 3) = \frac{4!}{(4-3)!} = 24$.

Example 4.4 Compute the number of distinct five-card hands tht can be dealt from a deck of 52 cards.

Solution: The number is $C(52, 5) = \frac{52!}{5!47!} = 2598960$, because the order in which the cards were dealt is irrelevant.

5. The Pigeonhole Principle

Some of the most profound and complicated results in modern combinatorial theory flow from a very simple propostition:

If n pigenoholes shelter $n + 1$ or more pigeons, at least 1 pigeonhole shelters at least 2 pigeons.

Note that the pigeonhole principle tells us nothing about how to locate the pigeonhole that contains two or more pigeons. It only asserts the existence of a pigeonhole containing two or more pigeons.

To apply the pigenonhole principle, we must decide which objects will play the roles of the pigeons and which objects will play the roles of the pigeonholes.

Example 5.1 Among any group of 367 people, there must be atleast two with the same birthday, because there are only 366 possible birthdays.

Example 5.2 If eight people are chosen in any way from some group, at least two of them will have been born on the same day of the week. Here each person (pigeon) is assigned to the day of the week (pigeonhole) on which he or she was born. Since there are eight people and only seven days of the week, the pigeonhole principle tells us atleast two people must be assigned to the same day of the week.

A generalization of the pigeonhole principle is as follows:

If K pigeons are assigned to n pigeonholes, then one of the pigenonholes must contain at least $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$ pigeons.

Note 5.1 If x is a real variable, the floor of x, denoted $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Example 5.3 Prove that if any 30 people are selected, then we may choose a subset of 5 so that all 5 were born on the same day of the week.

Solution: Assign each person to the day of the week on which she or he was born. Then 30 pigeons are being assigned to 7 pigeon holes. By the generalized pigeonhole principle with $K = 30$ and $n = 7$, at least $\left\lfloor \frac{30-1}{7} \right\rfloor + 1$ or 5 of the people must have been born on the same day of the week.

Example 5.4 Suppose there are 26 students and 7 cars to transport them. Then atleast one car must have 4 or more passengers.

Solution: Clearly, $K = 26, n = 7$

Now by generalized pigeonhole principle, at least $\left\lfloor \frac{26-1}{7} \right\rfloor + 1$ or 4 or more passengers should board atleast one car

Now using the $b_0 = 1 \Rightarrow 1 = -c_1 + \frac{4}{3} \Rightarrow c_1 = \frac{1}{3}$

$$\therefore \quad b_n = \frac{1}{3}[(-1)^n + 2(2^n)]$$

and hence

$$a_n = \frac{1}{3n}[(-1)^n + 2^{n+1}]$$

$$\therefore \quad b_n = \frac{1}{4}[5(1)^n - (-3)^n]$$

$$\text{then} \quad a_n = n!b_n = \left(\frac{n!}{4}\right)(5 - (-3)^n)$$

□

Example 7.4.3.5 Solve $\sqrt{a_n} = \sqrt{a_{n-1}} + 2\sqrt{a_{n-2}}, a_0 = a_1 = 1$

Solution:

$$a_n = \frac{[2^{n+1} + (-1)^n]^2}{3^2}$$

Example 7.4.3.6 Solve $a_n = (a_{n-1})^2(a_{n-2})^3$ where $a_0 = 4$ and $a_1 = 4$

Solution: Set $b_n = \log_2 a_n$

$$\therefore \quad b_n = 2b_{n-1} + 3b_{n-2} \quad \text{with} \quad b_0 = b_1 = 2$$

$$CE : r^2 - 2r - 3 = 0 \quad \Rightarrow \quad r = -1, \text{ and } 3$$

$$\therefore \quad b_n = c_1(-1)^n + c_2 3^n$$

$$\left.\begin{array}{l} b_0 = 2 \quad \Rightarrow \quad c_1 + c_2 = 2 \\ b_1 = 2 \quad \Rightarrow \quad -c_1 + 3c_2 = 2 \end{array}\right\} \Rightarrow \quad c_1 = c_2 = 1$$

$$\therefore \quad b_n = (-1)^n + 3^n$$

$$\therefore \quad a_n = 2^{(-1)^n} + 3^n$$

□

Example 7.4.3.7 Solve the recurrence relation

$$a_n = \sqrt{a_{n-1} + \sqrt{a_{n-2} + \sqrt{a_{n-3} + \sqrt{\dots}}}} \text{ with } a_0 = 4$$

Solution: Squaring the given equation

$$a_n^2 = a_{n-1} + \sqrt{a_{n-2} + \sqrt{a_{n-3} + \dots}}$$

$$a_n^2 = a_{n-1} + a_{n-1} \quad \left(\because \quad a_{n-1} = \sqrt{a_{n-2} + \sqrt{a_{n-3} + \dots}}\right)$$

$$a_n^2 = 2a_{n-1}$$

Example 7.4.3.3 Solve $a_n = -2na_{n-1} + 3n(n-1)a_{n-2}$ with initial conditions $a_0 = 1, a_1 = 2$

Solution: Set $b_n = \frac{a_n}{n!}$ then the given relation changes to

$$n!b_n = -2n(n-1)!b_{n-1} + 3n(n-1)(n-2)!b_{n-2}$$

$$\Rightarrow \quad n!b_n = -2n!b_{n-1} + 3n!b_{n-2}$$

$$\Rightarrow \quad b_n = -2b_{n-1} + 3b_{n-2}$$

$$\Rightarrow \quad b_n + 2b_{n-1} - 3b_{n-2} = 0$$

with $b_0 = 1, \quad b_1 = 2$

Characteristic *eq* of (2) is $r^2 + 2r - 3 = 0$

$$\Rightarrow \quad r = 1 \text{ and } -3$$

$$\therefore \quad b_n = c_1(1)^n + c_2(-3)^n$$

$$\left.\begin{array}{lll} b_0 = 1 & \Rightarrow & 1 = c_1 + c_2 \\ b_1 = 2 & \Rightarrow & 2 = c_1 - 9c_2 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = \frac{5}{4} \\ c_2 = \frac{-1}{4} \end{array}$$

Example 7.4.3.4 $na_n + (n-1)a_{n-1} = 2^n$ where $a_0 = 1$

Solution: Set $b_n = na_n$

$$\therefore \quad b_n + b_{n-1} = 2^n \quad \text{with} \quad b_0 = 1 \tag{1}$$

Characteristic equation of homogeneous Recurrence relation is $r^2 + r = 0 \quad \Rightarrow \quad r = 0, -1$

$\therefore \quad b_n^H = c_1(-1)^n$ is the solution of homogeneous part of equation (1)

Consider

$$b_n^P = D2^n$$

Substituting in (1)

$$D2^n + D2^{n-1} = 2^n$$

$$D + \frac{D}{2} = 1$$

$$\Rightarrow \quad D = \frac{2}{3}$$

$$b_n^P = \frac{2}{3}(2^n) \text{ is a particular solution}$$

Then $b_n = b_n^H + b_n^P = c_1(-1)^n + \frac{2}{3}(2^n)$

Solution: Consider $b_n = \log_2 a_n$

The the given equation becomes, by taking logarithms on both sides

$$\log_2 a_n = \frac{1}{2}(\log_2 a_{n-2}) - \frac{1}{2}(\log_2 a_{n-1})$$

$$\Rightarrow \quad 2\log_2 a_n = \log_2 a_{n-2} - \log_2 a_{n-1}$$

$$\Rightarrow \quad 2b_n = b_{n-2} - b_{n-1}$$

$$\Rightarrow \quad 2b_n + b_{n-1} - b_{n-2} = 0 \qquad (2)$$

where $b_0 = \log_2 a_0 = \log_2 8 = 3$

$$b_1 = \log_2 a_1 = \log_2 \frac{1}{2\sqrt{2}} = -\frac{3}{2}$$

$\therefore$ the characteristic equation of (2) is

$$2r^2 + r - 1 = 0$$

$$\Rightarrow \quad r = -1, \frac{1}{2}$$

$$\therefore \quad b_n = c_1(-1)^n + c_2\left(\frac{1}{2}\right)^n$$

$$\text{or} \quad b_n = c_1(-1)^n + c_2(2)^{-n}$$

Using $b_0 = 3 \Rightarrow 3 = c_1 + c_2$, $b_1 = -\frac{3}{2} \Rightarrow -\frac{3}{2} = -c_1 + \frac{c_2}{2}$ $\Big\} \Rightarrow$ $c_2 = 1$, $c_1 = 2$

$$\therefore \quad b_n = 2(-1)^n + (2)^{-n}$$

$$b_n = 2(-1)^n + (2)^{-n}$$

(or)

$$b_n = 2(-1)^n + \left(\frac{1}{2}\right)^n$$

.hen from $b_n = \log_2 a_n \Longleftrightarrow a_n = 2^{bn}$

$$\therefore \quad a_n = 2^{2(-1)^n + \left(\frac{1}{2}\right)^n}$$

□

From eq. (2):

$$\left[\frac{C_1 + C_2}{2}\right] + \left(\frac{C_1 - C_2}{2}\right)\sqrt{5} = 1$$

$$\left(\frac{C_1 - C_2}{2}\right)\sqrt{5} + \frac{1}{2} = 1 \qquad (\because C_1 + C_2 = 1)$$

$$C_1 - C_2 = \frac{1}{\sqrt{5}} \tag{3}$$

From (1) and (3):

$$C_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}, \; C_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

Hence the recurrence relation for the Fibonacci sequence explicitly is:

$$F_n = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}$$

7.4.3 Solving Non-linear Recurrence Relations:

Sometimes nonlinear recurrence relations can be solved by converting them into linear recurrence relation by a suitable substitution.

Example 7.4.3.1 Solve the recurrence relation $a_n^2 - 2a_{n-1}^2 = 0$ for $n \geq 1$ where $a_0 = 2$.

Solution: Let $b_n = a_n^2$, by this substitution the given nonlinear recurrence relation $a_n^2 - 2a_{n-1}^2 = 0$ changed to $b_n - 2\, b_{n-1} = 0$ we know clearly $b_n = c \cdot 2^n$

$$\text{Then} \quad a_n = \sqrt{c \cdot 2^n}$$

$$\text{But} \quad a_0 = 2$$

$$\therefore \quad c = 4$$

$$\therefore \quad a_n = \sqrt{4 \cdot 2^n} = 2 \cdot \sqrt{2^n}.$$

Example 7.4.3.2 Solve the recurrence relation

$$a_n = \sqrt{\frac{a_{n-2}}{a_{n-1}}} \quad \text{with initial conditions} \tag{1}$$

$$a_0 = 8, \quad a_1 = \frac{1}{2\sqrt{2}}$$

Exercise 3 Solve the following recurrence relations using the characteristic roots

(1) $a_n = 6\,a_{n-1} - 11\,a_{n-2} + 6\,a_{n-3}$ with the initial conditions $a_0 = 2, a_1 = 5$ and $a_2 = 15$

(2) $a_n - 3\,a_{n-1} - 4\,a_{n-2} = 0$ for $n \geq 2$ and $a_0 = a_1 = 1$

(3) $a_n - 4\,a_{n-1} + 4\,a_{n-2} = 0$ for $n \geq 2$ and $a_0 = \frac{5}{2},\ a_1 = 8$

(4) $a_n + 5\,a_{n-1} + 5\,a_{n-2} = 0$ and $a_0 = 0,\ a_1 = 2\sqrt{5}$

Answer: 3

(1) $a_n = 1 - 2^n + 2.3^n$

(2) $a_n = \frac{2}{5} \cdot 4^n + \frac{3}{5} \cdot (-1)^n$

(3) $a_n = \frac{5}{2} \cdot 2^n + \frac{3}{2} \cdot n \cdot 2^n$

(4) $a_n = 2\left(\frac{-5+\sqrt{5}}{2}\right)^n - 2\left(\frac{-5-\sqrt{5}}{2}\right)^n$

Finding explicit formula for the Fibonacci Relation:

Solve $F_n = F_{n-1} + F_{n-2},\ n \geq 2$ with Initial conditions $F_0 = F_1 = 1$

Using characteristic roots.

Solution: The characteristic equation is: $r^2 - r - 1 = 0$

Its roots are $\frac{1+\sqrt{5}}{2},\ \frac{1-\sqrt{5}}{2}$

Hence $F_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

Also From initial conditions:

$$C_1 + C_2 = 1 \tag{1}$$

$$C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \tag{2}$$

Solution: Characteristic equation of the given recurrence relation

$$r^3 - 8r^2 + 21r - 18 = 0$$

$$\Rightarrow \quad r = 2, 3, 3$$

$$a_n = c_1 2^n + c_2 3^n + c_3 n 3^n$$

Example 7.4.2.10 Solve $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0, \quad n \geq 3$

Solution: Characteristic equsation of the given recurrence relation

$$r^3 - 6r^2 + 12r - 8 = 0$$

$$\Rightarrow \quad r = 2, 2 \text{ and } 2$$

$$\therefore \quad a_n = c_1 2^n + c_2 n 2^n + c_3 n^2 2^n$$

From the initial conditions give the system of equations

$$c_1 + c_3 = 1$$
$$2c_1 + 2c_2 + 3c_3 = 4$$
$$4c_1 + 8c_2 + 9c_3 = 8$$

Solving the above system of equations (by elimination procedure)

$$c_1 = 5,\ c_2 = 3 \text{ and } c_3 = -4$$

Hence, the unique solution of the recurrence relation is

$$a_n = 5.2^n + 3.n\ 2^n - 4.3^n$$

Example 7.4.2.7 Solve $a_n - 7a_{n-1} + 10a_{n-2} = 0.\ \ n \geq 2\ \ a_0 = 10, a_1 = 41$
Solution: Characteristic equation of given recurrence relation is $r^2 - 7r + 10 = 0$

$$\Rightarrow \quad r = 2, 5$$
$$\therefore \quad a_n = c_1 2^n + c_2 5^n$$
$$\left.\begin{array}{l} a_0 = 10 \Rightarrow c_1 + c_2 = 10 \\ a_1 = 41 \Rightarrow 2c_1 + 5c_2 = 41 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 3 \\ c_2 = 7 \end{array}$$
$$\therefore \quad a_n = (3)2^n + (7)5^n$$

□

Example 7.4.2.8 Solve $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$ for $n \geq 3$ with $a_0 = 0, a_1 = 1$ and $a_2 = 10$
Solution: Characteristic equation of given recurrence relation is $r^3 - 9r^2 + 26r - 24 = 0$

$$r = 2, 3 \text{ and } 4$$
$$\therefore \quad a_n = c_1 2^n + c_2 3^n + c_3 4^n$$
$$a_0 = 0 \Rightarrow c_1 + c_2 + c_3 = 0$$
$$a_1 = 1 \Rightarrow 2c_1 + 3c_2 + 4c_3 = 1$$
$$a_2 = 10 \Rightarrow 4c_1 + 9c_2 + 16c_3 = 10$$

on solving $c_1 = \frac{3}{2}, c_2 = -4, c_3 = \frac{5}{2}$

$$\therefore \quad a_n = \frac{3}{2}(2^n) - 4(3^n) + \frac{5}{2}(4^n)$$

□

Example 7.4.2.9 Solve $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$ for $n \geq 3$

Example 7.4.2.2 Solve $a_n - 7\, a_{n-1} + 12\, a_{n-2} = 0$ for $n \geq 2$

Solution: The Characteristic equation is $r^2 - 7r + 12 = 0$

$$\Rightarrow (r-3)(r-4) = 0 \qquad \Rightarrow r = 3, 4$$

Thus, the general solution is $a_n = c_1\, 3^n + c_2\, 4^n$.

Example 7.4.2.3 Solve $a_n - 5\, a_{n-1} + 6\, a_{n-2} = 0$ where $a_0 = 2$ and $a_1 = 5$

Solution: The characteristic equation is: $r^2 - 5r + 6 = 0$

$$\Rightarrow (r-3)(r-2) = 0 \qquad \Rightarrow r = 2, 3$$

Then $a_n = C_1\, 2^n + C_2\, 3^n$.

From the initial conditions $a_0 = 2$ and $a_1 = 5$

We have the system of equations $c_1 + c_2 = 2$

$$2c_1 + 3c_2 = 5$$

Solving these equations: $C_1 = 1,\ C_2 = 1$

Thus $a_n = 2^n + 3^n$ for all integers $n \geq 0$.

Example 7.4.2.4 Write the general form of the solutions to $a_n - 6a_{n-1} + 9a_{n-2} = 0$

Solution: The characteristic equation is: $r^2 - 6r + 9 = 0$

$$\Rightarrow (r-3)^2 = 0 \qquad \Rightarrow r = 3, 3$$

Then the general solution is: $a_n = D_1\, 3^n + D_2\, n3^n$

Example 7.4.2.5 Suppose that the characteristic equation of a linear homogeneous recurrence relation is $(r-2)^3(r-3)^2(r-5)^3$.

Then the general solution is

$$a_n = (D_1 + D_2 n + D_3\, n^2)2^n + (D_4 + D_5 n)3^n + (D_6 + D_7 n + D_8 n^2)5^n.$$

Example 7.4.2.6 Solve the recurrence relation $a_n - 7\, a_{n-1} + 16\, a_{n-2} - 12\, a_{n-3} = 0$ for $n \geq 3$ with initial conditions $a_0 = 1,\ a_1 = 4$ and $a_2 = 8$.

Solution: The characteristic equation is:

$$t^3 - 7t^2 + 16t - 12 = (t-2)^2(t-3) = 0 \qquad \Rightarrow t = 2, 2, 3$$

Thus a_n may be written as $c_1\, 2^n + c_2\, n2^n + c_3\, 3^n$.

Exercise II Solve the following recurrence relations by substitution.

(a) $a_n = a_{n-1} + n^2$ where $a_0 = 7$
(b) $a_n = a_{n-1} + n^3$ where $a_0 = 5$
(c) $a_n = a_{n-1} + 2n + 1$ where $a_0 = 1$
(d) $a_n = a_{n-1} + 3n^2 + 3n + 1$ where $a_0 = 1$.

Answer (II)

(a) $a_n = \frac{n(n+1)(2n+1)}{6} + 7$
(b) $a_n = \frac{n^2(n+1)^2}{4}$
(c) $a_n = (n+1)^2$
(d) $a_n = (n+1)^3$

Note 7.4.1.1 In general, If a linear recurrence relation with constant coefficients of degree k has initial conditions fewer than k, then there will not be a unique solution.

7.4.2 Solving recurrence by the method of characteristic roots:

This method is some what general method to solve homogeneous linear recurrence relations of degree k. For this we require the definition of the characteristic equation of a homogeneous linear recurrence relation.

Definition 7.4.2.1 Let $a_n + c_1\, a_{n-1} + \ldots + c_k\, a_{n-k} = 0,\ n \geq k, c_k \neq 0$ be a linear recurrence relation of degree k. Then the equation

$$r^k + c_1\, r^{k-1} + \ldots + c_k = 0$$

is said to be the characteristic equation of the given recurrence relation.

Example 7.4.2.1 The characteristic equation of $a_n - 4\, a_{n-1} + 4\, a_{n-2} = 0$ is $r^2 - 4r + 4 = 0$.

Result 7.4.1 If the characteristic equation of a linear homogeneous recurrence relation of degree k has k distnct roots $r_1, r_2, \ldots r_k$ then $a_n = C_1\, r_1^n + C_2\, r_2^n + \ldots + C_k\, r_k^n$ for $n = 0, 1, 2 \ldots$ where $C_1,\ C_2,\ \ldots C_k$ are constants, is the general solution of the given recurrence relation

Result 7.4.2 If the characteristic equation of a linear homogeneous recurrence relation of degree k has a root 'r' repeated 'k' times, then $a_n = (D_1 + D_2\, n + D_3\, n^2 + \ldots + D_k\, n^{k-1})r^n$ where $D_1,\ D_2 \ldots D_k$ are constants, is the general solution of the given recurrence relation.

Example 7.4.1.4 Solve $a_n = a_{n-1} + 3^n, \; n \geq 1, \; a_0 = 1$ by substitution

Solution: Clearly

$$a_1 = a_0 + 3$$
$$a_2 = a_1 + 9 = a_0 + 3 + 9$$
$$a_3 = a_2 + 3^3 = a_0 + 3^1 + 3^2 + 3^3$$
$$\vdots$$

Hence
$$a_n = a_0 + \sum_{k=1}^{n} 3^k$$
$$= a_0 + \frac{3^{n+1} - 3}{2} \qquad \text{(by G.P)}$$
$$a_n = \frac{3^{n+1} - 1}{2}.$$

$$\therefore a_n = a_0 + \sum_{K=1}^{n} f(K).$$

Example 7.4.1.2 Solve the following Recurrence relation by substitution.

$$a_n = a_{n-1} + n,\ n \geq 1 \text{ where } a_0 = 2$$

Solution: Clearly

$$a_1 = a_0 + 1 = 1 + 1 = 2$$

$$a_2 = a_1 + 2 = a_0 + 3$$

$$a_3 = a_2 + 3 = a_0 + 6$$

$$\vdots$$

$$\Rightarrow \quad a_n = a_0 + \frac{n(n+1)}{2}$$

Since $a_0 = 2$

$$a_n = 2 + \frac{n(n+1)}{2}.$$

Example 7.4.1.3 Solve $a_n = a_{n-1} + \frac{1}{n(n+1)}$, $a_0 = 1$ by substitution.

Solution: Clearly

$$a_1 = a_0 + \frac{1}{1.2}$$

$$a_2 = a_1 + \frac{1}{2.3} = a_0 + \frac{1}{1.2} + \frac{1}{2.3}$$

$$a_3 = a_2 + \frac{1}{3.4} = a_0 + \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4}$$

$$\vdots$$

Hence $a_n = a_0 + \sum_{k=1}^{n} \frac{1}{k(k+1)}$

We Know that $\sum_{k=1}^{n} \left(\frac{1}{k(k+1)}\right) = 1 - \left(\frac{1}{n+1}\right)$

$$a_n = 1 + 1 - \frac{1}{n+1} = 2 - \frac{1}{n+1} \quad (\because a_0 = 1)$$

$$\Rightarrow \quad a_n = \frac{2n+1}{n+1}.$$

Note 7.4.1 Like in algebra, recurrence relations may have no solution. Equation $a_n^2 + (a_{n-1})^2 = -1$ has no solution, since there are no real-valued functions f such that $[f_n]^2 + [f_{n-1}]^2 = -1$. because the squares of real numbers are always non negative.

Note 7.4.2 We can compute a_n interms of $a_{n-1}, \ldots a_1, a_0$; then we can compute a_{n+1} interms of $a_n, a_{n-1}, \ldots, a_0$ and so on, provided the value of the sequence at one or more points is given so that the computation can be initiated. Therefore, we need some values of the sequence. Usually the values for $a_0, a_1, \ldots, a_{k-1}$ are given and then it would be appropriate to call these initial conditions

Example 7.4.2 $a_n = 3.2^n + 7.5^n$ is the solution of recurrence relation $a_n - 7\,a_{n-1} + 10\,a_{n-2} = 0,\ n \geq 2$ with initial conditions $a_0 = 10,\ a_1 = 41$,

Note 7.4.3 It is not possible to solve all recurrence relations. Also there are no general techniques to solve all recurrence relations. But there are techniques to solve linear recurrence relations with constant coefficients. Non linear recurrence relations can be solved by converting them to linear recurrence relations.

We are going to discuss two methods of solving recurrence relations. They are

1. Solving recurrence relations by substitution (or iteration),
2. Solving recurrence relations by characteristic roots.

Now, we shall study one by one:

7.4.1 Solving Recurrence relations by substitution:

In this substitution method, the given recurrence relation for a_n is used repeatedly to solve for a general expression for a_n interms of n, this expression does not contains any other terms of the sequence except those given by initial conditions.

This method can be illustrated by the following examples:

Example 7.4.1.1 Solve the recurrence relation $a_n = a_{n-1} + f(n), n \geq 1$ by substitution.

Solution: Given $a_n = a_{n-1} + f(n),\ n \geq 1$

then

$$a_1 = a_0 + f(1)$$

$$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$$

$$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$$

$$\vdots$$

$$a_n = a_0 + f(1) + f(2) + \ldots + f(n)$$

$$= a_0 + \sum_{K=1}^{n} f(k)$$

$$= F_{k+1}(F_k + F_{k-1})$$
$$= F_{k+1}^2$$

$\therefore$ Hence the result by induction. □

Result 7.3.2 Prove that $F_1 + F_3 + \ldots + F_{2n-1} = F_{2n} - 1, n \geq 1$

Proof: Using induction on n

When $n = 1$

$$F_1 = 1 = 2 - 1 = F_2 - 1$$

Now Assume that the result is true for $n = k$

$$F_1 + F_3 + \ldots + F_{2k-1} = F_{2k} - 1 \tag{1}$$

Consider

$$F_1 + F_3 + \ldots + F_{2k-1} + F_{2k+1} = F_{2k} - 1 + F_{2k+1} \quad \text{by (1)}$$
$$= F_{2k+2} - 1$$

Hence the result by induction on n.

Similarly, by using induction on n, we can also prove

$$F_0 - F_1 + F_2 + \ldots + (-1)^n F_n = (-1)^n F_{n-1} + 1$$
$$F_n F_{n+2} + (-1)^n = F_{n-1} F_{n+3} + 3$$

□

7.4 Solving Recurrence Relations

The process of finding an expression for the terms of a sequence from its recurrence relation is called solving recurrence relation.

Definition 7.4.1 The solution of a recurrence relation is an explicit formula for the general term a_n of the sequence $a_0, a_1, \ldots, a_{n-1}, a_n, \ldots$ (which are from the recurrence relation) satisfying the recurrence relation.

Examples 7.4.1

(i) For the recurrence relation $a_n = 2a_{n-1}$, $n \geq 1$, $a_n = 2^n$ is the solution.

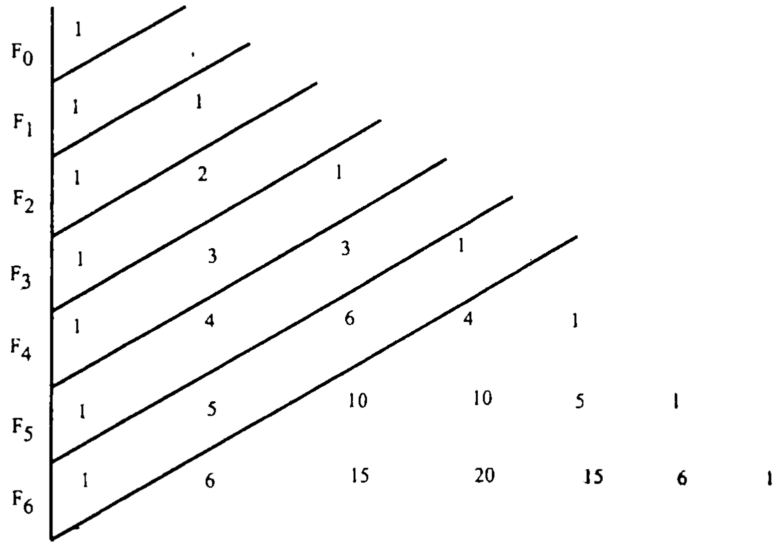
(ii) For the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $n \geq 2$, $a_n = C_1 2^n + C_2 5^n$ is the solution, where C_1 and C_2 are arbitrary constants.

(2) $F_0 + F_2 + F_4 + \ldots + F_{2n} = F_{2n+1}$

(3) $F_0^2 + F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$

The above properties (2) and (3) can be proved by mathematical induction.

Also, there is an interesting relation between pascal's triangle and the Fibonacci numbers, illustrated below:



Clearly, the sum of the elements lying on the diagonal running upward from the left are Fibonacci numbers.

With this introductory exposure about recurrence relations, our main aim is to solve the recurrence relations.

Result 7.3.1 Prove that $F_n^2 = F_{n-1} F_{n+1} + (-1)^n, n \geq 2$

Proof: Using induction on n

When $n = 2$

$$F_2^2 = 4 = 1.3 + 1 = F_1 F_3 + (-1)^2$$

Now Assume that the result is true for $n = k$

$$F_k^2 = F_{k-1} F_{k+1} + (-1)^k \tag{3}$$

Consider

$$\begin{aligned} F_k F_{k+2} + (-1)^{k+1} &= F_k(F_{k+1} + F_k) + (-1)^{k+1} \\ &= F_k F_{k+1} + F_k^2 + (-1)^{k+1} \\ &= F_k F_{k+1} + F_{k-1} F_{k+1} + (-1)^k + (-1)^{k+1} \quad (\because \quad (1)) \end{aligned}$$

Clearly, All the above examples are linear recurrence relations except (7), (8) and (9); the relation (7) is not linear because of the squared term $(a_{n-1})^2$.

The relations (1), (2), (3), (4), (6), and (10) are linear with constant coefficients.

Relations (2) and (10) have degree 1; (3), (4) and (5) have degree 2; (6) has degree 3.

Relations (1), (3), (5) and (10) are homogeneous.

Note 7.2.2 a_n can be represented as $a(n)$, a_{n-1} can be represented as $a(n-1) \ldots a_1$ can be represented as a(1).

The first example, which is linear homogeneous recurrence relation of degree 2 is very important recurrence relation known as:

7.3 The Fibonacci Recurrence Relation:

Definition 7.3.1 The Recurrence relation $F_n = F_{n-1} + F_{n-2}, n \geq 2$ with initial conditions $F_0 = F_1 = 1$ is known as is known as Fibonacci recurrence relation (or) simply Fibonacci relation.

The numbers F_n generated by the Fibonacci relation with initial conditions $F_0 = F_1 = 1$ are called the Fibonacci numbers.

The sequence of Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ is the Fibonacci sequence.

Some Properties of Fibonacci Numbers:

(1)

$$F_0 + F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$$

Explanation: We have

$$F_n = F_{n-1} + F_{n-2}, n \geq 2, F_0 = F_1 = 1 \qquad (1)$$

replace n by $n + 2$ in (1)

$$F_{n+2} = F_{n+1} + F_n$$

$$\Rightarrow \quad F_n = F_{n+2} - F_{n+1} \qquad (2)$$

From (2):

$$\begin{aligned} F_0 &= F_2 - F_1 \\ F_1 &= F_3 - F_2 \\ F_2 &= F_4 - F_3 \\ \vdots \quad & \quad \vdots \quad \vdots \\ F_n &= F_{n+2} - F_{n+1} \end{aligned}$$

Adding: ______________

$$F_0 + F_1 + \ldots F_n = F_{n+2} - F_1$$

$$= F_{n+2} - 1 (\because F_1 = 1)$$

7.2 Basic Definitions

Definition 7.2.1 A recurrence relation for the sequence $a_0, a_1, \dots, a_n, \dots$ is an equation that relates a_n to some of its previous terms $a_0, a_1, \dots, a_{n-1}$.

Examples of recurrence relations 7.2.1:

(1) The Fibonacci sequence is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}, n \geq 2$ with initial conditions $F_0 = F_1 = 1$. Here any $F_n (n \geq 2)$ can be obtained by using initial conditions and repeated application of the recurrence relation.

(2) $a_n = n + a_{n-1}$

(3) $a_n - 3a_{n-1} + 2a_{n-2} = 0$

(4) $a_n - 3a_{n-1} + 2a_{n-2} = n^2 - 1$

(5) $a_n - (n-1)a_{n-1} - (n-1)a_{n-2} = 0$

(6) $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 4^n$

(7) $a_n - 3(a_{n-1}^2 + 2a_{n-2} = n^2$

(8) $a_n = a_0a_{n-1} + a_1a_{n-2} + \dots + a_{n-1}a_0$.

(9) $a_n^2 + (a_{n-1})^2 = -1$

(10) $a_n = 3a_{n-1}, n \geq 1$

Definition 7.2.2 If n and K are nonnegative integers. A recurrence relation of the form

$$c_0(n)\, a_n + c_1(n)\, a_{n-1} + \dots + c_K(n)\, a_{n-k} = f(n) \text{ for } n \geq k \qquad \text{(I)}$$

where $c_0(n)$, $c_1(n), \dots, c_k(n)$, and $f(n)$ are functions of n, is said to be a linear recurrence relation.

Definition 7.2.3 If $c_0(n)$ and $c_k(n)$ are not identically zero in equation (I), then the recurrence relation is said to be a linear recurrence relation of degree k.

In other words, A recurrence relation is said to be of degree k if a_n is expressed as a function of $a_{n-1}, \dots a_{n-k}$ and a_{n-k} appears in the relation.

Definition 7.2.4 If $c_0(n), c_1(n), \dots, c_k(n)$ are constants, then the recurrence relation is known as a linear recurrence relation with constant coefficients.

Note 7.2.1 A linear recurrence relation with constant coefficients is simply called a linear relation.

Definition 7.2.5 If $f(n)$ is identically zero in equation (I) then the recurrence relation is said to be homogeneous; otherwise, it is nonhomogeneous (or) inhomogeneous

7. Recurrence Relations

> A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are two months old. After they are two months old, each pair of rabbits produces another pair each month. What is the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.
>
> Leonardo di Pisa

7.1 Introduction

The fundamental tools of combinatorics viz., permutations, combinations are inadequate to solve many combinatorial problems that the computer scientist should face. But some of the combinatorial problems that cant be solved by fundamental tools can be solved by finding relationships called recurrence relations.

Recurrence relations arise naturally in many counting problems and in analyzing programming problems. They occur in the analysis of certain discrete-time systems, analysis of algorithms, error correcting codes. Recurrence relations are useful to analyze .he recurrence relation, the time needed by the algorithm can be determined. Recurrence 'elations, recursive algorithms and mathematical induction are closely related. A Recur- 'ence relation uses prior values in a sequence to compute the current value. A Recursive ılgorithm means an algorithm interms of itself where it is interms of 'previous' values. Γhe inductive steps of mathematical induction assume the truth of prior instances of the ;tatement to prove the truth of the current statement.

The essential idea in a recurrence relation is that it expresses the general term of an un known) sequence as a (known) function of its earlier terms.

Recurrence is a way of giving information interms of prior knowledge.

In otherwords, the main reason for using recurrence relations is that some times it is :asier to determine the n^{th} term of a sequence interms of its predecessors than it is to ind an explicit formula for the n^{th} term interms of n.

Recurrence relations are to discrete mathematics where as differential equations are ɔ continuous mathematics. A recurrence relation is also called as difference equation, nd those two terms will be used interchangeably.

Now by the following problem, one can understand essence of the previous paragraphs.

:xample 7.1.1 The number of bacteria in a colony doubles every hour. If a colony egins with 4 bacteria, how many will be present in n hours.

Solution: Let a_n be the number of bacteria at the end of n hours. Since the number f bacteria doubles every hour, the relationship, together with the initial condition $a_0 = 4$, Iniquely determines a_n for all non negative integers n. We can find a formula for a_n 'om this information as follows: $a_n = 2a_{n-1}, a_0 = 4$.

ow, it is the time to study the concepts of recurrence relations one by one.

Exercise (I)

(1) A Label identifier for a computer program consists of one letter followed by three digits. If repetitions are allowed, how many distinct label identifiers are possible?

(2) How many different bit strings are there of length seven?

(3) A student can choose a computer project from one of three lists. The three lists contain 12, 18 and 24 possible projects respectively. How many possible projects are there to choose from?

(4) Suppose that either a member of mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 38 members of the mathematics faculty and 82 mathematics majors.

(5) A coin is tossed four times and the result of each toss is recorded. How many different sequences of heads and tails are possible?

(6) Currently, telephone area codes are three-digit numbers whose middle digit must be 0 or 1. Codes whose last two digits are 1's are being used for other purposes (Ex. 911). With these conditions, how many area codes are avaliable?

(7) In how many ways can a committee of three facutly members and two students be selected from seven faculty members and eight students?

(8) Suppose that there are eight runners in a race. the winner receives a gold medal, the second place finisher receives a silver medal, and the third place finisher receives a bronz medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

(9) How many ways one there to distribute 12 different books among 15 people if no person is to receive more than one book?

(10) Show that if seven colors are used to paint 50 bicycles, at least 8 bicycles will be same color.

Answers (I)

(1) 26000
(2) 128
(3) 54
(4) 120
(5) 16
(6) 190
(7) 980
(8) 336
(9) P(15, 12)
(10) atleast 8

Then $n(A_1) = \dfrac{250}{3} = 83 \qquad n(A_1 \cap A_2) = \dfrac{250}{3 \times 5} = 16$

$$n(A_2) = \frac{250}{5} = 50 \qquad n(A_1 \cap A_3) = \frac{250}{3 \times 7} = 11$$

$$n(A_3) = \frac{250}{7} = 35 \qquad n(A_2 \cap A_3) = \frac{250}{5 \times 7} = 7$$

$$n(A_1 \cap A_2 \cap A_3) = \frac{250}{3 \times 5 \times 7} = 2.$$

Number of integers divisible by 3 or 5

$$\text{i.e.,} \quad n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$$
$$= 83 + 50 - 16 = 117.$$

Number of integers divisible by 3 or 5 or 7

$$\begin{aligned} n(A_1 \cup A_2 \cup A_3) &= n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_1 \cap A_3) \\ &\quad - n(A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_3) \\ &= 83 + 50 + 35 - 16 - 11 - 7 + 2 \\ &= 136 \end{aligned}$$

Number of integers divisible by 3 or 7 but not by 5

$$\begin{aligned} &= n(A_1 \cup A_2 \cup A_3) - n(A_2) \\ &= 136 - 50 = 86 \end{aligned}$$

6. The Inclusion-Exclusion Principle:

We discussed the sum rule by which we can count the number of objects in the union of disjoint sets. However, if the sets are not disjoint we must refine the statement of the sum rule.

In other words, when two tasks can be done at the same time, we cannot use the sum rule to count the number of ways to do one of the two tasks. Adding the number of ways to do each task leads to an over count, since the ways to do both tasks are counted twice. To correctly count the number of ways to do one of the two tasks, we add the number of ways to do each of the two tasks and then subtract the number of ways to do both tasks. This technique is called the principle of inclusion-exclusion.

We can state the above principle in terms of sets. Let A_1 and A_2 be sets. Let T_1 be the task of choosing an element from A_1 and T_2 the task of choosing an element from A_2. There are $n(A_1)$ way to do T_1 and $n(A_2)$ ways to do T_2. The number of ways to do either T_1 or T_2 is the sum of the number of ways to do T_1 and the number of ways to do T_2, minus the number of ways to do both T_1 and T_2. Since there are $n(A_1 \cup A_2)$ ways to do either T_1 or T_2 and $n(A_1 \cap A_2)$ ways to do both T_1 and T_2, we have

$$n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$$

Example 6.1 From a group of 12 professors how many ways can a committee of 6 members be formed so that at least one of professor A and professor B will be included.

Solution: The total number of committees is $C(12, 6)$ among these committees. Let A_1 and A_2 be the set of committees that include Professor A and Professor B respectively. Since $n(A_1) = C(11, 5) = n(A_2)$ and $n(A_1 \cap A_2) = C(10, 4)$
Now $n(A_1 \cup A_2) = C(11, 5) + C(11, 5) - C(10, 4)$. □

The principle of inclusion-exclusion can be generalized to find the number of ways to do one of 'n' different tasks (or) equivalently, to find the number of elements in the union of n sets:
Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$n(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{1 \le i \le n} n(A_i) - \sum_{1 \le i < j \le n} n(A_i \cap A_j)$$
$$+ \sum_{1 \le < j < k < n} n(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1} n(A_1 \cap A_2 \cap \ldots \cap A_k).$$

Example 6.2 Consider a set of integers from 1 to 250. Find how many of these numbers are divisible by 3 or 5 or 7. Also indicate how many are divisible by 3 or 7 but not by 5 and divisible by 3 or 5.

Solution: Let A_1, A_2, A_3 denote the set of integers 1 to 250 divisible by 3, 5 and 7 respectively

$$\Rightarrow \quad a_n^2 - 2a_{n-1} = 0 \tag{1}$$

$$\text{Put} \quad b_n = \log_2 a_n$$

$$\Rightarrow \quad a_n = 2^{b_n}$$

From(1)

$$(2^{b_n})^2 - 2 \times 2^{b(n-1)} = 0$$

$$2^{2b_n} - 2^{b(n-1)+1} = 0$$

$$\Rightarrow \quad 2b_n - b_{n-1} + 1 = 0 \quad \text{with} \quad b_0 = \log_2 a_0 = \log_2^4 = 20 \tag{2}$$

Now Eq(2) is first onder linear non homogeneous recurrence relation

The characteristic equation of (2) is $2m - 1 = 0 \Rightarrow m = \frac{1}{2}$

$$\text{Therefore homogeneous solution is} \quad b_n^H = c_1 \left(\frac{1}{2}\right)^n \tag{3}$$

Consider $b_n^p = 0$ substituting in (2) $b_n^p = -1$

$$\therefore \quad b_n = b_n^H + b_n^p = c_1 \left(\frac{1}{2}\right)^n - 1$$

$$\therefore \quad \text{by using} \quad b_0 = 2 \quad \Rightarrow \quad b_n = 3\left(\frac{1}{2}\right)^n - 1$$

$$\therefore \quad a_n = 2^{3\left(\frac{1}{2}\right)^n - 1}$$

□

Example 7.4.3.8 Solve

$$\sqrt{a_n} - \sqrt{a_{n-1}} - 2\sqrt{a_{n-2}} = 0 \text{ where } a_0 = a_1 = 1 \tag{1}$$

Solution: Let $b_n = \sqrt{a_n}$

Then From (1): $b_n - b_{n-1} - 2\,b_{n-2} = 0$, $b_0 = b_1 = 1$

Characteristic equation is: $r^2 - r - 2 = 0$

its roots are: $r = 2, -1$

$$\therefore \quad b_n = C_1\, 2^n + C_2(-1)^n$$

From initial conditions $b_0 = b_1 = 1$

$$C_1 = \frac{2}{3},\ C_2 = \frac{1}{3}$$

then

$$b_n = \left[\frac{2^{n+1} + (-1)^n}{3} \right]$$

Thus

$$a_n = \frac{\left[2^{n+1} + (-1)^n\right]^2}{3^2}.$$

Solved problems

Problem 1 How may permutations are there using the letters of the word BOMMI?

Solution: There are 5 letters of which 2 letters are alike. Each of the remaining are distinct ∴ Number permutations $= \frac{5!}{2! \times 1! \times 1! \times 1!} = 60$

Problem 2 How many 3 digit numbers can be formed using the digits 5,7,9,1 if (i) a digit can not appear more than once in a number (ii) any digit may appear any number of times in a number

Solution: (i) $4 \times 3 \times 2 = 24$ (ii) $4 \times 4 \times 4 = 64$

Problem 3 How may 4 digit number divisible by 5 can be formed using the digit 3,7,1,5,6

(i) repetition of digits is permitted

(ii) repetition of digits is not permitted

Solution:

(i) Each of the 4 digit numbers that can be formed should be divisible by 5, satistying both these conditions, units place can be filled in 1 way (that is using 5 only) and the remaining ten's place, hundred's place and thousand's place, each can be filled in 5 ways
By the fundamental principle of counting, the total number of five digit number is $5 \times 5 \times 5 \times 1 = 125$

(ii) The unit's place can be filed in 1 way (since the number is to be divisible by 5) the remaining place in succession can be filled in 4, 3, 2 ways respectively

By the fundamental principle of counting the total number of 4 digits numbers each divisible by 5 is $2 \times 3 \times 4 \times 1 = 24$

Problem 4 How many 4 digit numbers, not beginning with zero, without repeating any digit, can be formed using 0, 1,2,3,4?

Solution: A number should not begin with zero is the condition. The remaining hundred's place, ten's place and unit's place can be filled in succession by 4,3,2 ways
By the fundamental principle of counding, the total number of 4 digit numbers is $4 \times 4 \times 3 \times 2 = 96$.

Problem 5 In how many different ways 5 gents and 3 ladies may be seated in a row such that two ladies were not seated together?

Solution: First, all the 5 gents may be seated in $5p_5 = 5!$ ways

Next, two ladies are not to be seated in adjacent seats. Hence, to satisfy this condition 3 ladies may be seated in 6 places. This can be done in $6p_3$ By the principle of counting, we have $5! \times 6P_3 = 14400$ ways

Problem 6 In how many ways can we select 3 white balls and 2 black balls from 5 white balls and 4 black balls?

Solution: : 3 white balls from 5 white balls can be selected in $5c_3$ ways
2 black balls from 4 black balls can be selected in $4c_2$ ways
$\therefore$ Total number of selection = $5c_3 \times 4c_2 = 60$

Problem 7 Committee of 5 students is formed selecting from 6 boys and 5 girls such that it contains atleast one boy and one girl. How many different committees can be formed?

Solution: As per the given conditions, a committee of 5 can be formed in the following combinations
1 Boy and 4 GIRLS
(or) 2 Boys and 3 GIRLS
(or) 3B and 2G
(or) 4B and 1 G
$\therefore$ The total number of combinations is

$$= 6c_1 \times 5c_4 + 6c_2 \times 5c_3 + 6c_3 \times 5c_2 + 6c_4 \times 5c_1$$
$$= 455(\text{check!})$$

Problem 8 In how many ways can nine students be particioned into three teams containing four, three, and two students respectively?

Solution: Since all the cells contain different number of studente, the number of unrecorded partitions equals the number of ordered partitions $\frac{9!}{4!3!2!} = 1260$.

Problem 9 Assume there are n distinct pairs of shoes in a closet, show that if you choose $n + 1$ single shoes at random from the closet, you are certain to have a pair

Solution: The n distinct pairs constitute n pigeonholes. The $n + 1$ single shoes correspond to $n + 1$ pigeons. Therefore, there must be atleast one pigeonhole with two shoes and thus you will certainly have drawn atleat one pair of shoes

Problem 10 Laet be al list (not necessarily in alphabetical order) of 26 letters in the English alphabet (a) show that L has a sublist consisting of four or more consecutive consonants. (b) Assuming L begins with a vowel, say A, show that L has a sublist consisting of five or more consecutive consonants.

Solution: (a) The five letters partition L into n=6 sublists (pigeonholes) of consecutive cousonants. Here $K + 1 = 4$ and so $K = 3$. Hence $nK + 1 = 6(3) + 1 = 19 < 21$ Hence some sublist has atleast four consecutive consonants.

(b) Since 2 begins with a vowel, the remainder of the vowels partion L into n = 5 sublists : Here $K + 1 = 5$ and so $K = 4$. Hence $nK + 1 = 21$. Thus some sublist has atleast five consecutive consonants.

Problem 11 Let A,B,C,D denote respectively, art, biology, chemistry and drama courses. Pind the number N of students in a dormitory given the data.

12 take A	5 take A and B	3 take A, B, C
20 take B	7 take A and C	2 take A, B, D
20 take C	4 take A and D	2 take B, C, D
8 take D	16 take B and C	3 take A, C, D
	4 take B and D	2 take all four
	3 take C and D	71 take none

Solution: Suppose we have any finite number of sets say $A_1, A_2, \ldots, A_m$. Let S_k be the sum of the cardinatities $n(A_i \cap Ai_2 \cap Ai_3 \ldots \cap Ai_k)$ of all possible K-tuple intersections of the given m sets. Then we have the following general inclusion - exclusion principle

$n(A_1 \cup A_2 \cup \ldots A_m) = S_1 - S_2 + S_3 \ldots + (-1)^{m-1} S_m$

Let E be the number of students who take atleast one course. By the inclusion - Enclusion principle

$$
\begin{aligned}
E &= S_1 - S_2 + S_3 - S_4 \\
\text{where } S_1 &= 12 + 20 + 8 = 60 \\
S_2 &= 5 + 7 + 4 + 16 + 4 + 3 = 39 \\
S_3 &= 3 + 2 + 2 + 3 = 10 \\
S_4 &= 2 \\
\therefore E &= 29, \\
N &= 71 + E = 100.
\end{aligned}
$$

Problem 12 solve the recurrence relation $a_n - 7a_{n-1} + 16a_{n-2} + 12a_{n-3} = o$ for n7,3 with the initial conditions $a_0 = 1, a_1 = 4$ and $9_2 = 8$.

Solution:

The characteristic pohynomial is $r^3 - 7r^2 + 16r - 12 = (r - 2)^2(r - 3)$. and the roots of characteristic equation are $r = 2, 2, 3$

$\therefore a_n = c_2 2^n + c_2 2^n. + c_3 3^n$ Now, By initial conditions

$$c_1 + c_3 = 1$$
$$2c_1 + 2c_2 + 3c_3 = 4$$
$$4c_1 + 8c_2 + 9c_3 = 8$$
$$\therefore \text{on solving} c_1 = 5$$
$$c_2 = 3 \text{ and } c_3 = -4$$

Hence the unique solution of the recurrence relation is $a_n = (5)(2^n) + 3(n2^n) - (4)(3^n)$

Problem 13 $a_n + 7a_{n-1} + 8a_{n-2} = 0, a_0 = 2, a_1 = -7$

Solution: Roots of characteristic equation are $r = \dfrac{-7 \pm \sqrt{17}}{2}$

$$\therefore a_n = c_1 \left(\frac{-7 + \sqrt{17}}{2}\right)^n + c_2 \left(\frac{-7 - \sqrt{17}}{2}\right)^n$$

Now by $a_0 = 2$ and $a_1 = -7$, $c_1 = 3$, $c_2 = -1$ (check!)

$$\therefore a_n = 3\left(\frac{-7 + \sqrt{17}}{2}\right)^n - 1\left(\frac{-7 - \sqrt{17}}{2}\right)^n$$

Problem 14 Find the solution of $S_k - 3S_{k-1} - 4S_{k-2} = 4^k$

Solution: The characteristic equation of the associated homogeneous relation is $^2 - 3r - 4 = o \Rightarrow r = -1, 4$

$$\therefore S_k^{(H)} = c_1(-1)^k + c_2(4)^k$$

Now $S_k^{(p)} = D.4^k$
substitute in (i)
$\therefore D.4^k - 3D.4^{k-1} - 4D4^{k-2} = 4^k \Rightarrow D - \dfrac{3D}{4} - \dfrac{D}{4} = 1, 0 = 1$, which is not possible

$\therefore$ A function of the form $D.4^K$ will not be a particular solution of the nonhomogeneous elation.

$$\text{Now} S_k^{(P)} = D.K.4^k$$

substitute in(1)

$$D = \frac{4}{5}$$

$$\therefore \quad S_k^{(P)} = \frac{4}{5}k.4^k$$

$\therefore$ The general solution

$$S_k = b_1(-1)^k + b_2(4)^k + \frac{4}{5}.K.4^k.$$

Problem 15 Solve $a_n = -2na_{n-1} + 3n(n-1)a_{n-2}$ with initial conditions $a_0 = 1, a_1 = 2$

Solution: By taking the substitution $b_n = \frac{a_n}{n!}$, the given Recurrence Relation changes to $b_n + 2b_{n-1} - 3b_{n-2} = 0$

$\therefore$ char equation: $r^2 + 2r - 3 = 0$

Roots: $1, -3$

$\therefore$ $b_n = c_1(1)^n + c_2(-3)^n$

Now by initial conditions $b_0 = 1, b_1 = 2$

$$\left.\begin{array}{l} c_1 + c_2 = 1 \\ c_1 - 3c_2 \end{array}\right\} \text{on solving} c_1 = \frac{5}{4}, c_2 = \frac{-1}{4}$$

$$b_n = \frac{1}{4}[5 - (-3)^n)], a_n = n!b_n$$

$$\therefore \quad a_n = \frac{n!}{4}[5 - (-3)^n]$$

is the complete solution of the given recumence relation

Problem 16 Show that $F_n \geq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$, $n \geq 1$ where F denotes the Fibonacc: sequence.

Solution: We prove the inequality by using induction on n when $n = 1$ and $n = 2$, it is easy to verify that the above inequality holds.

Now assume that the inequality is true for values less than $n + 1$.

Then

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\geq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \left(\frac{1+\sqrt{5}}{2} + 1\right) \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \left(\frac{1+\sqrt{5}}{2}\right)^{2} \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{n} \end{aligned}$$

$\therefore$ Hence the result by induction.

Quiz Questions

1. $n_{p_r} or p(n, r) =$ _______

   $\left(\text{Ans:} \dfrac{n!}{(n-r)!}\right)$

2. n_{c_r} or $c(n, r) =$ ____

   $\left(\text{Ans:} \dfrac{n!}{r!(n-r)!}\right)$

3. It K pigeons are assigned to n pigeonholes then one of the pigeonholes must contain atleast ____ pigeons

   $\left(\text{Ans:} \left\lfloor \dfrac{K-1}{n} \right\rfloor + 1\text{pigeons}\right)$

4. The solution of $a_n = 2a_{a-1}, n \geq 1$ is____

   $(Ans : a_n = 2^n)$

5. The solution of $F_n = F_{n-1} + F_{n-2}, n \geq 2$ with $F_0 = F_1 = 1$ is ____

   $\left(Ans : F_n = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} - \dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^{n+1}\right)$

6. $n_{c_{r-1}} + n_{c_r} = -$(Ans: $n + 1_{c_r}$)

7. $n_{c_r} = n_{c}$____(Ans: $n - r$)

8. The number of distinguishable permutations that can be formed from a collection of n objects in which the first object appears K_1 times, the second object K_2 times and so on is ____

   $\left(\text{Ans:} \dfrac{n!}{k_1!k_2!\ldots k_t!}\right)$

9. The number of distinguishable 'words' that can be formed from the letters of MISSISSIPPI is ____ $\left(Ans \dfrac{11!}{1!4!4!2!}\right)$

10. How many "words" of three distinct letters can be formed from the letters of the word MAST?

    $(Ans : 4_{P_3} = 24)$

11. There are ____ permutation of n distinct objects in a circle (Ans:$(n-1)!$)

12. $n_{p_r} =$ ____ $(r!n_{c_r})$

13. $r \times n_{c_r} = n - 1_{c_{r-1}} \times$ ____ (Ans: n)

14. $n_{p_r} =$ ____ $\times\, n - 1_{p_{r-1}}$ (Ans: n)
15. If A and B are disjoint sets, then $n(A \cup B) =$ ____

    (Ans:$n(A) + n(B)$)
16. Let A × B be the cartesian product of sets A and B then $n(A \times B) =$ ________

    (Ans:$n(A).n(B)$)
17. How many automobile license plates can be made if each plate contains two differents letters followed by three different digits?
    (Ans: 26. 25. 10. 9. 8 = 468000)
18. $n_{c_0} + n_{c_1} + \ldots + n_{c_n} =$ ____

    (Ans:2^n)
19. How many different bit strings are there of length seven?
    ($Ans : 2^7 = 128$)
20. How many functions are there from a set with m elements to one with n elements?
    (Ans: n^m)

First, we introduce the basic terminology of graph theory. We shall also discuss some of the basic results and theorems of graph theory.

The terminology used in graph theory is not standard. It is common to find several different terms being used as synonyms, because of the diversity of the fields in which graph theory is applied.

We will define a graph as an abstract mathematical system. In order to provide some motivation for the terminology used and also to develop, we shall represent graphs diagrammatically. Any such diagram will also be called as graph.

2. Basic Terminology

Definition 2.1 A graph G is an ordered triple $(V(G), E(G), \phi)$ consists of a non empty set V called the set of vertices (nodes, points) of the graph, E is said to be the set of edges of the graph, and ϕ is a mapping from the set of edges E to a set of order or unordered pairs of elements of V.

In general, $V(G)$ and $E(G)$ of a graph are finite. In some cases $E(G)$ may be empty.

If $e \in E$ is an edge and $\phi(e) = \{u, v\}$, then we say that e is an edge joining u and v, and the vertices u and v are called the ends (end vertices) of e.

That is, the definition of a graph implies that to every edge of the graph G we can associate an ordered or unordered pair of vertices of the graph.

The most common representation of a graph is by means of a diagram in which the vertices are represented as points and each edge as a line segment joining its end vertices. This diagram itself is referred to as the graph.

We denote the graph G as $G(V, E)$ or simply as G.

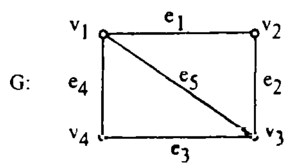
Example 2.1

$$G = (V(G), E(G), \phi)$$
$$\text{Where } V(G) = \{v_1, v_2, v_3, v_4, v_5\}$$
$$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$$

and ϕ is defined by

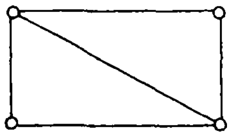
$$\phi(e_1) = \{v_1, v_2\}, \phi(e_2) = \{v_2, v_3\}$$
$$\phi(e_3) = \{v_3, v_4\}, \phi(e_4) = \{v_4, v_1\}$$
$$\phi(e_5) = \{v_1, v_3\}$$

Now the graph G can be diagramatically represented as follows:



We usually omit the names of the edges, when they have no intrinsic meaning. Also we omit the labels on vertices as well if the graphical representation is adequate for all discussion.

In fact, we will usually denote a graph by drawing its diagram rather than explicitly listing its vertices and edges



Note 2.1 In graphs, an edge should not pass through any points vertices other than the two end vertices of the edge.

Note 2.2 Generally a number of different diagrams may represent the same graph.

Example 2.2 $G = (V(G), E(G), \phi)$

$$\text{Where } V(G) = \{v_1, v_2, v_3, v_4, v_5\}$$
$$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

and ϕ is defined by

$$\phi(e_1) = \{v_1, v_2\},\ \phi(e_2) = \{v_2, v_3\}$$
$$\phi(e_3) = \{v_3, v_4\}$$
$$\phi(e_4) = \{v_4, v_5\}$$
$$\phi(e_5) = \{v_5, v_1\}$$
$$\phi(e_6) = \{v_5, v_1\}$$

graph is as shown:

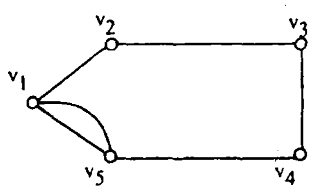


Fig 1

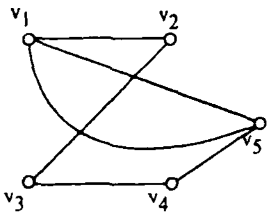


Fig 2

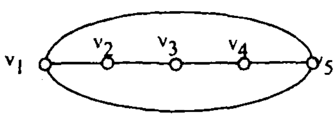
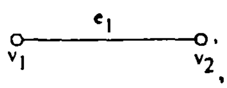


Fig 3

The figures 2 and 3 represent the same graph given in fig 1.

Definition 2.2 Any pair of vertices which are connected by an edge in a graph is callec adjacent vertices

Ex: 

v_1 and v_2 are adjacent vertices.

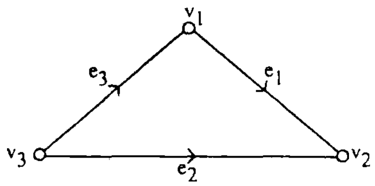
Definition 2.3 In a graph $G(V, E)$, an edge which is associated with an *ordered* pair of vertices is called a *directed edge* of graph G, while an edge which is associated with an unordered pair of vertices is called an *undirected edge*.

A graph in which every edge is directed is called a directed graph or simply a *digraph*

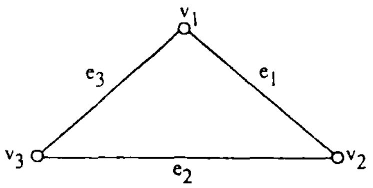
A graph in which every edge is undirected is called an *undirected graph*.

Note 2.3 The end vertices of an edge are said to be *incident* with the edge and vice versa

Example 2.3 The edge e_1 is incident with the vertices v_1 and v_2, also the vertex v_1 is incident with e_1 and e_4.



Directed graph

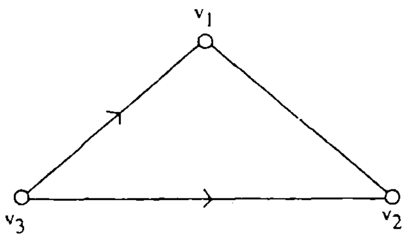


Undirected graph

Let $G(V, E)$ be a digraph and let $e \in E$ be a directed edge associated with the ordered pair of vertices (u, v). Then the edge e is said to be initiating (or) originating at the vertex u and terminating or ending at the vertex v. The vertices u and v are also called the initial and terminal vertices of the edge e.

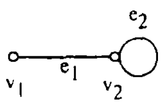
Definition 2.4 If some edges are directed and some are undirected in a graph, then the graph is a *mixed graph*

Example 2.4



Definition 2.5 An edge of a graph which joins a vertex to itself is called a loop.

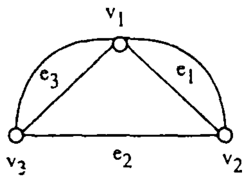
Example 2.5



here e_2 is a loop

Definition 2.6 In a graph, if some pairs of vertices are joined by more than one edge. Such edges are called *parallel edges*.

Example 2.6

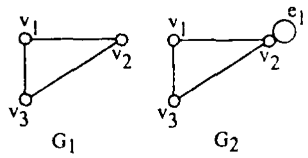


In this graph e_4& e_3 are parallel and
e_1& e_5 are parallel edges

Definition 2.7 Any graph which contains some parallel edges and loops is called a multigraph.

Definition 2.8 Any graph without parallel edges and loops is known as simple grapl

Example 2.7



G_1: This graph is a simple graph as it is not containing *any loops and parallel edges*

G_2· non simple graph since e_2 is a loop

Definition 2.9 A graph is finite if both its vertex set and the edge set are finit Otherwise it is an infinite graph.

Here after by a graph G means only a finite graph (for which $V(G)$ is a nonempl finite set).

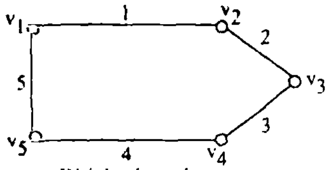
Definition 2.10 A finite graph with one vertex and no edges i.e., a single vertex called the trivial graph.

Definition 2.11 In any graph, a vertex which is not adjacent to any other vertex called an isolated vertex.

Definition 2.12 A graph with only isolated vertices is called a null graph In oth words, the set of edges in a null graph is empty.

Definition 2.13 A graph is which weights are assigned to every edge is called weighted graph

Example 2.8



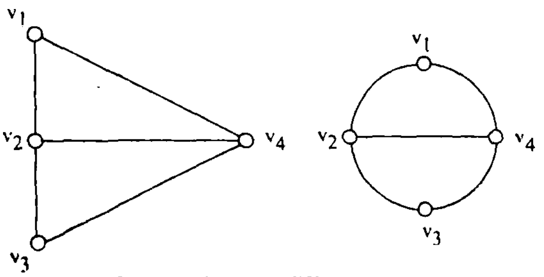
Weighted graph

here 1, 2, 3, 4, 5 are weights assigned to each edge respectively

Note 2.4 In drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short, what is important is the incidence between the edges and vertices.

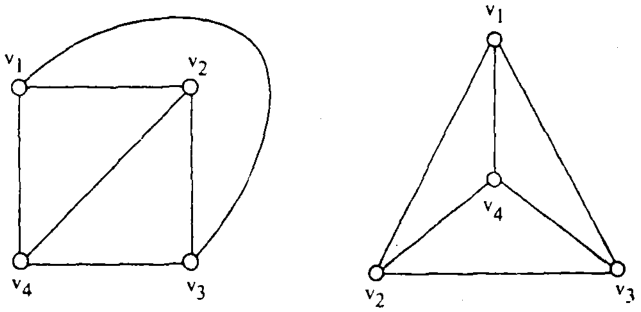
The definition of the graph contains no reference to the length or the shape and positioning of the edge joining any pair of vertices, nor does it prescribe any ordering of positions of the vertices. Therefore, for a given graph, there is no unique diagram which represents the graph. We can obtain a variety of diagrams by locating the vertices in an arbitrary number of different positions and also by showing the edges by arcs or lines of different shapes. Because of this arbitrariness it can happen that two diagrams which look entirely different from one another may represent the same graph, because incidence between edges and vertices is the same in both cases.

Example 2.9



Same graphs drawn differently

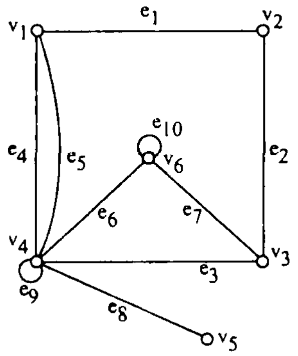
Example 2.10



same graphs drawn differently

Definition 2.14 If the graph G is finite, $|V|$ denotes the number of vertices of G, known as order of G and $|E|$ denotes the number of edges of G, known as size of G.

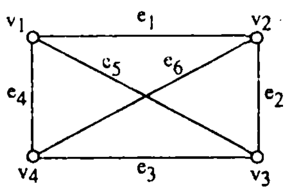
Example 2.11



For this graph:
$|V| = 6$
$|E| = 10$

Note 2.5 In a diagram of a graph, sometimes two edges may seem to intersect at a point that does not represent a vertex

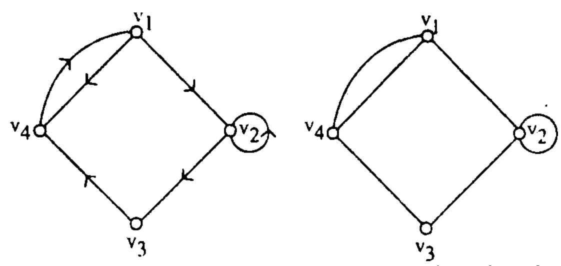
Example :



Edges e_5 and e_6 have no common point. Edges e_5 and e_6 in the above graph should be thought of as being in different planes and thus having no common point.

Definition 2.15 A graph obtained by ignoring the direction of edges in a directed graph is called underlying undirected graph

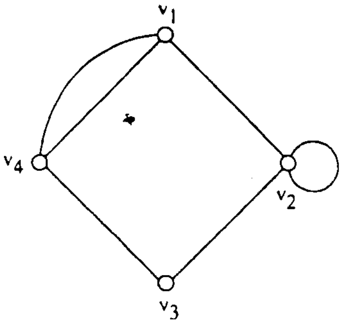
Example 2.12



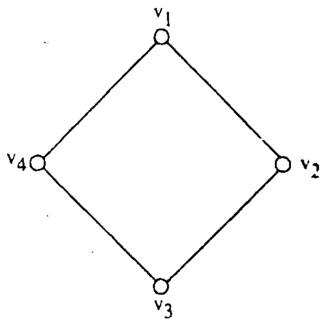
Digraph G Underlying undirected graph of G

Definition 2.16 A graph obtained by deleting all loops and parallel edges from a graph is called underlying simple graph

Example 2.13



Multigraph G



Underlying simple graph of G

3. Incidence and degree

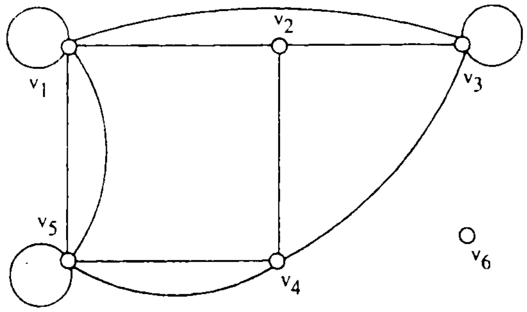
Let $G(V, E)$ be a graph and $e \in E, e = \{u, v\}$, then e is said to be incident with vertices u and v; and the vertices u and v are said to be incident with e. Two vertices u and v in V(G) are said to be adjacent if there is an edge $e \in E$ s.t $e = \{u, v\}$

It is natural to count the edges that are incident with a particular vertex i.e., given a vertex v, we can find the number of edges that are incident with v. If $e = \{u, v\}$ where $u \neq v$ is an edge, then e will be counted once while counting the edges that are incident with u, and again it will be counted once while counting the edges that are incident with v, with this in mind, we make a convention that a loop e will be counted twice when finding the number of edges that are incident with u.

Now we will give a name to the number obtained by counting all the edges that are incident with a vertex v.

Definition 3.1 Let v be a vertex in a graph G. Then the degree $d_G(v)$ of the vertex v in G is the number of edges of G that are incident with v (each loop is counted twice). The $d_G(v)$ can also be denoted by $deg_G(v)$ (or explicity, we use d(v) or deg(v) to denote the degree of v).

Example 3.1



deg $(v_1) = 6$
deg $(v_2) = 3$
deg $(v_3) = 5$
deg $(v_4) = 4$
deg $(v_5) = 6$
deg $(v_6) = 0$

The fundamental theorem of Graph theory

Theorem 3.1 Let G be an undirected graph with $|E|$ edges and $|V| = n$ vertices, then

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|.$$

Proof: Let G be a graph with $|E|$ edges and n vertices $v_1, v_2, \ldots, v_n$. When we sum over the degrees of all the vertices, we count each edge (v_i, v_j) twice: once when we count it as (v_i, v_j) in the degree of v_i and again when we count it as (v_j, v_i) in the degree of v_j. Then the conclusion follows:

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|.$$

□

Corollary 3.2 In any graph, the number of vertices of odd degree is even.

Proof: Let V_1 and V_2 be the sets of vertices of odd and even degree respectively in G. Then

$$V = V_1 \cup V_2$$
$$V_1 \cap V_2 = \phi \qquad \text{and}$$

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v) \tag{I}$$

By previous theorem, $\sum_{v \in V} \deg(v) = 2.|E|$,an even number.

The sum $\sum_{v \in V_2} \deg(v)$is even, since $\deg(v)$ is even for $v \in V_2$.

From (I) it follows that

$$\sum_{v \in V_1} d(v) = \text{an even number} \tag{II}$$

From (II), each term deg(v) (as$v \in V_1$) is odd and so the total number of terms is the sum must be even to make the sum an even number. Thus $|V_1|$ = no. of vertices of odd degree is even.

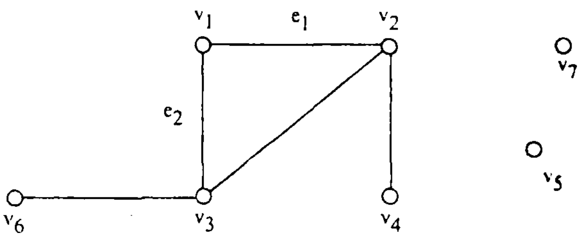
□

Definition 3.2 A vertex of degree one is called a pendant or end vertex in G

Definition 3.3 A vertex of degree zero is called an isolated vertex in G.

Definition 3.4 Two adjacent edges are said to be in series if their common vertex is of degree two.

Example 3.2



The vertices v_4, v_6 are pendant vertices

The vertices v_5, v_7 are isolated vertices

The edges e_1 and e_2 are in series, since $d(v_1) = 2$.

Definition 3.5 In a directed graph G, an edge $e =< u, v >$ is said to be *incident out* of the vertex u and *incident into* the vertex v. The number of edges incident out of a vertex v is called the out degree of v, denoted by $deg_G^-(v)$. The number of edges incident into v_i is called the indegree of v denoted by $deg_G^+(v)$. Also the sum of the out degree and indegree of a vertex v is called its total degree, denoted by $deg_G(v)$.

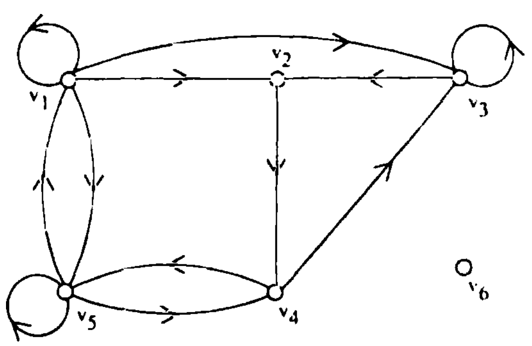
Corollary 3.3 If G is a directed graph, then

$$\sum_{i=1}^{n} \deg^+(v_i) = \sum_{i=1}^{n} \deg^-(v_i) = |E|.$$

Proof: The result follows from the fact that each edge has an initial vertex and a terminal vertex, therefore each edge contributes one outdegree to its initial vertex and one indegree to its terminal vertex.

Thus the sum of the indegrees and the sum of the out degrees of all vertices in a directed graph are same. □

Example 3.3

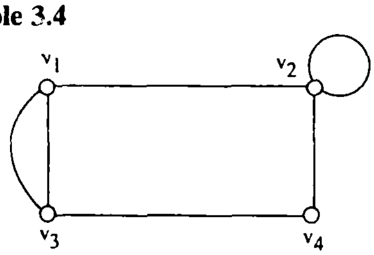


deg	outdegree : $\deg^-$	indegnee : $\deg^+$
v_1	4	2
v_2	1	2
v_3	2	3
v_4	2	2
v_5	3	3
v_6	0	0

Definition 3.6 The minimum of all the degrees of the vertices of a graph G is denoted by $\delta(G)$, and the maximum of all the degrees of the vertices of G is denoted by $\Delta(G)$.
In other words

$$\delta(G) = \min\{\deg(v)/v \in V(G)\} \text{ and}$$
$$\Delta(G) = \max\{\deg(v)/v \in V(G)\}$$

Example 3.4



$d(v_1) = d(v_3) = 3$
$d(v_2) = 4$
$d(v_4) = 2$
$\therefore \delta(G) = 2 \quad \Delta(G) = 4$

clearly $\delta(G) \leq \Delta(G)$.

Note 3.1 For a simple graph G with $|V| = n$, we have

$$0 \leq \delta(G) \leq \deg(v) \leq \Delta(G) \leq n - 1$$

Corollary 3.4 $\delta(G) \leq \dfrac{2|E|}{|V|} \leq \Delta(G)$

Proof: For a graph G, we have $\delta(G) \leq \deg(v) \leq \Delta(G) \quad \forall\, v \in V(G)$

Hence $\sum \delta(G) \leq \sum_{v \in V} \deg(v) \leq \sum \Delta(G)$

When $|V| = n$, we have

$$n\delta(G) \leq \sum_{v \in V} \deg(v) \leq n \cdot \Delta(G)$$

$$\delta(G) \leq \left[\frac{\sum\limits_{v\in V} \deg(v)}{n}\right] \leq \Delta(G)$$

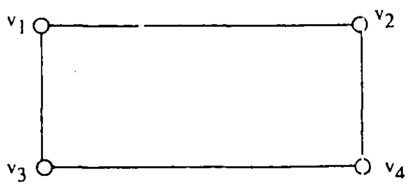
From theorem 3.1 $\sum\limits_{v\in V} \deg(v) = 2|E|$

$$\therefore \delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G) \qquad (\because |V| = n)$$

□

Definition 3.7 If $\delta(G) = \Delta(G) = K$ i.e., if each vertex of a graph G has degree K, then G is said to be K-regular or regular graph of degree K.

Example 3.5:



$$d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$$

The given graph is 2-regular (or) regular graph of degree 2.

Corollary 3.5 If G is a K-regular graph, then $\delta = \frac{2.|E|}{|V|} = \Delta$

Proof: By the definition of K-regular graph

$$\delta(G) = \Delta(G) = K$$

Now by the previous corollary:

$$K \leq \frac{2|E|}{|V|} \leq K$$

$$\Rightarrow \frac{2|E|}{|V|} = K$$

$$\therefore \delta(G) = \frac{2.|E|}{|V|} = \Delta(G).$$

□

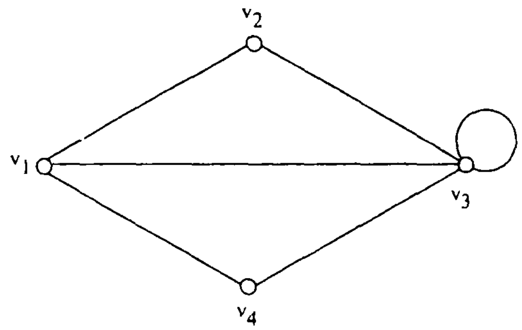
Definition 3.8 If $v_1, v_2, \ldots, v_n$ are the n vertices of G, then the sequence $(d_1, d_2, \ldots, d_n)$ where $d_i = \deg(v_i)$ *is the degree sequence* of G.

In general, we order the vertices so that the degree sequence is montonically increasing i.e.,

$$\delta(G) = d_1 \leq d_2 \leq \ldots \leq d_n = \Delta(G).$$

Example 3.5:

G



degree sequence of G:
(2, 2, 3, 5)

Definition 3.9 A degree sequence $d = (d_1, d_2, \ldots d_n)$ is graphic if there is a simple undirected graph with degree sequence d

Problem 3.1 Is there a graph with degree sequence (1, 3, 3, 3, 5, 6, 6)?

Solution: No, because, the no. of vertices with odd degree is odd, *a* contradiction to corollary: the number of vertices of odd degree must be even.

Result 3.1 (Havel Hakimi) Consider the following two squences and assume the sequence (i) is in descending order

(i) $s, t_1, t_2, \ldots t_s, d_1, \ldots, d_n$
(ii) $t_1 - 1, t_2 - 1, \ldots t_s - 1, d_1, \ldots, d_n$

Then sequence (i) is graphical if and only if (ii) is graphical.

Example 3.6 Show that the sequence (6, 6, 6, 6, 4, 3, 3, 0) is not graphical

Solution: By Havel hakimi theorem, we prove the above

Given sequence is 6, 6, 6, 6, 4, 3, 3, 0

First term of the sequence is 6, thus eliminating the first term and reducing the numbers in the next 6 terms by one, we get the sequence

$$5, 5, 5, 3, 2, 2, 0$$

This sequence is in descending order. The first term of the sequence is 5, thus eliminating the first term and reducing the numbers in the next 5 terms by one, we get the sequence

$$4, 4, 2, 1, 1, 0$$

The first term of the sequence is 4, thus eliminating the first term and reducing the numbers in the next 4 terms by one, we get the sequence

$$3, 1, 0, 0, 0$$

There exists no graph having one vertex of degree three and other vertex of degree one. Therefore the last sequence is not graphical. Hence the given sequence is also not graphical.

Example 3.7 Show that the following sequence is graphical: 6, 5, 5, 4, 3, 3, 2, 2, 2

Solution: Reducing the sequence as follows:

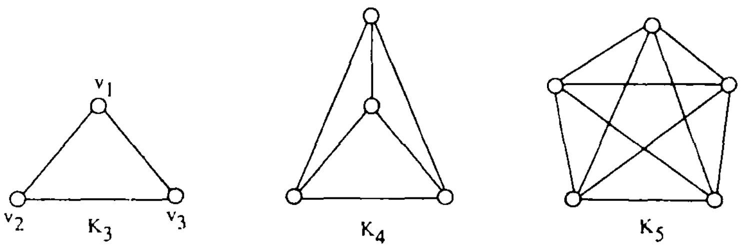
6	5	5	4	3	3	2	2	2
	4	4	3	2	2	1	2	2
	4	4	3	2	2	2	2	1
		3	2	1	1	2	2	1
		3	2	2	2	1	1	1
			1	1	1	1	1	1

Sequence (1, 1, 1, 1, 1, 1) is graphical

□

Definition 3.10 A simple graph in which every pair of distinct vertices are adjacent is called a complete graph. If G has n vertices then the complete graph will be denoted by K_n

Example 3.8:



Note 3.2 If G is a simple graph with n vertices then $0 \leq \deg(V) \leq n - 1$.

Note 3.3 Now onwards, denote ordered pair of vertices as (u, v) in a digraph and denote unordered pair of vertices as $\{u, v\}$ in a undirected graph

Definition 3.11 An edge labelling of a graph G is a function f from E to D i.e., $f : E \to D$, where D is some domain of labels.

Similarly A vertex labelling of a graph G is a function $g : V \to D$.

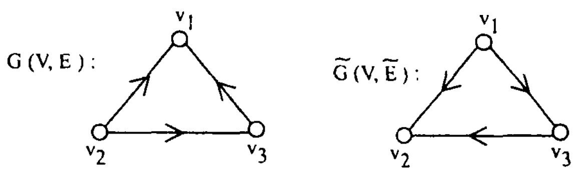
Example 3.9 The weight function which assigns weight to each edge in a graph is a edge labelling

The degree function which assigns degree to each vertex is a vertex labelling

Definition 3.12 The converse of a directed graph $G(V, E)$ is a directed graph denoted by $\tilde{G}(V, \tilde{E})$, where $\tilde{E}$ is the converse of E. The diagram of $\tilde{G}$ is obtained from G by simply reversing the directions of the edges in G.

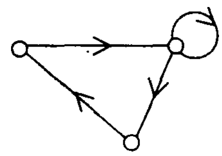
The converse $\tilde{G}$ is also called the reversal graph (or) directional dual graph of G

Example 3.10



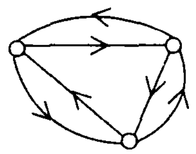
Definition 3.13 (a) A digraph that has atmost one directed edge between a pair of vertices (loops are allowed) is called asymmetric graph

Example 3.11



(b) If G is a digraph s.t whenever $e = (u, v)$ there is an edge $e' = (v, u)$, then G is said to be symmetric digraph

Example 3.12



(c) A digraph which is both simple and symmetric is called a simple symmetric graph

(d) A digraph which is both simple and asymmetric is called a simple asymmetric graph.

4. Paths, Reachability, and connectedness

Definition 4.1 (a) In a nondirected graph G a sequence 'P' of zero or more edges of the form $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}$ or $v_0 — v_1 — \ldots — v_n$ is called a path from v_0 to v_n. Where v_0 is the initial vertex and v_n is the terminal vertex of the path P.

We denote path P as a $v_0 - v_n$ path. In the definition of the path, vertices and edges may be repeated.

(b) If $v_0 = v_n$, then P is called a closed path. On the other hand if $v_0 \neq v_n$, then P is an open path

Observations 4.1 The path P is a graph itself where $V(P) = \{v_0, \ldots, v_n\} \subseteq V(G)$ and $E(P) \subseteq E(G)$

Also $1 \leq |V(P)| \leq n + 1$ and $0 \leq |E(P)| \leq n$

(c) In a directed graph G, a sequence 'P' of zero or more edges of the form $(v_0, v_1), (v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$ or $v_0 — v_1 — v_2 — \ldots — v_{n-1} — v_n$ is called a path from v_0 to v_n

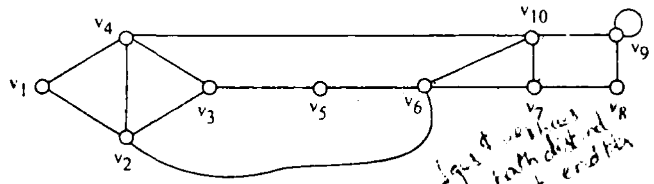
A path is said to be traversed through the vertices appearing in the sequence, originating in the initial vertex of the first edge and ending in the terminal vertex of the last edge in the sequence.

(d) The number of edges appearing in the sequence of the path 'P' is called the length of the path P

(e) If the length of the path P is zero i.e., the path P have no edges at all, it contains only a single vertex is called a trivial path

(f) A path P is said to be *simple* if all edges and vertices in the path are distinct except possibly the end points.

Example 4.1 For the following graph



we have

Path	length	Simple path?	Closed path?	Circuit?	Cycle?
$v_1 - v_4 - v_3 - v_5 - v_6 - v_{10} - v_4 - v_1$	7	no	yes	no	no
$v_2 - v_3 - v_5 - v_6 - v_7 - v_{10} - v_6 - v_2$	7	no	yes	yes	no
$v_1 - v_2 - v_1$	2	no	yes	no	no
$v_1 - v_4 - v_3 - v_2 - v_1$	4	yes	yes	yes	yes
$v_9 - v_9$	1	yes	yes	yes	yes
v_1	0	yes	yes	no	no
$v_5 - v_6 - v_7 - v_{10} - v_6 - v_2$	5	no	no	no	no
$v_4 - v_2 - v_3 - v_4$	3	yes	yes	yes	yes

Note 4.1 (a) : The trivial path is taken to be a simple closed path of zero.

(b) : A simple path is certainly a path and although the converse statement need not be true.

Definition 4.2 (a) A path of length ≥ 1 with no repeated edges and whose end vertices are same is called a *circuit*. A circuit may have repeated vertices other than the end vertices

(b) A cycle is a circuit with no other repeated vertices except its end vertices.

Example : Previous example 4.1

Note 4.2 A cycle is a simple circuit, a loop is a cycle of length 1. Clearly, In a graph a cycle that is not a loop must have length at least 3, but there may be cycles of length 2 in a multigraph.

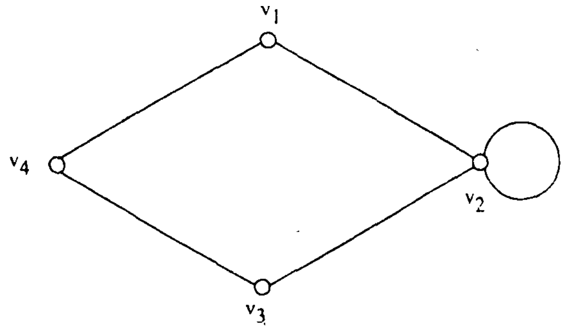
Result 4.1 Prove that any circuit in a graph must contain a cycle and that any circuit which is not a cycle contains atleast two cycles.

Proof: Suppose the vertices of the circuit $v_0, v_1, \ldots, v_n, v_0$ consider all subcircuits of the form $v_i, v_{i+1}, v_{i+2} \ldots, v_i$. (there is atleast one such, taking $i = 0$). That subcircuit $v_i, v_{i+1}, v_{i+2}, \ldots, v_i$ which uses the fewest number of vertices is a cycle. If the original circuit was not a cycle, then those vertices not on the first chosen subcircuit $v_i, v_{i+1}, v_{i+2}, \ldots v_i$ together with v_i from another subcircuit and the same arguement as before shows that this contains a second cycle. □

Definition 4.3 Two paths in a graph are said to be edge-disjoint if they have no common edges; they are vertex-disjoint if they have no common vertices.

Example 4.2

(i)



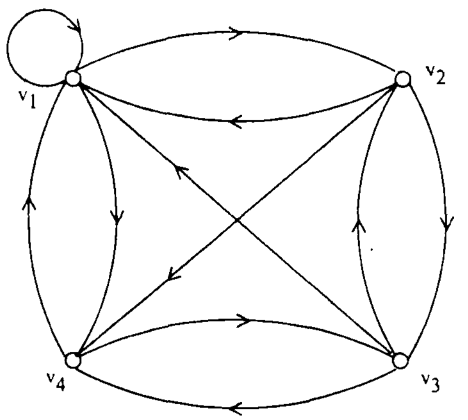
$\{v_2, v_2\}$: cycle of length 1

$\{\{v_4, v_3\}, \{v_3, v_2\}, \{v_2, v_4\}\}$: cycle of length 3

$\{\{v_4, v_1\}, \{v_1, v_2\}, \{v_2, v_2\}, \{v_2, v_4\}\}$: circuit of length 4

$\{\{v_4, v_1\}, \{v_1, v_2\}, \{v_2, v_4\}, \{v_4, v_3\}, \{v_3, v_2\}, \{v_2, v_4\}\}$: closed path of length 6

(ii)



Paths: $P_1 : ((v_1, v_2), (v_2, v_3))$
$P_2 : ((v_1, v_4), (v_4, v_3))$
$P_3 : ((v_1, v_2), (v_2, v_4), (v_4, v_3))$
$P_4 : ((v_1, v_2), (v_2, v_4), (v_4, v_1),$
$(v_1, v_2), (v_2, v_3))$
$P_5 : ((v_1, v_2), (v_2, v_4), (v_4, v_1),$
$(v_1, v_4), (v_4, v_3))$
$P_6 : ((v_1, v_1), (v_1, v_1), \ldots, (v_1, v_2)$
$(v_2, v_3))$

Cycles:
$C_1 : ((v_1, v_1))$
$C_2 : ((v_1, v_2), (v_2, v_1))$
$C_3 : ((v_1, v_2), (v_2, v_3), (v_3, v_1))$
$C_4 : ((v_1, v_4), (v_4, v_3), (v_3, v_1))$
$C_5 : ((v_1, v_4), (v_4, v_3), (v_3, v_2), (v_2, v_1))$

Definition 4.4 A simple digraph having no cycles is called acyclic graph

Note 4.3 An acyclic graph cannot have any loops. Since loop is a cycle of length one.

Theorem 4.1 In a graph G, every $u - v$ path contains a simple $u - v$ path.

Proof: If a path is a closed path, then it certainly contains the trivial path i.e., only single vertex.

Then, assume P is an $u - v$ path. We will prove this theorem by induction on the length n of P.

If P has length one, then P is itself a simple path. Suppose that all open $u - v$ paths of length K, where $1 \leq K \leq n$ contains a simple $u - v$ path. Now suppose that P is the open $u - v$ path $\{v_0, v_1\}, \ldots, \{v_n, v_{n+1}\}$ where $u = v_0$ and $v = v_{n+1}$. Of course, it may be that P has repeated vertices but if not, then P is a simple $u - v$ path and we are done. If on the other hai ., there are repeated vertices in P, Let i and j be distinct positive integers where $i < j$ and $v_i = v_j$. If the closed path $v_i - v_j$ is removed from P, an open path P' is obtained having length $\leq n$ since at least one edge $\{v_i, v_{i+1}\}$ was deleted from P. Thus by the inductive hypothesis, P' contains a simple $u - v$ path and so P. □

Definition 4.6 (a) A vertex v of a simple digraph is said to be reachable from a vertex u of the same digraph, if there exists a path from u to v.

Note 4.4 The concept of reachability is independent of the number of alternate paths from u to v and also of their lengths. For the sake of completeness we shall assume that every vertex is reachable from itself.

(b) If a vertex v is reachable from the vertex u, then a path of minimum length from u to v is called geodesic

(c) The length of a geodesic from the vertex u to the vertex v is called the distance and is denoted by $d(u, v)$.
Also $d(u, u) = 0$ for any vertex u.

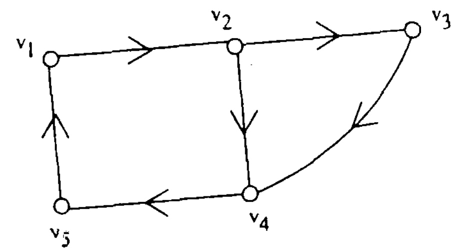
Observation 4.2 Reachability is a binary relation on the set of vertices of a simple graph.

Reachability is reflexive, transitive, not symmetric, not antisymmetric.

Note 4.5 If v is not reachable from u, then it is customary to write $d(u, v) = \infty$.

If v is reachable from u and u is reachable from v, then $d(u, v)$ is not necessarily equal to $d(v, u)$.

Example 4.3

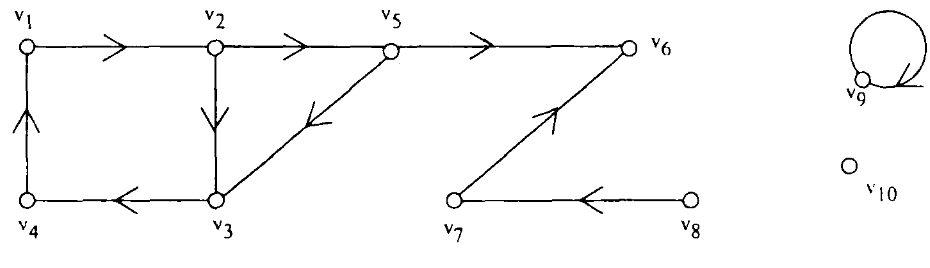


here $v_4 - v_5 - v_1$ is a path $\therefore$ v_1 is reachable from v_4.

Definition 4.7 If G is a directed graph, the set of vertices which are reachable from a given vertex v is said to be the reachable set of v, denote by $R(v)$.

For any subset $S \subseteq V$, the reachable set of S is the set of vertices which are reachable from any vertex of S, denoted by $R(S)$.

Example 4.4



$$R(v_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$
$$R(v_2) = \{v_1, v_2, v_3, v_4, v_5, v_6\} = R(v_3) = R(v_4) = R(v_5)$$
$$R(v_6) = \{v_6\}, R\{v_7\} = \{v_6, v_7\}, R(v_8) = \{v_6, v_7, v_8\},$$
$$R(v_9) = \{v_9\}, R\{v_{10}\} = \{v_{10}\}$$

$R(v_5, v_8, v_9, v_{10}) = V$ i.e., if $S = \{v_5, v_8, v_9, v_{10}\}$, $R(s) = V$

$R(v_1, v_8, v_9, v_{10}) = V$

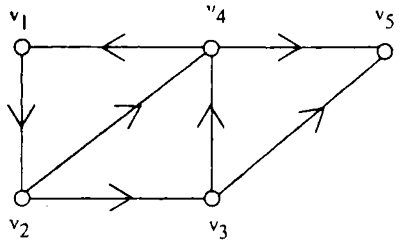
Definition 4.8 In a digraph $G(V, E)$, a subset $W \subseteq V$ is called vertex base (nodebase) if its reachable set is V and if no proper subset of W has this property.

Example 4.5 From the previous example, the sets $\{v_1, v_8, v_9, v_{10}\}$ and $\{v_5, v_8, v_9, v_{10}\}$ are vertex bases.

Observations 4.2 (i) Every isolated vertex of a digraph must be present in a vertex base.

(ii) Any vertex whose in degree is zero must be present in any vertex base

Problem 4.1 Find the vertex base and reachable sets of $\{v_1, v_4\}$, $\{v_4, v_5\}$, $\{v_3\}$ for the given digraph



Solution:

$$R(v_1, v_4) = \{v_1, v_2, v_3, v_4, v_5\} = V$$

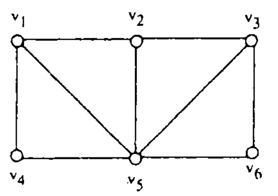
$$R(v_4, v_5) = R(v_3) = \{v_1, v_2, v_3, v_4, v_5\} = V$$

Given three sets are vertex bases as their Reachable set is the vertex set V.

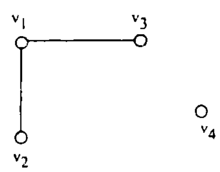
Definition 4.10 A graph G is said to be connected if there is a path between any two of its vertices.

In other words, in a undirected graph if any pair of vertices are reachable from one another, then the graph G is connected.

Example 4.6



Connected graph



Not Connected graph

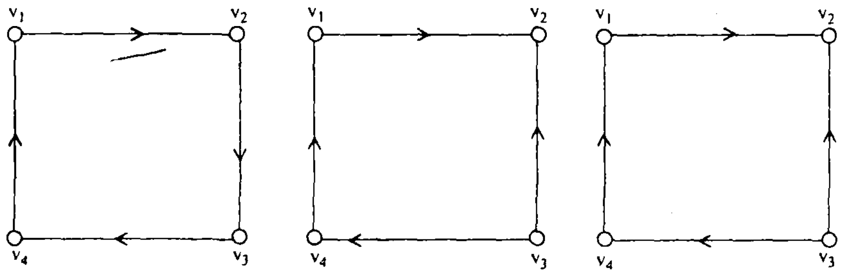
The above definition cannot be applied to directed graphs without some further modifications, because in a directed graph if a vertex u is reachable from another vertex v, the vertex v may not be reachable from u.

Definition 4.11 A simple digraph is said to be *Unilaterally Connected* if for any pair of vertices of the graph at least one of the vertices of the pair is reachable from the other vertex.

Definition 4.12 If for any pair of vertices of the graph both the vertices of the pair are reachable from one another, then the graph is called *strongly connected.*

Definition 4.13 A simple digraph is said to be weakly connected if its underlying undirected graph is connected.

Example 4.7



(a) Strongly Connected

(b) Unilaterally Connected but not strongly connected

(c) Weakly Connected but not unilaterally connected

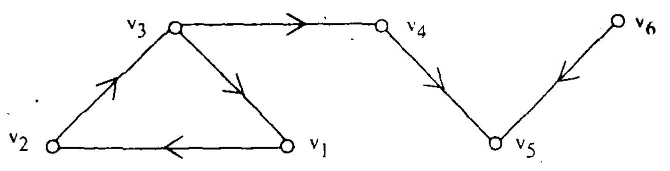
Observations 4.3 (i) A Unilaterally connected digraph is weakly connected but a weakly connected digraph is not necessarily unilaterally connected

(ii) A strongly connected digraph is both Unilaterally and weakly connected.

Definition 4.14 A subgraph G_1 of a graph G is said to be (weakly, Unilaterally or strongly) connected component if it is a maximal (weakly, Unilaterally or strongly) connected subgraph; that is there is no (weakly, Unilaterally or strongly) connected subgraph of G that properly contains G_1.

Definition 4.15 For a simple digraph, a maximal strongly connected subgraph is called a *strong component.* Similarly, a maximal Unilaterally connected or maximal weakly connected subgraph is called a Unilateral or weak component respectively.

Example 4.8



$\{v_1, v_2, v_3\}, \{v_4\}, \{v_5\}, \{v_6\}$ are the strong components

$\{v_1, v_2, v_3, v_4, v_5\}, \{v_6\}$ are the unilateral components

$\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is the weak component because the graph is weakly connected

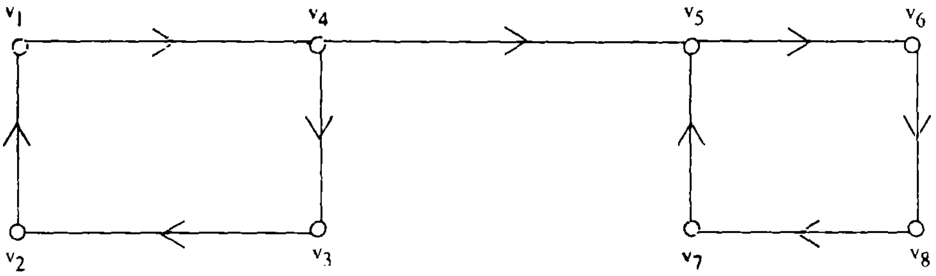
Theorem 4.3 In a simple digraph $G(V, E)$, every vertex of the diagraph lies in exactly one strong component.

Proof: Let $v \in V$ and S be the set of all vertices of G which are mutually reachable with v. The set S naturally contains v and is a strong component of G. This shows that every vertex of G is contained in a strong component.

Now, Assume that a node v is in two strong components. It imply that any node in one strong component which contains v is reachable from any vertex in the other strong component which also contains v, because every such path is easily established through v. This however is impossible. Hence every vertex is contained in exactly one strong component. Thus the strong components partition V. □

Observations 4.4 (i) It is not necessary that every edge of the digraph lies in one strong component.

Example 4.9 Consider the digraph G



$\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6 v_7, v_8\}$ are two strong components of G. But the edge $\{v_4, v_5\}$ is not contained in any of these two strong components.

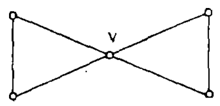
(ii) Every node and edge of a simple digraph is contained in exactly one weak component

(iii) Every node and edge of a simple digraph lies in atleast one Unilateral component.

(iv) The strong components partition the vertex set V of the simple digraph

Definition 4.16 [cut vertex] Let G be a connected graph. If v is a vertex of G such that $G - v$ is *not connected* ie., disconnected then the vertex v is called a cut vertex.

Example 4.10



v is the cut vertex.

If v is a cut vertex of G, then the removal of the vector v increase the number of components in G.

A cut vertex is also called a cut point

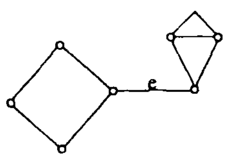
Theorem 4.4 A vertex v in a connected graph G is a cut vertex if and only if there exist vertices u and v distinct from v such that every path connecting u and w contains the vertex v.

Proof: Let v be a cut vertex in a connected graph G. Then $G - v$ is disconnected and $G - v$ contains atleast two components say A and B. Let u be a vertex of A and w be a vertex of B. There is no path in $G - v$ connecting u and w. Since G is connected there exists a path P from u to w in G. If the path does not contain v, then the removal of v from G will not disconnect the vertices u and w which is contradiction to the fact that u and v lie in two different components of $G - v$.

Conversely, if every path from y to w contains the vertex v, then removal of v from the graph G disconnects u and w. Hence u and w lie in different components of G. Which shows that $G - v$ is a disconnected graph. Thus v is a cut vertex of G. □

Definition 4.17 **[cut edge or bridge]** Let G be a connected graph. If e is an edge of G, such that $G - e$ is disconnected, the edge e is called a cut edge (or bridge)

Example 4.11



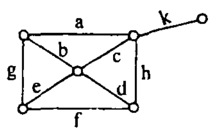
In the above graph 'e' is a cut edge

Definition 4.18 **[cut set]** Let G be a connected graph. A cut set in G is a set of edges whose removal from G leaves the graph G disconnected provided no proper subset of these edges disconnects the graph G.

A cut set in a graph always breaks the graph G into two parts. (Every edge in a tree is a cut set since the removal of any edge from a tree breaks the tree into two parts).

Cut sets are of great importance in studying the properties of networks.

Example 4.12



$\{a, b, c, f\}$ is a cut set. The other cut sets are $\{a, b, g\}\{a, b, e, f\}$. But $\{a, c, h, d\}$ is not a cut set because one of its proper subsets $\{a, c, h\}$ is a cut set.

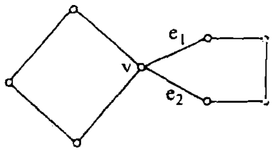
Edge connectivity

Definition 4.19 Let G be a connected graph. The edge connectivity of G is the minimum number of edges whose removal results in a disconnected. In otherwords, the number of edges in the smallest cut set is defined as the edge connectivity of G.

The edge connectivity of a connected graph G is denoted by $\lambda(G)$. If G is a disconnected graph then $\lambda(G) = 0$.

If G is a connected graph and has a bridge, then the edge connectivity of G is one.

Example 4.13 Find the edge connectivity of the graph G



Clearly $G - \{e_1, e_2\}$ is a disconnected graph
$\lambda(G) = 2$.

Vertex connectivity

Definition 4.20 Let G be a connected graph. The minimum number of vertices whose removal results in a disconnected is called the vertex connectivity of G.

The vertex connectivity of G is denoted by $K(G)$.

If $K(G) = 1$, then G has a vertex v such that $G - v$ is not connected and the vertex v is called a cut vertex.

If $G = K_n$ then

$$K(G) = n - 1$$
$$K(C_n) = 2 (n \geq 4)$$

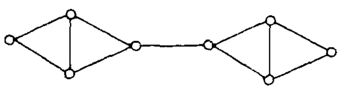
If a graph G has a bridge then the vertex connectivity of G i.e., $K(G) = 1$.

Theorem 4.5 The edge connectivity of a connected graph G can't exceed the minimum degree of G i.e., $\lambda(G) \leq \delta(G)$

Proof: Let G be a connected graph and v be a vertex of minimum degree in G. Then the removal of edges incident with the vertex v disconnects the vertex v from the graph G. Thus the set of all edges incident with the vertex v forms a cutset of G. But from the definition edge connectivity is the minimum number of edges in G whose removal disconnects G. This implies that the edge connectivity $\lambda(G)$ is always less than the number of edges incident with the vertex v. Hence $\lambda(G) \leq \delta(G)$. □

Note 4.6 The $\lambda(G)$ need not be equal to $\delta(G)$

Example 4.14



$\lambda(G) = 1$ but $\delta(G) = 2$.

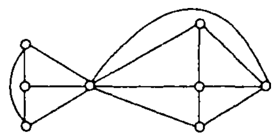
Theorem 4.6 The vertex connectivity of a graph G is always less than or equal to the edge connectivity of G i.e., $K(G) \leq \lambda(G)$

Proof: If G is disconnected then $K(G) = \lambda(G) = 0$. If G is connected, Let $\lambda = \lambda(G)$ be the edge connectivity. Then there exists a cut set S in G with λ edges. Let V_1 and V_2 be the partition of the vertex set of G with respect to S. Then the edges in S are the edges of G between V_1 and V_2. If no two edges in S have the same end vertex in the set V_1 or V_2. Then removal of all the end vertices of the edges in S disconnects G, otherwise the number of vertices required to disconnect G is less than the number of edges in S (since removal of a common end vertex of more than one edge removes all the edges incident with it).

Hence the theorem. □

Observation 4.6 For any graph G, $K(G) \leq \lambda(G) \leq \delta(G)$

Example 4.15 Find the edge connectivity and the vertex connectivity of the following graph



Solution: The minimum number of edges removal disconnects the graph is 3.

$$\therefore \quad \lambda(G) = 3$$

The minimum number of vertices required to disconnect the graph is 1.

$$\therefore \quad K(G) = 1$$

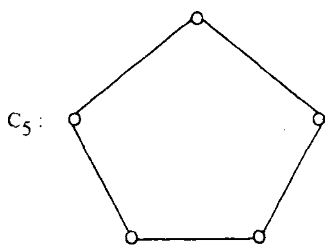
Definition 4.21 A connected graph with atleast one cut vertex is called a separate graph.

In otherwords, A connected graph is said to be a separable graph if its vertex connectivity is one. otherwise i. is called nonseparable graph.

5. Some Special Graphs

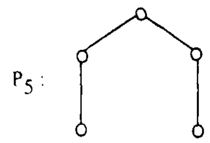
Cycle graph 5.1 A cycle graph of order 'n' is a connected graph whose edges form a cycle of length 'n' and denoted by C_n.

Example 5.1



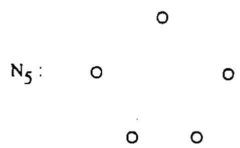
Path graph 5.2 A path graph of order 'n' is obtained by removing on edge from a C_n graph, denoted by P_n

Example 5.2



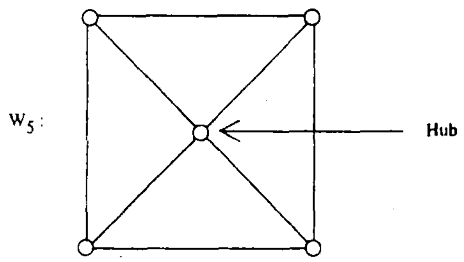
Null graph 5.3 A null graph of order n is a graph with n vertices and no edges. Null graphs of order n are denoted by N_n.

Example 5.3



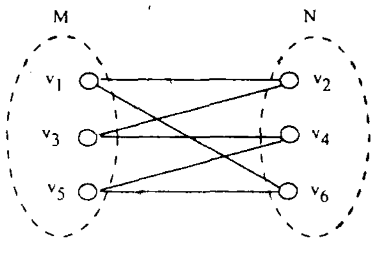
Wheel graph 5.4 A wheel graph of order n is obtained by joining a new vertex called 'Hub' to each vertex of a cycle graph of order $n - 1$, denoted by W_n

Example 5.4



Bipartite graph 5.5 A bipartite graph is an undirected graph whose set of vertices can be partitioned into two sets M and N is such a way that each edge joins a vertex in M to a vertex in N and no edge joins either two vertices in M or two vertices in N.

Example 5.5



here $V = M \cup N$

$M \cap N = \phi$

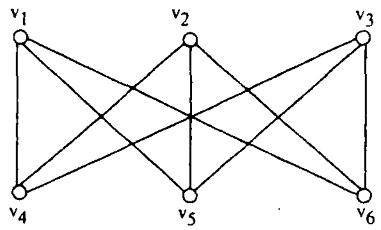
$M = \{v_1, v_3, v_5\}$

$N = \{v_2, v_4, v_6\}$

Complete bipartite graph 5.6 A complete bipartite graph is a bipartite graph in which every vertex of M is adjacent to every vertex of N. The complete bipartite graphs that may be partitioned into sets M and N as above s.t $M = m$ and $|N| = n$ are denoted by $K_{m,n}$

Example 5.6

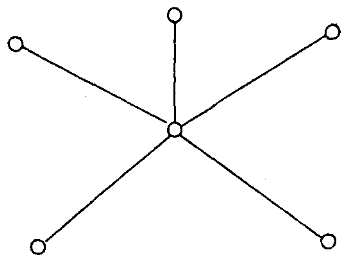
$K_{3,3}$:



Star graph 5.7 Any graph that is $K_{1,n}$ is called a star graph

Example 5.7

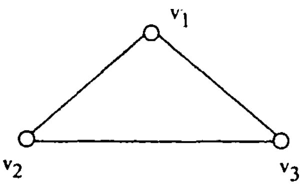
$K_{1,5}$:



Problem 5.1 Is K_3 is bipartite?

Solution: No, the complete graph K_3 is not bipartite

K_3 :



Explanation : If we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices.

If the graph is bipartite, these two vertices should not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge.

$\therefore$ K_3 is not bipartite.

6. Matrix Representation of Graphs

A diagrammatic representation of a graph has limited usefulness. Further, such a representation is only possible when the number of nodes and edges is reasonably small. In this context, A matrix is a convenient and useful way of representing a graph. Many known results of matrix algebra can be applied to study the properties of graphs and to calculate paths, cycles and other characteristics of a graph. Now, we are going to study the adjacency matrix, the incidence matrix and the path matrix of a graph.

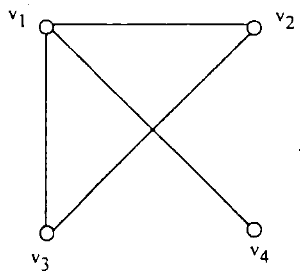
6.1 Adjacency Matrix

Let $G(V, E)$ be a simple graph with n vertices ordered from v_1 to v_n, then the adjacency matrix $A = [a_{ij}]_{n \times n}$ of G is an $n \times n$ symmetric matrix defined by the elements

$$a_{ij} = \begin{cases} 1 & \text{when } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

It is denoted by $A(G)$ or A_G

Example 6.1.1 A graph G and its adjacency matrix A_G are shown below:



$$A_G = \begin{matrix} & v_1 & v_2 & v_3 & v_4 \\ v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 0 \\ v_4 & 1 & 0 & 0 & 0 \end{matrix}$$

6.1.1 Properties of adjacency matrix (of undirected simple graph)

(1) An Adjacency matrix completely defines a simple graph

(2) The Adjacency matrix is symmetric

(3) Any element of the adjacency matrix is either 0 or 1, therefore it is also called as, bit matrix or boolean matrix

(4) The i^{th} row in the adjacency matrix is determined by the edges which originate in the node v_i.

(5) If the graph G is simple, the degree of the vertex v_i equals the number of 1's in the i^{th} row (or i^{th} column) of A_G

(6) Given an $n \times n$ symmetric boolean matrix A, we can find a simple graph G s.t A is the adjacency matrix of G.

(7) The Adjacency matrix A_G depends upon the ordering of the elements of $V(G)$. For different ordering of the elements of V. We get different $(n!)$ adjacency matrices of the same graph. But any one of the adjacency matrices of G can be obtained from another adjacency matrix of the same graph G by interchanging some of the rows and the corresponding columns of the matrix. Hence we can take any adjacency matrix of the graph G.

(8) G is null $\Longleftrightarrow A(G)$ is the zero matrix of order n.

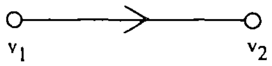
6.1.2 In case of Directed graphs

Let $G(V, E)$ be a simple digraph with $V = \{v_1, v_2, \ldots, v_n\}$ and the vertices are assumed to be ordered from v_1 to v_n. An $n \times n$ matrix A whose elements a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

is called the adjacency matrix of the graph G.

Example 6.1.2



, $A_G = \begin{matrix} & v_1 & v_2 \\ v_1 & 0 & 1 \\ v_2 & 0 & 0 \end{matrix}$

$$A_G = \begin{array}{c} v_1 \\ v_2 \end{array}\overset{\begin{array}{cc} v_1 & v_2 \end{array}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

Observation 6.3

(i) The number of elements in the i^{th} row whose value is 1 is equal to the out degree of the vertex v_i.

(ii) The Adjacency matrix for a directed graph may or may not be a symmetric matrix, since there may not be an edge from v_j to v_i when there is an edge from v_i to v_j.

(iii) An adjacency matrix completely defines a simple diagraph.

(iv) Some of the properties of a simple digraph are immediately seen from its adjacency matrix.

If a diagraph is reflexive, then the diagonal elements of the adjacency matrix are 1s.

If a digraph is symmetric, the adjacencymatrix is also symmetric, i.e., $a_{ij} = a_{ji} \forall\, i$ and j.

If a digraph is assymmetric then $a_{ij} = 1 \Rightarrow a_{ji} = 0$ and $a_{ij} = 0 \Rightarrow a_{ji} = 1 \forall\, i$ and j.

(v) If G is a simple digraph whose adjacency matrix is A then the adjacency matrix of $\tilde{G}$, the converse of G is the transpose of A i.e., A^T.

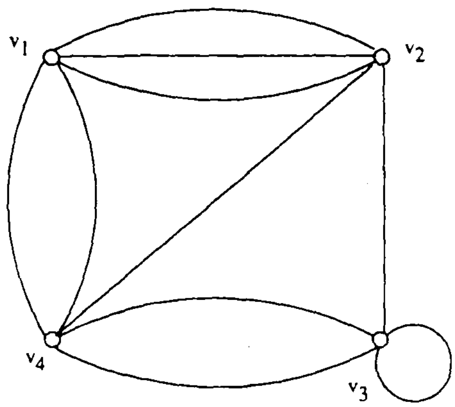
(vi) The diagonal elements of $A \cdot A^T$ show that out degree of the vertices. The diagonal entries of $A^T \cdot A$ shows the indegree of the vertices

Adjacency matrices can also be used to represent directed multigraphs or multigraphs. Clearly these matrices are not bit matrices, since a_{ij} is the number of edges that are associated to $\{v_i, v_j\}$ or (v_i, v_j).

(vii) All undirected graphs including multigraphs have symmetric adjacent matrices.

(viii) If G has self loop at every vertex and has no other edge then A_G is the identity matrix I_n of order n.

Example 6.1.3



$$A_G = \begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

Theorem 6.1.1 Let A be the adjacency matrix of a digraph G. Then element in the i^{th} row and j^{th} column of A^n (n is a non negative integer) is equal to the number of paths of length n from the i^{th} vertex to the j^{th} vertex.

Proof: The theorem will be proved using mathematical induction. Let G be a graph with adjacency matrix A (with ordered vertices $v_1, v_2, \ldots, v_n$ of G). The number of paths from v_i to v_j of length 1 is the ij^{th} entry of A, since this entry is the number of edges from v_i to v_j.

Assume that the ij^{th} entry of A^r is the number of different paths of length r from v_i .o v_j. This is the induction hypothesis.

Since $A^{r+1} = A^r A$, the ij^{th} entry of A^{r+1} equals to

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \ldots + b_{in}a_{nj}$$

vhere b_{ik} is the ik^{th} entry of A^r. By induction hypothesis b_{ik} is the number of paths of ength r from v_i to v_k.

A path of length $r+1$ from v_i to v_j is made up of a path length r from v_i to some ntermediate vertex v_k and an edge from v_k to v_j.

By product rule for counting, the number of such paths is the product of the number ›f paths of length r from v_i to v_k namely b_{ik}, and the number of edges from v_k to v_j, amely a_{kj}. When these products are added for all possible intermediate vertices v_k, the j^{th} entry in the matrix A^{r+1} gives the no of different paths having length $(r+1)$ from $_i$ to v_j.

Hence the proof. □

'ote 6.1.1 Let A be the adjacency matrix of a *simple* graph G. Then the ij^{th} entry in ^{r} is the number of different paths of length r between the vertices v_i and v_j

ote 6.1.2 If $B_n = A + A^2 + \ldots + A^n$

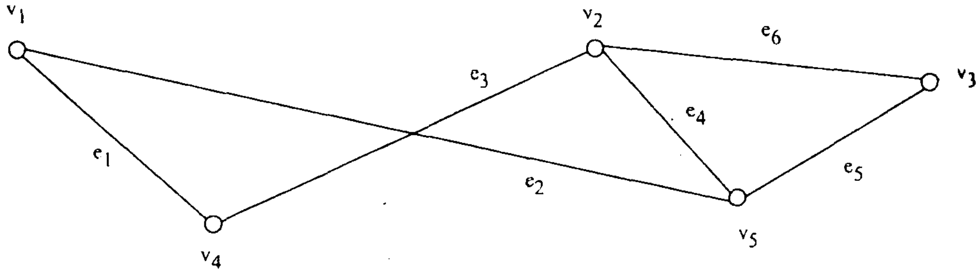
The element in the i^{th} row and j^{th} column of B_n shows the number of paths of length n or less which exist from v_i to v_j. If this element is non zero, then it is clear that v_j is reachable from v_i. Therefore, B_n provides the information about the reachability of any vertex of the graph from any other vertex.

6.2 Incidence Matrix

Let G be a graph with n vertices. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Define $n \times m$ matrix $I_G = [m_{ij}]_{n \times m}$ Where

$$m_{ij} = \begin{cases} 1 & \text{when } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

Example 6.2.1 Give the incidence matrix for the graph



$I_G =$

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

Observations 6.2.1

(i) The Incidence matrix contains only 0 and 1.

(ii) The number of 1s in each row equals to the degree of the corresponding vertex.

(iii) A row with all zeros represents an isolated vertex.

(iv) Every edge is incident on exactly two vertices, each column of the incidence matrix has exactly two ones except the loop.

(v) The parallel edges in a graph produce identical columns in its incidence matrix.

(vi) Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and the edges of the same graph.

Problem 6.2.1 Is there exists a graph G corresponding to following incidence matrix? Justify

$$I(G) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution: No, because the last column contains no 1's (each column should contains exactly two 1's)

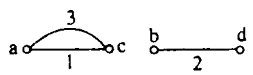
Problem 6.2.2 Let $I(G) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Show that there exists no connected graph G corresponding to this incidence matrix.

Solution: Performing the operations $R_2 \leftrightarrow R_3, C_3 \leftrightarrow C_2$ on $I(G)$
We get

$$I(G) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} [A(G_1)]_{2\times 2} & 0 \\ 0 & [A(G_2)]_{2\times 1} \end{bmatrix}$$

Hence the graph is disconnected.
Further G has two components G_1 and G_2 they are



6.3 Path Matrix (Reachability Matrix)

Let G be a simple digraph having no parallel directed edges and $V = \{v_1, v_2, \ldots, v_n\}$ be its vertex set. An $n \times n$ matrix $P = [p_{ij}]_{n\times n}$ is given by

$$p_{ij} = \begin{cases} 1 & \text{if there is a path from } v_i \text{ to } v_j \\ 0 & \text{otherwise} \end{cases}$$

is called the path matrix of G.

Remark 6.3.1 The above definition is valid for undirected graphs which have no parallel edges.

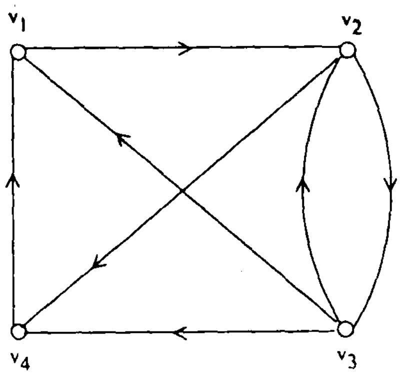
Observation 6.4 (i) The path matrix only shows the presence or absence of atleast one path between a pair of vertices and also the presence or absence of a cycle at any vertex.

(ii) However, path matrix does not show all the paths that may exist. i.e., a path matrix does not give complete information about a graph.

Remark 6.3.2 The path matrix can be calculated from the matrix $B_n = A + A^2 + \ldots + A^n$ by choosing

$$p_{ij} = \begin{cases} 1 & \text{if the element in the } i^{th} \text{ row and } j^{th} \text{ column of } B_n \text{ is non zero} \\ 0 & \text{otherwise} \end{cases}$$

Example 6.3.1 Consider the graph
Find path matrix.



Solution: Adjacency matrix of the graph $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

then

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 2 & 3 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

Now, $B_4 = A + A^2 + A^3 + A^4$

$$\text{Then } B_4 = \begin{pmatrix} 3 & 4 & 2 & 3 \\ 5 & 5 & 4 & 6 \\ 7 & 7 & 4 & 7 \\ 3 & 2 & 1 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Note 6.3.1 The method of calculating the path matrix P of a graph by calculating first $A, A^2, \ldots, A^n$ and then B_n is cumbersome we shall now describe another method based upon similar idea but which is more efficient.

Another method to calculate the path matrix from the Adjacency matrix

Before explaining this method, we first define two operations on matrices with entries 0 and 1.

A matrix with entries 0 and 1 is called a Boolean matrix. In the set {0, 1} we define two binary operations $\wedge$ and $\vee$ by the following table:

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

Binary operations $\wedge$ and $\vee$ on {0, 1}

For any two $n \times n$ Boolean matrices A and B, the boolean sum and the boolean product of A and B are written as $A \vee B$ and $A \wedge B$ which are also Boolean matrices, say C and D.

The elements of C and D are given by

$$c_{ij} = a_{ij} \bigvee b_{ij} \quad \text{and} \quad d_{ij} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj}) \forall\, i, j = 1, 2, \ldots, n$$

Note that the element d_{ij} is easily obtained by scanning the i^{th} row of A from left to right and simultaneously the j^{th} column of B from top to bottom. If, for any K, the K^{th} element in the row and K^{th} element in the column are both 1, then $d_{ij} = 1$, otherwise $d_{ij} = 0$.

Let us denote $A \wedge A = A^{(2)}$, $A \wedge A^{(r-1)} = A^{(r)}$ for any $r = 2, 3, \ldots$

Now the pathmatrix is given by

$$P = A \vee A^{(2)} \vee A^{(3)} \vee \ldots A^{(n)} = \bigvee_{k=1}^{n} A^{(K)}$$

Example 6.3.2 From previous example

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Now

$$A^{(2)} = A \wedge A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A^{(3)} = A \wedge A^{(2)} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$A^{(4)} = A \wedge A^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$P = A \vee A^{(2)} \vee A^{(3)} \vee A^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

In other words,

In a simple digraph $G(V, E)$, $E \subseteq V \times V$ so that E can be interpreted as a relation in V. The adjacency matrix A is the relation matrix of the relation E.

Similarly $A^{(2)}$ is the relation matrix of the relation $EoE = E^2$. Ingeneral for a given relation E in V, a relation E^+, called the transitive closure of E given by

$$E^+ = E \cup E^2 \cup \ldots$$

clearly, the relation matrix A^+ of E^+ is given by

$$A^+ = A \vee A^{(2)} \vee A^{(3)} \vee \ldots$$

whose A is the relation matrix of E.

If the number of elements in $\vee$ is n, then no path or cycle exceeds n in length; therefore A^+ can be obtained by simply considering the sum up to $A^{(n)}$ for powers higher than n will not change A^+. Therefore

$$A^+ = A \vee A^{(2)} \vee A^{(3)} \vee \ldots \vee A^{(n)} = P$$

The matrix A^+ is the same as the path matrix.

This method of obtaining the transitive closure of the relation as well as the path matrix of a simple digraph can easily be programmed by using the following algorithm by warshall.

6.4 Warshall's Algorithm

Given the adjacency matrix A of a simple digraph, then the following steps produce the path matrix $P(\text{or} A^+)$:

Step 1: $P^{[0]} = A$

Step 2: $K = 1$

Step 3: $i = 1$

Step 4: $p_{ij}^{[K]} = p_{ij}^{[K-1]} \vee (p_{ik}^{[K-1]} \wedge p_{kj}^{[K-1]}) \forall j = 1$ to n

Step 5: $i = i + 1$. If $i \leq n$, go to step4

Step 6: $K = K + 1$ If $K \leq n$, go to step3; otherwise, stop.

Remark 6.4.1 A different algorithm due to warshall which also permits the calculation of the path matrix P from a given adjacency matrix is obtained from algorithm WARSHALL by replacing Step4 by Step4′:

Step 4′: If $p_{ik} = 1$, then $p_{ij}^{[K]} = p_{ij}^{[K-1]} \vee p_{kj}^{[K-1]} \quad \forall j = 1$ to n.

The other steps remain the same.

Algorithm WARSHALL can be modified further to obtain a matrix which gives the lengths of shortest paths between the vertices.

For this purpose, let A be the adjacency matrix of the graph. Replace all those elements of A which are zero by ∞, which shows that there is no edge between the vertices in question. The following algorithm produces the required matrix which shows the lengths of minimum paths.

6.5 Minima Algorithm

Start with the adjacency matrix. Replace the zero elements in the adjacency matrix by infinity or by some very large number. Let this matrix be denoted by M. The matrix C produced by the following steps shows the minimum lengths of paths between the vertices

Step 1: $C^{[0]} = M$

Step 2: $K = 1$

Step 3: $i = 1$

Step 4: $C_{ij}^{[k]} = \min\left\{C_{ij}^{[k-1]}, C_{ik}^{[k-1]} + C_{kj}^{[k-1]}\right\} \forall j = 1 \text{ to } n$

Step 5: $i = i + 1$. If $i \leq n$, go to step 4

Step 6: $k = k + 1$. If $k \leq n$, go to step 3; otherwise, stop.

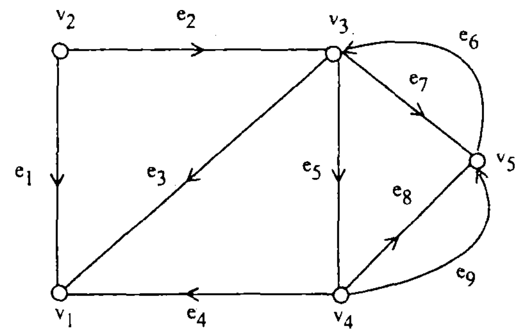
Note 6.5.1 + in Step 4 means the ordinary adding of integers.

6.6 Incidence Matrix of a digraph

The incidence matrix with the vertex set $\{v_1, v_2, \ldots, v_n\}$ the edge set $\{e_1, e_2, \ldots, e_m\}$ and with no self-loops is an $n \times m$ matrix $B = (b_{ij})$ defined by

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the initial vertex of the edge } e_j \\ -1 & \text{if } v_i \text{ is the final vertex of } e_j \\ 0 & \text{otherwise.} \end{cases}$$

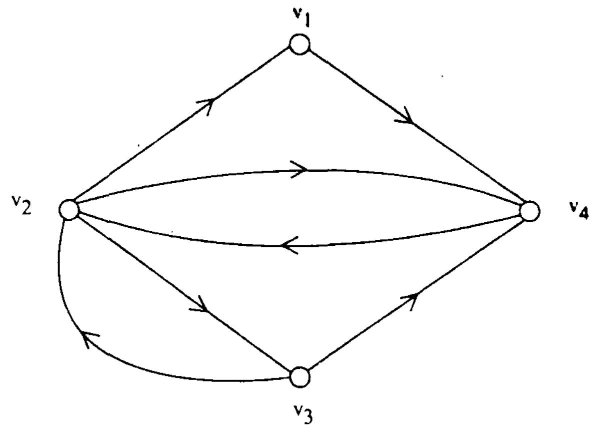
Example 6.6.1 A digraph G and its incidence matrix B_G are given as follows:



Solution:
$$B_{(G)} = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Note 6.6.1 If the graph is not directed then change all -1 entries into 1.

Example 6.6.2 Obtain the adjacency matrix A of digraph given below: Find the elementary paths of length 1 and 2 from v_1 to v_4.

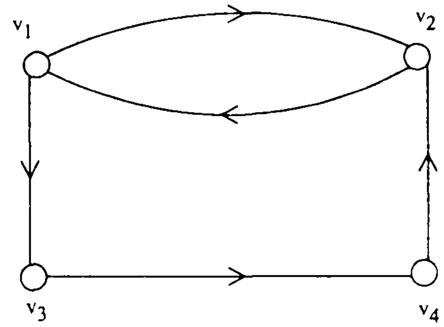


Solution: $A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$

Elementary paths of length 1 from v_1 to v_4 should be directed edge v_1v_4. It exists.

An elementary path of length 2 from v_1 to v_4 should be of the form v_1vv_4 where v_1v and vv_4 are directed edges. There is no elementary path of length 2 from v_1 to v_4

Example 6.6.3 Consider the following digraph. Use its adjacency matrix to find how many paths of length 3 exists from v_1 to v_2.



Solution: $A_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

To find the number of paths of length 3 from v_1 to v_2, we have to find A^3 (not $A^{(3)}$) and the entry (1,2) in A^3.

$$A^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

As $(1, 2)^{th}$ entry in A^3 is 2, there are two directed paths of length 3 from v_1 to v_2

They are $v_1 \to v_2 \to v_1 \to v_2$
$v_1 \to v_3 \to v_4 \to v_2$

Example 6.6.4 Calculate the path matrix P from the adjacency matrix (by warshall's algorithm).

Given Adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Solution:

$$P = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \qquad \text{(Check!)}$$

7. Subgraphs and Isomorphic Graphs

Definition 7.1 A subgraph H if a graph $G(V, E)$ is a graph H(V(H), E(H)) where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

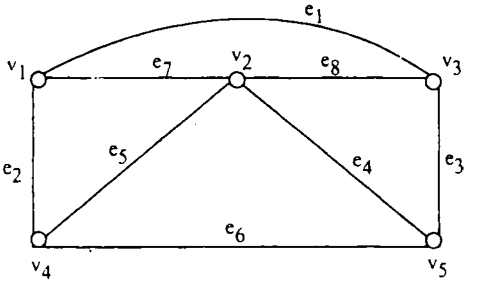
If H is a subgraph of G, we write $H \subseteq G$.

If $H \subseteq G$, then G is said to be a super graph of H

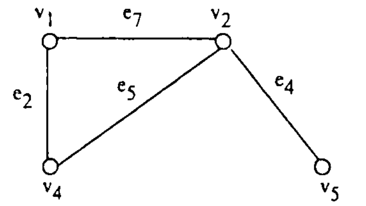
Definition 7.2 A subgraph H of G is called a spanning subgraph of G if and only if $V(H) = V(G)$

Definition 7.3 If W is any subset of vertex set of G, then the subgraph generated (or) induced by W is the subgraph H of G obtained by taking $V(H) = W$ and $E(H)$ to be those edges of G that joins pair of vertices in W

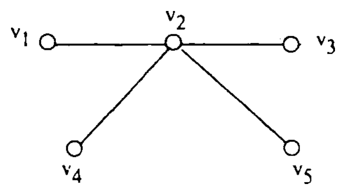
Example 7.1



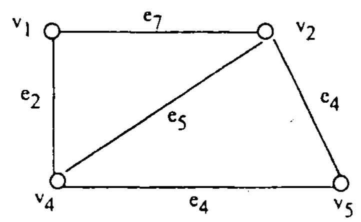
(a) Graph G



(b) a Subgraph of G



(c) a Spanning Subgraph of G



(d) Subgraph induced by $W = \{v_1, v_2, v_4, v_5\}$

Note 7.1 The graph G and the null graph are subgraphs of G.

Note 7.2 Subgraphs, Spanning subgraphs, induced subdigraphs of a digraph G can be defined similarly.

Definition : Isomorphic Graphs 7.4 Two graphs G and G' are isomorphic if there is a function $f : V(G) \to V(G')$ from the vertices of G to the vertices of G' such that

(i) f is one-to-one

(ii) f is onto and

(iii) For each pair of vertices u, and v of G

$$\{u, v\} \in E(G) \Longleftrightarrow \{f(u), f(v)\} \in E(G')$$

Any function f with the above three properties is called an isomorphism from G to G'.

The condition (iii) says that vertices u and v are adjacent $\Leftrightarrow$ $f(u)$ and $f(v)$ are adjacent in G'.

In otherwords, we say that the function 'f preserves adjacency'

If the graphs G and G' are isomorphic and f is an isomorphism of G to G', then intuitively the only difference between the graphs is the names of the vertices. Indeed, if we were to change the names of the vertices of G' from $f(v)$ to v for each $v \in V(G)$, then G' with the newly named vertices would be identical to the graph G, for then they both would have the same lists of vertices and edges.

Of Course, if G and G' are isomorphic graphs the isomorphism f is by no means unique, there may be several isomorphisms from G to G'. But if such an isomorphism f exists, then there are several conclusions we can make, namely.

1. $|V(G)| = |V(G')|$
2. $|E(G)| = |E(G')|$
3. If $v \in V(G)$, then $\deg_G(v) = \deg_G(f(v))$, and, thus, the degree sequences of G and G' are the same.
4. If $\{v, v\}$ is a loop in G, then $\{f(v), f(v)\}$ is a loop in G', and more generally, if $v_0 - v_1 - v_2 - \ldots - v_{k-1} - v_k (= v_0)$ is a cycle of length k in G, then $f(v_0) - f(v_1) - f(v_2) - \ldots - f(v_{k-1}) - f(v_k)$ is a cycle of length k in G'. In particular, the cycle vectors of G and G' are equal, where the cycle vector of G is by definition the vector $(c_1, c_2, \cdots, c_n)$ where c_i is the number of cycles in G of length i. Also $c_1 = 0$ for simple graphs and c_2 is nonzero only for multigraphs.

Result 7.1 Suppose that G and G' are two graphs and that $f : V(G) \to V(G')$ is a one-to-one onto function. Let A be adjacency matrix for the vertex ordering $v_1, v_2, \ldots, v_n$ of the vertices of G. Let A' be the adjacency matrix for the vertex ordering $f(v_1), f(v_2), \ldots, f(v_n)$. Then f is an isomorphism from $V(G)$ to $V(G') \iff$ the adjacency matrices A and A' are equal.

But, if the adjacency matrices A and A' are not equal, then all that proves is that the function f itself is not an isomorphism; it may still be the case that graphs G and G' are *isomorphic under some other function*. The fact that an isomorphism must map a vertex of G of degree d to a vertex of G' of degree d, will shorten the search for isomorphisms considerably.

In some cases atleast, the degree sequence of a graph can be used to shorten the search for isomorphisms.

Determining when Graphs are Not isomorphic

We can show that two graphs are not isomorphic by showing that they do not share a property that isomorphic graphs must both have. Such property is called an *invariant* with respect to the isomorphism of graphs.

The invariants are

(i) The number of vertices

(ii) The number of edges and

(iii) The degree sequences of the two graphs

If any of these Quantities differ in two graphs, those graphs can't be isomorphic. However, when these invariants are the same it doesn't mean that the two graphs are isomorphic. Apart from these invariants, we need a one-to-one and onto function which preserves adjacency of the vertices in simple graph and preserves the direction of edges in digraphs.

Observations 7.1 (i) The above definition is valid for digraphs also

(ii) Isomorphism preserves the degrees

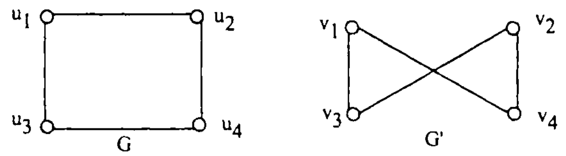
(iii) If G and G' are isomorphic, then G is simple $\iff$ G' is simple.

(iv) "has a vertex of degree d", "has a simple cycle of length k" are also invariants w.r.to graphs and isomorphism of graphs

(v) It is easy to test whether a pair of graphs is isomorphic if we can find a small number of easily checked invariants that isomorphic graphs and only isomorphic graphs share. Unfortunately, no one has succeeded in finding such a set of invariants.

Example 7.2 Show that the following graphs G and G' are isomorphic

Solution:



The two graphs have the same number of vertices, same number of edges and same degree sequences.
Now, define a function f as follows:

$$f(u_1) = v_1,\ f(u_2) = v_4,\ f(u_3) = v_3,\ f(u_4) = v_2$$

$$A(G) = \begin{matrix} & u_1 & u_4 & u_3 & u_2 \\ u_1 & 0 & 1 & 1 & 0 \\ u_4 & 1 & 0 & 0 & 1 \\ u_3 & 1 & 0 & 0 & 1 \\ u_2 & 0 & 1 & 1 & 0 \end{matrix}$$

Now

$$A'(G') = \begin{matrix} & v_1 & v_4 & v_3 & v_2 \\ v_1 & 0 & 1 & 1 & 0 \\ v_4 & 1 & 0 & 0 & 1 \\ v_3 & 1 & 0 & 0 & 1 \\ v_2 & 0 & 1 & 1 & 0 \end{matrix}$$

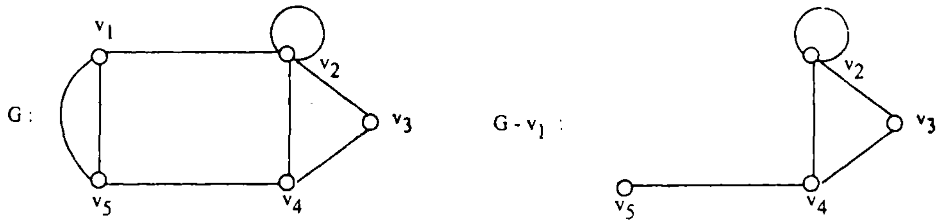
$\Rightarrow$ f is one to one and onto also it is preserving the adjacency of the vertices.
$\therefore$ The two given graphs are isomorphic to each other.

8. Operations on Graphs

Definition 8.1 **(Deleting a vertex)** Let G be a graph with $V(G) = \{v_1, v_2, v_3, v_4, \ldots v_n\}$ then $G - v_i$ is the graph obtained deleting or removing the vertex v_i from G together with all edges incident on v_i.

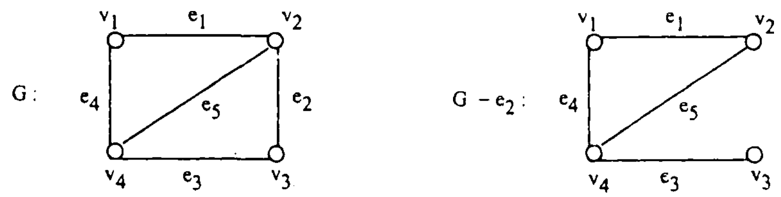
More generally, we write $G - \{v_1, \ldots, v_k\}$ for the graph obtained by deleting the vertices $v_1, \ldots v_k$ and all edges incident on any of them.

Example 8.1



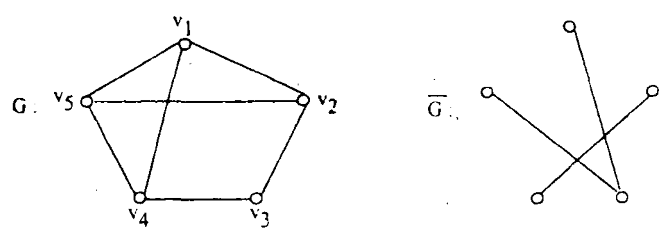
Definition 8.2 **(Deleting an edge)** Let G be a graph with $E(G) = \{e_1, e_2, \ldots e_n\}$ then $G - e_i$ is the graph obtained by removing or deleting the edge e_i without deleting the vertices which are incidented with e_i.

Example 8.2



Definition 8.3 **(Complement of a graph)** The complement of a graph G is the graph $\overline{G}$ with the same vertices as G. An edge exists in $\overline{G} \iff$ it does not exist in G, in otherwords, two vertices adjacent in $\overline{G} \iff$ they are not adjacent in G. ie., $\overline{G}(V, \overline{E})$ where $\overline{E} = V \times V - E$

Example 8.3



A graph and its Complement

Note 8.1

(i) If G is simple, then either G or $\bar{G}$ is connected

(ii) A simple graph G is self-complementary if G and $\bar{G}$ are isomorphic

(iii) Let G_1 and G_2 be simple graphs. Then G_1 and G_2 are isomorphic if and only if $\bar{G}_1$ and $\bar{G}_2$ are isomorphic.

Problem 8.1 Show that a graph G is self complementary if it has $4n$ or $4n + 1$ vertices (n is a non negative integer)

Solution: Let $G(V, E)$ be a self complementary graph with m vertices
Since G is self complementary, G is isomorphic to G^c

$$\therefore \quad |E(G)| = |E(G^c)|$$

$$|E(G)| + |E(G^c)| = \frac{m(m-1)}{2}$$

$$\therefore \quad 2|E(G)| = \frac{m(m-1)}{2}$$

$$|E(G^c)| = \frac{m(m-1)}{4}$$

$\frac{m(m-1)}{4}$ is an integer and one of m or $(m - 1)$ is odd
$\therefore$ m or $(m - 1)$ is a multiple of 4
Hence n is of the form $4n$ or $4n + 1$

Note: From the above problem, A graph G with n vertices is isomorphic to its complement, where n or $n - 1$ is a multiple of 4 and number edges in $G^c = \frac{n(n-1)}{4}$.

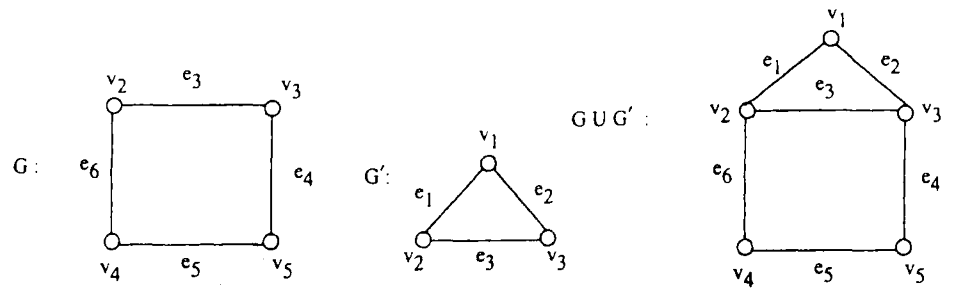
Problem 8.2 Can a graph with seven vertices be isomorphic to its complement

Solution: Hence $n = 7, n - 1 = 7 - 1 = 6$
Neither 7 or 6 is a multiple of 4
$\therefore$ A graph with $|V| = 7$ can't be isomorphic to its complement

Definition 8.4 **(Union of the two graphs)** Let G and G' be two graphs the union of G and G' is the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$.

$$\text{Hence} \quad G \cup G' = (V \cup V',\ E \cup E')$$

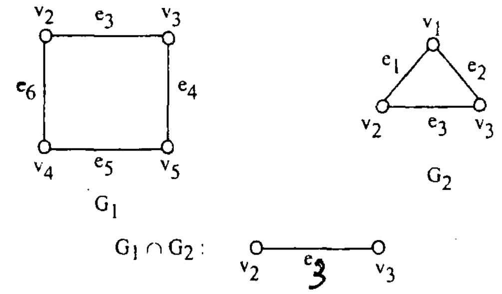
Example 8.4



Definition 8.5 (Intersection of two graphs) The intersection of G_1 and G_2 is the graph consisting only those vertices and edges that are both in G_1 and G_2

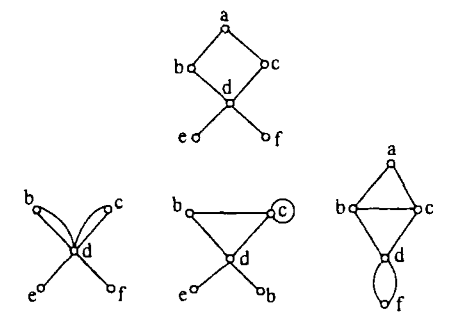
$$\Longrightarrow G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$$

Example 8.5



Definition 8.6 [Fusion] Fusion of two vertices a and b in a graph G is an operation G on which two vertices a and b are fused (merged) together without deletion of any edge of G.

Example 8.6



Fusion of the vertices a and b | Fusion of the vertices a and c | Fusion of the vertices e and f

Note 8.1 Let v be the vertex obtained by the fusion of two vertices a and b in G and G' be the graph obtained after the fusion. Then $\deg_g(v) = \deg_{G'}(a) + \deg_G(b)$ and $\deg_{G'}(x) = \deg_G(x)$ for all vertices of G which are not a or b.

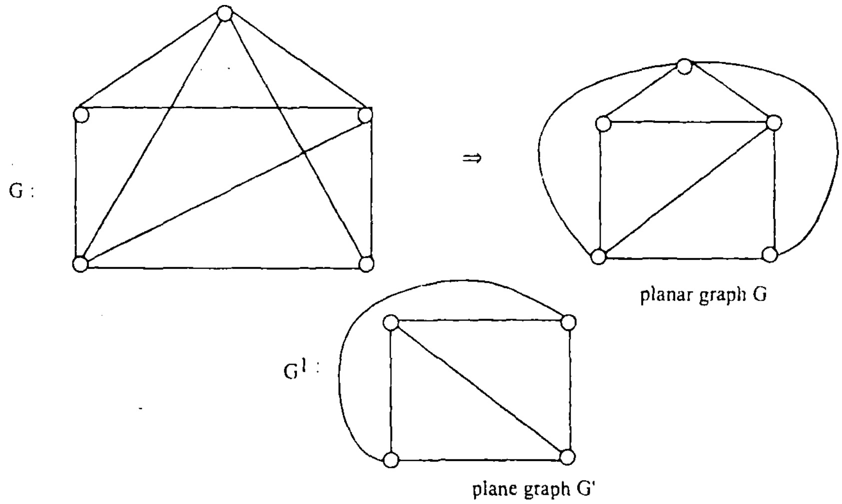
Note 8.2 Fusion of two adjacent vertices always produce a loop at the point of fusion and the number of loops is equal to the number of edges between the vertices which are fused together.

9. Planarity

Definition 9.1 A graph G is said to be planar if it can be drawn in the plane without its edges crossing. Otherwise G is nonplanar.

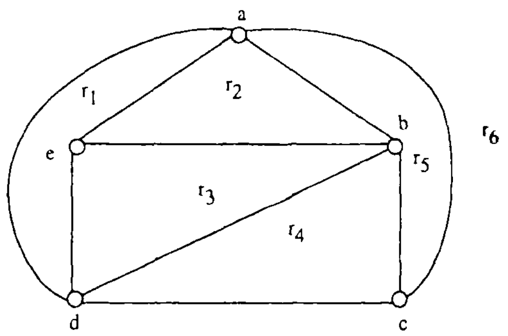
Remark 9.2 A graph may be planar even if it is usually drawn with edge crossings, since it may be possible to draw it in a different way without any edge crossings. We say that a planar graph is a plane graph if it is already drawn in the plane without edge crossings.

Example 9.1



A plane graph divides the plane into regions. A *region* is characterized by the cycle that forms its boundary. These regions are connected portions of the plane.

Example 9.2



In each plane, plane graph G determines a region of infinite area called the exterior region of G. In the above example r_6 is the exterior region. The vertices and edges of G incident with a region r make up the boundary of the region r.

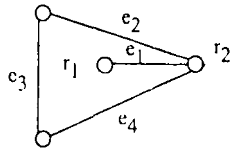
Example: a-e-d-a is the boundary of the region r_1. The degree of the region 'r' is the length of its boundary, denoted by deg (r).

Example:

$$\deg(r_4) = 3$$

$$\deg(r_6) = 3 \quad \text{etc.,}$$

Example 9.3



deg $(r_2) = 3$

deg $(r_1) = 5$, Since the boundary of r_1 includes edge e_1 twice.

Definition : (Dual of a graph) 9.2 Given a plane graph G, we can define another multigraph G^* as follows: Corresponding to each region r of G there is a Vertex r^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices r^* and s^* are joined by the edge e^* in G^* if and only if their corresponding regions r and s are seperated by the edge e in G. In particular, a loop is added at a vertex r^* of G^* for each *cut-edge* of G that belongs to the boundary of the region r. The multigraph G^* is called the *dual* of G.

Observations on (geometric) dual of a graph

1. An edge forming a self-loop in G yields a pendant edge in G^*
2. A pendant edge in G yields a self loop in G^*
3. Edges that are in series in G produce parallel edges in G^*
4. Edges that are parallel in G produce series edges in G^*
5. $|E^*| = |E|$
6. $|R^*| = |V|$
7. The dual of the dual graph is the graph itself. Thus G and G^* are duals to each other
8. For a given planar group G, the graph G^* need not be unique.
9. G^* is planar
10. The nullity of $G(G^*)$ = The rank of $G^*(G)$
11. $\deg_{G^*}(r^*) = \deg_G(r)$ for each region v of G where $r^* \in v^*(G^*)$ and $r \in R(G)$.
12. A necessary and sufficient condition for two planar graphs G_1 and G_2 to be duals of each other is as follows:
    There is a one to one correspondence between the edges in G_1 and the edges in G_2 such that a set of edges in G_1 forms a cycle $\Longleftrightarrow$ the corresponding set in G_2 forms a cut set.

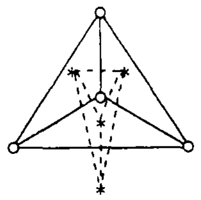
Theorem: A graph G has a dual $\Longleftrightarrow$ it is planar

Proof: We have to show that no non planar graph has a dual.

Let G be a non planar graph. Then G contains a subgraph homeomorphic to k_5 or $k_{3,3}$. But k_5 and $k_{3,3}$ have no dual. Moreover, series edges of a graph yields only parallel edges in the dual graph, and hence the parallel edges in the homeomorphic image of k_5 or $k_{3,3}$ does not support to write the dual. Hence G has no dual. □

Definition: A planar graph isomorphic to its own dual is called a self dual graph.

Examples: k_4 is a self dual graph



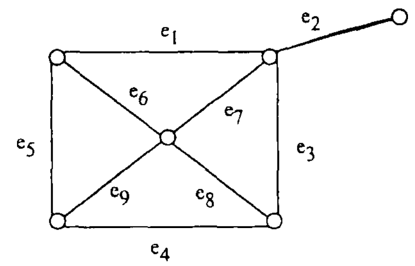
Cut-edge (Definition) 9.3 If a graph G is connected and e is an edge such that G-e is not connected, then e is said to be a *bridge* or a cut edge.

Similarly

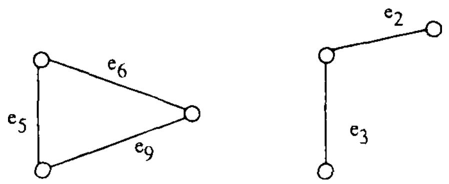
Cut-vertex 9.4 If v is a vertex of G such that G-v is not connected, then v is a cut vertex.

Cut-set 9.5 In a connected graph G, a *cut-set* is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G. A cut-set always 'cuts' a graph into two.

Example 9.4



Now remove the edge e_1, e_7, e_8, e_4 then G is



clearly $\{e_1, e_7, e_8, e_4\}$ is a cut-set.

Example (dual graph) 9.5 *Drawing* G^*:

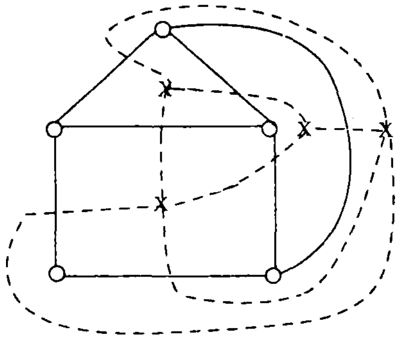
(i) Place r^* in the corresponding region r of G

(ii) Draw each e^* in such away that it crosses the corresponding edge e of G exactly once and crosses no other edge of G.

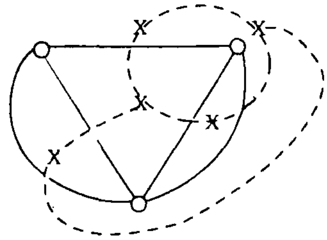
(iii) Dual edges are indicated by dashed lines and the dual vertices by asterisks

(iv) If e is a loop of G, the e^* is a cut edge of G^* and conversely

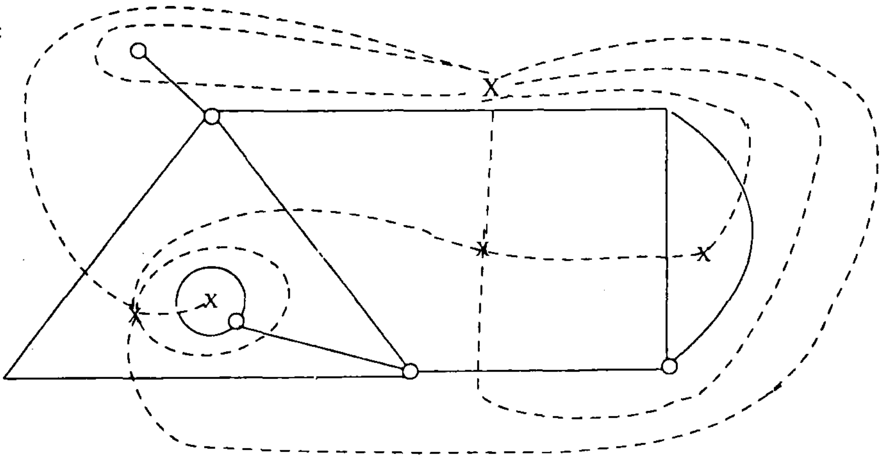
EX1 :



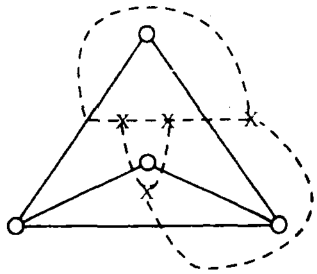
EX2 :



EX4:



EX3



Observations 9.1 Let $|E^*|$, $|R^*|$, $|V^*|$ and $|E|$, $|R|$, $|V|$ denote the number of edges, regions and vertices of G^* and G respectively. Then the following relations are direct consequences of the definition of G^*. For all plane graphs G,

(i) $|E^*| = |E|$

(ii) $|V^*| = |R|$ and

(iii) $\deg_{G^*}(r^*) = \deg_G(r)$ for each region r of G.

We mean that the degree of the vertex r^* in G^* is the same as the degree of the corresponding region r determined by G.

Moreover, if G is connected

(iv) $|R^*| = |V|$

Theorem 9.1 If G is a plane graph, then the sum of degrees of the regions determined by G is $2|E|$, where $|E|$ is the number of degree of G.

Proof:

$$\sum_{r \in R(G)} \deg(r) = \text{Sum of the degrees of the all the regions determined by G.}$$

$$\sum_{r^* \in V(G^*)} \deg(r^*) = \text{Sum of the degrees of the vertices of } G^*.$$

where G^* is the dual of G.

Then

$$\sum_{r \in R(G)} \deg(r) = \sum_{r^* \in V(G^*)} \deg(r^*) \text{ by observations (iii) and (iv)}$$

$$= 2|E^*| \left(\text{by } \sum_{\text{all} v} \deg(v) = 2|E|\right)$$

$$= 2|E| \left(\text{by}(i)|E| = |E^*|\right)$$

∴ Hence the theorem. □

10. Euler's Formula

If G is a connected planar graph, then any drawing of G in the plane as a plane graph will always form $|R| = |E| - |V| + 2$ regions, including the exterior region, where $|R|$, $|E|$ and $|V|$ denote respectively, the number of regions edges and vertices of G.

Theorem (Euler's Formula) 10.1 If G is a connected plane graph then $|V| - |E| + |R| = 2$

Proof: We prove this result by induction on the number of regions 'k' determined by G.

It is obvious when $k = 1$.

Assume *the result for* $k > 1$ and suppose that G is a connected plane graph having $(k + 1)$ regions.

Delete an edge common to both the regions. The resulting graph G' has the same number of vertices, one fewer edge, but also one fewer region since two previous regions have been combined by the removal of the edge.

$$\therefore |E'| = |E| - 1; |R'| = |R| - 1; |V'| = |V|$$

where $|E'|$, $|R'|$ and $|V'|$ are number of edges, regions, vertices of G'

Then

$$\begin{aligned} |V'| - |E'| + |R'| &= |V| - |E| + 1 + |R| - 1 \\ &= |V| - |E| + |R| \\ &= 2 \quad (\text{ By induction hypothesis assumed on G}) \end{aligned}$$

Hence the theorem is proved by mathematical induction □

Note 10.1 (i) The above theorem allows the graph to have loops

(ii) However, we assume throughout this section that the graph is simple and $|E| > 1$; thus we are assuming that the degree of each region is greater than or equal to 3.

(iii) If a plane graph has K components, then $n - e + r = k + 1$.

Definition 10.1 A connected plane graph is *polyhedral* if $\deg(r) \geq 3$ for each region $r \in R(G)$; and if, in addition $\deg(v) \geq 3$ for each vertex $v \in V(G)$.

Result 10.1 In a plane graph G, if the degree of each region is $\geq k$ then $k|R| \leq 2|E|$. In particular, we have $3|R| \leq 2|E|$.

Proof: We have

$$\sum_{r \in R(G)} \deg(r) = 2|E|$$

Since degree $(r) \geq k$ for each region $r \in R(G)$

$$\Rightarrow \quad 2|E| \geq k|R|.$$

□

Corollary 10.1 In a connected plane simple graph G, with $|E| > 1$

(i) $|E| \leq 3|V| - 6$ and

(ii) there is a vertex v of G such that degree $(v) \leq 5$.

Proof: (i) By Euler's formula $|R|+|V| = |E|+2$ and since G is simple, it has no loops and no parallel edges. So boundary of each region contains at least three edges. That is, the degree of each region ≥ 3.

$$\therefore \quad 3|R| \leq 2|E| \quad \text{or} \quad |R| \leq \frac{2}{3}|E|.$$

Hence $$\frac{2}{3}|E| + |V| \geq |R| + |V| = |E| + 2$$

Thus $$|V| - 2 \geq \frac{1}{3}|E|$$

or $$3|V| - 6 \geq |E|.$$

$$\Rightarrow \quad |E| \leq 3|V| - 6$$

(ii) Suppose if each vertex has degree ≥ 6, then $6|V| \leq 2|E|$ or $|V| \leq \frac{|E|}{3}$,

Since $$\sum_{v \in V(G)} \deg(v) = 2|E|$$

Similarly $$|R| \leq \frac{2}{3}|E|$$

Now, from $$|R| + |V| = |E| + 2$$

We have $$\frac{2}{3}|E| + \frac{1}{3}|E| \geq |R| + |V| = |E| + 2$$

$$\Rightarrow \quad |E| \geq |E| + 2$$

$$\Rightarrow \quad 0 \geq 2$$

an obvious contradiction

$\therefore$ our supposition is wrong

$\therefore$ there is a vertex v of G s.t degree $(v) \leq 5$.

□

Theorem 10.2 If G is a connected plane graph with $|V| \geq 3$ and $|R|$ regions, show that $|R| \leq 2|V| - 4$

Proof: Using $|E| \leq 3|V| - 6$

Substituting in $|E| = |V| + |R| - 2$

$\Rightarrow \quad |V| + |R| - 2 \leq 3|V| - 6$

$\Rightarrow \quad |R| \leq 2|V| - 4$

Hence the theorem. □

Theorem 10.3 Prove that if G is a planar graph with k connected components, each component having atleast three vertices then $|E| \leq 3|V| - 6k$

Proof: Let $G_1, G_2, \ldots G_k$ be the connected components of G. Since G_i has atleast three vertices then

$|E_{G_i}| \leq 3|V_{G_i}| - 6$ for each i

$\Rightarrow \sum |E_{G_i}| \leq 3 \sum |V_{G_i}| - 6k$

So $|E| \leq 3|V| - 6k$

Hence the theorem □

Theorem 10.4 If a plane graph has k components then $|V| - |E| + |R| = k + 1$

Proof: For each component i (1 to k)

$$|V_i| - |E_i| + |R_i| = 2$$

taking summation

$$\sum |V_i| - \sum |E_i| + \sum |R_i| = 2k$$

Since

$$\sum |V_i| = |V|$$
$$\sum |E_i| = |E|$$

but $\sum |R_i| = |R| = |R| + (k-1)$ since the exterior region is common to all components

Thus

$$|V| - |E| + |R| + k - 1 = 2k$$
$$\Rightarrow \quad |V| - |E| + |R| = k + 1$$

Hence the theorem □

Theorem 10.5 A complete graph K_n is planar if and only if $n \leq 4$.

Proof: It is easy to see that K_n is planar for $n = 1, 2, 3, 4$.

Now, we have to show that when $n \leq 4$, K_n is planar.

For this, it is sufficient to show K_n is non-planar when $n \geq 5$. in other words, we prove this by an indirect argument.

Now, Assume that K_5 is planar.

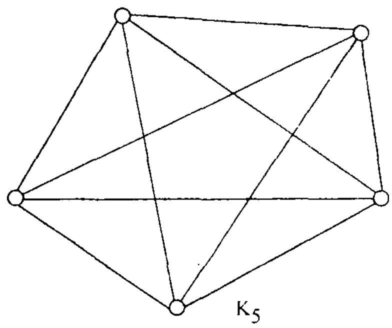
then $|R| = |E| - |V| + 2$
$= 10 - 5 + 2$
$= 7$

$\left[\text{the number of edges of } K_n = \dfrac{n(n-1)}{2} = 10 \right]$

Since K_n is simple and loop free, we have

$$3|R| \le 2|E|$$
$$\Rightarrow \quad 3.7 \le 2.10$$
$$21 \le 20$$

an obvious contradiction.



$\therefore$ Our assumption that K_5 is planar is wrong

$\therefore$ Hence the theorem. □

Note 10.2 We can also get the contradiction also by using $|E| \le 3|V| - 6$

Theorem 10.6 A complete graph $K_{m,n}$ is planar $\Longleftrightarrow$ $m \le 2$ or $n \le 2$.

Proof: It is clear that $K_{m,n}$ is planar if $m \le 2$ or $n \le 2$.

Now let $m \ge 3$ and $n \ge 3$. To prove that $K_{m,n}$ is nonplanar it sufficient to prove that $K_{3,3}$ is nonplanar.

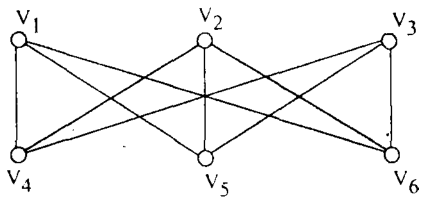
Since $K_{3,3}$ has six vertices and nine edges, if $K_{3,3}$ is planar, By Euler's formula,

$$|R| = |E| - |V| + 2$$
$$= 9 - 6 + 2$$
$$= 5$$

Since $K_{3,3}$ is bipartite there can be no cycles of odd length.

Hence each cycle has length ≥ 4 and thus the degree of each region would have to be greater than or equal to 4. then we have $4|R| \le 2|E|$

K3,3 :



or $\quad 20 = 4.5 \le 2.9 = 18$

a contradiction.

$\therefore$ our assumption that $K_{3,3}$ is planar is wrong

$\therefore$ when $m \geq 3$ and $n \geq 3$, $K_{m,n}$ is nonplanar

So, A complete bipartite graph $K_{m,n}$ is planar $\Longleftrightarrow$ $m \leq 2$ or $n \leq 2$. □

Problem 10.1 Prove that there does not exist a polyhedral graph with exactly seven edges

Solution: Assume that there is such a polyhedral graph with $|E| = 7$.

then $$3|R| \leq 2|E| = 14 \qquad (\because \deg(r) \geq 3)$$

also, each vertex has degree ≥ 3

$$\Rightarrow \quad 3|V| \leq 2|E| = 14$$

$$\Rightarrow \quad |R| \leq 4 \text{ and } |V| \leq 4$$

By Euler's formula: $$|R| + |V| = |E| + 2 = 9$$

and then $$8 \geq |R| + |V| = |E| + 2 = 9$$

an obvious contradiction

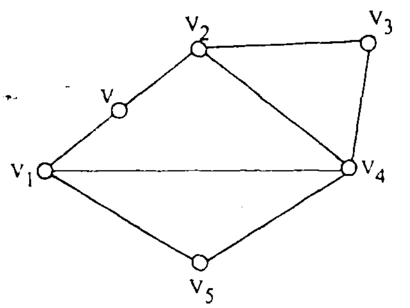
$\therefore$ A polyhedral graph with 7 edges does not exist.

Note 10.3 If a connected planar simple graph is triangle free then $|E| \leq 2|V| - 4$.

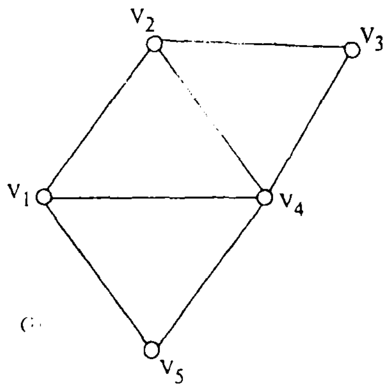
Definition 10.2 If a graph G has a vertex v of degree 2 and edges (v, v_1) and (v, v_2) with $v_1 \neq v_2$, we say that the edges (v, v_1) and (v, v_2) are in series.

A series reduction consists of deleting the vertex v from the graph G and replacing the edges (v, v_1) and (v, v_2) by the edge (v_1, v_2). The resulting graph G' is said to be obtained G by a series reduction. By convention G is said to be obtainable from itself by a series reduction.

Example 10.1



G



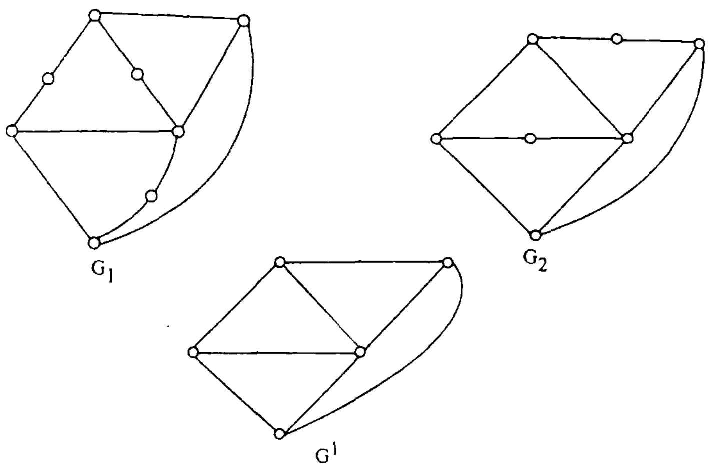
G'

In the graph G, the edges (v, v_1) and (v, v_2) are in series. The graph G' is obtained rom G by a series reduction.

)efinition 10.3 Two Graphs G_1 and G_2 are *homeomorphic* if G_1 and G_2 can be educed to isomorphic graphs by performing a sequence of series reductions.

By the above two definitions, any graph is homeomorphic to itself. Also, graphs G_1 nd G_2 are homeomorphic if G_1 can be reduced to a graph isomorphic to G_2 or if G_2 an be reduced to a graph isomorphic to G_1.

ːxample 10.2



The graphs G_1 and G_2 are homeomorphic, since they can both be reduced to the graph ;$'$ by a sequence of series reductions.

lote 10.4 If we define a relation R on a set of graphs by the rule $G_1\ R\ G_2$ if G_1 and ;$_2$ are homeomorphic, R is an equivalence Relation. Each equivalence class consists of set of mutually homeomorphic graphs.

Now, we state a necessary and sufficient condition for a graph to be planar.

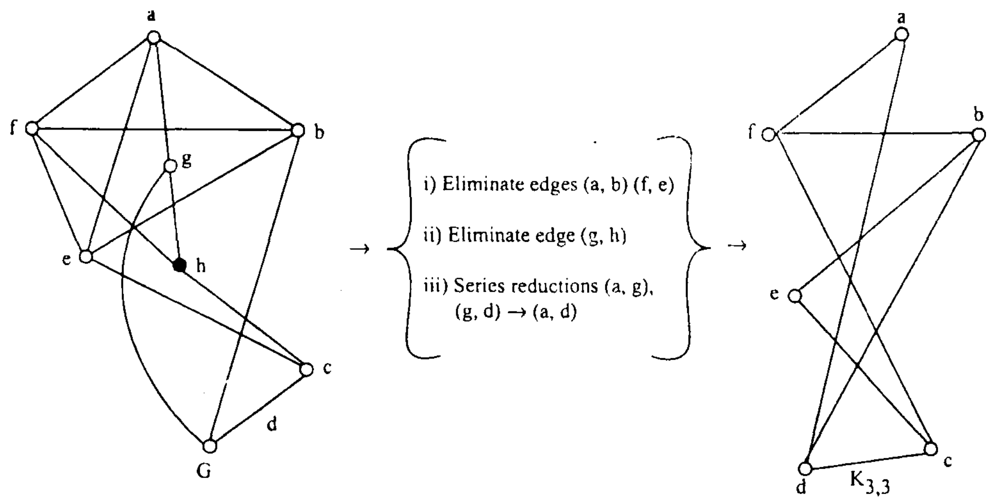
'heorem 10.4 A graph G is planar if and only if G does not contain a subgraph omeomorphic to K_5 or $K_{3,3}$.

(OR) A graph G is non-planar if and only if it contains a subgraph homeomorphic to $_{3,3}$ *or* K_5.

xample 10.3

Show that the graph G is not planar by using kuratowski's theorem.

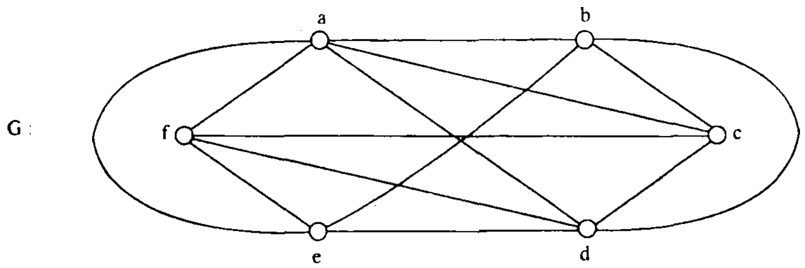
Note that, the vertices a, b, f and e each have degree 4. In $K_{3,3}$ each vertex has degree so let us eliminate the edges (a,b) and (f,e) so that all vertices have degree 3. We note at if we eliminate one more edge, we will obtain two vertices of degree 2 and we can en carry out two series reductions. The resulting graph will have nine edges and since



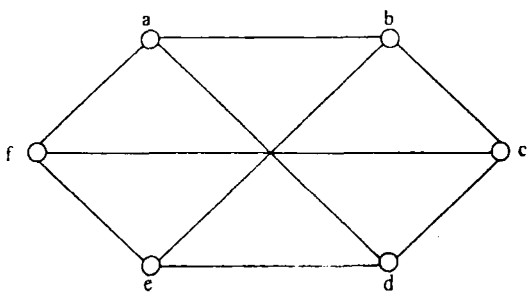
Elimination of an edge to obtain a subgraph, followed by its series reductions

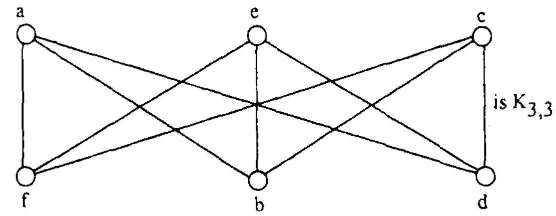
$K_{3,3}$ has nine edges, we finally see that if we eliminate edge (g,h) and carry out the series reductions, we obtain an isomorphic copy of $K_{3,3}$. Therefore, the graph G is not planar, since it Contains a subgraph homeomorphic to $K_{3,3}$.

Problem 10.2 Show the following graph is not planar by finding a subgraph homeographic to either K_5 or $K_{3,3}$



Solution: Delete the edges: $\{a, e\}$, $\{b, d\}$, $\{a, c\}$, $\{d, f\}$ from G



$\Rightarrow$ 

$\therefore$ The given graph is not planar.

Definition 10.4 A pair of vertices in a digraph are weakly connected if there is a nondirected path between them. They are unilaterally connected if there is a directed path between them. They are strongly connected if there is a directed path from u to v and a directed from v to u.

A graph is (weakly, unilaterally, strongly) connected if every pair of vertices in the graph is (weakly unilaterally, or strongly) connected.

A subgraph G' of a graph A is a (weakly, unilaterally or strongly) *connected component* if it is a maximal (weakly, unilaterally or strongly) connected subgraph that is, there is no (weakly, unilaterally or strongly) connected subgraph of G that properly contains G'.

Definition 10.5 Two vertices a and b of a graph are said to be connected $\iff$ there is a nondirected path from a to b in G, and then the graph G is connected $\iff$ each pair of its vertices is connected.

In general, if we define the relation R on the vertices of a graph G by aRb $\iff$ a and b are connected, then R is an equivalence relation. Consequently the vertices of G can be partitioned into disjoint nonempty sets $V_1, V_2, \ldots, V_K$ and the subgraphs $H_1, H_2, \ldots, H_K$ of G induced by $V_1, V_2, \ldots,$ and V_K respectively, are called the *Connected components* of G or simply the components of G. Usually we denote the number of components of G by C(G) and $C(G) = 1 \iff G$ is connected.

Equivalently, a component of a graph G is a connected subgraph of G not properly contained in any other connected subgraph of G. That is, a component of G is a subgraph of G that is maximal w.r. to the property of being connected. In otherwords, a connected subgraph H of a graph G is a component of G if for each connected subgraph F of G where $H \subseteq F \subset G$, $V(H) \subseteq V(F)$ and $E(H) \subseteq E(F)$, then it follows that $H = F$.

Theorem 10.5 A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof: Let G be a simple graph of order n. Let $G_1, G_2 \ldots G_k$ be the components of G. Let the number of vertices in i^{th} component be n_i

$$|V(G_i)| = |V_i| = n_i, 1 \leq i \leq k$$

Then $\sum_{i=1}^{k} n_i = n_1 + n_2 + \ldots + n_k = n = |V|$ and $n_i > 1$.

Now maximum possible edges in i^{th} components cannot exceed $n_{i_{C_2}} = \dfrac{n_i(n_i - 1)}{2}$

$$\Rightarrow \quad \max|E(G_i)| = \max|E_i| \leq \frac{n_i(n_i - 1)}{2} \quad 1 \leq i \leq k$$

Hence

$$\begin{aligned}
|E(G)| &\leq \sum_{i=1}^{k} \max|E_i| \\
&= \sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2} \\
&= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right] \\
&= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right]
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{i=1}^{k}(n_i - 1) &= (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) \\
&= (n_1 + n_2 + \ldots + n_k) - (1 + 1 + \ldots + k \text{ times}) \\
&= n - k
\end{aligned}$$

Squaring on both sides

$$\left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = (n - k)^2 = n^2 + k^2 - 2nk \tag{4.1}$$

$$\Rightarrow \quad \sum_{i=1}^{k}(n_i - 1)^2 + 2 \text{ (non negative terms)} = n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k}(n_i - 1)^2 = n^2 + k^2 - 2nk - 2 \text{ (non negative terms)}$$

$$\sum_{i=1}^{k}(n_i - 1)^2 \leq n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \leq n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 + k - 2n \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \quad \sum_{i=1}^{k} n_i^2 - n \leq n^2 - nk + n - nk + k^2 - k$$

$$= n(n-k+1) - k(n-k+1)$$

$$= (n-k)(n-k+1)$$

$$\sum_{i=1}^{k} n_i^2 - n \leq (n-k)(n-k+1) \tag{4.2}$$

From (1) and (2), we get

$$|E(G)| \leq \frac{1}{2}(n-k)(n-k+1)$$

Hence proved. □

Theorem 10.6 If G is not connected then $\bar{G}(G^c)$ is connected

Proof: Let G be disconnected graph. Then G has more than one component

Let u, v be any two vertices of G. The theorem is proved if we show that there is a $u - v$ path in $\bar{G}$. If u, v are in different components of G, then u, v are not adjacent in G.

Hence they are adjacent in $\bar{G}$

If u, v are in the same components of G, choose a vertex w in a different component of G.

Then $u - w - v$ is a $u - v$ path in $\bar{G}$

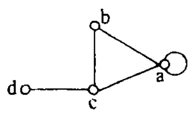
Hence $\bar{G}$ is connected. □

Distance and centers

Definition 10.6 The distance between two vertices u and v of a graph is the length of the shortest path (path of minimum length) between them and is denoted by $d_G(u, v)$.

If the vertices u and v lie in different components of G then we define $d_G(u, v) = \infty$.

Example 10.4



$$d(a,a) = d(b,b) = d(c,c) = 0$$
$$d(a,b) = d(b,a) = 1$$
$$d(a,c) = d(c,a) = 1$$
$$d(a,d) = d(d,a) = 2$$
$$d(b,c) = d(c,b) = 1$$
$$d(b,d) = d(d,b) = 2$$
$$d(c,d) = d(d,c) = 1$$

A function $f : A \times A \xrightarrow{\text{into}} R$ is said to be a matrix if it satisfies the following axioms

1. $f(a, b) \geq 0$ (non negativity) and $f(a, b) = 0 \Longleftrightarrow a = b$
2. $f(a, b) = f(b, a)$ (symmetry)
3. $f(a, b) + f(b, c) \geq f(a, c)$ (triangular inequality), clearly the distance $d_G(u, v)$ satisfies the above three axioms, hence distance in a graph is a metric.

Definition 10.7 **[Eccentricity]** The maximum distance from a vertex u to any vertex of G is called eccentricity of the vertex v and is denoted by $e(v)$.

That is, $e(v) = \max\{d(v, u) : u \in V(G)\}$

Definition 10.8 **[Radius]** The radius of a graph G is the minimum of eccentricity of all its vertices and is denoted by r.

That is $r = \min\{e(v) : v \in V(G)\}$

Definition 10.9 **[diameter]** The diameter of a graph G is the maximum of eccentricity of all its vertices and is denoted by d.

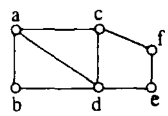
That is $d = \max\{e(v) : v \in V(G)\}$

Definition 10.10 **[central vertex]** Among all the vertices of a graph, the one which is having minimum eccentricity is called a central vertex of G. (OR) A vertex of a graph having eccentricity equal to the radius of the graph is called central vertex.

Definition 11.11 **[center]** The set of all central vertices of a graph is called the center of the graph.

Center $C = \{v \in V(G) | e(v) = r, \text{ the radius of } G\}$

Example 10.5



$d(a, a) = 0$	$d(b, a) = 1$	$d(c, a) = 1$	$d(d, a) = 1$
$d(a, b) = 1$	$d(b, b) = 0$	$d(c, b) = 2$	$d(d, b) = 1$
$d(a, c) = 1$	$d(b, c) = 2$	$d(c, c) = 0$	$d(d, c) = 1$
$d(a, d) = 1$	$d(b, d) = 1$	$d(c, d) = 1$	$d(d, d) = 0$
$d(a, e) = 1$	$d(b, e) = 2$	$d(c, e) = 2$	$d(d, e) = 1$
$d(a, f) = 2$	$d(b, f) = 3$	$d(c, f) = 1$	$d(d, f) = 2$

$d(e, a) = 1$	$d(f, a) = 2$
$d(e, b) = 2$	$d(f, b) = 3$
$d(e, c) = 2$	$d(f, c) = 1$
$d(e, d) = 1$	$d(f, d) = 2$
$d(e, e) = 0$	$d(f, e) = 1$
$d(e, f) = 1$	$d(f, f) = 0$

$$\begin{aligned} e(a) &= \max\{d(a,u)/u \in V(G)\} \\ &= \max\{d(a,a), d(a,b), d(a,c), d(a,d), d(a,e)d(a,f)\} \\ &= \max\{0,1,1,1,1,2\} \\ &= 2 \\ e(b) &= \max\{1,0,2,1,2,3\} = 3 \\ e(c) &= \max\{1,2,0,1,2,1\} = 2 \\ e(d) &= \max\{1,1,1,0,1,2\} = 2 \\ e(e) &= \max\{1,2,2,1,0,1\} = 2 \\ e(f) &= \max\{2,3,1,2,1,0\} = 3 \end{aligned}$$

r = radius of $G = \min\{e(a), e(b), e(c), e(d), e(e), e(d)\}$

$= \min\{2,3,2,2,2,3\} = 2$

d = diameter of $G = \max\{2,3,2,2,2,3\} = 3$ (diameter need not be twice the radius in a graph)

Central vertices are: a, c, d and e

Therefore center of $G = \{v \in V(G) | r = e(v)\} = \{a, c, d, e\}$.

Theorem 10.7 If r is the radius and d is the diameter of a connected graph G then $r \leq d \leq 2r$

Proof: From definition of 'r' and 'd' we have

$$r \leq d \tag{4.1}$$

Let u, v be the ends of a diametral path and w be the central vertex then

$$D = d(u,v) \leq d(u,v) + d(w,v) \leq r \tag{4.2}$$

$$\Rightarrow \quad d \leq 2r$$

$\therefore \quad r \leq d \leq 2r$ from (1) and (2) □

11. Eulerian Graphs:

Definition 11.1 An *Euler path* in a multigraph is a path that includes each edge of the multigraph exactly once and includes or intersects each vertex of the multigraph at least once.

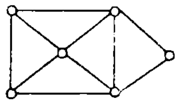
A multigraph is said to be traversable if it has an Euler path.

Definition 11.2 An *Euler circuit* is an Euler path whose end points are identical. That is if an Euler path is a sequence of edges $e_1, e_2, \ldots e_K$ corresponding to the sequence of pairs of vertices $(v_1, v_2), (v_2, v_3) \ldots (v_{K-1}, v_K)$, then e_i's are all distinct and $v_1 = v_K$.

Definition 11.3 A multigraph is said to be an *Eulerian multigraph* if it has an Euler circuit.

Definition 11.4 **[Semi-Eulerian graph]** A non Eulerian graph G is semi-eulerian if there exists a path of distinct edges (a trail) containing every edge of G

Example 11.1



Result 11.1 A connected graph is semi-eulerian if and only if it has exactly two vertices of odd degree.

Proof: In a Semi Eulerian graph, any semi-Eulerian trail have one vertex of odd degree as its initial vertex and the other as its final vertex.

Note also that, by the fundamental theory of graph theory a graph cannot have exactly one vertex of odd degree. □

Hence the result.

Theorem 11.1 A nondirected multigraph has an Euler path if and only if it is connected and has 0 or exactly two vertices of odd degree.

In the Latter case, the two vertices of odd degree are the end points of every Euler path in the multigraph

Proof: (Necessary). Assuming G has an Euler Path. It is clear that G must be connected. Moreover, every time the Euler Path meets vertex it traverses two edges which are incident on the vertex and which have not been traced before. Except for the two end points coincide, their degrees are even and the path becomes an Euler circuit.

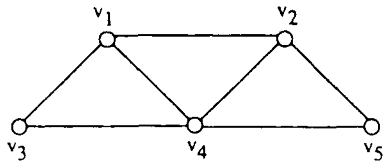
(Sufficient) Let us construct an Eulerpath by starting at one of the vertices of odd degree and traversing each edge of G exactly once. If there are no vertices of odd degree we will start at an arbitary vertex. For every vertex of even degree the path will enter the vertex and leave the vertex by tracing an edge that was not traced before. Thus the construction will terminate at a vertex with an odd degree, or return to the vertex where it started. This tracing will produce an Euler path if all edges in G are traced exactly once this way.

If not all edges in G are traced, we will remove those edges that have been traced and obtain the subgraph G' induced by the remaining edges. The degrees of all vertices in this subgraph must be even and atleast one vertex must intersect with the path, since G is connected. Starting from one of these vertices, we can now construct a new path which in this case will be a cycle, since all degrees are now even. This path will be joined into the previous path. The argument can be repeated until a path that traverses all edges in G is obtained. □

Corollary 11.1 A nondirected multigraph has an Euler circuit if and only if it is connected and all of its vertices are of even degree.

Example 11.2

(i)



$\deg(v_1) = \deg(v_2) = 3$
$\deg(v_3) = \deg(v_5) = 2$
$\deg(v_4) = 4$

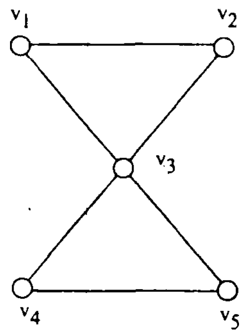
Clearly the given graph has two vertices of odd degree v_1 and v_2 the remaining vertices all are of even degree, Also G is connected.

Now, $v_1 - v_3 - v_4 - v_5 - v_2 - v_4 - v_1 - v_2$ is Euler path, since it is started with v_1 and ended with v_2, the two vertices are of degree 3.

By the above corollary, Euler circuit is not there in the graph, since all the vertices of the graph are not of even degree.

$\therefore$ The given graph has Euler path only

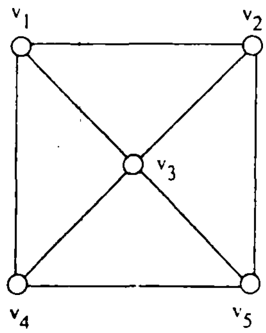
(ii)



$\deg(v_1) = \deg(v_2) = \deg(v_4) = \deg(v_5) = 2$
$\deg(v_3) = 4$

Clearly G is connected and all the vertices are having even degree. Therefore G has Euler Circuit: $v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1$

(iii)



$\deg(v_1) = \deg(v_2) = \deg(v_4) = \deg(v_5) = 3$

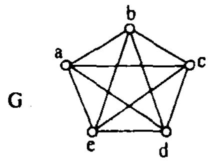
$\deg(v_3) = 4$

By the theorem on Eulerpath, corollary on Euler circuit, G has no Eulerpath no Euler circuit.

(iv) For which values of $n > 1$, if any is K_n Eulerian?

Solution: The complete graph k_n is Eulerian if and only if n is odd

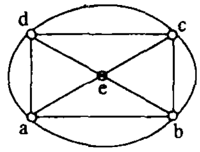
(v) Show that the graph G is Eulerian



Solution: Each vertex is of even degree in G

$\therefore$ G is Eulerian with EC: abdeacebda

(vi) Show that the graph G is not Eulerian



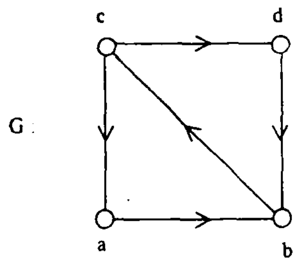
Solution: There are four vertices of degree 5 in the graph a, b, c, d
Therefore G is not Eulerian.

Corollary 11.2 A directed multigraph G has an Euler path if and only if it is unilaterally connected and the indegree of each vertex is equal to its out degree with the possible exception of two end vertices, for which it may be that the indegree of one is one larger than its out degree and the indegree of the other is oneless than its out degree.

Corollary 11.3 A directed multigraph G has an Euler circuit if and only if G unilaterally connected and the indegree of every vertex in G is equal to its out-degree

Example 11.2

(i)



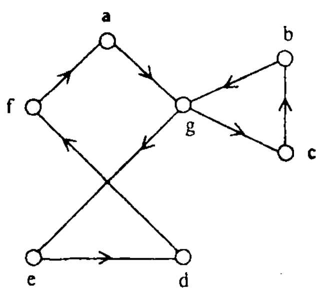
V(G)	indegree	outdegree
a	1	1
b	2	1
c	1	2
d	1	1

The graph G is unilaterally connected and clearly the degrees of the vertices are satisfying the corollary on Euler path for directed multi graphs. Therefore it has Euler path: $c - a - b - c - d - b$

As the degrees of the vertices are not satisfying the corollary on Euler circuit therefore the graph has no Euler circuit.

$\therefore$ The given graph G has only Euler path

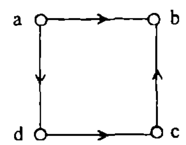
(ii)



V(G)	indegree	outdegree
a	1	1
b	1	1
c	1	1
d	1	1
e	1	1
f	1	1
g	2	2

As the graph G is unilaterally connected and the degrees of all vertices are satisfying the corollary on Euler circuit for dimultigraphs. Therefore given graph has Euller circuit: $a - g - c - b - g - e - d - f - a$.

(iii)

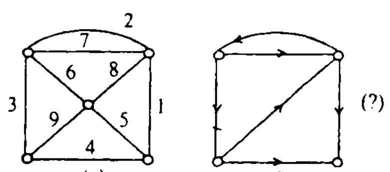


clearly, d is not reachable from b. That is the given graph is not Unilaterally connected.

Therefore the given graph has no Euler path and no Euler circuit.

Example 11.3 The problem of drawing a multigraph on paper with a pencil without lifting the pencil or repeating any lines is clearly a problem of finding an Euler path in the multigraph.

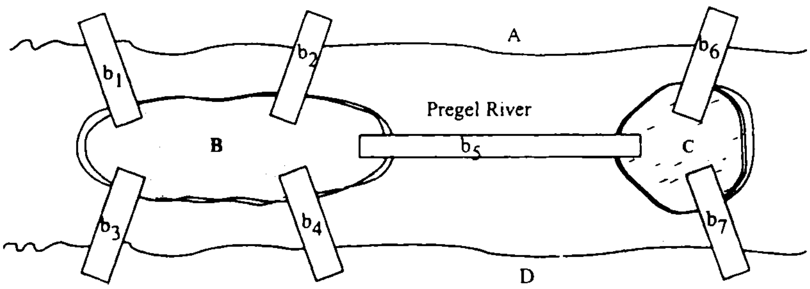
A multigraph can be drawn in the above way $\Longleftrightarrow$ it has an Euler path.



(a) (b)

The multigraph (a) in the above graphs can be drawn in the above said way with each edge being traced exactly once. While the directed multigraph (b) cannot be drawn.

Example 11.4 Can anyone cross all the bridges shown in the following map exactly once and return to the starting point? (Konigsberg bridge problem) where $b_1, b_2, b_3, b_4, b_5, b_6, b_7$ are bridges and B, C and D, A are two islands and two river banks respectively



Solution: The bridge configuration can be modeled as a graph as shown below

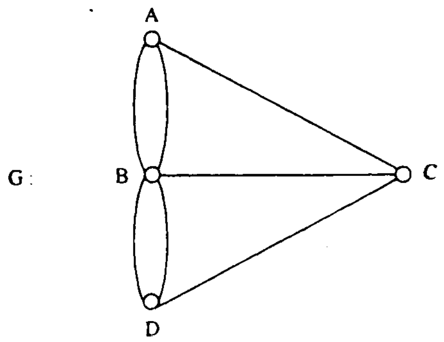


Fig. 4.1 *Graph model of the bridges of Konigsberg*

The vertices represent the locations and the edges represent the bridges Now, The Konigsberg bridge problem is reduced to finding a circuit in the graph that includes all the edges and all the vertices exactly once except the end vertices (Euler circuit). From the corollary of Euler circuit, there is no Euler circuit in the graph G because there are four vertices of odd degree.

$\therefore$ There is no way to start at a given point cross each bridge exactly once and return to the starting point

Note 11.1 (i) A connected multigraph has Euler circuit $\Longleftrightarrow$ each of its vertices has even degree

(ii) A connected multigraph has an Euler path but not an Euler circuit $\Longleftrightarrow$ it has exactly two vertices of odd degree

Theorem 11.4 A connected graph G is Eulerian if and only if it can be decomposed into cycles.

Proof: If G can be decomposed into cycles, then G is a union of edge disjoint cycles. Since the degree of every vertex in a cycle is two, the degree of every vertex in G is even. Hence G is Eulerian. Conversely, Let G be Eulerian. As degree of every vertex is even, minimum degree $\delta(G) \geq 2$ So G contains a cycle C_1. Again degree of every vertex is even in $G - C_1$. If $G - C_1$ has atleast one edge, then it has a connected component in which minimum degree ≥ 2. So $G - C_1$ contains a cycle C_2. If G is not the union of C_1 and C_2. Consider the graph $G - (C_1 \cup C_2)$. As degree of every vertex is even in $G - (C_1 \cup C_2)$ it contains a cycle C_3. Continue this process until no edge is left. At the final step we obtain a decomposition of G into a union of edge disjoint cycles □

12. Hamiltonian Graphs:

Definition 12.1 A cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is Known as *Hamiltonian cycle*.

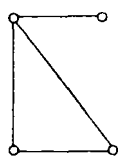
Definition 12.2 A Hamiltonian graph is a graph containing a Hamiltonian cycle.

Definition 12.3 A Hamiltonian path is a simple path that contains all vertices of G but where the end points may be distinct.

Definition 12.4 **[Semi - hamiltonian graph]** A non hamiltonian graph G is semi hamiltonian if there exists a path (of distinct vertices) passing through every vertex

Example:

(i)



(ii) The complete bipartite graph $k_{2,3}$

Note 12.1 A graph is Hamiltonian $\Longleftrightarrow$ its underlying simple graph is Hamiltonian. We consider this discussion to simple graphs.

Observations 12.1 (i) Euler circuit is a circuit that traverses each edge exactly once and, therefore each vertex at least once, a Hamiltonian cycle traverses each vertex exactly once (and hence may miss some edges altogether) Thus, there is a striking Similarity between the concepts of Eulerian graph and Hamiltonian graph. But till now there is no characterization of Hamiltonian graphs as in the case of Eulerian graphs.

(ii) A Hamiltonian cycle always provides a Hamiltonian path, by deleting any edge. On the other hand, a Hamiltonian path may not lead to a Hamiltonian cycle (it depends on whether or not the end points of the path happen to be joined by an edge in the graph)

(iii) The problem of proving that no Hamiltonian cycle (or) path exists in a given graph is very difficult.

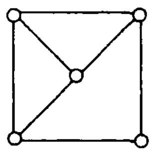
(iv) To prove the non existence of a HP or HC, we must begin building parts of a HP and show that the construction always fail, that is, we cannot visit all vertices without visiting some vertices at least twice.

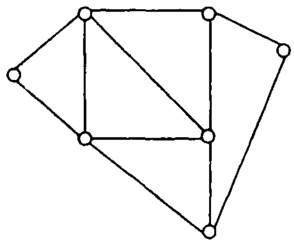
(v) The idea underlying is that a HC must contain exactly two edges incident at each vertex and a HP must contain atleast one of the edges.

Note 12.2 Unlike the situation for Euler cycles no easily verified necessary and sufficient conditions are known for the existence of a HC in a graph.

Example 12.1

The following two graphs are hamiltonian graphs

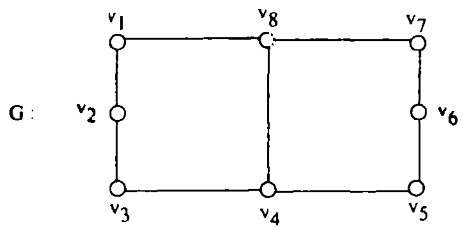




The complete graph $K_n (n \geq 3)$ is hamiltonian

The complete bipartite graph $K_{m,n}$ is hamiltonian if and only if $m = n$ and $n > 1$.

Example 12.2



G is Hamiltonian, HC is $v_1\ v_2\ v_3\ v_4\ v_5\ v_6\ v_7\ v_8\ v_1$

Some Basic Rules for Constructing Hamiltonian Paths and Cycles

Rule1: If G has n vertices, then a Hamiltonian path must contain exactly $n - 1$ edges and a Hamiltonian cycle must contain exactly n edges.

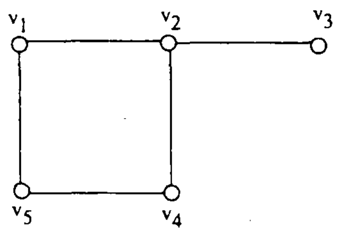
Rule2: If a vertex v in G has degree K, then a Hamiltonian path must contain atleast one edge incident on v and atmost two edges incident on v. A HC contains exactly two edges incident on v. In particular, both edges incident on a vertex of degree two will be contained in every HC. Finally, there cannot be three or mode edges incident with one vertex in a HC.

Rule3: No cycle that does not contain all the vertices of G can be formed when building a HP or HC.

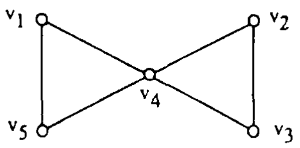
Rule4: Once the HC we are building has passed through a vertex v, then all other unused edges incident on v can be deleted because only two edges incident on v can be included in a HC.

Example 12.3

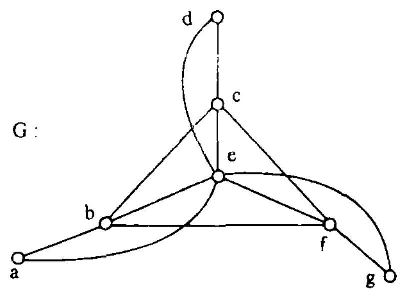
(i) HP: $V_1 - V_5 - V_4 - V_2 - V_3$ No HC since the graph has a vertex V_3 with degree 1.



(ii) Since deg$(V_4) = 4$, it does not have a HC

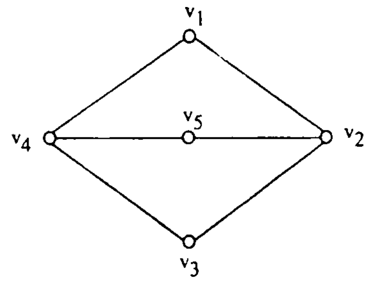


(iii) The path through the vertices of G in order of appearance in the English alphaset forms a HP. However G has no HC since if so, any HC must contain the edges $\{a, b\}$, $\{a, e\}\{c, d\}$, $\{d, e\}$, $\{f, g\}$ and $\{e, g\}$ But there are more than three edges of the cycle incident on the vertex 'e'



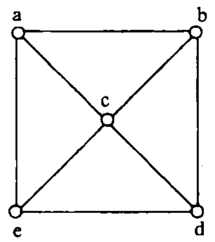
Example 12.4

(i)



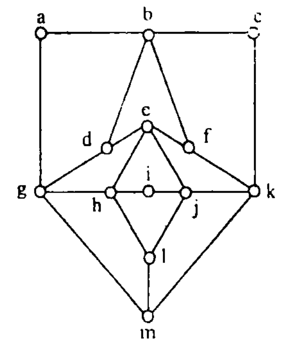
Since there are five vertices, aHc must have five edges. Suppose that we can eliminate edges from the graph leaving just a HC. then, we need to eliminate one edge incident at v_2 and one edge incident at v_4, since each vertex in a HC has degree 2. But this leaves only four edges not enough for a HC of length 5 Therefore the above graph does not contain a HC.

(ii)



A graph with a HC

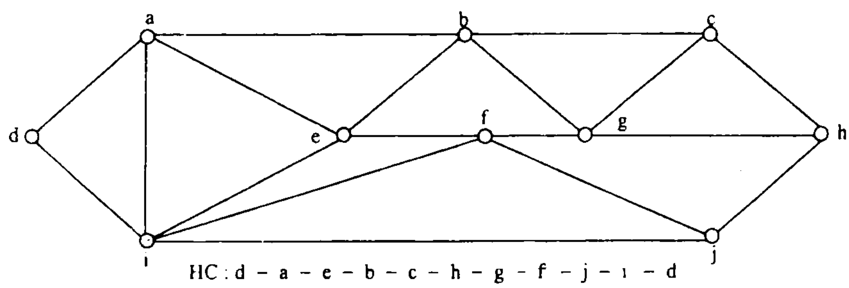
(iii)



For (iii)

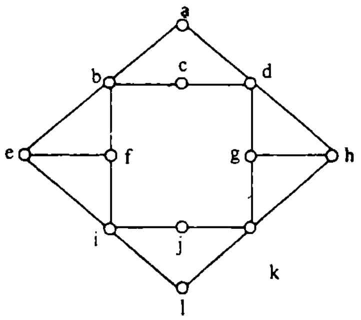
Suppose that G has a HC 'C'. The edges (a,b), (a,g), (b,c) and (c,k) must be in C since each vertex in a HC has degree 2. Thus edges (b,d) and (b,f) are not in C. Therefore edges (g,d) (d,e), (e,f) and (f,k) are in C. The edges now known to be in C form a cycle C_1. Adding an additional edge to C_1 will give some vertex in C degree greater than 2. This contradiction shows that G does not have a HC.

(iv)



HC : d – a – e – b – c – h – g – f – j – i – d

(v)



We need to eliminate two edges each at b, d, i and k, leaving 19 − 8 = 11 edges.

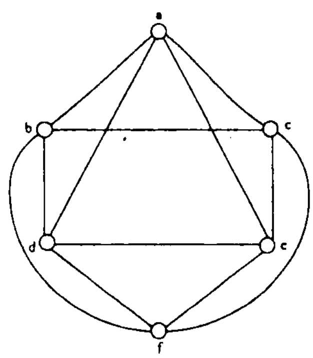
A HC should have 12 edges.

∴ No HC is in the given graph

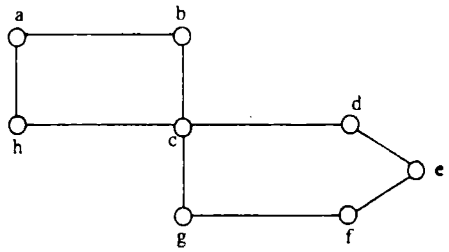
Example 12.5

Give examples for

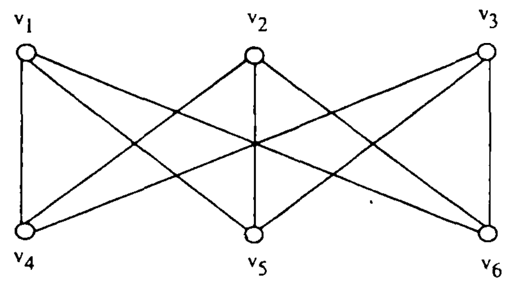
(i) A graph with an EC and HC



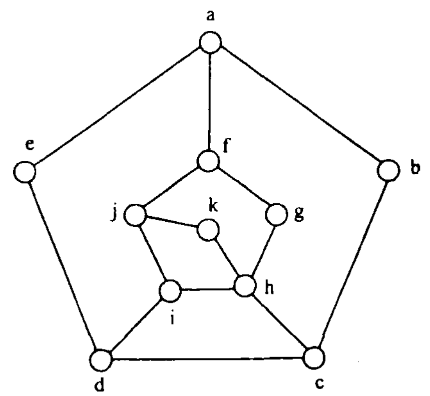
(ii) A graph with an EC but no HC,



(iii) A graph with HC but no EC



(iv) A graph with no EC and no HC



Traveling salesperson problem:

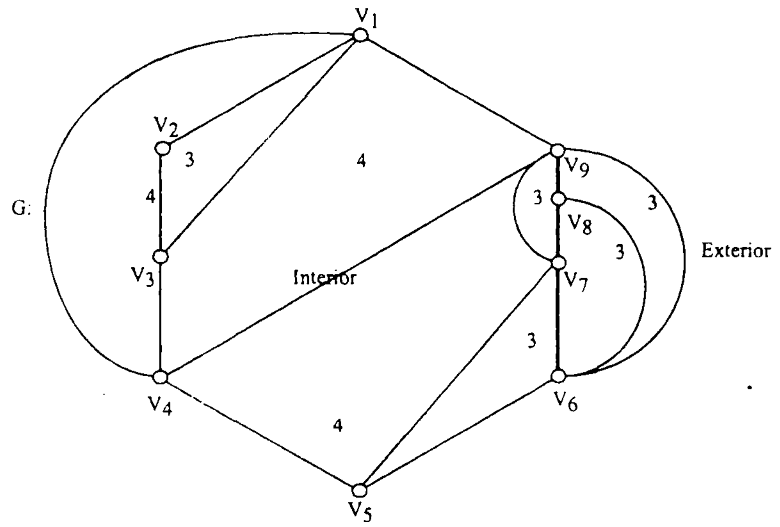
The traveling salesperson problem is related to the problem of finding a HC in a graph.

The Problem: Given a weighted graph G, find a minimum length HC in G. If we think of the vertices in a weighted graph as cities and the edge weights as distances, the traveling salesperson problem is to find a shortest route in which the salesperson can visit each city one time, starting and ending at the same city. No efficient algorithm for solving this problem is known.

Definition 12.4 Let G be a plane Hamiltonian graph within vertices suppose that C is a fixed HC in G. With respect to this cycle, a *diagonal* is an edge of G that does not lie on C.

Let r_i ($i = 3, 4, \ldots, n$) denote the number of regions of G in the interior of C whose boundary contains exactly i edges. That is r_i is the number of regions of degree i in the interior of C. Similarly, let r_i' denote the number of regions of degree i in the exterior of

Example 12.6



HC in G is C : $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_1$ clearly $r_3 = 3$, $r_3' = 2$, $r_4 = 2$, $r_4' = 1$, $r_5 = 0$ and $r_5' = 1$. $\{v_1, v_3\}$, $\{v_4, v_9\}$, $\{v_5, v_7\}$ and $\{v_7, v_9\}$ are diagonals in the interior of C while $\{v_1, v_4\}$ is a diagonal in the exterior of C.

Theorem (Grinberg) 12.1 Let G be a simple plane graph with n vertices suppose that C is a Hamiltonian cycle in G. Then with respect to this cycle C,

$$\sum_{i=3}^{n}(i-2)(r_i - r_i') = 0$$

Proof: First consider the interior of C.

Suppose that exactly d diagonals occur there.

Since G is a plane graph, none of its edges intersect.

Thus a diagonal splits the region through which it passes into two parts. Thinking of putting in the diagonals one at a time, we see that the insertion of a diagonal increases by one the number of regions inside the cycle.

Consequently 'd' diagonals divide the interior of C into $d + 1$ regions. Therefore

$$\sum_{i=3}^{n} r_i = d + 1 \quad \text{and} \quad d = \sum_{i=3}^{n} r_i - 1$$

Let N denote the sum of the degrees of the regions interior to C.

Then

$$N = \sum_{i=3}^{n} i\, r_i$$

However, N counts each diagonal twice (since each diagonal bounds two of the regions interior to C) and each edge of C once (since each bounds only one region interior to C). Thus

$$N = \sum_{i=3}^{n} i\, r_i = 2d + n$$

Substituting for d, we have

$$\sum_{i=3}^{n} i\, r_i = 2\sum_{i=3}^{n} r_i - 2 + n$$

so that

$$\sum_{i=3}^{n} (i-2) r_i = n - 2$$

By considering the exterior of C we conclude in a similar fashion that

$$\sum_{i=3}^{n} (i-2)\, r_i' = n - 2$$

∴ Combining the two results gives

$$\sum_{i=3}^{n} (i-2)(r_i - r_i') = 0$$

□

Example 12.7

Show that the following graph has no HC.

Solution:

In the above graph, there are three regions of degree 4 and six regions of degree 6. Thus, by Grinberg's theorem of a HC existed, when we will have

$$r_4 + r_4' = 3,\ r_6 + r_6' = 6$$

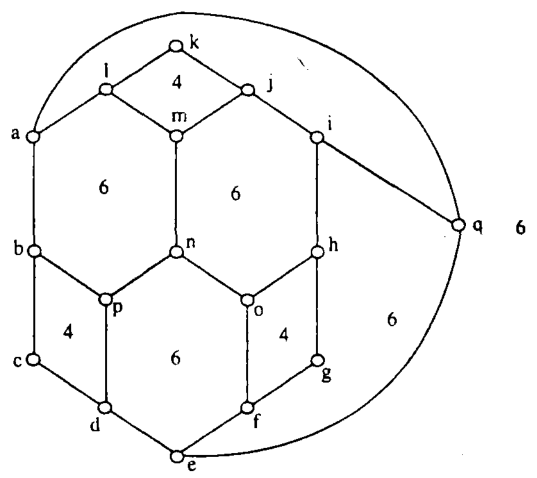
and $$2(r_4 - r_4') + 4(r_6 - r_6') = 0$$

or $$(r_4 - r_4') = -2(r_6 - r_6')$$

then $(r_4 - r_4')$ must be an even integer.

However, since $r_4 + r_4' = 3$, the only possibilities for r_4 and r_4' are 0 and 3, and 1 and 2. Neither of the possibilities is such that their difference is even

Therefore, the assumption that there was a HC for the graph led to a contradiction, and the given graph has no HC.



Dirac's theorem 12.2 A simple graph with n vertices ($n \geq 3$) and $\delta(G) \geq \frac{n}{2}$, has a HC.

Corollary 12.1 If G is a complete simple graph on n-vertices ($n \geq 3$), then G has a HC.

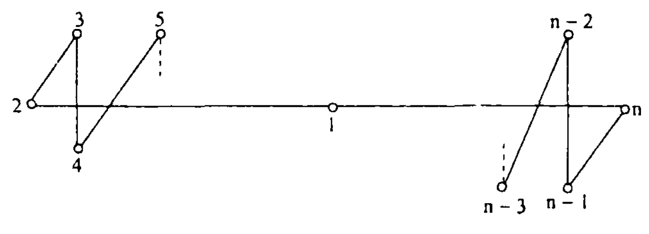
This corollary is true even for directed graphs.

Theorem 12.3 In a complete graph with n vertices there are $\left(\frac{n-1}{2}\right)$ edge disjoint hamiltonian cycles, if n is odd number ≥ 3.

Proof: Let G be a complete graph with n vertices where n is odd and $n \geq 3$. Then G has $\frac{n(n-1)}{2}$ edges. A hamiltonian cycle in G contains n edges. Therefore the number of edge disjoint hamiltonian cycle in G cannot exceed $\frac{n-1}{2}$.

Now consider to show that the hamiltonian cycles are $\left(\frac{n-1}{2}\right)$.

The subgraph of G, shown in the following figure, is a hamiltonian cycle



Keeping the vertices fixed on a cycle, rotate the polygonal pattern clockwise by

$$\frac{360}{n-1}, \left(2 \cdot \frac{360}{n-1}\right), \ldots \left[\left(\frac{n-3}{2}\right)\left(\frac{360}{n-1}\right)\right] \text{ degrees}$$

observe that each rotation produces a hamiltonian circuit that has no edge in common with any of the previous ones.

Thus we have $\frac{n-3}{2}$ new hamiltonian cycles all edges disjoint from the one in the figure given, and also edge disjoint among themselves. Therefore the number of edge disjoint hamiltonian cycles in G is $\frac{n-3}{2}+1=\frac{n-1}{2}$

Hence the theorem. □

Theorem 12.4 Let G be a graph with n vertices and with no loops and multiple edges. G has a HC if for any two vertices u and v of G, which are not adjacent. Then

$$\deg(u)+\deg(v)\geq n$$

Corollary 12.2 G has a HC if $\deg(u)\geq\frac{n}{2}$ for each vertex u of G

Theorem 12.5 G has a HC if $m\geq\frac{1}{2}(n^2-3n+6)$, where m is the number of edges and n is the number of vertices in G.

Proof: Given $m\geq\frac{1}{2}(n^2-3n+6)$

Let u and v be two vertices of G which are not adjacent.

Let H be the subgraph of G obtained by removing u and v and the edges incident at u and v.

The number of edges in H are $m-\deg(u)-\deg(v)$

The maximum number of edges H can have is $(n-2)_{C_2}=n^2-5n+6$.

Now

$$m-\deg(u)-\deg(v)\leq\frac{1}{2}(n^2-5n+6)$$

that is

$$\deg(u)+\deg(v)\geq m-\frac{1}{2}(n^2-5n+6)$$

$$\Rightarrow\quad \deg(u)+\deg(v)\geq\frac{1}{2}(n^2-3n+6)-\frac{1}{2}(n^2-5n+6)=n$$

Hence by theorem 12.4 G has a HC. □

Theorem 12.6 Let G be any hamiltonian graph. For every non empty proper subset S of $V(G)$, $k(G-S)\leq|S|$. Where $G-S$ is the graph G with the vertices in S and their associated edges removed.

For any graph G, $k(G)$ is the number of components of G

Proof: Let G be a hamiltonian and C be a HC of G. Let S be a non empty proper subset of $V(G)$. Then if we remove the set of S vertices from C, the cycle will fall into atmost $|S|$ pieces. That is $k(C-S)\leq|S|$

However $C-S$ is a subgraph that includes every vertex of $G-S$ and so $G-S$ cannot have more components than $C-S$. Thus

$$k(G-S)\leq k(C-S)$$

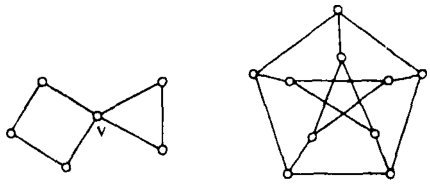
Hence

$$k(G - S) \leq |S|$$

□

Note: We can use this necessary condition to show that certain graphs are not hamiltonian. However, because the condition is not sufficient it is possible to find a non-hamiltonian graph for which the removal of any set of vertices S leaves no more components than the number of vertices in S.

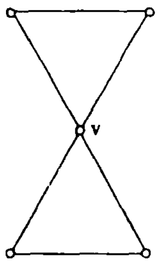
Example 12.8 Are the two graphs shown below hamiltonian?



Solution: In the first graph G, if we let $S = \{v\}$ then $G - S$ has two components i.e., $k(G - S) = 2$ and $|S| = 1$. Hence by previous theorem G is not hamiltonian.
By trial and error we can see that the petersen graph is not hamiltonian. However, there is no set of vertices whose removal leaves a graph with more components than the number of vertices removed.

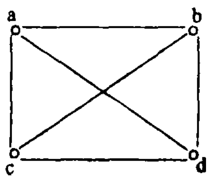
Observation 12.1 By definition, a hamiltonian graph contains a cycle containing all the vertices. So hamiltonian graph can't have cut vertices and pendant vertices.

Example 12.9 The graph is not hamiltonian

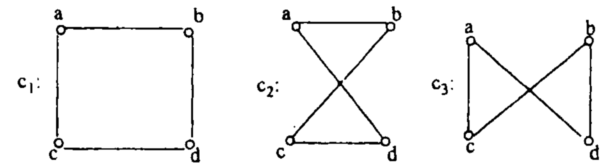


Since 'v' is the cut vertex.

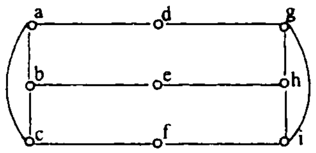
Example 12.10 Find three distinct hamiltonian circuits in the following graph



Solution: 'The circuit C_1, C_2 and C_3 are three distinct HCs

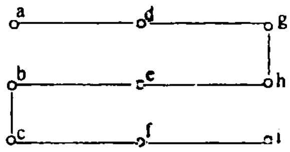


Example 12.11 Show that the graph shown below has no HC. But the graph has a hamiltonian path

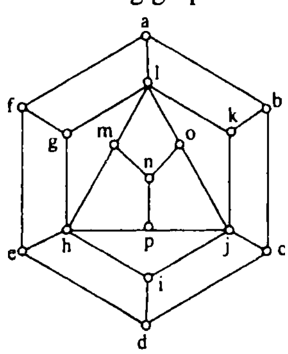


Solution: G has 9 vertices and 12 edges.

∴ The number of edge in any hamiltonian cycle in G is 9. There are three vertices of degree two in G. Therefore all the edges incident on the vertices d, e, f must be included in any HC. The edges $\{a, d\}$, $\{d, g\}$, $\{b, e\}$, $\{c, f\}\{f, i\}$ must be included in constructing a HC. The degree of b is 3 when the edges given above are included and we include the edge { a, b } in the HC, we delete the edge { b, c }. Then we must include { a, c } in the HC. Thus the degree of a would be 3. Hence no HC exists in G. However, there exists a HP in G $\{a, d\}$, $\{d, g\}$, $\{g, h\}$, $\{e, b\}$, $\{b, c\}$, $\{c, f\}\{f, i\}$ when included will give us a HP in G i.e., $a - d - g - h - e - b - c - f - i$ is a hamiltonian path in G.

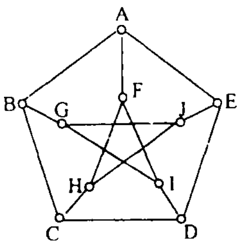


Example 12.12 Show that the following graph is not hamiltonian



Solution: G has 27 edges and 16 vertices. Therefore by hamiltonian cycle in G should have 16 edges and hamiltonian path in G should have 16 edges and hamiltonian path in G must contain exactly 15 edges. In the graph G we have $\deg(l) = \deg(h) = \deg(j) = 5$ i.e., there are three vertices of degree 5, therefore atleast three edges on l can't be included in any HP. The same is true for the vertices h and j. There are 13 vertices of degree 3 in that consider the vertices b, d, f and n. Each of these vertices is of degree three. Atleast one of the three edges incident in each of three vertices can't be included in any hamiltonian path. Thus atleast $3 + 3 + 3 + 4 = 13$ edges can't be included in any hamiltonian path. The number of remaining edges is $27 - 13 = 14$. But any HP it is not possible to construct a hamiltonian with the remaining 14 edges. Thus G has no hamiltonian path in G and G is not hamiltonian

Example 12.13 Show that petersen graph is not hamiltonian

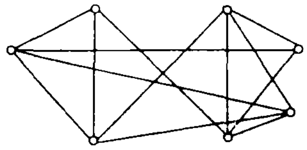


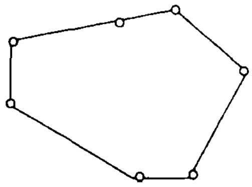
Solution: Suppose 'H' is a hamiltonian cycle in the petersen graph. Then H must contain at least one of the five edges connecting the outer to the inner vertices. Since the graph is symmetric there is no loss of generality in assuming that AF is part of H. According to the construction of hamiltonian cycle precisely one of the two edges FH and FI is in H. Again by symmetry, we may assume FH is part of the cycle while FI is not.

Since FI is not in H, but two edges incident with I must be in H, IG and ID are in H. Now precisely one of the edges GB, GJ is in H.

Suppose first that GB is in and GJ is out. Because precisely two edges incident with j are in H and JG is not, both JH and JE are part of H. Thus CH is out and both BC and CD are in H. At this point, however H contains the proper cycle $BCDIGB$, a contradiction we conclude that GB can't be part of H and hence that GJ is. An argument similar to the one just given now leads again to the false conclusion that H contains a proper cycle.

Example 12.14 Show that the two graphs shown below are Hamiltonian





Solution: Using Dirac's theorem:

The first graph has seven vertices and every vertex has degree atleast four. So Dirac's theorem tells us that this graph is hamiltonian.

The second graph is clearly hamiltonian by inspection, but there are seven vertices and the degree of each vertex is only two. Hence Dirac's theorem is sufficient but not necessary.

Example 12.15 Show that the graph k_n has a Hamiltonian cycle whenever $n \geq 3$

Solution: A HC in k_n can be formed starting at any vertex, such a HC can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other exactly once, as it is possible in k_n.

Also there are $\frac{(n-1)!}{2}$. Hamiltonian cycles.

Example 12.16 How many edges must a HC in k_n contain?

Solution: n edges

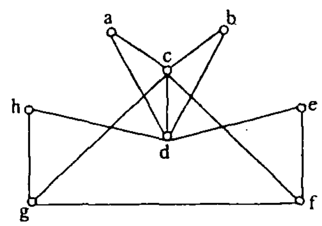
Example 12.17 Given an example of a graph which has a HP but no HC

Solution:

Example 12.18 What is the maximum number of edge disjoint hamiltonian cycles in k_n?

Solution: k_n has $\frac{n(n-1)}{2}$ edges, so the maximum is $\frac{n-1}{2}$ edge disjoint cycle

Example 12.19 Is the following graph Hamiltonian? Is it Eulerian? Explain you answers



Solution: The given graph is not hamiltonian. To see this, assume that 'H' is a hamiltonian cycle. Since vertices a and b have degree 2, the two edges incident with each of these vertices would be in H. Thus H would contain the cycle $acbda$, which can't be the case since this does not contain all vertices of the graph. The graph is not Eulerian because it contain vertices of odd degree.

Example 12.20 Show that there are no hamiltonian planar graphs with regions of degree 5, 8, 9 and 11 with exactly one region of degree 9.

Solution: From Grinberg's theorem we have

$$3(r_5 - r_5^1) + 6(r_8 - r_8^1) + 7(r_9 - r_9^1) + 9(r_{11} - r_{11}^1) = 0$$

but there is exactly one region of degree 9

$$\therefore \quad 3(r_5 - r_5^1) + 6(r_8 - r_8^1) + 7(\pm 1) + 9(r_{11} - r_{11}^1) = 0$$

$$\Rightarrow \quad 3[(r_5 - r_5^1) + 2(r_8 - r_8^1) + 3(r_{11} - r_{11}^1)] = \pm 7$$

$\therefore$ 3 is a divisor of $\pm$, a contradiction

$\therefore$ There can't be hamiltonian planar graphs with regions of degree 5, 8, 9 and 11 with exactly one region of degree 9

Graph coloring

In this section we describe the coloring of a graph and its chromatic number.

Discussions related to coloring of vertices, edges and regions are considered.

A vertex coloring is called proper coloring if no two adjacent vertices of the graph receive the same color and the graph is then called *properly colored* graph.

Painting each vertex of G of order n with n distinct colors is a proper coloring of G. The number of colors requires for a proper coloring of a graph can't exceed the order of the graph.

Definition 12.5 [Edge coloring] Assignment of colors to the edges of a graph G so that no two adjacent edges receive the same color is called an edge coloring of G.

k - edge coloring of a graph G is an assignment of k - colors to the edges of G such that no two edges of G receive the same color.

A graph G is said to be k - edge colorable, if there exist k - edge coloring of G

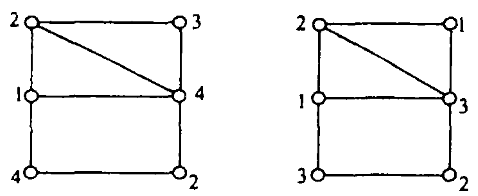
13. Chromatic Numbers

Definition (Vertex coloring) 13.1 The assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are assigned different colors is called vertex coloring.

An n-coloring of G is a coloring of G using n-colors. If G has an n-coloring, then G is said to be n-colorable.

Example 13.1

4-coloring as well as a 3-coloring of graph G, let 1,2,3 and 4 represent colors.

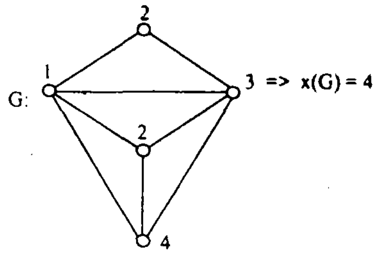


Definition (Chromatic Number) 13.2 The chromatic Number of a graph G is the minimum number of colors to color the vertices of the graph G.

We denote chromatic number of G by $\chi(G)$
If $\chi(G) = K$, then G is K- chromatic.

Example 13.2

(i)



(ii) The chromatic Number of a cycle is 2 or 3 depending on whether its length is even or odd.

(iii) $\chi(K_n) = n$

(iv) $\chi(K_{m,n}) = 2$

(v) $\chi(W_n) = \begin{cases} 3 & \text{n is even} \\ 4 & \text{n is odd} \end{cases}$ where W_n is the wheel graph with n- vertices.

In general, it is extremely difficult to determine the chromatic number of a graph. For graphs with a small number of vertices it is not too difficult to guess the chromatic number.

In investigating the chromatic number of a graph, we shall restrict ourselves to simple graphs.

We list few rules that may be helpful in finding the chromatic number of a graph G

Rule 1: $\chi(G) \leq |V|$, where $|V|$ is the number of vertices of G

Rule 2: A triangle always requires three colors, i.e., $\chi(K_3) = 3$; More generally $\chi(K_n) = n$, where K_n is the complete graph on n vertices.

Rule 3: If some subgraph of G requires K colors then $\chi(G) \geq K$.

Rule 4: If degree $(v) = d$, then atmost 'd' colors are required to color the vertices adjacent to v.

Rule 5: $\chi(G) = \text{Max}\{\chi(C) | C \text{ is a connected component of } G\}$

Rule 6: Every K-chromatic graph has atleast K-vertices v such that degree $(v) \geq K - 1$.

Rule 7: For any graph G, $\chi(G) \leq 1 + \Delta(G)$, where $\Delta(G)$ is the largest degree of any vertex of G.

Rule 8: When building a K-coloring of a graph G, we may delete all vertices of degree less than K (along with their incident edges) Ingeneral, when attempting to build a K-coloring of a graph, it is desirable to start by K-coloring a complete subgraph of K vertices and then successively finding vertices to $K-1$ different colors there by forcing the color choice of such vertices

Rule 9: The following statements are equivalent

i. A graph G is 2-colorable
ii. G is bipartite
iii. Every cycle of G has even length

i.e., $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

Rule 10: If $\delta(G)$ is the minimum degree of any vertex of G then $\chi(G) \geq \frac{|V|}{|V|-\delta(G)}$, where $|V|$ is the number of vertices of G.

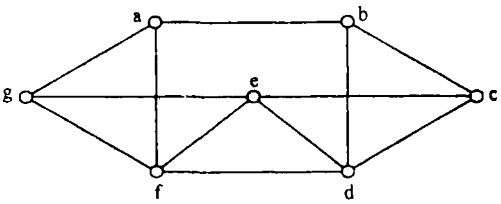
We need to perform two steps to show that the chromatic number of a graph is n.

Step 1: We must show that the graph can be colored with n-colors. This can be done by constructing such a coloring

Step 2: We must show that the graph cannot be colored Using fewer than n-colors.

Example 13.3 Find the chromatic number of the following graphs

(i)



$\deg(a) = \deg(b) = \deg(g) = \deg(c) = 3$

$\deg(e) = \deg(f) = \deg(d) = 4 \Rightarrow \Delta(G) = 4$

$$\therefore \text{ By } \chi(G) \leq 1 + \Delta(G) \quad \text{(Rule 7)}$$
$$\chi(G) \leq 5$$

Since G has a triangle subgraph $\chi(G) \geq 3$

$$\therefore \quad 3 \leq \chi(G) \leq 5$$

Suppose $\chi(G) = 5$ then G should have 5 vertices with degree atleast 4 (Rule 6), But there are only 3 vertices in G

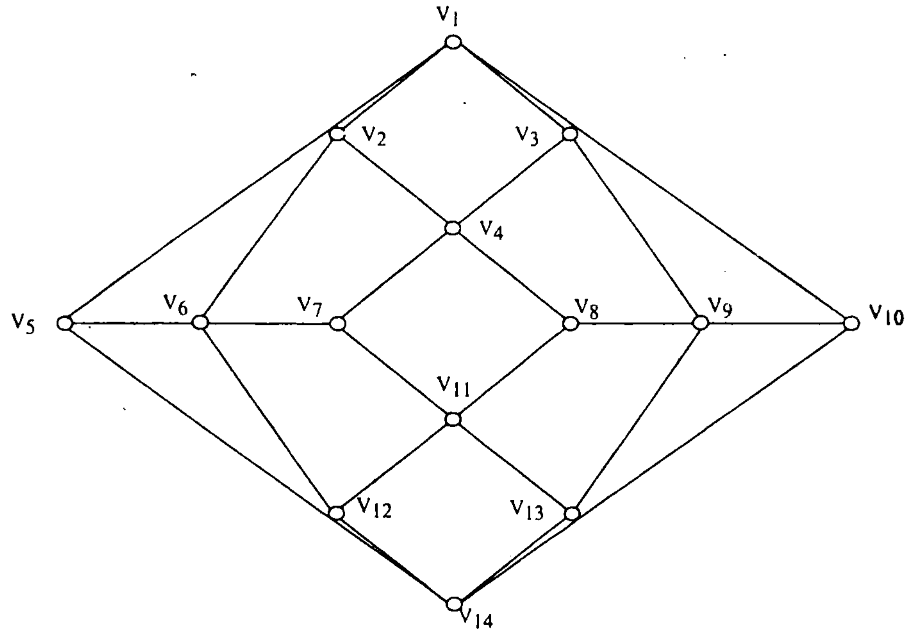
$\therefore \quad \chi(G) = 5$ is not possible

$$\therefore \quad 3 \leq \chi(G) \leq 4$$

Now consider 3-coloring of the graph, clearly 3-coloring is not possible.

$$\therefore \quad \chi(G) = 4$$

(ii)



Solution:

$$\deg(v_1) = \deg(v_4) = \deg(v_6)\deg(v_9) = \deg(v_{11}) = \deg(v_{14}) = 4$$

$$\deg(v_2) = \deg(v_3) = \deg(v_5) = \deg(v_7) = \deg(v_8) = \deg(v_{10}) = \deg(v_{12})$$
$$= \deg(v_{13}) = 3$$

clearly

$$\Delta(G) = 4$$
$$\therefore \quad \chi(G) \leq 1 + \Delta(G)$$
$$\therefore \quad \chi(G) \leq 5$$

By observing the above graph, it is clear that the graph G is connected and any subgraph of G can be colored with 2 colors only.

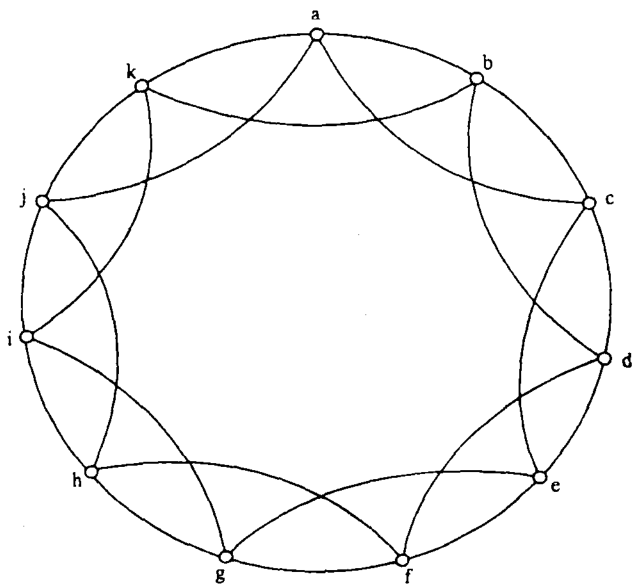
Also the above graph is special type of bipartite graph that is the vertex set of graph G can be partitioned into two sets in such a way that each edge joins a vertex of the first set to a vertex of the second set.

$$V = V_1 \cup V_2, \ V_1 \cap V_2 = \phi$$

$$V_1 = \{v_1, v_4, v_{11}, v_{14}\}, \ V_2 = \{v_2, v_3, v_5, v_{10}, v_7, v_8, v_9, v_6, v_{12}, v_{13}\}$$

∴ The above graph is 2-colorable since it is bipartite.

(iii)



clearly, Degree of each vertex is 4.

$$\Rightarrow \chi(G) \leq 5$$

By Examing the above graph, it has subgraphs whose chromatic number is 3.

$$\therefore \quad \chi(G) \geq 3$$
$$\Rightarrow \quad 3 \leq \chi(G) \leq 5$$

Also, the graph is connected

By Rule 4, If degree $(v) = 4$, then *at most* 4 colors are required to color the vertices adjacent to v.

Since all the vertices are of degree 4.

$$\Rightarrow 3 \leq \chi(G) \leq 4$$

Now, let us attempt to build a 3-coloring of G,

Since degree of each vertex is 4, then atmost 4 colors are required to color the vertices adjacent to v.

$$\therefore \quad \chi(G) = 4$$

Note 13.1 When an attempting to give a K-coloring of a graph G, order the vertices of G according to decreasing degrees. Assign the first color to the first vertex and then in sequential order. Assign the same first color to each vertex which is not adjacent to a previous vertex was assigned first color and repeat the same procedure in a sequential order for the remaining vertices. The above procedure may or may not give the chromatic number of G. But it is useful to give the coloring to the graph without any confusion.

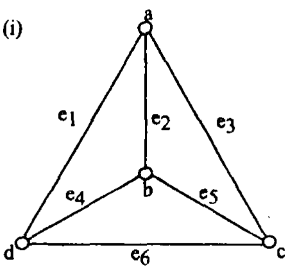
Definition : (edge chromatic Number) 13.3 The edge chromatic Number of G is the minimum number of colors to color all the edges of G, so that edges with common end points are colored different colors.

Note 13.2 If $\Delta(G)$ is the largest degree of the vertices of G, then $\Delta(G)$ is less than or equal to the edge chromatic number of G.

Example 13.4

Give the edge chromatic number for the following graph

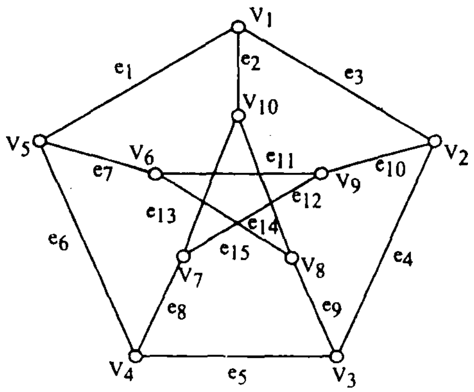
(i)



edge	color
e_1	1
e_2	2
e_3	3
e_4	3
e_5	1
e_6	2

$\therefore$ edge chromatic Number is 3

(ii)



This graph has 15 edges and 10 vertices $\Delta(G) = 3$

$\therefore \quad 3 \leq$ edge chromatic Number of G

By assigning similar to the above problem

$\therefore$ edge chromatic number is 4.

Theorem 13.1 The Four-color theorem: If G is any planar graph then $\chi(G) \leq 4$.

A proof of the four color theorem is far beyond the scope of this text, but we can prove a theorem that is only 25 percent inferior.

Theorem 13.2 The Five color theorem: If G is a planar graph is 5-colorable.

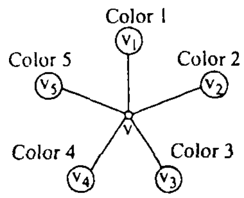
or

Every planar graph is 5-colorable

Proof: Proof by induction on the number of vertices in the Graph.

Clearly, a graph with one vertex has chromatic number of 1. Assume that all planar graphs with $n - 1$ vertices have a chromatic number of 5 or less. Let G be a planar graph by theorem 10.1 there exists a vertex v with deg $(v) \leq 5$. Let $G - v$ be the planar graph obtained by deleting v and all edges that connect v to other vertices in G. By the induction hypothesis, $G - v$ has a 5-coloring. Assume that the colors used are color 1, color 2, color 3, color 4 and color5. If deg $(v) < 5$, then we can produce a 5-coloring of G by selecting a color that is not used in coloring the vertices that are connected to v with an edge in G.

If $\deg(v) = 5$, then we can use the same approach if five vertices that are adjacent to v are not all colored differently. we are now left with possibility that v_1, v_2, v_3, v_4 and v_5 are all connected to v by an edge and they are all colored differently. Assume that they are colored color 1, color 2, color 3, color 4 and color 5 respectively suppose that v_1 and v_3 are not connected to one another using only color 1 and color 3 vertices in $G - v$. If we consider all paths that start at v_1 and go through only color 1 and color 3 vertices, then we can't reach v_3. If we exchange the colors of the vertices in these paths, including v_1, we still have a 5-coloring of $G - v$. since v_1 is now color 3, we can color v color 1.



Now suppose that v_1 is connected to v_3 using only color 1 and color 3 vertices. Then a path from v_1 to v_3 by using color 1 and color 2 vertices followed by the edges (v_3, v) and (v, v_1) completes a circuit that either encloses v_2 or encloses v_4 and v_5. Therefore no path from v_2 to v_4 exists using only color 2 and color 4 vertices. We can then repeat the same process as in the previous paragraph with v_2, and v_4, which will allow us to color v color 2.

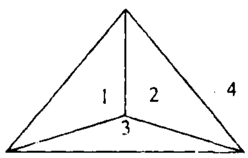
Four color conjecture

so for we have discussed proper coloring of vertices and proper coloring of vertices of edges of a graph. Now, briefly consider the proper coloring of *regions* in planar graphs such that no two adjacent region receive the same color.

The four color problem states that every plane map however complex, can be colored with four colors in such a way that two adjacent regions get different colors. This problem was solved by Appel and Hanken in 1976. However this problem is infact equivalent to the statement of conjecture.

Four color conjecture: Every planar graph is 4-colorable □

Example 13.5 The graph K_4 is a planar graph and K_4 is 4-colorable.



Theorem 13.3 A graph G is bipartite $\Longleftrightarrow$ it doesnot contain a odd cycle

Proof: Let G be bipartite. Then the vertex set G can be partitioned into two subsets V and V_2 such that $xy \in E(G) \Rightarrow x \in V_1$ and $y \in V_2$.

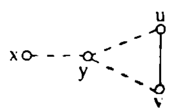
Suppose G contains a cycle. Let v be a vertex of this cycle. Then to trace the cycle starting from v we have to travel on the edges of G. The edges of G are the only edge between V_1 and V_2. Thus starting from v to come back to v along the cycle of G we have to travel exactly even number of times between V_1 and V_2. That is the number of edge in C is even, thus, the length of C is even.

Conversely, without loss of generality we assume G is connected (otherwise conside each component of the graph follow the proof for each component)

Let G does not contain a odd cycle. Choose a vertex x of G. color the vertex by the color black. Color all the vertices that are at odd distances from x with the color red. colo all the vertices that are at even distances from x with color black. Since every distance is either a odd or even integer (but not both), every vertex of G is now colored.

We show that the graph G is now properly colored. suppose G is not properly colored then G contains two adjacent vertices say u and v, colored with the same color. Then distances from the vertex x to both the vertices u and v is odd. Let P_1 and P_2 be shortest paths from x to u and x to v respectively. Then the edge uv is not included in P_1 or P_2 (observe). Let y be the last vertex common to P_1 and P_2 (i.e the path from y to u and path y to v along P_1 and P_2 are distinct; y may be x). Then $d(x, y)$ along P_1 is same along P_2 (since both P_1 and P_2 are shortest paths) otherwise if the $d(x, y)$ along P_1 is smaller than that of P_2, then the path from x to y along P_1 with the path from y to v along P_2 is shorter than P_2 which is a contradiction to the fact that P_2 is shortest.

Since $d(x, u) = d(x, v)$ we have $d(y, u)$ and $d(y, v)$ are both either odd or even and hence the sum is even. Further. the circuit formed due to these paths together with the edge uv is of odd length. which is a contradiction



Thus we concluded that the coloring is proper. Now consider the set V_1 of all vertices of G colored by the color Red. These sets are the partition of G such that no two vertices in the same set are adjacent, Hence G is bipartite.

□

Theorem 13.4 A graph G is 2-chromatic if and only if G is bipartite

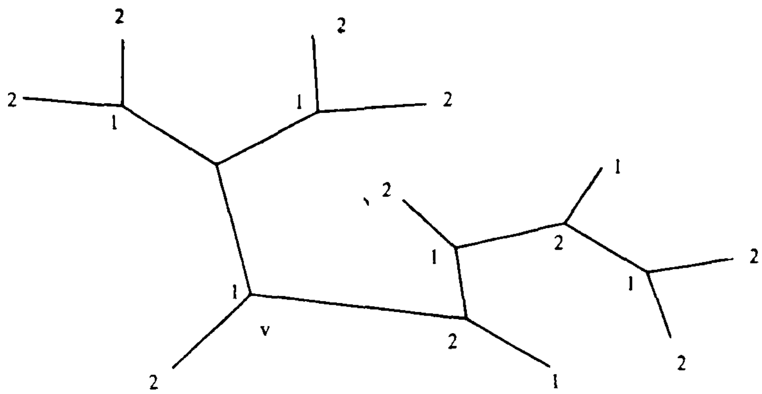
Proof: Let chromatic index of a graph G be two. Let G be properly colored with two colors 1 and 2. consider the set of vertices colored with the color 1 and the set of all vertices colored with the color 2. These sets are precisely partition of the vertex set such that no two of the vertices of the same set are adjacent hence G is bipartite.

Conversely, Let G be a bipartite graph. The vertex set $V(G)$ can be partitimed into two independent set V_1 and V_2. we can use color c_1 to point the vertices of V_1 and use color 2 to point the vertices of V_2. Hence G is 2- chromatic. □

Note 13.3 If a graph G is bipartite, then it does not imply that $\chi(G) = 2$. For example $\bar{K}_2$ is bipartite.

Theorem 13.5 Every Tree with two or more vertices is 2-chromatic.

Proof: Let T denote a given Tree and v be any vertex in T. Assign color 1 to the vertex v. Assign all the vertices adjacent to v with color 2 and assign color 1 to all vertices adjacent to the vertices of color 2. Continue this process till every vertex in T is colored. Now we find that all the vertices at odd distances from v have color 2 and the vertices which are at even distance from v have color 1.



Since T is connected and has atleast two vertices it contains ie., at least one edge (and there is one and only one path between two vertices in T) hence T is not 1-colorable.

Hence T can be properly colored with 2 colors. ∴ $\chi(T) = 2$. □

Theorem 13.6 A graph G containing atleast one edge is 2-chromatic if and only if it contains no odd cycles.

Proof: Let G be a connected graph with cycles of only even lengths. Consider a spanning tree T of G. we know that T can be properly colored with two colors. Now add edges (of G which are not in T) one by one. Since G has no cycles of odd length the end vertices of every edge being replaced are differently colored in T. Thus G is properly colored with two colors ie., G is 2-chromatic.

Conversely, Let G be 2-colored graph with atleast one edge. If G has cycle of odd length we would need atleast three colors just for that cycle. But G is 2-chromatic, hence G contains no odd cycles. □

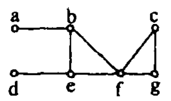
Covering

A vertex and an edge are said to cover each other if they are incident (ie., the vertex is the end vertex of the edge)

The set of vertices which covers all the edges of a graph G is called a vertex cover of G, while a set of edges which covers all the vertices is an edge cover of G. we consider only edge covers and write simlply a cover of G

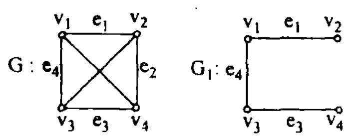
Definition (Cover) 13.4

A cover of a graph is the subset S of its edge set E such that every vertex of the graph is an end vertex of some edges in S. The set of edges that covers the graph G is called covering or *edge covering* or a covering subgraph of G. A covering of a graph such that not proper subsets of it is a cover of G is called *minimal covering* of G. The smallest number of edges in any covering of G is called *edge covering number* of G



The sets $\{ab, de, fg, gc\}$, $\{ab, de, bf, gc\}$, $\{ab, de, fc, fg\}$ ... are edge coverings of G. These coverings are minimal coverings of G. The smallest number of edges in a covering of G is four. Hence edge covering number of this graph is 4.

Example 13.6 $\therefore$ G_1 is an edge covering graph



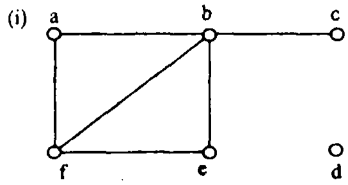
Observations on edge covering of a graph 13.1

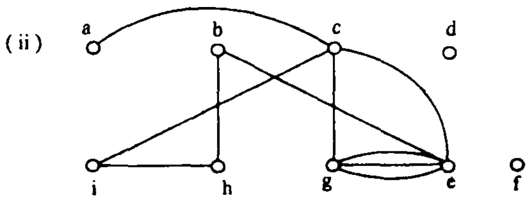
1. A covering exists for a graph if and only if it should not contain any isolated vertices. Hence covering exists for every connected graph

2. since the pendent vertex is incident with only pendent edge, every covering of the graph should include all the pendent edges of the graph
3. Since each vertex of the graph must be an end vertex of an edge in the covering of the graph it follows that each edge covering of a graph must contain at least $\lceil \frac{n}{2} \rceil$ edges, where n denote the number of vertices of the graph.
4. Every covering contains a minimal covering (since minimal covering is a proper subset of a covering of G having the property)
5. If S be a covering of a graph G, then as each vertex of G is adjacent to atleast one edge in S, we have $deg_{G-s}(v) \leq deg_G(v) - 1$ and vice verse
6. If a covering S contains all the edges of a cycle of a graph G, then we can remove one of the edges of the cycle from S without destroying its covering property. Thus a minimal covering of a G does not contain a circuit of G and hence a minimal covering may have atmost $n - 1$ edges of G.
7. A graph in general may have many minimal coverings of different cardinality (Example 1) The line covering number of a graph is equal to the minimum number of edges present in a minimal covering among all the minimal coverings of the graph.
8. The (line) edge covering number of a complete graph on n vertices is $\frac{n+1}{2}$
9. If a covering contains a subgraph that is a path of length three, then one of its middle vertex of this path can be delected without loosing the covering property and hence a minimal covering of a graph does not contain a path of length more than three. conversely, if a covering contains no path of length more than three, then each component of the covering is a star graph and from the star graph no edges can be delected without loosing its covering property, hence the covering is minimal.
10. All spanning trees and hamiltonian cycles are edge covering graphs.

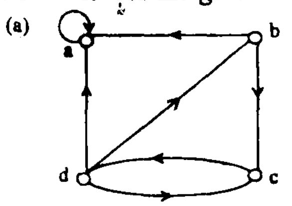
EXERCISE (I)

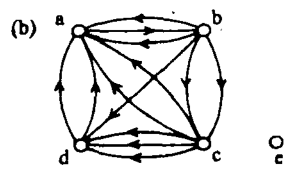
(1). Find the number of vertices, the number of edges and the degree of each vertex in the given undirected graph. Identify all isolated and pendent vertices



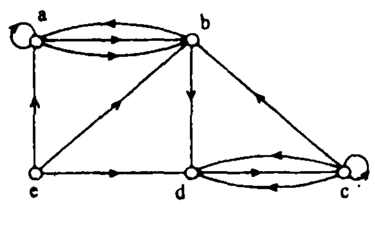


(2) Determine the number of vertices and edges and find the indegree and outdegree of each vertex for the given directed multigraph

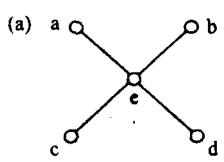


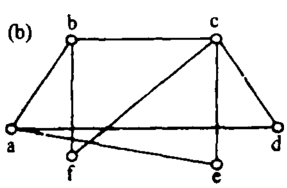


(3) Construct the underlying undirected graph for the graph with directed edges for the following graph



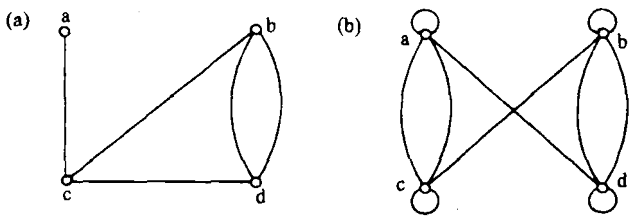
(4) Determine whether the graph is bipartite





(5) If the simple graph G has n vertices and 'e' edges how many edges does G^c have?

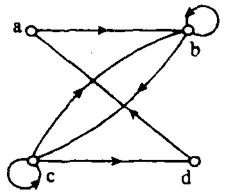
(6) Represent the given graph using an adjacency matrix



(7) Draw the graphs with given adjacency matrix

$$(a)\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad (b)\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

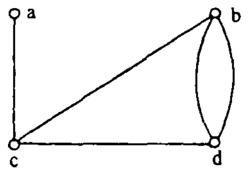
(8) Find the adjacency matrix of the given directed multigraph



(9) Draw the graph represented by the given adjacency matrix

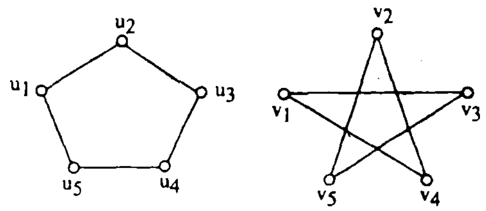
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

(10) Use an incidence matrix to represent the following graph

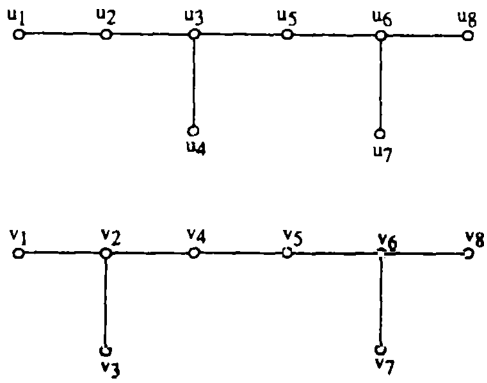


(11) determine whether the given pair of graphs is isomorphic

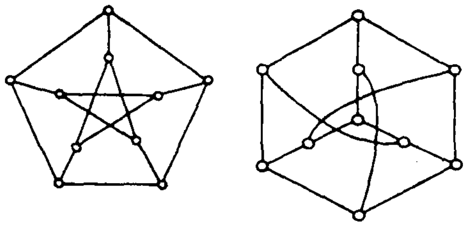
(a)



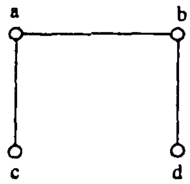
(b)



(c)

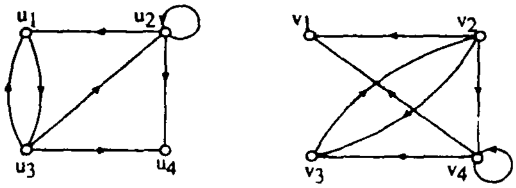


(12) Show that the following graph is self-complementary



(13) Determine whether the given pair of directed graphs is isomorphic

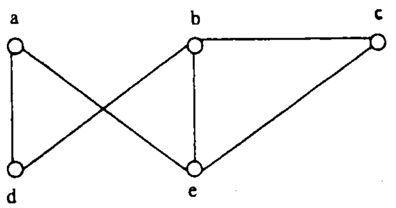
(a)



(14) Does each of the following lists of vertices form a path in the following graph? which paths are simple? which are circuits? what are the lengths of those that are paths

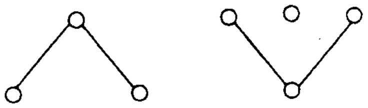
a) *a*, *e*, *b*, *c*, *b* (*c*) *e*, *b*, *a*, *d*, *b*, *e*

b) *a*, *e*, *a*, *d*, *b*, *c*, *a* (*d*) *c*, *b*, *d*, *a*, *e*, *c*

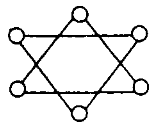


(15) Determine whether the given graph is connected

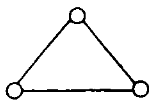
(a)



(b)

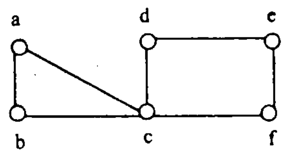


(c)

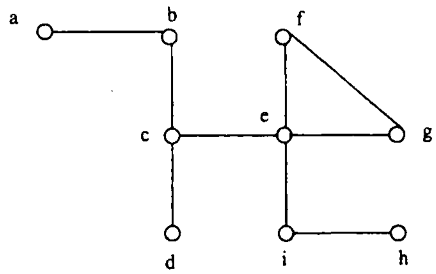


(16) Find all the cut vertices of the given graph

(a)

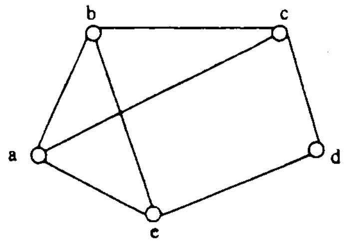


(b)

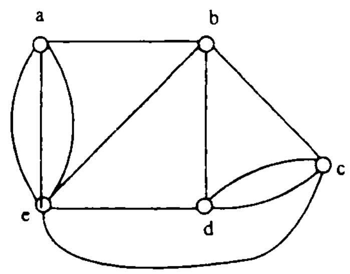


(17) Determine whether each graph has an Euler circuit. Construct such a circuit when one exists

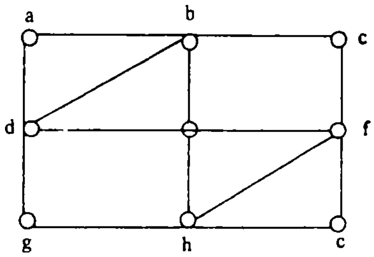
(a)



(b)

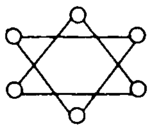


(18) Determine whether the graph has an Euler path construct such a path if it exists

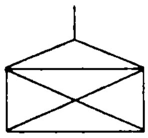


(19) Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

(a)

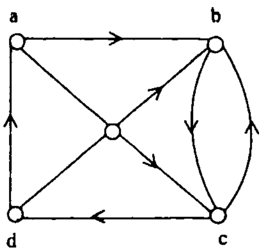


(b)

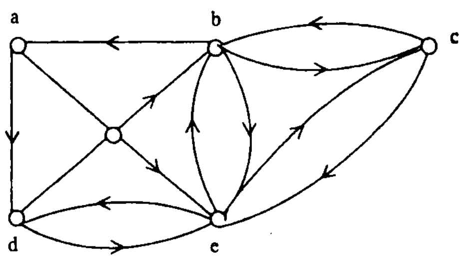


(20) Determine whether the directed graph shown has an Euler circuit construct an Euler circuit if it exists

(a)



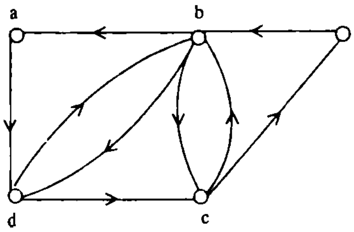
(b)



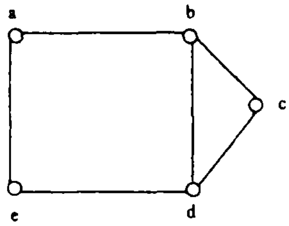
(21) Determine whether the directed graphs in the following figures has an Euler path. Construct an Euler path if it exists

(a) 20(a)

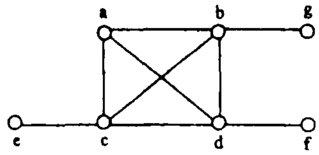
(b)



(22) Determine whether the given graph has a Hamiltonian circuit. If it does, find such a circuit. If it does not, given an arguement to show why no such circuit exists.

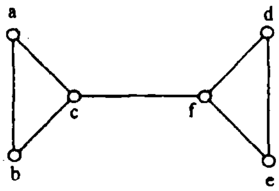


(b)

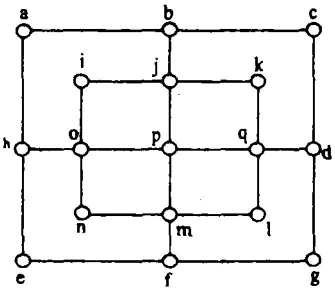


(23) Does the following graphs have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists

(a)



(b)

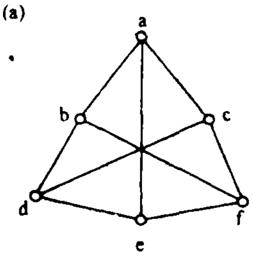


(24) For which values of m and n does the complete bipartite graph $K_{m,n}$ have a HC?

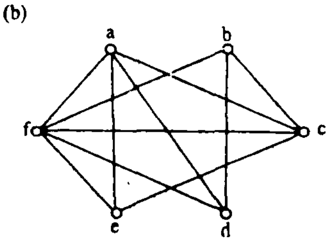
(25) Show that a bipartite graph with an odd number of vertices does not have a HC

(26) Determine whether the given graph is planar. If so draw it so that no edges cross.

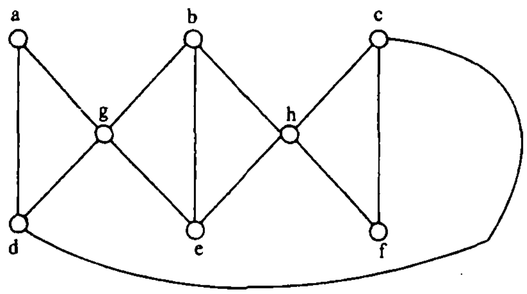
(a)



(b)

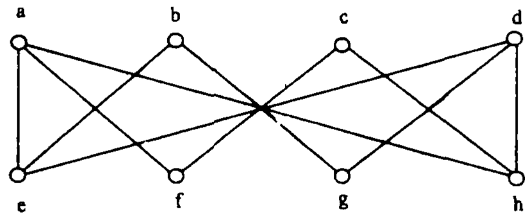


(27) Determine whether the given graph is homeomorphic to $K_{3,3}$

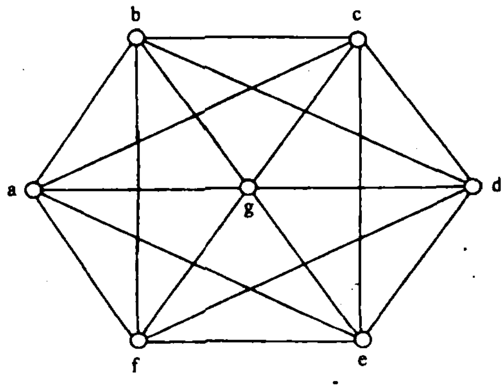


(28) Use Kurotowski's theorem to determine whether the given graph is planar.

(a)

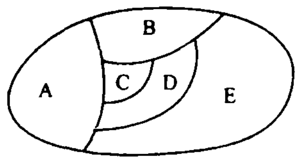


(b)

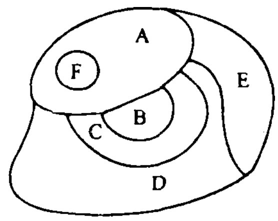


(29) Construct the dual graphs for each of the following maps

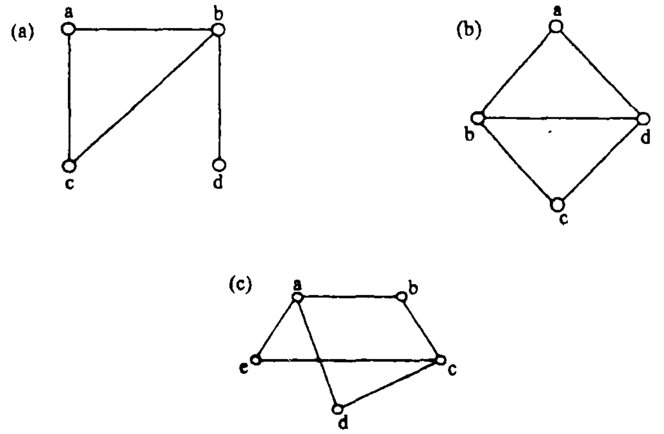
(a)



(b)



(30) Find the Chromatic number of the given graph



Answers

(1) (a) $v = 6$; $e = 6$; deg $(a) = 2$, deg $(b) = 4$, deg $(c) = 1$, deg $(d) = 0$; deg $(e) = 2$, deg $(f) = 3$, c is pendent; d is isolated.

(1) (b) $v = 9, e = 12$; deg $(a) = 3$, deg $(b) = 2$, deg $(c) = 4$, deg $(d) = 0$, deg $(e) = 6$, deg $(f) = 0$, deg $(g) = 4$, deg $(h) = 2$, deg $(i) = 3$, d and f are isolated.

(2) (a) $v = 4$; $e = 7$; $\deg^-(a) = 3$, $\deg^-(b) = 1$, $\deg^-(c) = 2$, $\deg^-(d) = 1$, $\deg^+(a) = 1$, $\deg^+(b) = 2$, $\deg^+(c) = 1$, $\deg^+(d) = 3$

(b) $v = 4, e = 13$, $\deg^-(a) = 6$, $\deg^+(a) = 1$, $\deg^-(b) = 1$, $\deg^+(b) = 5$, $\deg^-(c) = 2$, $\deg^+(c) = 5$, $\deg^-(d) = 4$, $\deg^+(d) = 2$, $\deg^-(e) = 0$, $\deg^+(e) = 0$.

(3) Draw the graph without directions

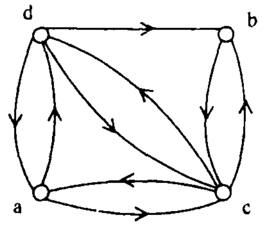
(4) (a) Bipartite (b) Not bipartite

(5) $\dfrac{n(n-1)}{2} - e$

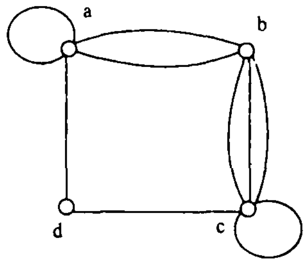
(6) $(a)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ $(b)\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

(7)

7(a)



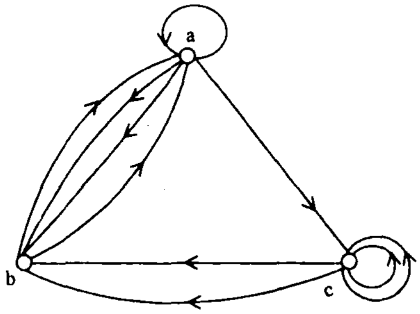
7(b)



(8)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(9)



(10) $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(11) (a) Isomorphic (b) Not isomorphic (c) Isomorphic

(13) (a) isomorphic (b) not isomorphic

(14) (a) path of length 4; not a circuit; not simple
(b) not a path
(c) not a path
(d) simple circuit of length 5

(15) (a) not (b) not (c) connected

(16) (a) *c* (b) *b*, *c*, *e*, *i*

(17) (a) No (b) *a*, *b*, *c*, *d*, *c*, *e*, *d*, *b*, *e*, *a*, *e*, *a*

(18) An Enter path exists; *f*, *a*, *b*, *c*, *d*, *e*, *f*, *b*, *d* is one such Euler path

(19) (a) Yes (b) No

(20) (a) No (b) No

(21) (a) *abdbcdcad* (b) *adbdebecba*

(22) (a) *abcdea* is a HC

(b) No HC exists, because once a circuit has reached e it would have no where to go

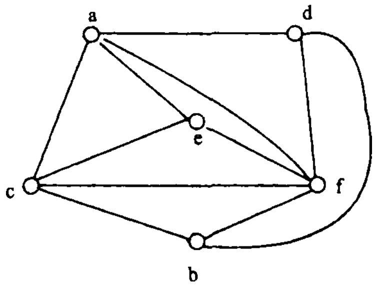
(23) (a) *abcfde* is a HP

(b) No HP exists. There are eight vertices of degree 2 and only two of them can be end vertices of a path. For each of the other six, their two incident edges must be in the path. It is not hard to see that if there is to be a Hamilton path, exactly one of the inside Corner Vertices must be an end, and that this is impossible.

(24) $m = n \geq 2$

(25) Suppose G is a bipartite graph with $V = V_1 \cup V_2$, where no edge connects a vertex in V_1 and a vertex in V_2. Suppose that V has a HC. Such a circuit must be of the form $a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_1$ where $a_i \in V_1$ and $b_i \in V_2$ for $i = 1$ to k. Since the HC visits each vertex exactly once, except for V_1, where it begins and ends, the number of vertices in the graph equals $2k$, an even number. Hence a bipartite graph with an odd number of vertices cannot have a HC.

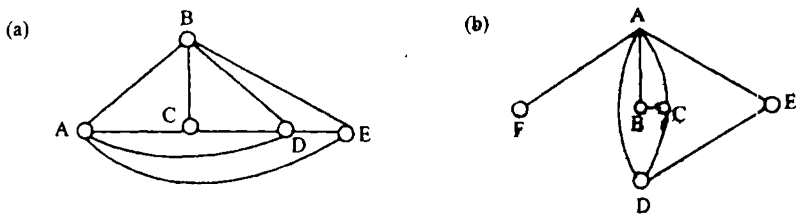
(26) (a) No (b) Yes



(27) Not

(28) (a) planar (b) not planar

(29)



(30) (a) 3 (b) 3 (c) 2

Solved Problems

Problem 1 Is there a graph with degree sequence (2, 3, 3, 4, 4, 5)

Solution: No, sum of degrees is odd (or) there is an odd number of vertices of odd degree

Problem 2 Is the degree sequence (2, 3, 3, 4, 5, 6, 7) graphic?

Solution: No, 'a' simple graph of order 7 cannot have a vertex of degree 7.

Problem 3 Is the degree sequence (1, 3, 3, 3) graphic?

Solution: Suppose that G is a graph with this degree sequence. Each vertex of degree 3 has an edge leading to each other vertex. Let a, b, c, d be the vertices where $\deg(a) = 1$. Since b, c, d have degree 3, there must be an edge joining b to a one joining c to a and one joining d to a. Hence a has degree 3 or more, a contradiction. Therefore (1, 3, 3, 3) is not graphic.

Problem 4 Is there a simple graph with degree sequence (1, 1, 3, 3, 3, 4, 6, 7)?

Solution: Assume there is such a graph. Then the vertex of degree 7 is adjacent to all other vertices, so inparticular it must be adjacent to both vertices of degree 1. Hence the vertex v of degree 6 cannot be adjacent to either of the two vertices of degree 1. But this leaves, is only six vertices (including v itself) to which the vertex v is adjacent. Since it is assumed that the graph is simple. v can't be adjacent to itself and therefore, there can be only five vertices adjacent to v. But then v can't have degree 6. The contradiction shows that there is no simple graph with the given degree sequence.

Problem 5 A degree sequence with all distinct elements canot be a graphic sequence.

Solution: Let $\{d_1, d_2, \ldots, d_n\}$ be a degree sequence with all distinct elements and $0 \leq d(v_i) \leq n - 1$
Suppose if it is a graphic sequence, there exists a simple graph with this sequence as its degree sequence. Now it follows that the sequence is a permutation of $\{0, 1, 2, \ldots n - 1\}$
Suppose there exists integers $i, j \in \{1, \ldots n\}$ s.t $d_i = 0$ or $d_j = n - 1$.
Suppose $d_i = 0$ then the corresponding vertex v_i is not adjacent with the remaining vertices and if $d_j = n - 1$, then the corresponding vertex v_j is adjacent to all the remaining vertices $\Rightarrow v_j$ is adjacent to v_i. which is a contradiction
$\therefore$ The sequence is not graphic

Problem 6 Let G be a simple graph, then $|E| \leq |V|_{c_2}$ in G.

Solution: Let G be a simple graph with $|V|$ vertices and |E| edges since G is simple, any two vertices of G determine atleast one edge. $\therefore$ The number of distinct edges that can be formed with $|V|$ vertices is $|V|_{c_2}$
Thus, the total number of edges atmost will be $|V|_{c_2}$

$$\Rightarrow |E| \leq |V|_{c_2}$$
$$|E| \leq \frac{|V|(|V|-1)}{2}$$

Problem 7 Let G be a simple graph, show that $|E| = |V|_{c_2} \Longleftrightarrow G$ is complete.

Solution: Given G is a simple graph with $|V|$ vertices and $|E|$ edges. Assume G is complete, So any pair of vertices of G, are adjacent in G, then the number of edges that can be formed with $|V|$ vertices is $|E| = |V|_{c_2}$
Conversely, Suppose $|E| = |V|_{c_2}$, then any pair of district vertices are adjacent in G. Thus G is complete.

Problem 8 Prove that every complete graph is regular

Solution: Let G be a complete graph, so it is a simple graph with $|V|$ vertices. Clearly, in a complete graph $G = K_{|V|}$, each vertex is adjacent to remaining $|V| - 1$ vertices Therefore, the degree of each vertex is $|V| - 1$. Hence the complete graph $K_{|V|}$ is a $(|V| - 1)$-regular graph.
Thus every complete graph is regular.

Note 13.1 Converse of the above result need not be true.

Problem 9 How many vertices will the follwoing graphs have?

(i) if it contain 16 edges and all vertices of degree 2

Solution: $\therefore \delta(G) = \Delta(G) = 2$, From the known result $\delta = \frac{2.|E|}{|V|}$

$$\Rightarrow \delta|V| = 2 \cdot |E|$$
$$2|V| = 2 \cdot |E| = 2 \times 16$$
$$\therefore |V| = 16.$$

(ii) 21 edges, 3 vertices of degree 4, other vertices of degree 3

Solution: By using $\sum \deg(v_i) = 2 \cdot |E|$

$$\Longrightarrow 3 \times 4 + x \times 3 = 2 \times 21$$

Where x is the number of remaining vertices

$$3x = 30$$
$$x = 10$$

∴ Total number of vertices $(10 + 3) = 13$.

Problem 10 What is the largest possible number of vertices in a graph with 35 edges and all vertices of degree atleast 3?

Solution: We have $\delta(G) \leq \dfrac{2|E|}{|V|} \Rightarrow |V| \leq \dfrac{2|E|}{\delta}$

Given $\delta = 3$, $|E| = 35$ then $|V| \leq \dfrac{2 \times 35}{3} \leq 23.3$

∴ largest possible number of vertices is 23.

Problem 11 Suppose G is an undirected graph with 12 edges and it has 6 vertices of degree 3 and the rest have degree less than 3 determine *minimum number* of vertices.

Solution: We know $\delta(G) \leq \dfrac{2|E|}{|V|} \leq \Delta(G)$

$$\Rightarrow \frac{2|E|}{\Delta} \leq |V| \leq \frac{2|E|}{\delta} \Rightarrow 1 \leq \frac{24}{|V|} \leq 3 \qquad \text{(i)}$$

consider $\delta(G) = 1$, $\Delta(G) = 3$

$$\Rightarrow \frac{24}{3} \leq |V| \leq \frac{24}{1}$$

$$\Rightarrow 8 \leq |V| \leq 24 \qquad \text{(ii)}$$

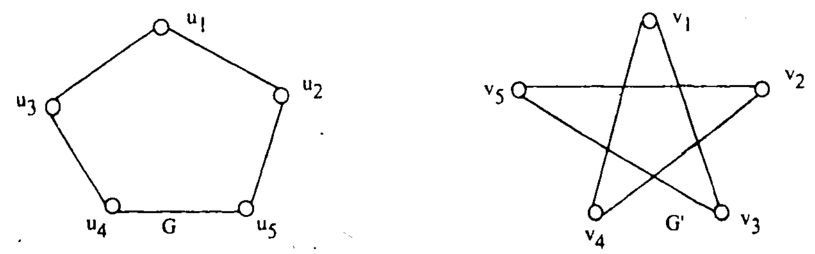
From (i) and (ii): possible values of $|V|$ are 8, 12, 24 if $|V| = 8$, we have vertices of degree 3, then remaining vertices are two, which must contribute $(2.12 - 18 = 6)$ degree 6. Possible degrees for the two vertices are: 2, 4; 4, 2; 5, 1; 1, 5; 3, 3 clearly a contradiction to the hypothesis of the problem that the degree of remaining vertices should be less than 3.

So $|V| = 8$ is not possible.

Then the next minimum value of $|V| = 12$.

Note 13.2 If G is a simple graph with n vertices then $0 \leq \deg(V) \leq n - 1$.

Example 12: Prove that the graphs G and G' given below are isomorphic



Solution: The two graphs have the same number of vertices same number of edges and same degree sequence consider the function f.

$$f(u_1) = v_1,\ f(u_2) = v_3,\ f(u_3) = v_4,\ f(u_4) = v_2,\ f(u_5) = v_5$$

then the adjacency matrices of the two graphs corresponding to 'f' are

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

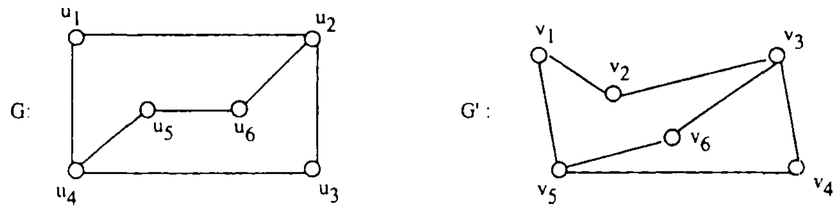
$$A'(G') = \begin{array}{c} \\ v_1 \\ v_3 \\ v_4 \\ v_2 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_3 & v_4 & v_2 & v_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$\therefore$ $A(G) = A'(G')$

$\therefore$ G and G' are isomorphic to each other □

Example 13:

Determine whether the graphs G and G' are isomorphic



Solution: Both G and G' have 6 vertices 7 edges, Both have 4 vertices of degree 2 and 2 vertices of degree 3 i.e., have same degree sequences.

Since G and G' agree w.r.to these invariants.

Now, it is reasonable to find an isomorphism f which shows isomorphism between G and G'.

Now define a function f and we will determine whether it is an isomorphism. Since $d(u_1) = 2$ and since u_1 is not adjacent to any other vertex of degree 2, the image of u_1 must be either V_4 or V_6, the only vertices of degree 2 in G' not adjacent to a vertex of degree 2.

Choose arbitrarily $f(u_1) = V_6$.

Since u_2 is adjacent to u_1, the possible images of u_2 are V_3 and V_5. We arbitrarily set $f(u_2) = v_3$.

Continuing in this way, using adjacency of vertices and degrees as a guide, we set

$$f(u_3) = v_4,\ f(u_4) = v_5,\ f(u_5) = v_1 \quad \text{and} \quad f(u_6) = v_2$$

We now have a one-to-one correspondence between the vertex set of G and the vertex set of G'.

Namely, $$f(u_1) = v_6;\ f(u_2) = v_3;\ f(u_3) = v_4$$

$$f(u_4) = u_5;\ f(u_5) = v_1;\ f(u_6) = v_2$$

To see whether f preserves edges, we examine the adjacency matrix of G and G' with rows and columns labelled by the images of the corresponding vertices in G.

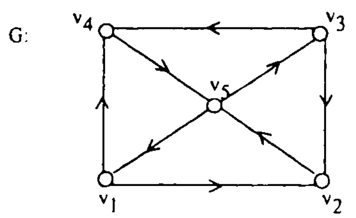
$$A_G = \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} \ ; \ A'_G = \begin{array}{c} \begin{matrix} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{matrix}$$

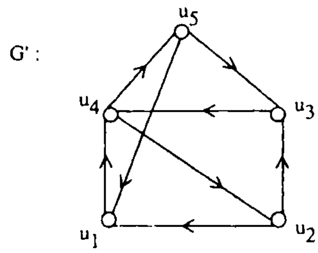
$\therefore$ $A_G = A'_G$, it follows that f preserves edges.

$\therefore$ f is an isomorphism, so that G and G' are isomorphic

Note 13.3 Suppose if f is not an isomorphism, we should not say that G and G' are not isomorphic, since another correspondence of the vertices in G and G' may be isomorphism

Example 14: Show that the Digraphs are isomorphic.





Solution: G and G' are having 5 vertices and 8 edges.
Consider indegree and out degree of the vertices if G and G'

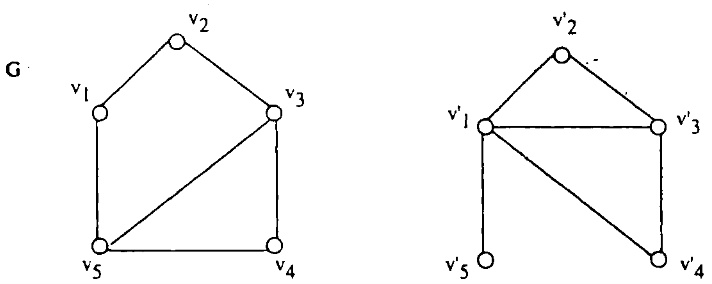
G	deg + in degree	deg − out degree
v_1	1	2
v_2	2	1
v_3	1	2
v_4	2	1
v_5	2	2

G'	deg + in degree	deg − out degree
u_1	2	1
u_2	1	2
u_3	2	1
u_4	2	2
u_5	1	2

Now set $f(v_1) = u_5$, $f(v_2) = u_1$, $f(v_3) = u_2$
$f(v_4) = u_3$, $f(v_5) = u_4$
clearly f is one to one and onto
Also $A_G = A_{G'}$ under this mapping f (check !)
$\therefore$ G and G' are isomorphic.

Example 15: Show that the graphs are not is isomorphic

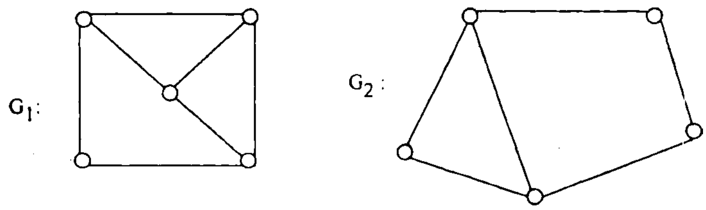
Solution:



The vertices and edges are same in both graphs. G and G' have 5 vertices and 6 edges. But degree sequences $d_G = (2, 2, 2, 3, 3)$, $d_{G'} = (1, 2, 2, 3, 4)$ are not same
$\therefore$ The two graphs are not isomorphic.

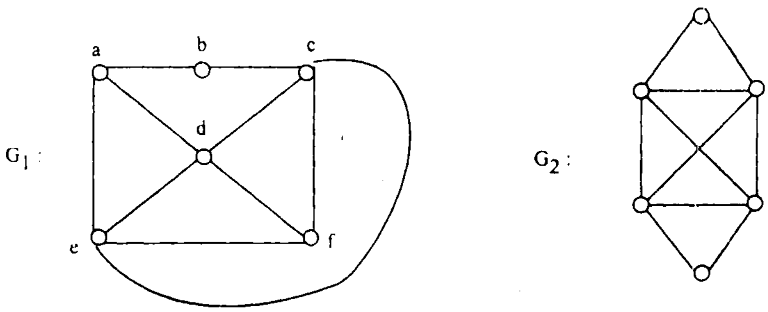
Example 16 Determine whether the following graphs are isomorphic

1.



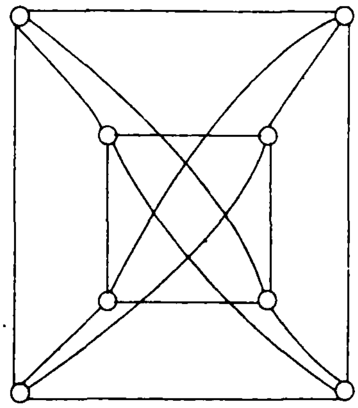
The graphs G_1 and G_2 are not isomorphic, since G_1 has 7 edges and G_2 has 6 edges and "has seven edges" is an invariant.

2.

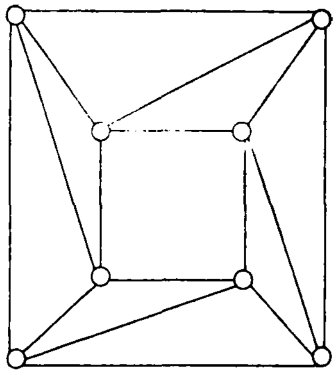


• G_1 and G_2 are non isomorphic graphs, since G_1 vertices of degree 3, but G_2 has no vertices of degree 3. The graph G_2 has a simple cycle of length 3, but all simple cycles

3.



G_2

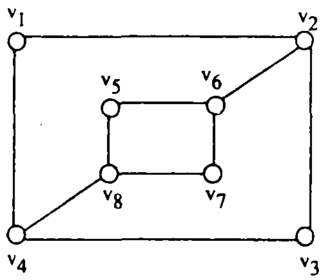


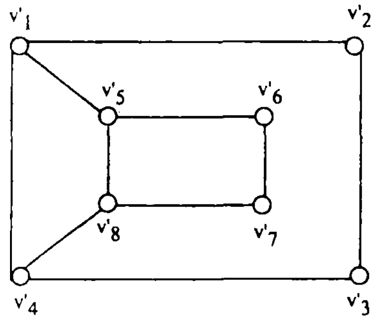
G_2

in G_1 have lenght atleast 4. Note that G_1 and G_2 have the same numbers of edges and vertices and that every vertex in G_1 or G_2 has degree 4.

Example 17:

Determine whether the following graphs are isomorphic





Solution: G and G' have 8 vertices and 10 edges.

Degree sequences of both G and G' are also same (check!).

Now consider $\deg(v_1) = 2$ in G.

$\therefore$ v_1 must correspond to either v_2', v_3', v_6', v_7', since these are the vertices of degree 2 in G'

However, each of these four vertices in G' is adjacent to another vertex of degree 2 in G' which is not true for v_1 in G.

$\therefore$ G and G' are not isomorphic.

Problem 18 Suppose that a connected plane graph has 20 vertices each of degree 3. Into how many regions, does a representation of this planar graph split the plane.

Solution: Given $|V| = 20$ and degree of each vertex is 3

$$\therefore \ \Sigma \deg(v) = 3 \times 20 = 60$$

We have $\Sigma \deg(v) = 2|E| = 60 \Rightarrow |E| = 30$
Now, By Euler's formula

$$|R| = |E| - |V| + 2$$
$$= 30 - 20 + 2 = 12$$

$\therefore$ The regions are 12.

Problem 19 Prove that there is no polyhedral graph with exactly 30 edges and 11 regions.

Solution: By Euler's formula: $|V| = |E| - |R| + 2 = 30 - 11 + 2 = 21$

$$\text{By} \quad 3|V| \le 2|E| \Rightarrow 3(21) \le 2(30)$$
$$\Rightarrow 63 \le 60$$

an obvious contradiction
$\therefore$ there is no polyhedral graph with 30 edges and 11 regions.

Problem 20 If G is a polyhedral graph with 12 vertices and 30 edges prove that degree of each region is 3.

Solution: Given $\begin{matrix} |E| = 30 \\ |V| = 12 \end{matrix}$ then by Euler's formula

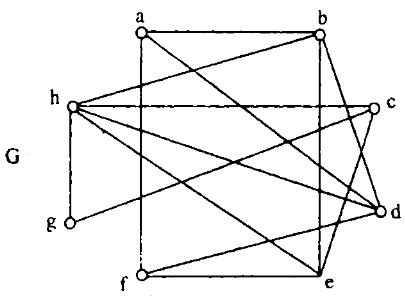
$$|R| = |E| - |V| + 2 = 20$$

When each region 'r' has degree K

$$\sum_{r \in R(G)} \deg(r) = K|R| = 2|E|$$
$$K \cdot 20 = 60$$
$$K = 3$$

$\therefore$ each region has degree = 3.

Problem 21 Prove the following graph is not planar by finding a subgraph homeomorphic to either K_5 or $K_{3,3}$



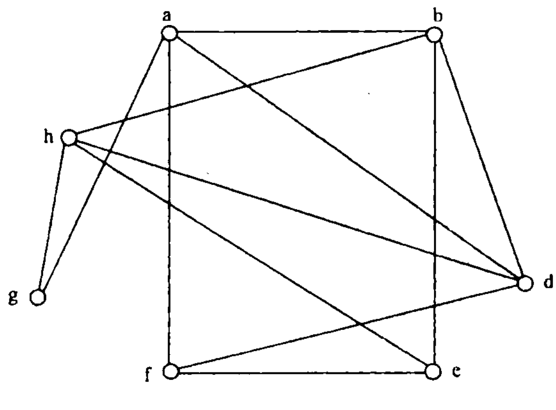
Solution: In K_5 each vertex has degree 4 only

K_5 has 5 vertices and 10 edges

In the graph G, note that Vertex 'h' has degree 5 at first make the series reduction $\{\{h, g\}, \{g, a\}\} \to \{h, a\}$; $\{\{a, f\}, \{f, e\}\} \to \{a, e\}$; $\{\{d, f\}, \{f, e\}\} \to \{d, e\}$

Delete the edges $\{h, c\}, \{g, c\}, \{e, c\}$

Then the given graph is as follows:

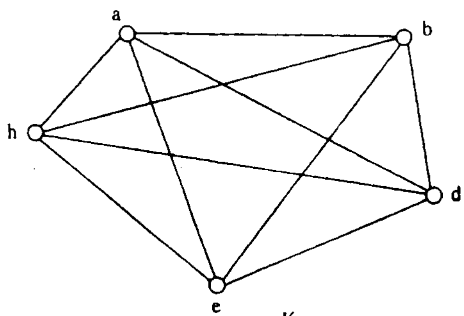


Now make the series Reductions:

$h - g - g - a :\to h - a$

$a - f - f - e :\to a - e$

$d - f - f - e :\to d - e$



K_5

$\therefore$ The given graph is not planar, since it contains a subgraph homeomorphic to K_5

Problem 22 How many vertices will the graph with 24 edges and all vertices of the same degree

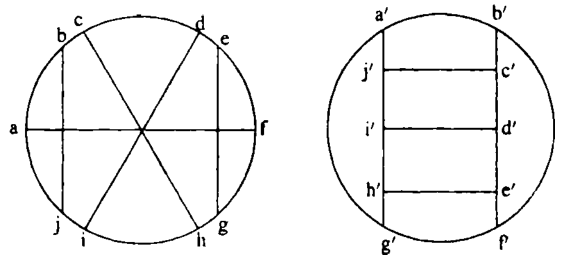
Solution: $\sum_{i=1}^{|V|} d(v_i) = 2|E| = 2 \times 24 = 48$

$|V|$ (degrees of all vertices) $= 48$

$$|V| = \text{Number of vertices} = \frac{48}{\text{(degrees of all vertices)}}$$

$|V| =$ divisors of 48.

Problem 23 Show that the following graphs are not isomorphic



Solution: Basic Invariants are same for the two graphs.
But the first graph is having a cycle of length 3 ie, $a - b - j - a$, but the second graph contains no cycle of length 3.
$\therefore$ two graphs are not isomorphic.

Problem 24 Let G be a simple graph with $|E| = 2|V| - 3$ and all vertices of degree 3. What can be said about G?

Solution: $\sum_{i=1}^{|V|} d(v_i) = 2|E|$

$$3|V| = 2|E| = 2(2|V| - 3)$$

$$\Rightarrow \quad |V| = 6$$

Number of vertices in $G = 6$

$\therefore$ G is isomorphic to K_{33}

Problem 25 Can a simple graph with 7 vertices be isomorphic to its complement?

Solution: A graph with 7 vertices can have a maximum number if edges $= \frac{7(7-1)}{2} = 21$ edges it is not possible to split 21 edges into two integers.

Therefore G and G^c cannot have equal number of edges. Hence a graph with 7 vertices can't be isomorphic to its complement.

Problem 26 Prove that for any polyhedral graph $3|R| - 6 \geq |E|$

Solution: Since the graph is polyhedral.

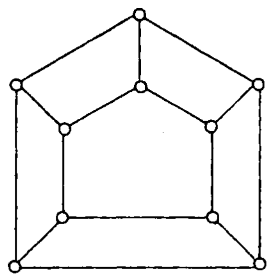
We can have the following four inequalities $3|R| \leq 2|E|$, $3|V| \leq 2|E|$, $|R| = |E| - |V| + 2$, $|E| \leq 3|V| - 6$

Now $3|V| \leq 2|E|$

$$3[|E| - |R| + 2] \leq 2|E|$$

$$\Rightarrow |E| \leq 3|R| - 6$$

Problem 27 The graph below does posses Hamiltonian Cycle. Show that any such cycle containing one of the edge e, e' must avoid the other.



Solution: There are 5 regions of degree 4 and two regions of degree 5.
Thus for any Hamiltonian cycle, by Grinberg's theorem

$$\sum_{i=3}^{n} (i - 2)(r_i - r_i') = 0$$

$$(4 - 2)(r_4 - r_4') + (5 - 2)(r_5 - r_5') = 0$$

$$\Rightarrow 2(r_4 - r_4') = -3(r_5 - r_5')$$

clearly 3 divides $r_4 - r_4'$.

But $r_4 - r_4' = 5$, the only possible values of r_4 and r_4' are 4 and 1, making $r_4 - r_4'$ either 3 or -3.

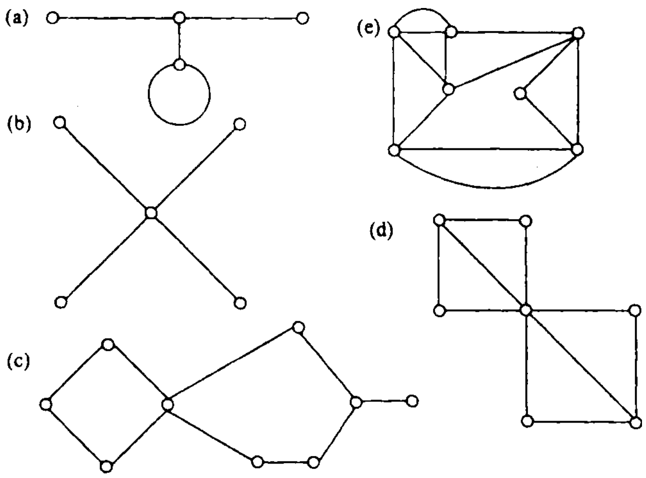
Now each of the edge e and e' seperates a pair of regions of degree 4.

Thus a HC would have one of e' Quadrilaterals inside and the other outside.

Similarly a HC containing the edge e' would split quadrilaterals.

If both e and e' belong to HC then three would be atleast two regions of degree 4 on the inside, and atleast two regions on outside.

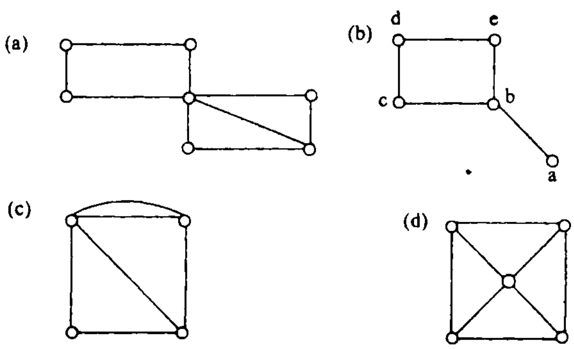
Problem 28 Tell whether the graph has an Euler circuit, an Euler path but no Euler circuit or neither.



Solution:

(a) Neither, there are four vertices of odd degree
(b) Neither, there are four vertices of odd degree
(c) Euler path only, since exactly two vertices have odd degree
(d) There is no Euler Circuit, but there must be an Euler path, since graph has exactly two vertices of odd degree
(e) every vertex has even degree, thus the graph must have an Euler Circuit.

Problem 29 Determine whether the graph shown has a Hamiltonian Circuit, a Hamiltonian path but no Hamiltonian circuit or neither. If the graph has a HC, give the circuit.

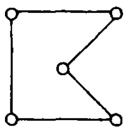


Solution:

(a) Neither

(b) The path abcde is a Hamiltonian path because it contains each vertex exactly once; however no HC.

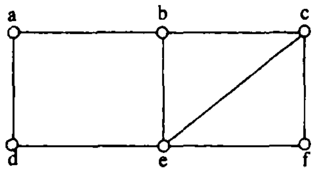
(c) The path adcba is a HC

(d) HC 

Problem 30 What is the maximum number of edges possible in a planar graph with eight vertices?

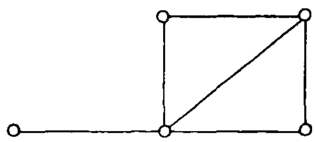
Solution: From the theorem, If G be a connected planar graph (not multigraph) with $|V|$ vertices and $|E|$ edges, where $|V| \geq 3$.
Then $|E| \leq 3|V| - 6$
For a planar graph with $|V| = 8$, the maximum number of edges possible in a planar graph with eight vertices ($|E| \leq 3 \times 8 - 6 = 18$) is 18.

Problem 31 Let G be the graph. Does G have any cut points?

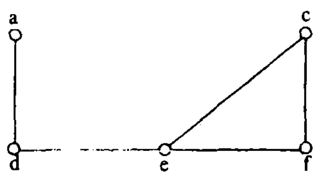


Solution:

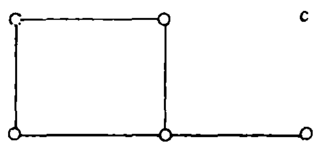
(G − a) :



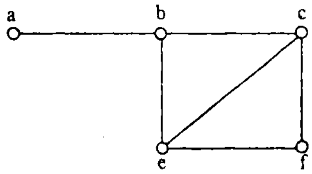
(G − b) :



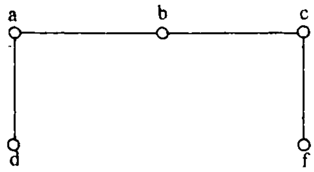
(G − c) :



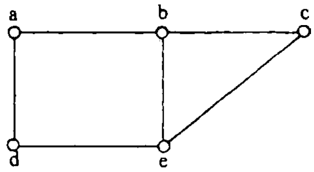
(G - d) :



(G - c) :

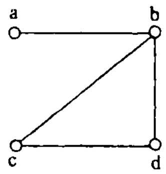


(G - f) :

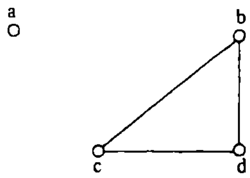


Above figures show that G - v is connected for any vertex v of G. Thus *G* has no cut points.

Problem 32



Let *G* be the above graph. Does *G* have any cut edge $G - \{a, b\}$:

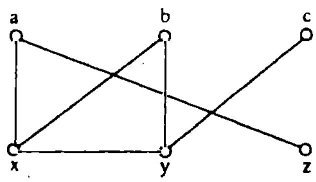


shows only

Solution: $G - \{a, b\}$ is disconnected.

Hence {a, b} is a cur edge and the only cut edge for *G*.

Problem 33

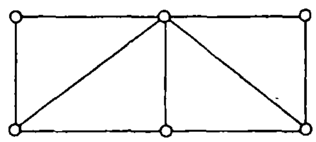


Does G have any bridges?

Solution: G has three cut edges {a, x}, {a, z} and {y, c}deleting any other edge o G does not disconnect G.

Problem 34 Draw a graph with six vertices which is hamiltonian but not Eulerian

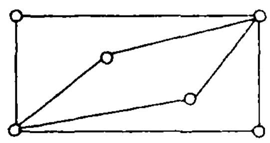
Solution:



Hamiltonian and non eulerian

There are many possible solutions to this problem and one of these is shown above. Every solution however, must have a cycle that includes every vertex exactly once (hamiltonian), but must not have a closed trail that uses every edge exactly once (eulerian). Note that when a candidate hamiltonian graph has been identified, one can easily determine if it is eulerian by looking for vertices of odd degree, should atleast one such vertex exist, the graph is not eulerian.

Problem 35 Draw a graph with six vertices which is eulerian but not hamiltonian.



Eulerian and nonhamiltonian

Solution: There are many possible solutions, one of which is shown above. Every solution, however, must have a closed trail that uses every edge exactly once (eulerian), but must not have a cycle that includes every vertex exactly once (hamiltonian). From Euler's theorem we know that any graph with all vertices of even degree is eulerian. But once a candidate eulerian graph has been identified, there is no simple criterion for determining whether or not the graph is hamiltonian.

Problem 36 Show that every cubic graph has even number of vertices.

Solution: If G is a regular graph of degree 3, it is called a cubic graph.

Let G be a cubic graph with $|v|$ vertices.

$$\text{Then } \Sigma \deg(v_i) = 3|V| \tag{4.1}$$

LHS of 1 is even $\therefore 3|V|$ is even

$\Rightarrow |V|$ is even.

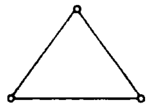
Problem 37 Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Solution: consider a simple graph on n vertices. choose $n-1$ vertices $v_1, v_2 \ldots v_{n-1}$ of G.

Clearly the maximum number of edges only can be drawn between these vertices is $(n-1)_{c_2} = \frac{(n-1)(n-2)}{2}$ Thus if we have more than $\frac{(n-1)(n-2)}{2}$ edges atleast one edge should be drawn between the nth vertex v_n to some vertex v_i, $1 \le i \le n-1$ of G.

Hence G must be connected.

Problem 38 Give an example of a connected planar graph for which $|E| = 3|V| - 6$



Solution: Since $E = 3 = 3(3) - 6 = 3|V| - 6$

Problem 39 A Connected planar graph G has 20 vertices Prove that G atmost 54 edges.

Solution: From $|E| \le 3|V| - 6$
then $|E| \le 3(20) - 6 = 54$

Problem 40 suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Solution:

$$|V| = 20$$
$$2|E| = \Sigma \deg(v_i) = 3 \times 20 = 60$$
$$\Rightarrow |E| = 30$$
$$\therefore |R| = |E| - |V| + 2$$
$$= 30 - 20 + 2$$
$$= 12$$

Note: For $K_{3,3}$

$$|E| = 9 \le 12 = 3(6) - 6$$

that is $|E| \le 3|V| - 6$ is satisfied

Consequently, the fact that the inequality $|E| \le 3|V| - 6$ is satisfied *does not* imply that a graph is planar.

Problem 41 Show that $K_{3,3}$ is nonplanar

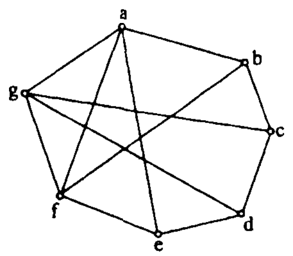
Solution: Since $k_{3,3}$ has no circuits of length 3, Also $k_{3,3}$ has six vertices and nine edges.

$$|E| = 9, 2|V| - 4 = 2 \times 6 - 4 = 8$$
$$\Rightarrow \text{from} |E| \leq 2|V| - 4$$
$$9 \leq 8$$

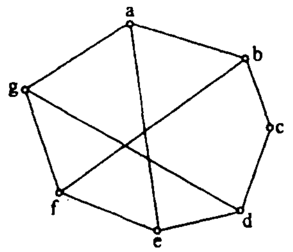
a contradiction

$\therefore$ $K_{3,3}$ is nonplanar.

Problem 42 show that the following graph is not planar.



Solution: By deleting two edges (g, c), (a, f) we have the following graph.



Problem 43 Prove that the number of edges in a bipartite graph with n vertices is at most $\frac{n^2}{4}$.

Solution: Let x be the number of vertices in one of bipartition sets. Then $n - x$ is the number of vertices in the other. The largest number of edges occurs when all x vertices in one set are joined to all $n - x$ vertices in the other.

So the number of edges is atmost $x(n - x)$.

The function $f(x) = x(n - x)$ (whose graph is a parabola) has a unique maximum at $\left(\frac{n}{2}, \frac{n^2}{4}\right)$ so $x(n - x) \leq \frac{n^2}{4}$ for all x

Hence the result.

Problem 44 Prove that in any graph with more than one vertex there must exist two vertices of the same degree

Solution: There are n vertices and n possible degrees for the vertices; namely 0,1,2, ... n-1. If however we have vertex of degree 0, then it is not possible to have another vertex of degree $n-1$. Hence there are really only $n-1$ possible 'holes' into which the n vertices can fit. Hence some vertex degree is repeated.

Problem 45 suppose all vertices in a graph have odd degree k show that total number of edges in G is a multiple of k.

Solution: Let G be a graph with all vertices of odd degree k.

$$\therefore \Sigma \deg(v_i) = K|V|$$

By Fundamental theorem of graph theory

$\Sigma \deg(v_i) = 2|E|$

$\Rightarrow$ $2|E| = k|V|$

$\Rightarrow$ $k$divides $2|E|$

But k is odd

Therefore k divides $|E|$.

Hence number of edges in G is a multiple of k.

Problem 46 Explain why any graph is isomorphic to a subgraph of some complete graph?

Solution: Any graph G with n vertices is a subgraph of K_n. Therefore we can have a graph from G by joining any pair of vertices of G where there is not already on edge, and this new graph is isomorphic to G.

Problem 47 Show that If G is a simple graph with n vertices, then the union of G and G^c is K_n.

Solution: The union of G and G^c contains an edge between each pair of the n vertices Hence this union is K_n.

Problem 48 show that a simple graph G with n vertices is connected if it has more than $\dfrac{(n-1)(n-2)}{2}$ edges.

Solution: Suppose that G is not conn. d. Then it has a component of k vertices for some k, $1 \leq k \leq n-1$. The most dges G could have is $k_{c_2} + (n-k)_{c_2} = \frac{1}{2}[k(k-1) + (n-k)(n-k-1)] = k^2 - nk + \dfrac{n^2-n}{2}$.

This function $f(k) = k^2 - nk + \dfrac{n^2-n}{2}$ is minimized at $k = \frac{n}{2}$ and maximized at $k = 1$ or $k = n-1$. Hence if G is not connected, the number of edges does not exceed the values of this function at 1 and at $n-1$ namely $\dfrac{(n-1)(n-2)}{2}$.

Problem 49 show that a graph G is self dual if $|E| = 2n - 2$ where n is the number of vertices in G.

Solution: Let G^* denote the self dual of G

Since G and G^* are isomorphic
we have

$$|E| = |E^*|, |R^*| = n, |V| = |V^*| = n$$

by Euler's formula

$$\begin{aligned}|E^*| &= |V^*| + |R^*| - 2\\ &= n + n - 2\\ &= 2n + 2\end{aligned}$$

Problem 50 If f denotes the number of regions in a graph G, then shown that $n \geq 2 + \frac{f}{2}$

Solution: We know that

$$\begin{aligned}3f &\leq 2|E|\\ &\Rightarrow |E| \geq \frac{3}{2}f\end{aligned}$$

By Euler's formula

$$|v| = |E| - |R| + 2 \geq \frac{3f}{2} - f + 2$$

$$\Rightarrow n \geq 2 + \frac{f}{2}$$

□

Quiz Questions

1. $\sum_{i=1}^{n} deg(V_i) =$ ——— in an undirected graph G with n vertices and $|E|$ edges

   (Ans: $2|E|$)

2. The number of vertices of odd degree is ———

   (Ans: even)

3. $\sum deg^+(v) = \Sigma\ deg^-(v) =$ ———

   (Ans: $|E|$)

4. In a simple graph G, ——— $\leq deg(v) \leq$ ———

   (Ans: 0, n - 1)

5. $\delta(G) \leq$ ——— $\leq \Delta(G)$

   $\left(\text{Ans: } \frac{2|E|}{|V|}\right)$

6. In a K–regular graph, $\delta(G) = \Delta(G) =$ ———

   (Ans: K)

7. In a Regular graph $\delta(G) = \Delta(G) =$ ———

   $\left(\text{Ans: } \frac{2|E|}{|V|}\right)$

8. Is the degree sequence (1, 2, 3, 4) is graphic?

   (Ans: No)

9. In a simple graph $|E| \leq$ ———

   (Ans: $|V|_{c_2}$)

10. In K_n, $|E| =$ ———

    $\left(\text{Ans: } \frac{|V|(|V|-1)}{2}\right)$

11. Every complete graph is ———

    (Ans: regular)

12. K_3 is ——— bipartite

    (Ans: not)

13. $A(G) = A'(G') \Longleftrightarrow G$ and G' are ———

    (Ans: isomorphic to each other)

14. Euler's formula: $|V| - |E| + |R| =$ ———

(Ans: 2)

15. A complete graph K_n is planar $\Longleftrightarrow$ $n \leq$ ———

(Ans: 4)

16. A complete graph $K_{m,n}$ is planar $\Longleftrightarrow$ $m \leq$ - - - - - - - - - - - - - or $n \leq$ - - - - - - - - - - - -

(Ans: 2)

17. A graph G is non planar $\Longleftrightarrow$ if it contains a subgraph ——— to $K_{3,3}$ or K_5

(Ans: homeomorphic)

18. A nondirected multigraph has an Euler path $\Longleftrightarrow$ it is conncected and has — or exactly ——— vertices of ——— degree

(Ans: 0, 2, odd)

19. A nondirected multigraph has an Euler circuit $\Longleftrightarrow$ it is connected and all of its vertices are of ——— degree

(Ans: even)

20. In Grin berg's theorem $\sum_{i=3}^{n}(i-2)(r_i - r_i') =$ ———

(Ans: 0)

21. $\chi(K_n) =$ ———

(Ans: n)

22. $\chi(K_{m,n}) =$ ———

(Ans: 2)

23. $\chi(G) \leq$ ——— always

(Ans: $|V|$)

24. $\chi(G) \leq$ ——— always

(Ans: $1 + \Delta(G)$)

25. $\chi(K_3) =$ ———

(Ans: 3)

26. $K_{m,r}$ has ——— edges

    (A: mn)

27. If A is the adjacency matrix of the digraph G then for each $K \geq 0$, A^K is the adjacency matrix of the diagraph of ——— of G

    (Ans: K-stage paths or paths of length K)

28. Let A be the adjacency matrix of a graph G with m vertices and Let

$$B_m = \sum_{i=1}^{m} A^i$$

    Then G is ——— $\Longleftrightarrow$ if B_m has no zero entries

    (Ans: Strongly connected)

29. A graph is bipartite $\Longleftrightarrow$ if it can be coloured with ——— colours

    (Ans: two)

30. A graph is bipartite $\Longleftrightarrow$ it has no circuit of ——— length

    (Ans: odd)

5. Trees

1. Introduction

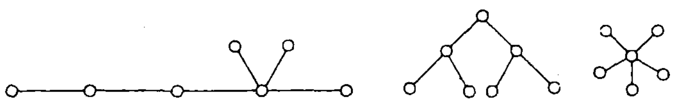
Trees form one of the most widely used subclasses of graphs. In Computer Science, trees are useful in organizing and relating data in a data base. Trees arise in many practical applications; frequently they occur in situations where many elements are to be organized into some sort of hierarchy that expresses what is more important, what must be done first, or what is more desirable. The concept of Tree is very important in Computer Science, in maintaining the files and directories by OS, in system Software for parsing the expressions. Trees were discovered by kirchoff while investigating electrical networks. Cayley used them to enumerate isomers of saturated hydrocarbons. Apart from all these, Trees have many applications.

Now, we introduce the basic terminology of trees.

2. Trees and their properties

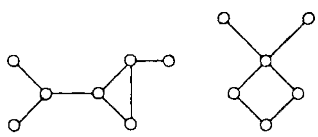
Definition 2.1 A tree is a connected, undirected, simple acyclic graph. In other words, a tree is a simple graph G such that there is a unique simple undirected path between each pair of vertices of G.

Example 2.1



G_1 G_2 G_3

G_1, G_2 and G_3 are trees



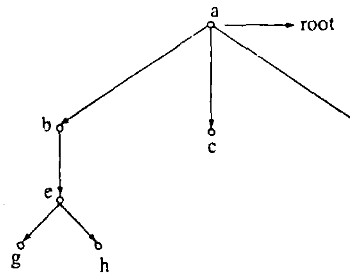
G_4 G_5

G_4 and G_5 are not trees, Since they have cycles.

Definition 2.2 A rooted tree is a tree in which a particular vertex is designated as the root.

Definition 2.3 A rooted tree is a directed tree if there is a root from which there is a directed path to each vertex of the tree.

Example 2.2

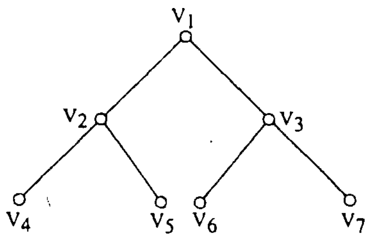


A directed, as well as rooted tree.

Definition 2.4 The *level* of a vertex v in a (rooted) tree is the length of the simple path from the root.

The height of a rooted tree is the maximum level number that occurs.

Example 2.3



The vertices v_1, v_2, v_3, v_4, v_5, v_6, v_7 in the rooted tree are on levels 0, 1, 1, 2, 2, 2, 2 respectively. The height of the tree is 2.

Definition 2.5 A tree T with only one vertex is called a trivial tree otherwise T is a nontrivial tree

Note 2.1 In a directed root, there is exactly one root. In a tree any vertex may be designated as root.

The definition of tree mentioned above applies to undirected graphs as well as directed graphs

Trees have many equivalent characterizations, any of which can be taken as the definition. Such characterizations are useful because we need only to verify that a graph satisfies any one of them to prove that it is a tree

Now, we discuss a few interesting properties of trees

Theorem 2.1 A simple undirected graph G is a tree if and only if G is connected and contains no cycles.

Proof: Suppose that G is a tree. Since each pair of vertices are joined by a path, G is connected. If G contains a cycle containing distinct vertices u and v, then u and v are joined by at least two simple paths, the one along one portion of the cycle and the other path completing the cycle. This contradicts the hypothesis that there is a unique simple path between u and v, and thus a tree has no cycles.

Conversely, Suppose that G is connected and contains no cycles. Let a and b be any pair of vertices of G. If there are 2 different simple paths P_1 and P_2 from a to b, then we can find a cycle in G as follows: Since P_1 and P_2 are different paths there must be a vertex v_1 (possibly $v_1 = a$) on both paths such that the vertex following v_1 on P_1 is not the same as the vertex following v_1 on P_2. Since P_1 and P_2 terminate at b, there is a first vertex after v_1, call it v_2, which P_1 and P_2 have in common (possibly $v_2 = b$). Thus, the part of P_1 from v_1 to v_2 together with that part of P_2 from v_1 to v_2 form a cycle in G. This contradicts the assumption that G has no cycles. Therefore, G has exactly one path joining a and b. □

Corollary 2.1 If G is a nontrivial tree then G contains at least 2 verties of degree 1.

Proof: Let n = the number of vertices of G. By the sum of degrees formula,

$$\sum_{i=1}^{n} \deg(v_i) = 2|E| = 2(n-1) = 2n-2.$$

Now if there is only one vertex, say v_1 degree 1, then

$$\deg(v_i) \geq 2 \text{ for } i = 2, \ldots, n$$

and

$$\sum_{i=1}^{n} \deg(v_i) = 1 + \sum_{i=2}^{n} \deg(v_i) \geq 1 + 2n - 2 = 2n - 1$$

$$\Rightarrow \quad 2n - 2 \geq 2n - 1$$

or

$$-2 \geq -1$$

a contradiction.

Hence the theorem. □

Corollary 2.2 If 2 nonadjacent vertices of a tree T are connected by adding an edge, then the resulting graph will contain a cycle

Theorem 2.2 A graph G is a tree $\Longleftrightarrow$ G has no cycles and $|E| = |V| - 1$.

Proof: From previous theorem, we already proved one half of the theorem. To prove the other half we need only show that if G has no cycles and $|E| = |V| - 1$, then G is connected. Assume that G is not connected

Denote by $G_1, G_2, \ldots, G_k$ the components of G, where $K \geq 1$.

Let $|V_i| =$ the number of vertices of G_i.

Now each G_i is a tree, for G_i is connected and G_i contains no cycles, since G does not.

Thus G_i has $|V_i| - 1$ edges.

Hence G has $(|V_1| - 1) + (|V_2| - 1) + \ldots + (|V_k| - 1)$

$$= |V_1| + |V_2| + \ldots + |V_k| - K$$
$$= |V| - K \text{ edges}$$

By hypothesis, G has $|V| - 1$ edges.

Thus $k = 1$ i.e., $C(G) = 1$ $\therefore$ G is connected. □

All the above theorems can be stated as follows

Let T be a graph with n vertices. The following are equivalent

(a) T is a tree

(b) T is connected and a cyclic

(c) T is connected and has $n - 1$ edges

(d) T is a cyclic and has $n - 1$ edges

i.e., $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$.

Theorem 2.3 Every tree contains exactly one central vertex or two adjacent central vertices.

Proof: Let T be a tree on n vertices. Theorem will be proved by induction on n If $n = 1$ or 2, then the result is trivial.

Assume that the statement of the theorem holds good for lesser values of $n (n > 2)$.

Every tree contains atleast two end vertices.

The distance from any vertex x to the vertex y is maximum if and only if y is an end vertex of T. Moreover, there is a unique path between x and y in T.

Hence removal of all the pendent vertices of the T reduces the eccentricities of every other vertices exactly by one. Let T^1 be a tree obtained by removing all the pendent vertices of T. Then central vertices of T^1 are the same as that of T. Repeating the process of deletion we get a nested trees $T^1, T^{11} \ldots$ Finally we get K_2 graph. The end vertices of K_2 are the central vertices of the tree T. As these are adjacent, we conclude that the central vertices are adjacent. □

Definition 2.3 A connected graph G is said to be minimally connected if removal of any one edge from G disconnects it.

Theorem 2.4 A graph is a tree $\Longleftrightarrow$ it is minimally connected.

Proof: Let G be a tree

To prove G is minimally connected, let us assume that G is not minimally connected, that is removal of the edges from the graph does not disconnect it.

⇒ The edge is in some cycle, which is a contradiction to the statement that the graph G is a tree.

Conversely, G is minimally connected.

To prove G is a tree.

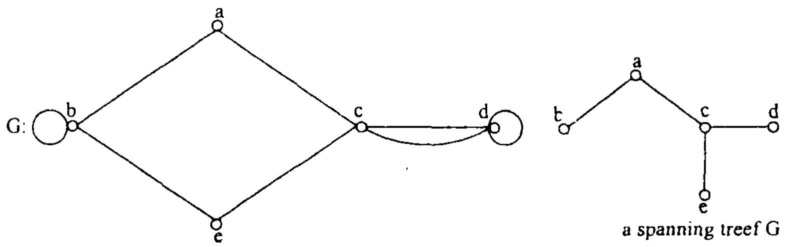
Since G is minimally connected ⇒ G is connected and removed of one edge disconnects the graph.

⇒ G is a tree. □

3. Spanning Trees

Definition 3.1 A tree T is a *spanning tree of a graph* G is T is a subgraph of G that contains all of the vertices of G.

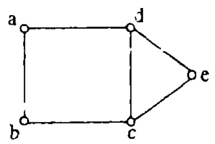
Example 3.1



A spanning tree that is a directed tree is called a directed spanning tree of G

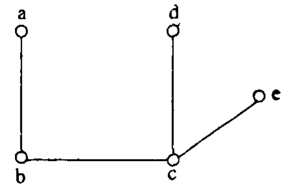
Definition 3.2 In general, if G is connected graph with n verties and m edges, a spanning tree of G must have $n - 1$ edges. Hence the number of edges that must be removed before a spanning tree is obtained must be $m - (n - 1) = m - n + 1$. This number is called the *Circuit rank* of G

Example 3.2



Circuit rank $= 6 - 5 + 1 = 2$ i.e., to form a spanning tree two edges will be deleted such that the obtained graph does not contain any cycles further.

$\therefore$ deleting $\{a, d\}$, $\{d, e\}$ edges, we have the spanning tree



Kirchhoff Theorem 3.1

Let A be the adjacency matrix of a connected graph G and M be the matrix obtained from the adjacency matrix of a connected graph G by changing all I's to -1 and each diagonal 0 to the degree of the corresponding vertex. Then the number of spanning trees of G is equal to the value of any cofactor of M.

Example 3.3 Consider the graph whose adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

The matrix specified in Kirchhoff's theorem is

$$M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

The $(1, 1)$ cofactor of M is $+det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = 3(4) + 1(-2) + (-1)2 = 8$

Kirchhoff's theorem guarantees that all the cofactors of M equal 8. Again (2, 3) factor of M is 8.

Thus there are eight spanning Trees in the graph

Theorem 3.2, An undirected graph G has a spanning tree if and only if G is connected.

Proof: Suppose that a graph G has a spanning tree T.

Let a and b be verties of G. Since a and b are also verties in T and T is a tree, there is a path P from a to b.

However, P also serves as a path from a to b in G; thus G is connected.

Conversely, Suppose that G is connected. If G is acyclic, then G is a tree.

Suppose that G contains a cycle. We remove an edge (but no vertices) from this cycle. The graph produced is still connected. If it is acyclic, we stop. If it contains a cycle, we remove an edge from this cycle. Continuing in this way. We eventually produce an acyclic, connected subgraph T. By known theorem, T is a tree. Since T contains all the vertices of G, T is a spanning tree of G. □

Note 3.1 The complete graph K_n has n^{n-2} different spanning trees.

Definition 3.3 Let T be a rooted tree with designated root v_0. Suppose that u and v are vertices in T and that $v_0 - v_1 \ldots - v_n$ is a simple path in T. Then

(a) v_{n-1} is the *parent* of v_n

(b) $v_0, v_1, \ldots, v_{n-1}$ are the *ancostors* of v_n

(c) v_n is a *child* of v_{n-1}

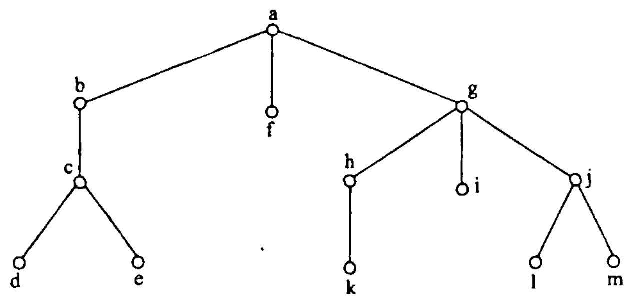
(d) If u is an *ancostor* of v, then v is a *descendant* of u.

(e) If u has no children, then u is a *leaf* of T

(f) If v is not a leaf of T, then v is an *internal vertex* of T

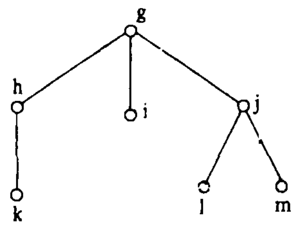
(g) The subgraph of T consisting of v and all its descendants with v designated as a root, is the *subtree* of T rooted at v.

Example 3.3



The parent of c is b. The Children of g are h, i and j. The siblings of h are i and j. The ancestors of e are c, b & a. The descendants of b are c, d & e. Internal vertices are: a, b, c, g, h and j leaves (or) terminal vertices are: d, e, f, i, k, l and m.

The subtree rooted at g:



4. Breadth-First & Depth - First Spanning Trees

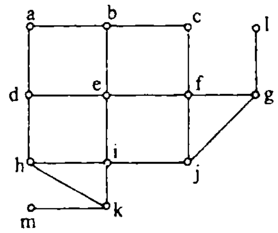
By the previous theorem, an algorithm for finding spanning trees by removing edges from simple cycles, is inefficient, Since it requires that simple cycles to be ideified. Instead of constructing spanning trees by removing edges, Spanning trees can be built by sucessively adding edges. Two algorithms based on this principle will be presented here. They are Breath-First search (BFS) and Depth-first search (DFS) algorithms.

4.1 BFS algorithm

The idea of BFS is to visit all vertices sequentially on a given level before going onto the next level

Procedure Arbitrarity choose a root from the vertices of a graph. Then add all edges incident to this vertex, such that the addition of edges does not produce any cycle. The new vertices added at this stage become the vertices at Level 1 in the spanning tree, arbitrarity order them. Next, for each vertex at Level 1, visited in order, add each edge incident to this vertex to the tree as long as it does not produce any cycle. Arbitrarity order the children of each vertex at level 1. This produces the vertices at Level 2 in the tree. Follow the same procedure until all the vertices in the tree have been added. The procedure ends, since there are only a finite number of edges in the graph. A spanning tree is produced since we have produced a tree containing every vertex of the graph.

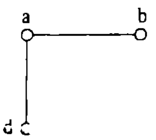
Example 4.1 Use BFS to find a Spanning tree for the graph



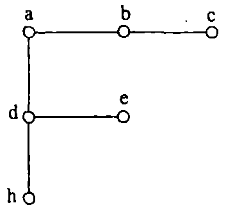
Solution:

(i) Choose the vertex 'a' as the root.

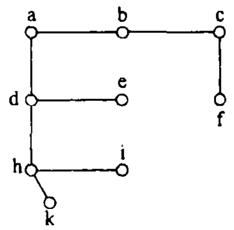
(ii) Then add all edges incident with all vertices adjacent to a, so that edges $\{a, b\}$, $\{a, d\}$ are added. The vertices b, d are at Level 1 in the tree



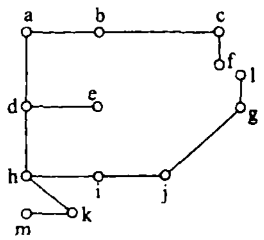
(iii) Next, add the edges from these vertices at level 1 to adjacent vertices not already in the tree. Hence, the edges $\{b, c\}$, $\{d, h\}$ $\{d, e\}$ are added. Now c, h, e are at the level 2.



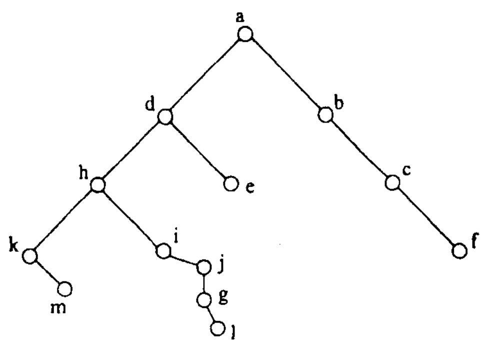
(iv) Next, add the edges from these vertices to adjacent vertices not already in the graph. Hence the edges $\{c, f\}$, $\{h, i\}$ $\{h, k\}$ are added, don't add $\{e, i\}$, $\{e, f\}$. Since adding this edge produce a cycle. Now i, f, k are at level 3.



(v) Add $\{i, j\}$, $\{k, m\}$, Now j is at the level 4, add $\{j, g\}$, g at level 5 add $\{g, l\}$.



∴ The spanning tree is



4.2 DFS algorithm

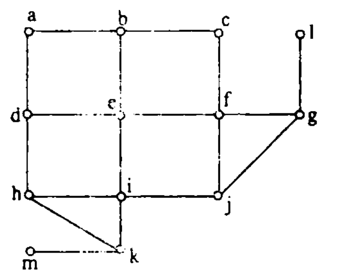
The idea of DFS is proceeding to higher levels successively in the first opportunity. Later we back track and add the vertices which are not visited.

Procedure Choose a vertex as the root of the spanning tree T arbitrarily. Form a path starting at this vertex by successively adding edges where each new edge is incident with the last vertex in the path and the vertex not already in the path. Continue this process by adding edges to this path as long as possible without producing any cycles.

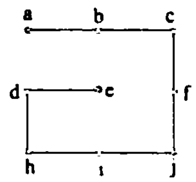
If the path goes through all the vertices in the graph, then the tree is a spanning tree. Otherwise more edges must be added, move back to next vertex to last vertex in the path and if possible, form a new path starting at this vertex passing through vertices that are not already visited. By this also, if all the vertices are not visited, move back to another vertex and try again. Repeat this process beginning at the last vertex visited. Moving back the path one vertex at a time forming new paths that are as long as possible.

Until no more edges can be added. This process ends at the production of a spanning tree.

Example 4.2 Use DFS to construct a spanning tree

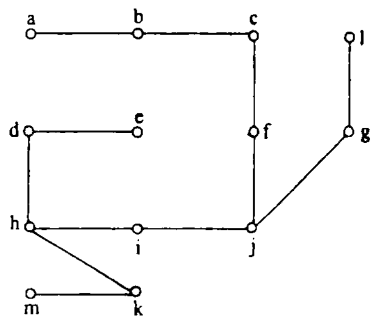


(i) Start with the vertex a, build a path by successively adding edges incident with vertices not already in the path, as long as this is possible.



This produces a path $a - b - c - f - j - i - h - d - e$

(ii) Now back track to d. There is no path beginning at d containing vertices not already visited. So move back track to h and form the path $h - k - m$. Now back track to K, and h, and i and j then form the path $j - g - l$. This produces the spanning tree



4.3 Algorithm BFS

BFS for a spanning tree T.
Input : A connected graph G with vertices labelled $v_1, v_2, \dots v_n$
Output : A Spanning tree T for G
Step 1 : Let v_1 be the root of G. For the set $V = \{v_1\}$.
Step 2 : Adding the new edges. Consider the vertices of the graph in order, consistent with the original Labelling, then for each vertex $x \in V$ add the new edge $\{x, v_k\}$ to T, where k is the minimum index such that adding the edge $\{x, v_k\}$ does not produce any cycle.
If no edge can be added, then STOP. Then T is a spanning tree for G.
After all the vertices of V have been considered in order, go to step 3.
Step 3 : Replace V by all the children v in T of the vertices x of v where the edges $\{x, v\}$ were added in Step 2. Go back and repeat step 2 for the new set V.

4.4 Algorithm DFS

DFS for a spanning Tree T.
Input : A connected graph G with vertices labeled $v_1, v_2, \dots v_n$
output : A spanning tree T for G
Step 1 : (visit a vertex) Let v_1 be the root of T and set $L = v_1$ (The name L stands for the vertex Last Visited)
Step 2 : (Find an unexamined edge and an unvisited vertex adjacent to L). For all vertices adjacent to L, choose the edge $\{L, v_k\}$, where K is the minimum index such that adding $\{L, v_K\}$ to T does not create a cycle. If no such edge exists, go to step 3; otherwise add edge $\{L, v_K\}$ to T and set $L = v_K$; repeat step2 at the new value for L.
Step 3 : (Back track or terminate) If x is the parent of L in T set $L = x$ and apply step 2 at the new value of L. If on the otherhand, L has no parent in T (so that $L = V_1$) the DFS terminates and T is a spanning tree of G.

Note 4.1 whenever the given connected graph G is with labeled vertices $v_1, v_2, \dots, v_n$; v_1 be the root of the spanning tree T, and selecting the next vertex with minimum index, we will get a *unique* spanning tree either by DFS or BFS.

5. Minimal Spanning Trees (MST)

Let us consider simple connected graph G, in which each edge e has been assigned some real number 'w' called its weight. Then G together with these weights on its edges is called a connected weighted graph

Definition 5.1 A minimal spanning tree of G is a spanning tree of G with minimum weight.

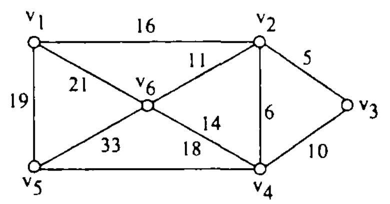
Now, we describe an algorithm to find or construct a MST. The algorithm is known as Kruskal's Algorithm.

5.1 Kruskal's Algorithm For finding a MST

Input : A connected weighted graph G
Output : A minimal spanning tree for G
Step 1 : Select any edge of minimal value that is not a loop. This is the first edge of T (If here is more than one edge of minimal valule, arbitrarity choose one of these edges).
Step 2 : Select any remaining edge of G having minimal value that does not form a circuit with the edges already included in T
Step 3 : Continue step 2 until T contains $n - 1$ edges, where n is the number of vertices of G.

Note 5.1 Simply, choose an edge in the graph with minimum weight, add the edges successively having minimum weight that do not form any cycles. Stop after $(n - 1)$ edges have been selected.

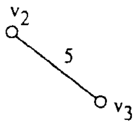
Example 5.1.1 Use Kruskal's Algorithm to find MST



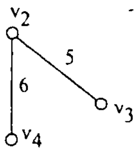
Solution:

edge	cost
$v_2 - v_3$	5
$v_2 - v_4$	6
$v_4 - v_3$	10
$v_2 - v_6$	11
$v_4 - v_6$	14
$v_2 - v_1$	16
$v_4 - v_5$	18
$v_5 - v_1$	19
$v_1 - v_6$	21
$v_5 - v_6$	23

(i) Choose the edge $v_2 - v_3$

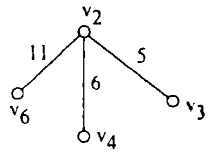


(ii) Add the Next edge with min. wt.



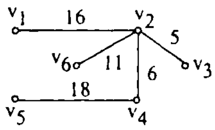
(iii) Reject the edge $v_4 - v_3$, because it will forms cycle

(iv) Add the edge $v_2 - v_6$



Reject $v_4 - v_6$, since it forms cycle

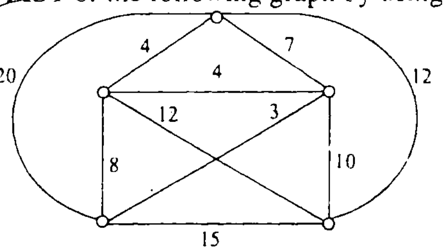
(v) Similarly, adding the edges, $v_2 - v_1$, $v_4 - v_5$



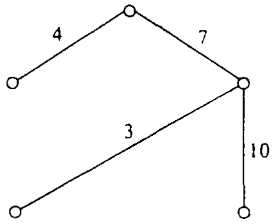
All the vertices of G are covered, therefore we stop the algorithm

$$\text{Cost (or) Weight of MST} = 5 + 6 + 11 + 16 + 18$$
$$= 56$$

Example 5.1.2 Give a MST of the following graph by using Krus' il's algorithm



Solution:



Minimum cost of MST = 3+4+7+10
= 24

6. Tree Traversals

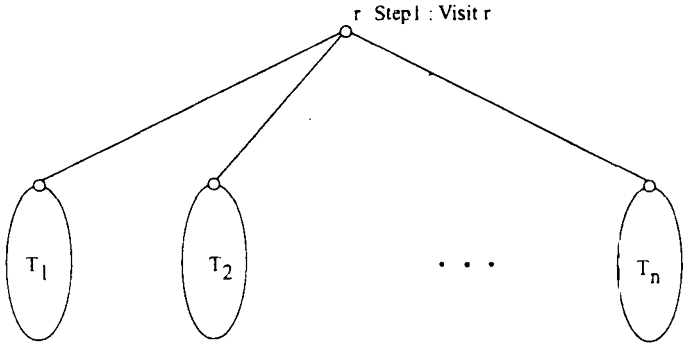
We can use trees to store information in a computer. Therefore, some procedures are needed for a accessing the information easily (or for visiting each vertex easily).

BFS and DFS provide ways to walk a true, that is traverse a tree in a systematic way sc that each vertex is visited exactly once. Now, we consider three additional tree traversal methods. We define these traversals recursively. They are :

(1) Pre-order traversal
(2) In-order traversal
(3) Post-order traversal

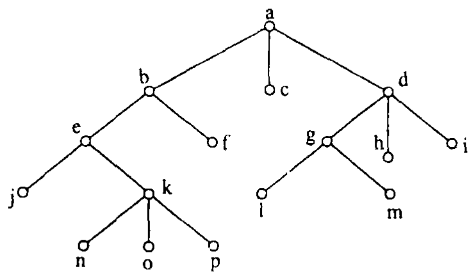
6.1 Pre-order traversal

If T is an ordered rooted tree with root r. If T consists only of r, then r is the preorder traversal of T. Otherwise Suppose that $T_1, T_2, \ldots, T_n$ are the substrees at r from left to right in T. The preorder traversal begins by visiting r. It continues by traversing T_1 in preorder, then T_2 in preorder and so on until T_n is traversed in preorder.

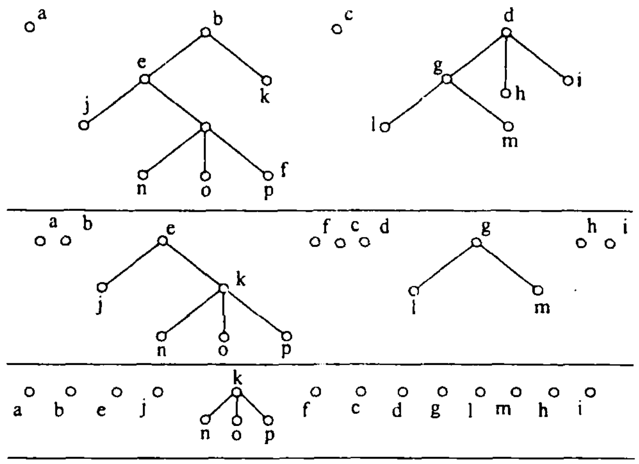


Preorder traversal : visit root, visit substress left to right in preorder

- **Example 6.1.1** Visit the following in preorder



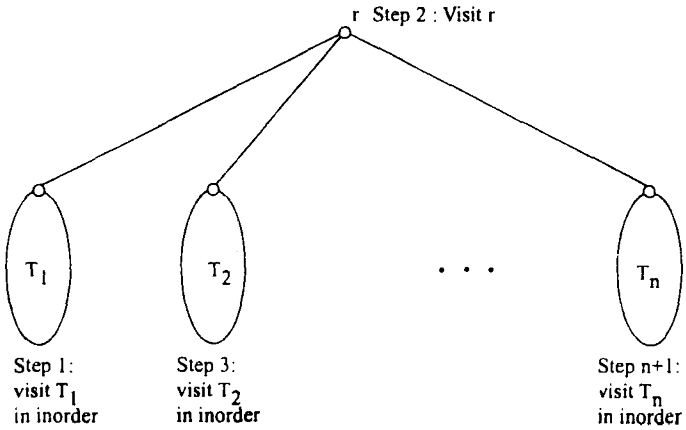
Solution:



a b e j k n o p f c d g l m h i : Preorder traversal of T.

6.2 Inorder traversal

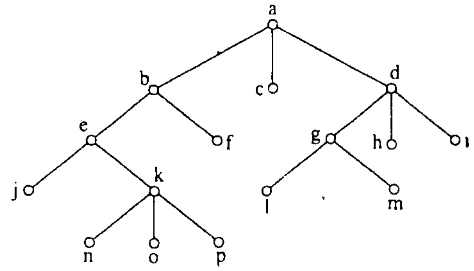
Let T be an ordered rooted tree with r. **If** T consists only of r, then r is the inorder traversal of T. otherwise suppose $T_1, T_2, \ldots, T_n$ are the subtrees at r from left to right.



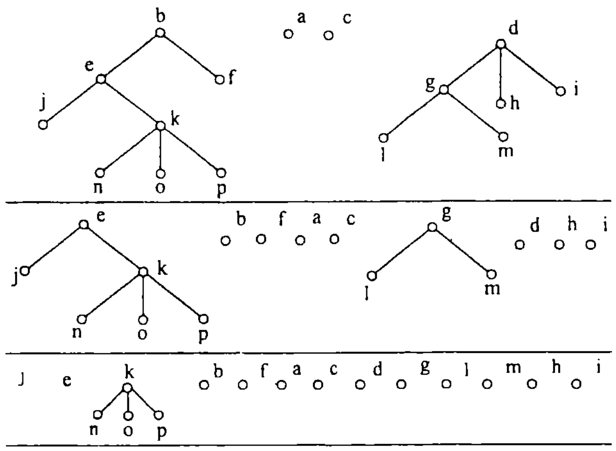
The inorder traversal begins by traversing T_1 in inorder, then visiting r. It continues by traversing T_2 in inorder, then T_3 in inorder, $\cdots$ and finally T_n in inorder.

In order traversal: Visit left most subtree, visit root, visit other subtrees left to right.

Example 6.2.1 Visit the following Tree in inorder



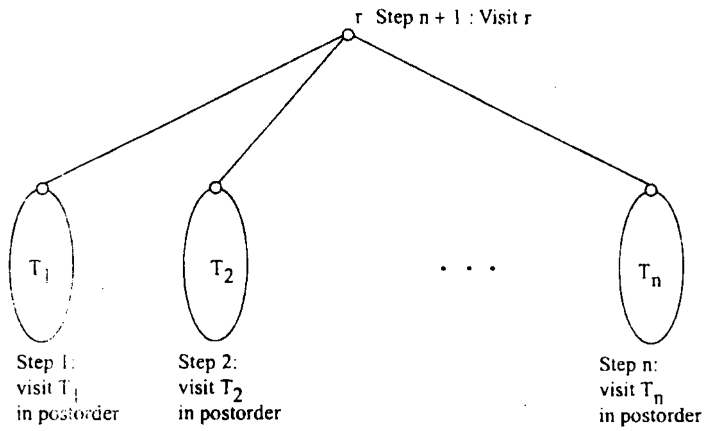
Solution:



j e n k o p b f a c l g m d h i : Inorder traversal of T.

6.3 Post order traversal

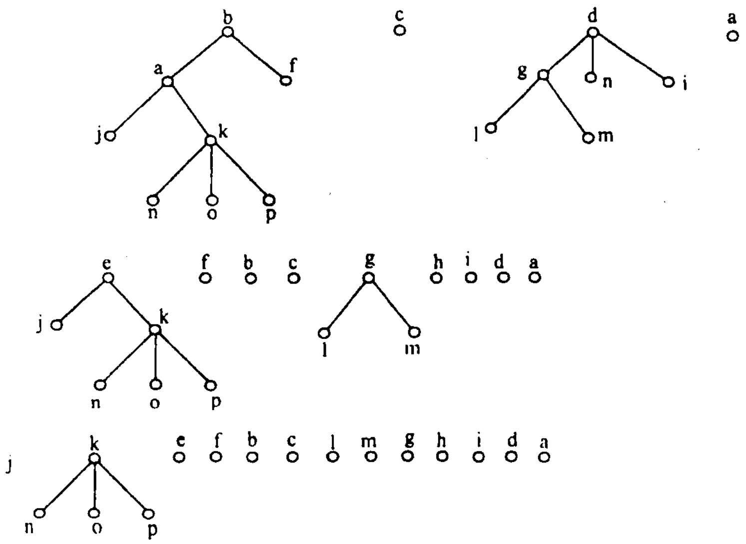
Let T be an ordered rooted tree with root r. If T consists only of r, then r is the post order traversal of T. Otherwise, Suppose that $T_1, T_2, \ldots, T_n$ are the subtrees at r from left to right. The post order traversal begins by traversing T_1 in post order, then T_2 in post order ... then T_n in post order, and ends by visiting r.



Post order traversal: visit subtrees left to right, visit root

Example 6.3.1 Visit the Tree in the previous example in post order.

Solution:



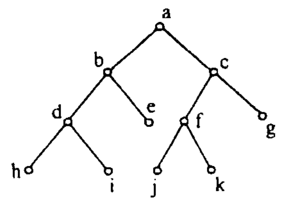
j n o p k e f b c l m g h i d a : post order traversal of T

Example 6.3.2 Write the preorder, inorder, post order traversal of the following Tree

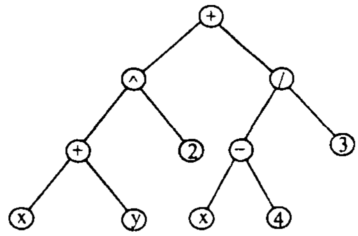
Preorder: a b d h i e c f j k g

Inorder: h d i b e a j f k c g

Post order: h i d e b j k f g c a



Example 6.3.3 Write the preorder, inorder, post order traversal of the following Tree

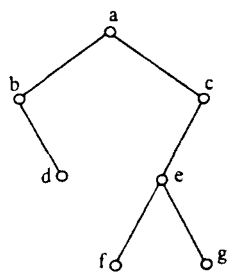


preorder:	$+ \wedge +xy2/-x43$
Inorder:	$\big((x+y)\wedge 2\big)+\big((x-4)/3\big)$
post order:	$xy+2\wedge x4-3/+$

7. Special Trees

Definition 7.1 A binary tree is a rooted tree in which each vertex has either no children, one child or two children. If a vertex has one child, that child is designated as either a left child or a right child (but not both). If a vertex has two children, one child is designated as left child and the other child is designated as right child.

Example 7.1



A binary Tree

Vertex b is the left child of vertex a and vertex c is the right child of vertex a. Vertex d is the right child of vertex b; vertex b has no left child. Vertex e is the left child of vertex e; vertex c has no right child.

Note: If a binary tree of height h has t terminal vertices, then

$$\log_2(t) \le h$$

i.e., if $h = 3$ and the number of **terminals** $t = 8$ **(maximum).**

Definition 7.2 A full binary tree is a binary tree in which each vertex has either two children of zero children.

Theorem 7.1 If T is a full binary tree with i internal vertices then T has $i+1$ terminal vertices and $2i+1$ total vertices.

Proof: The vertices of T consist of the vertices that are children (of some parent) and the vertices that are not children (of any parent). There is one nonchild-the root. Since there are i internal vertices, each having two children, there are $2i$ children. Thus the total number of vertices of T is $2i+1$ and the number of terminal vertices is $(2i+1)-i = i+1$. □

Binary trees are used extensively in computer science to store elements from an ordered set such as a set of numbers or a set of strings.

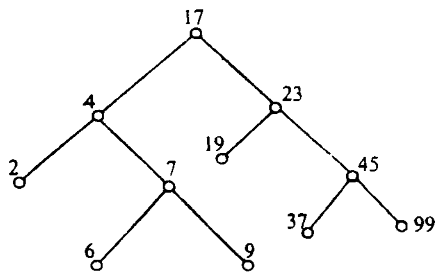
If data item $d(v)$ is stored in vertex v and data item $d(w)$ is stored in vertex w, then if v is a left child (or right child) of w, Some ordering relationship will be guaranteed to exist between $d(v)$ and $d(w)$.

Definition 7.3 A binary search tree is a binary tree T in which data are associated with the vertices. The data are arranged so that, for each vertex v in T, each data item in the left subtree of v is less than the data item in v and each data item in the right subtree of v is greater than the data item in v.

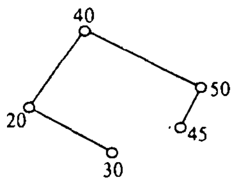
Binary search trees are useful for Locating data. That is, given a data item D, we can easily determine if D is in a binary search tree and if it is present, where it is Located.

Example 7.2 Draw the BST for the data given below taken in that order: (i) 17, 23, 4, 7, 9, 19, 45, 6, 2, 37, 99

Solution:



Example 7.3 (ii) 40, 20, 30, 50, 45



8. Some Applications of Trees, Binary Trees

8.1 Representation of algebraic expressions by Binary Tree

We can represent algebraic or arithmetic expressions involving the operators $+$, $-$, $*$, $/$ and $\uparrow$ (exponentiation) by Binary Trees i.e., (ordered rooted) Binary Trees. Where the internal vertices represent operations, and the leaves represent the variables (or) numbers. Each operation operates on its left and right subtrees (in that order).

i.e., a (operation) b can be diagrammed as

Note 8.1.1 When the operation is not '+', a (operation) b is different from b (operation) a.

A binary tree for an algebraic expression can be built from bottom up scheme.

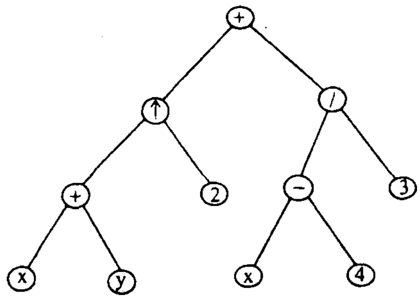
Example 8.1.1 What is the Binary tree that represents the expression

$$\big((x + y) \uparrow 2\big) + \big((x - 4)/3\big)?$$

Solution: The binary tree for the above expression can be built from bottom up scheme. First, a subtree for the expression $x + y$ is constructed. Then this is incorporated as part of the larger subtree representing $(x + y) \uparrow 2$. Also, a subtree for $x - 4$ is constructed and then this is incorporated into a subtree representing $(x - 4)/3$. Finally the subtrees representing $(x + y) \uparrow 2$ and $(x - 4)/3$ are combined to form the binary ·ee (ordered rooted tree) representing

$$\big((x + y) \uparrow 2\big) + \big((x - 4)/3\big).$$

Now the binary tree is:



8.2 Prefix, Infix and Post fix Notations of an arithemetic expression

We can represent expressions in three different ways. They are Infix, Prefix and Post fix forms of an expression.

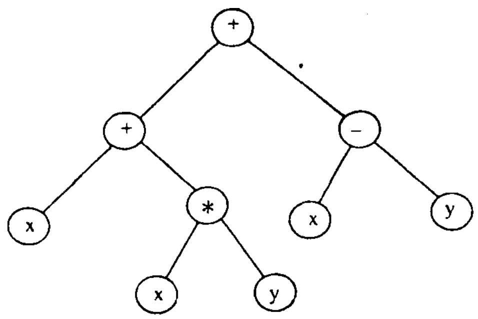
8.2.1 Infix Notation

The Infix form of an algebraic expression is the inorder traversal of the binary tree representing the expression. It gives the original expression with the elements and operations in the same order as they originally occured. That is, in the infix form of an expression, an operator appears between its operands.

Example 8.2.1.1 Give the infix form of the expression

$$(x + xy) + \left(\frac{x}{y}\right)$$

Solution: The corresponding Binary tree is



Infix form: $\left((x + (x * y)) + \left(\frac{x}{y}\right)\right)$

Note 8.2.1.1 To make the infix forms of an expression Unambiguous it is necessary to include parentheses in the inorder traversal whenever we encounter an operation.

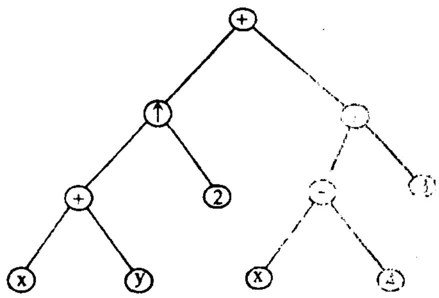
8.2.2 Prefix Notation

The Prefix form of an expression is the preorder traversal of the binary tree representing the given expression. Expressions written in prefix form are said to be in polish Notation. The expressions in prefix notation are unambiguous, so that no parentheses are needed in such expressions.

Example 8.2.2.1 Give the prefix form of the expression

$$((x + y) \uparrow 2) + ((x - 4)/3)$$

Solution: We can get the prefix form for this expression by traversing the binary tree that represents it:



Prefix form of the above expression is $+ \uparrow +xy2/ - x43$.

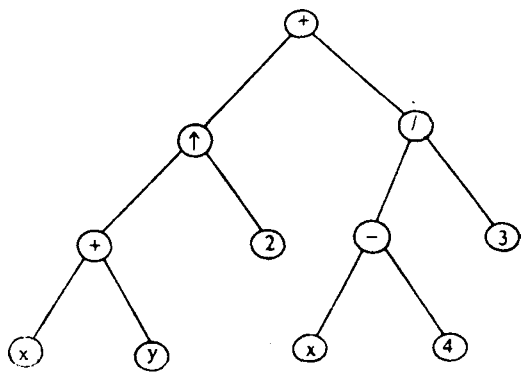
8.2.3 Postfix Notation

The post fix form of an expression is the post order traversal of the binary tree representing the given expression. Expressions written in post fix form are said to be in reverse polish Notation. Expressions in reverse polish Notation are Unambiguous, so that parentheses are not needed.

Example 8.2.3.1 What is the post fix form of the expression

$$((x + y) \uparrow 2) + ((x - 4)/3)$$

Solution: The Binary tree representing the above expression is:



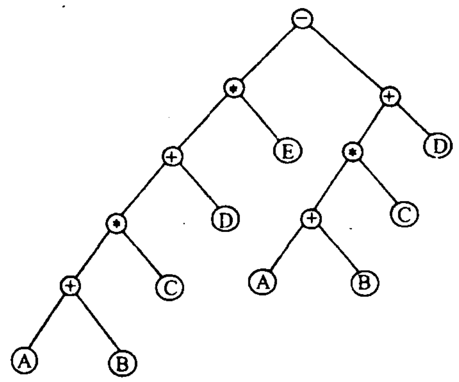
Now, By traversing the above binary tree in postorder, we have the **post fix** notation of the given above expression:

$$xy + 2 \uparrow x4 - 3/ + .$$

Example 8.2.3.2 Represent the expression as a binary tree and write the prefix and post fix forms of the expression

$$\big[[(A + B) * C + D] * E\big] - [(A + B) * C - D].$$

Solution: Binary Tree:



Prefix: $- * + * + ABCDE + * + ABCD$
Postfix: $AB + C * D + E * AB + C * D + -$

8.2.4 Evaluating postfix form and prefix form of an expression

(i) In the prefix form of an expression, operator precedes its two operands. Hence we can evaluate an expression in prefix form by working from **right to left**. When we encounter an operator, we perform the corresponding operation with the two operands immediately to the right of this operation. Also whenever an operation is performed, we consider the result as a new operand.

Example 8.2.4.1 What is the value of the prefix expression

$$+ - *235/ \uparrow 234?$$

Solution: The steps used to evaluate this expression by working right to left and performing operations, using the operands on the right

$$+ - * 235/ \underbrace{\uparrow 23}_{2\uparrow 3=8}\ 4$$

$$+ - * 235\ \underbrace{/84}_{\frac{8}{4}=2}$$

$$+ - \underbrace{*23}_{2*3=6}\ 52$$

$$+ \underbrace{-65}_{6-5=1}\ 2$$

$$\underbrace{+12}_{1+2=3}$$

∴ value of expression is 3.

(ii) In the post fix form of an expression, a binary operator follows its two operands. So, to evaluate an expression from its post fix form, work *left to right*, Carrying out operations. Whenever an operator follows two operands. After an operation is carried out, the result of this operation becomes a new operand.

Example 8.2.4.2 What is the value of the post fix expression

$$723 * -4 \uparrow 93/+?$$

Solution: The Steps used to evaluate this expression by starting at the left and Carrying out operations, when two operands are followed by an operator are shown in:

$$7\ \underbrace{23*}_{2*3=6}\ -4 \uparrow 93/+$$

$$\underbrace{76-}_{7-6=1}\ 4 \uparrow 93/+$$

$$\underbrace{14 \uparrow}_{1^4=1} 93/+ \qquad (\uparrow \rightarrow \text{exponentiation})$$

$$1\ \underbrace{93/}\ +$$

$$\underbrace{13+}$$

$$1+3=4$$

Value of the expression is 4.

8.3 Parsing Trees

To every statement formula, we can associate a tree, called a parsing tree. This is a tree whose vertices are labelled by a formula. Before giving the definition of the parsing tree of a formula, we explain the process of drawing the parsing tree.

Procedure to draw the parsing tree:

Step 0: If ϕ is the given formula take a vertex and label it as ϕ.

Step 1: If ϕ is $(\neg\phi_1)$ for some formula ϕ_1, then take a vertex such that the new vertex is vertically down to the vertex ϕ_1. And Label the new vertex by ϕ_1 and join the vertices ϕ *and* ϕ_1 *by an edge (going downwards).*

Step 2: If ϕ is $(\phi_1 \wedge \phi_2)$ or $(\phi_1 \vee \phi_2)$ or $(\phi_1 \rightarrow \phi_2)$ or $(\phi_1 \rightleftarrows \phi_2)$ then take two new vertices, both of them are down to the vertex ϕ, but one new vertex is to the left of the vertex ϕ and the other new vertex is to the right of the vertex ϕ. The new vertex which is to the left of ϕ is labelled ϕ_1 and the other new vertex is labelled ϕ_2. The vertex ϕ is joined to the vertices ϕ_1 and ϕ_2 by edges (i.e., by one edge going down and to the left of ϕ and the other edge going down and to the right of ϕ). Mark the angle at ϕ with $\wedge$ or $\vee$ or $\rightarrow$ or $\rightleftarrows$ respectively.

Step 3: Repeat the steps 1 and 2 for ϕ_1 and ϕ_2 till we get vertices for all the atomic variables that occur in the given formula.

This process will automatically give a tree.

Example 8.3.1 Draw the parsing tree for the formula

$$\left[(P \rightarrow (\neg Q)) \rightarrow (P \wedge Q)\right]$$

Solution:

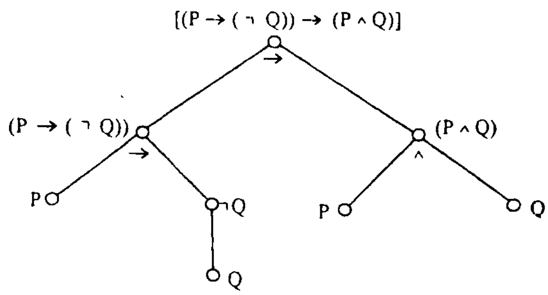
$$\phi : \left[(P \rightarrow (\neg Q)) \rightarrow (P \wedge Q)\right]$$

ϕ is $(\phi_1 \rightarrow \phi_2)$ where $\phi_1 : (P \rightarrow (\neg Q))$ and
$\phi_2 : (P \wedge Q)$
Connective $: \rightarrow$

ϕ_1 is $(\phi_3 \rightarrow \phi_4)$ where $\phi_3 : P$
$\phi_4 : \neg Q$
Connective $: \rightarrow$

ϕ_2 is $(\phi_5 \wedge \phi_6)$ where $\phi_5 : P$
$\phi_6 : Q$
Connective $: \wedge$

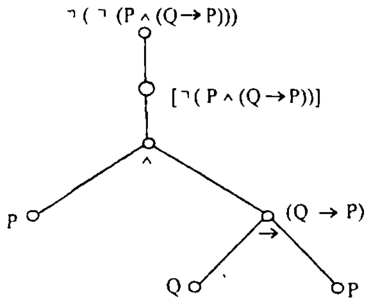
Now parsing tree for $\left[(P \rightarrow (\neg Q)) \rightarrow (P \wedge Q)\right]$



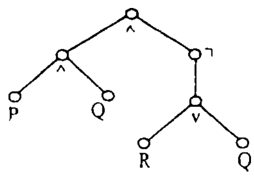
Example 8.3.2 Draw the parsing tree for the formula

$$\neg(\neg(P \wedge (Q \rightarrow P)))$$

solution



Example 8.3.3 Find the statement formula whose parsing tree is given below:



Statement formula: $(P \wedge Q) \wedge [\neg(R \vee Q)]$.

Note 8.3.1 A Labelled tree is a parse tree if

i) there are atmost two downward edges from every vertex v and exactly two downward edges from v if the label of v is $\wedge$, $\vee$, $\rightarrow$ or $\rightleftarrows$.
ii) there is atmost one upedge from any vertex and exactly one upedge from v if the label of v is $\neg$.
iii) the label of a bottom most vertex is an atomic statement variable.

8.4 Conversion of a tree to Binary tree

Every tree can be uniquely represented by a binary tree. Now, we show how one can obtain a binary tree which represents a given ordered tree.

Step 1: Delete all the branches except the leftmost branch in all the vertices.

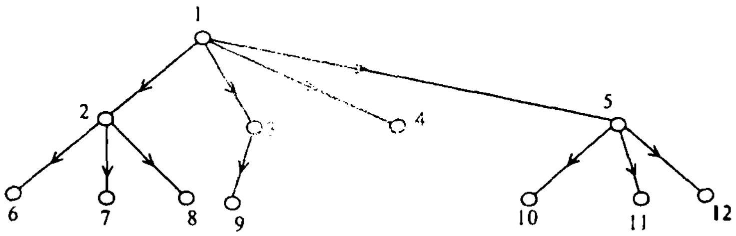
Step 2: Draw edges from a vertex to the vertex on the right, of the same parent if any which is at the same level.

Step 3: Once this is done, then for any particular vertex, we choose its left and right child in the following manner

(i) The left child is immediately below the given vertex and

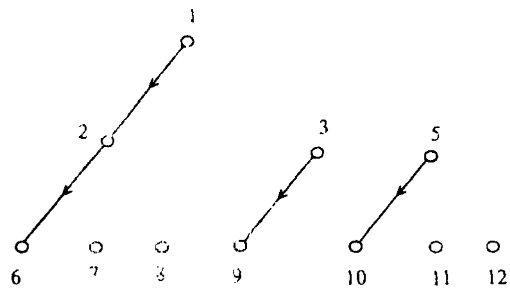
(ii) the right child is on the same horizontal line.

Example 8.4.1 Convert the following tree into binary tree

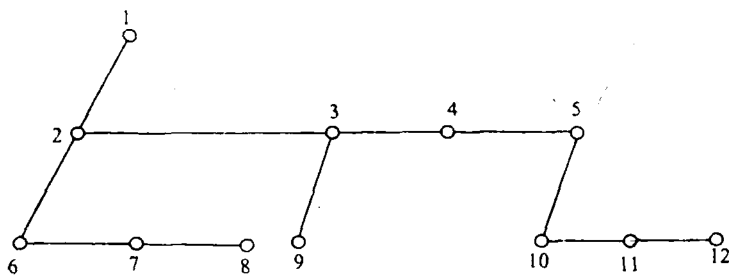


Solution:

Step 1:

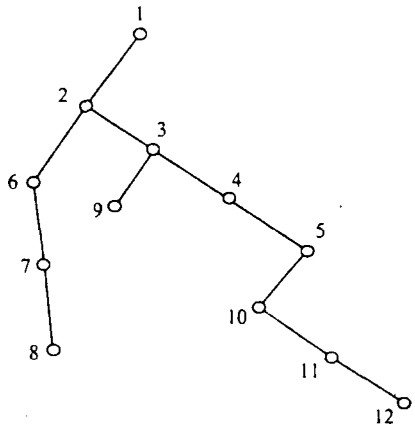


Step 2: Connecting the vertices at the same level we have

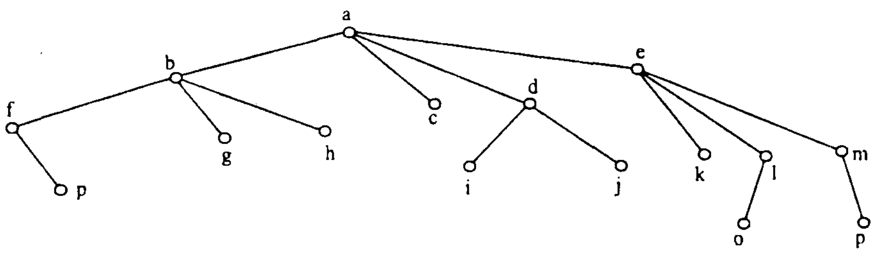


Here we can't connect 8 to 9, 9 to 10 because, their parents are different, eventhough they are all at same level.

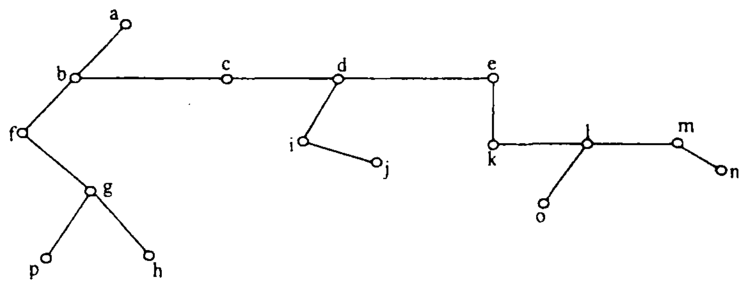
Step 3:



Example 8.4.2 Convert the following tree into binary tree



Solution:



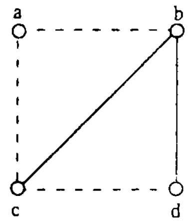
9. Fundamental circuits

Let T be a spanning tree of a graph G. Then the edges of G that are in T are called *branches* of G. The edge of G that is not in T is called a *chord* of G with respect to T. The set of all chords of G is called *chord set*.

We have T is an acyclic graph and addition of an edge between any two vertices of T gives exactly one circuit.

Definition 9.1 Any circuit of G that can be obtained by adding a chord to the spanning tree is called a *fundamental circuit* of G.

Example 9.1



The branches of G are : ab, ac, cd
The chords of G are : bc, bd
The circuits of G are : $abca$, $abdca$ and $cdbc$

Addition of the chord cb to the tree T yields the circuit $acba$
Addition of the chord bd to the tree T yields the circuit $acdba$
$\therefore$ Fundamental circuits are $acba$ and $acbda$

Theorem 9.1 A nonempty intersection of two fundamental circuits is always a path.

Proof: With respect to some specified spanning tree T. Let e_1 and e_2 be two chords forming fundamental circuits C_1 and C_2 respectively. Then if $C_1 \cap C_2$ contains two edges x and y not connected in $C_1 \cap C_2$ there is a path P_1 and C_1 between x and y (that is a path between one of the end vertices of edge x and one of the end vertices of edge y); and this path does not contain a chord e_1. Similarly, there is a path P_2 in C_2 between x and y that does contain chord e_2. Then the subgraph $P_1 \cup P_2 \cup \{x, y\}$ contains a circuit without containing any chord, which is impossible. □

Corollary 9.1.1 In a graph G if edges a and b belong to a fundamental circuit C_i and if edges b and c belong to another fundamental circuit C_j such that $a \notin C_j$ and $c \notin C_i$, then there exists some circuit C in G such that a and c both are in C.

Definition 9.2 For any graph G of order n and size e, the number of edges to be removed to make it acyclic is called *Nullity* of the graph and is equal to $\mu = e - n + 1$.

Definition 9.3 The complement of nullity is called rank (r) i.e., the number of chords in any spanning true of the graph G and $r = n - 1$.

Thus Rank of G + nullity of G = total number of edges of G.

Theorem 9.2 A set S of edges is a cutset of a connected graph G if and only if S is minimal and contains atleast one branch of every spanning tree of G.

Proof: Let S be a cut set of G. Let T be a spanning tree of G. Suppose S does not contain any branch of T. Then all the edges of T present in $G - S$. Thus $G - S$ is a connected graph, a contradiction to the fact that S is a cut set.

Conversely, Let S be the set of edges of G such that S contain atleast one branch of every spanning tree T. having minimum number of edges. Suppose $G - S$ is connected, then we have $G - S$ contains a spanning Tree T. Further, no edges of T present in S, a contradiction. Since S is minimal $G - S + e$ is a connected graph, for all $e \in S$. Hence S is a cutset. □

Definition 9.4 Let T be a spanning tree of graph G. Then the cutset containing exactly one branch of T is called *fundamental cutset* of G with respect to T.

Theorem 9.3 Every circuit of G has an even number of edges in common with any cutset.

Proof: Let S be a cutset of G. Let C be a circuit of G. Then the graph $G - S$ is a disconnected graph, and hence the vertex set V of $G - S$ can be partitioned into two subsets V_1 and V_2 such that edge in S is the edges between a vertex in V_1 and a vertex in V_2 in G. Thus if all the vertices of C are in V_1 only (or in V_2 only) then $E(S) \cap E(C) = \emptyset$.

If some vertices are in V_1 and some are in V_2, then starting from a vertex of C in V_1, to trace the entire circuit C of G, we have to travel exactly even number of times between the sets V_1 and V_2. These are possible only through the edges in S. Thus the number of edges common to C and S should be even.

Hence the proof. □

Theorem 9.4 Let T be a spanning tree and S be the fundamental cutset of G, determined by branch e with respect to T. Then the edge e contained in a fundamental circuit C of G with respect to T if and only if C is the fundamental circuit of G determined by a chord in S.

Proof: Let T be a spanning tree and S be the fundamental cutset of G, determined by branch e with respect G. Then $S = \{e, e_1, e_2, \ldots e_k\}$, where $e_1, e_2 \ldots e_k$ are the chords with respect to T. Let e_c be a chord of G with respect to T. Let C be the fundamental circuit of G determined by the chord e_c. Then $E(C) = \{e_c, e'_1, e'_2 \ldots e'_j\}$ where $e'_1, e'_2, \ldots e'_j$ are branches of T.

Let the branch $e \in E(C)$. Then $e \in E(C) \cap S$ and $e = e'_i$ for some i, $1 \le i \le j$. But the number of edges common to a cutset S and a circuit of G is always even (see previous theorem) it implies that there exists one more edge of G common to C and S other than e. However the only possible common edge in S and C other than e is a chord e_c (observe: chord cannot be a branch). Thus $e_c \in S$. Thus C is the fundamental circuit determined by the chord e_c belonging to S.

Conversely, Let T be a spanning tree and S be the fundamental cutset of G, determined by branche with respect to T. Then $S = \{e, e_1, e_2, \ldots e_k\}$ where $e_1, e_2 \ldots e_k$ are the chords with respect to T. Let C be the fundamental circuit of G with respect to T, determined by a chord e_i in S, for some i, $1 \le i \le k$. Then $E(c) = \{e_i, e'_1, \ldots e'_j\}$ where $e'_1, \ldots e'_j$ are the branches of G with respect to T.

If possible let e is not in C. But, again as a number of edges common to C and S is even, there exists an edge in common with S and C other than e_t and e. This implies that $e_l = e'_m$ for some $e_l \in S$ and $e'_m \in C$, $1 \leq l \leq k$, $1 \leq m \leq j$ which is contradiction, since chord e_l can't be a branch e'_m of G. Hence $e \in E(c)$, this proves the theorem. □

Theorem 9.5 Let T be a spanning tree and C be the fundamental circuit set of G, determined by chord e with respect to T. Then the chord e contained in a fundamental cutset S of $G \iff S$ is the fundamental cutset of G determined by branches in C.

Proof: Let T be a spanning tree and C be the fundamental circuit of G, determined by chord e with respect to T. Then $E(C) = \{e, e_1, e_2, \ldots, e_k\}$ where $e_1, e_2, \ldots e_j$ are the branches with respect to T. Let e_b be *a* branch of T. Let S be the fundamental cutset of G determined by the branch e_b. Then $S = \{e_b, e'_1, e'_2, \ldots e'_j\}$ where $e'_1, e'_2 \ldots e'_j$ are chords of G with respect to T.

Let the chord $e \in S$. Then $e \in E(C) \cap S$ and $e = e'_i$ for some i, $1 \leq i \leq j$. But the number of edges common to any cutset and a circuit of G is always even, it implies that there exist one more edge of G common to C and S other than e. But the only possible common edge in S and C other than e is a branch e_b (observe). Thus $e_b \in E(C)$. Thus S is the fundamental cutset determined by the branch e_b belonging to C.

Conversely, Let T be a spanning tree and C be the fundamental circuit of G, determined by chord e with respect to T. Then $E(C) = \{e, e_1, e_2, \ldots, e_k\}$ where $e_1, e_2, \ldots, e_k$ are the branches with respect to T. Let S be the fundamental circuit of G with respect to T, determined by a branch e_i in C, for some i, $1 \leq i \leq k$. Then $S = \{e_i, e'_1, \ldots e'_j\}$ where $e'_1, \ldots e'_j$ are chords of G with respect to T.

If possible Let e is not in S. But then, as a number of edges common to a cutset and a circuit of G is even there exists on edge in common with S and C other than e_t and e. This implies that $e_l = e'_m$ for some $e_l \in S$ and $e'_m \in C$, $1 \leq l \leq k$, $1 \leq m \leq j$, which is a contradiction, since branch e_l cannot be a chord $'e_m$ of G. Hence $e \in S$. This proves the theorem. □

Example 9.1 Consider the graph shown below. The spanning tree T is shown with dotted lines (broken lines)

The branches of G are: $e_1, e_2, e_4, e_5, e_7, e_{10}, e_{11}, e_{12}$ and e_{13}

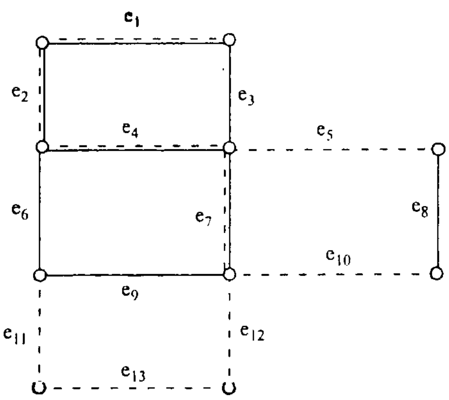
The chords of G are: e_3, e_6, e_8 and e_9

The edge set of the fundamental circuit due to chord e_3 is $C_1 : \{e_1, e_2, e_4, e_3\}$

The edge set of the fundamental circuit due to chord e_6 is $C_2 : \{e_4, e_7, e_{12}, e_{13}, e_{11}, e_6\}$

The edge set of the fundamental circuit due to chord e_8 is $C_3 : \{e_5, e_7, e_8, e_{10}\}$

The edge set of the fundamental circuit due to chord e_9 is $C_4 : \{e_9, e_{11}, e_{12}, e_{13}\}$



The fundamental cutset due to branch e_1 is S_1 : $\{e_1, e_3\}$

The fundamental cutset due to branch e_2 is S_2 : $\{e_2, e_3\}$

The fundamental cutset due to branch e_4 is S_3 : $\{e_4, e_3, e_6\}$

The fundamental cutset due to branch e_5 is S_4 : $\{e_5, e_8\}$

The fundamental cutset due to branch e_7 is S_5 : $\{e_7, e_6, e_8\}$

The fundamental cutset due to branch e_{10} is S_6 : $\{e_{10}, e_8\}$

The fundamental cutset due to branch e_{11} is S_7 : $\{e_{11}, e_6, e_9\}$

The fundamental cutset due to branch e_{12} is S_8 : $\{e_{12}, e_6, e_9\}$

The fundamental cutset due to branch e_{13} is S_9 : $\{e_{13}, e_6, e_9\}$

Now the observations are

(i) e_3 is the chord in C_1 and e_3 lies in the fundamental cutset S_1, S_2 and S_3 determined by the branches e_1, e_2 and e_4 of C_1 respectively. Further e_3 is not an edge of any other cutsets of G w.r. to T.

(ii) e_{12} is a branch in S_8 and e_{12} lies in the fundamental circuits C_2 and C_4 determined by the chords e_6 and e_9 of S_8 respectively. Further e_{12} is not an edge of any other fundamental circuits of G with respect to T.
Reader is advised to verify the theorems for the other possible chords as well as branches of G.

□

Theorem 9.6 A cycle and the complement of any any spanning tree must have atleast one edge in common.

Proof: If there is a circuit that has no common edge with the complement of a spanning tree, the circuit is contained in the spanning Tree. Which is not possiblé, since a tree can't contain a circuit. □

Theorem 9.7 A cutset and any spanning tree must have atleast one edge common.

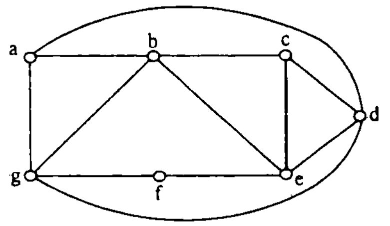
Proof: If there is a cutset that has no common edge with a spanning tree, the removal of the cutset will leave the spanning tree intact. This means that the removal of the cutset will not separate the graph into two components which is a contradiction to the definition of a cutset. □

Exercise 2 (1) write the expression $((x + 2) \uparrow 3) * (y - (3 + x)) - 5$ in
(a) Prefix (b) postfix (c) infix notation

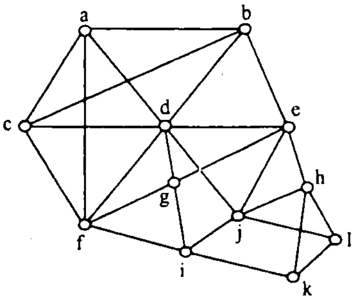
(2) What is the value of each of the following perfix expressions?
(a) $\uparrow - * 33 * 425$
(b) $* + 3 + 3 \uparrow 3 + 333$

(3) Find a Spanning tree for the graph shown by removing edges in simple circuits.

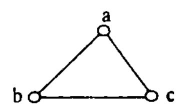
(a)



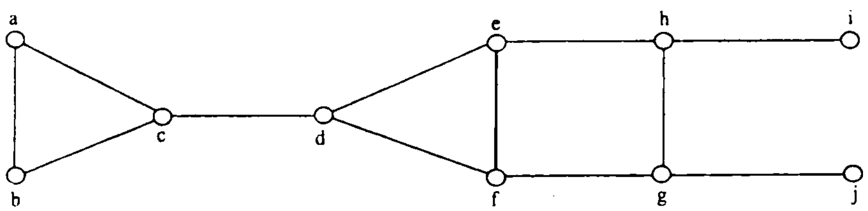
(b)



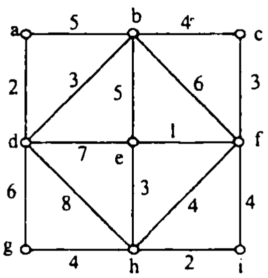
(4) Draw all the Spanning trees of the given simple graph



(5) Use DFS to produce a Spanning tree for the given simple graph (Choose 'a' as the root)



(6) Use Kruskal's algorithm to find a minimum spanning tree for weighted graph

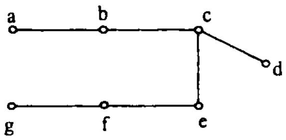


Answer: 2

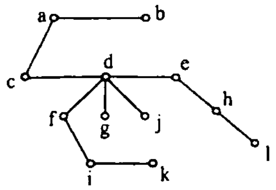
1. (a) $- * \uparrow + x23 - y + 3x5$
   (b) $x2 + 3 \uparrow y3x + - * 5 -$
   (c) $((((x + 2) \uparrow 3) * (y - (3 + x))) - 5)$

2 (a) 1 (d) 2205

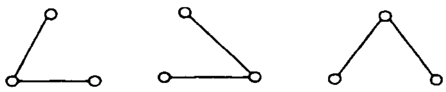
3(a)



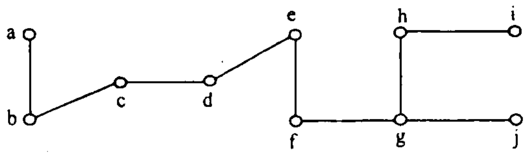
3(b)



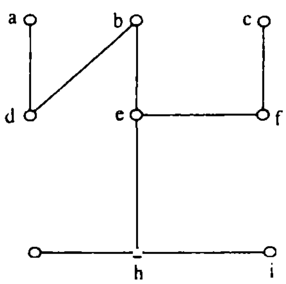
4.



5.



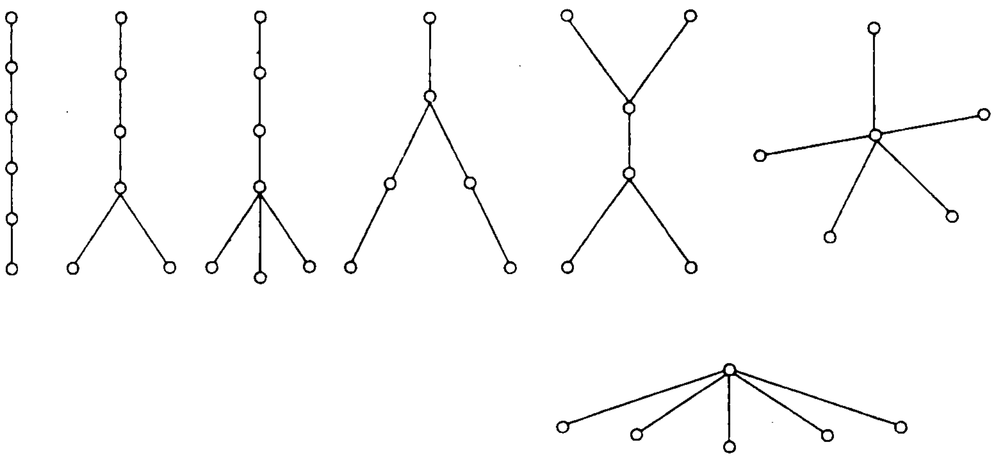
6.



Solved Problem

Problem 1 Draw all possible non isomorphic trees of order 6

Solution:



Problem 2 Prove that it G is a connected graph

$$|E| \geqslant |V| - 1$$

Solution: Given that G is connected, then we can have two cases :

Case: 1 G is connected with no circuits

ie., G is a connected, with no circuits

$\because$ G is a tree

$\therefore$ $|E| = |V| - 1$

Case:2 G is connected with circuits

then G has an extra edge or more edges

So on with relative to the tree

$\therefore$ $|E| > |V| - 1$

From case 1 and case 2

$$|E| \geqslant |V| - 1$$

Problem 3 Prove that a connected graph G is a tree $\Longleftrightarrow$ each edge is a cut edge.

Solution: Let G be a tree, and we know that there is unique path between any two vertices. Hence deletion of any edge in this path makes the graph disconnected.

$\therefore$ Each edge in G is a cut edge

Conversely, Let each edge in the graph be a cut edge which implies G has a unique path between each pair of vertices

$\therefore$ G is a tree

Problem 4 Suppose that a tree has d_1 vertices of degree 1, 2 vertices of degree 2, 4 vertices of degree 3 and 3 vertices of degree 4. Find d_1

Solution:

$$\sum_{i=1}^{|V|} d(V_i) = 2|E|$$

$$d_1 + 4 + 12 + 12 = 2|E|$$

$$d_1 + 28 = 2(d_1 + 9 - 1)$$

$$\Longrightarrow d_1 = 12$$

Problem 5 Is there a tree with $|V| = 5$ and 2 vertices of degree 3?

Solution: No, Since $\sum_{i=1}^{|V|} d(V_i) = 2|V| - 2$ will not be satisfied by the above data

Problem 6 Let G be a graph with K components, where each component is a tree. Obtain a formula for $|E|$ interms of $|V|$ and K.

Solution: Clearly $|E| = \sum_{i=1}^{K} |E_i|$ and $|V| = \sum_{i=1}^{K} |V_i|$ also given that each component is a tree

$$\therefore\ |E_i| = |V_i| - 1 \text{ for each } i$$

$$\therefore\ |E| = \sum_{i=1}^{K} (|V_i| - 1)$$

$$= \sum_{i=1}^{K} |V_i| - \sum_{i=1}^{K} 1$$

$$\Longrightarrow |E| = |V| - K$$

Problem 7 Show that a vertex V in a tree T is a cut vertex of $T \Longleftrightarrow \deg(V) > 1$.

Solution: It a vertex V in T is a cut vertex, then by deleting it the number of components of T increases by 1, therefore it is not a leaf vertex. $\therefore$ deg $(V) > 1$ conversely deg $(V) > 1$, obviously by deleting it, the number of components of T increases by 1, then V is a cut vertex.

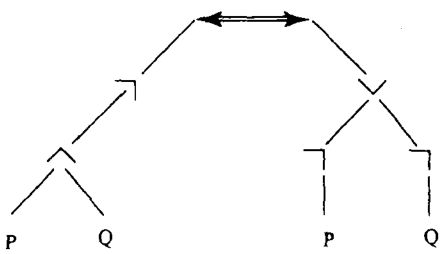
Problem 8 Show that a tree is a bipartite graph

Solution: Let T be a tree. Root T at some arbitrary vertex. Le V be the set of vertices on even levels and let W be the set of vertices on odd levels. Since each edge is incident on a vertex in V and a vertex in W, T is a bipartite graph.

Problem 9 Find the ordered rooted tree representing the compound proposition.

$$\neg(P \wedge Q) \Longleftrightarrow \neg P \vee \neg Q$$

Solution:

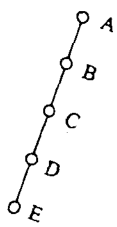


Prefix : $\Longleftrightarrow \ \neg \wedge PQ \vee \neg P \neg Q$

Postfix : $PQ \wedge \neg P \neg Q \neg \wedge \Longleftrightarrow$

Infix : $(\neg(P \wedge Q)) \Longleftrightarrow ((\neg P) V (\neg Q))$

Problem 10 List the order in which the vertices are processed using preorder, inorder and post order traversal



Solution:

Preorder : ABCDE

Inorder : EDCBA

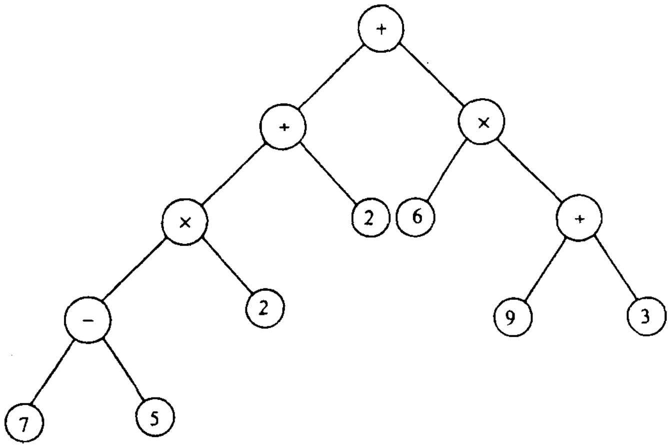
Postorder : EDCBA

Problem 11 Consider the algebraic expression

$$\left[\left((7-5) \times 2\right) \div 2\right] + \left(6 \times (9 \div 3)\right)$$

Draw a corresponding binary tree

Solution:



Problem 12 Show that a graph G with n vertices and fewer than $n - 1$ edges is not connected.

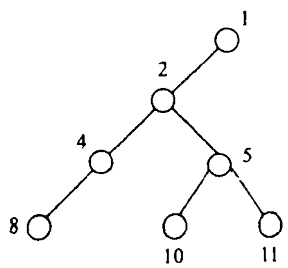
Solution: Suppose G is connected. Add parallel edges untill the resulting graph G' has $n - 1$ edges. Since G' is connected and has $n - 1$ edges, than G' is acyclic. But adding an edge in parallel introduces a cycle contradiction.

Problem 13 A full many tree with i internal vertices contains $n = mi + 1$ vertices.

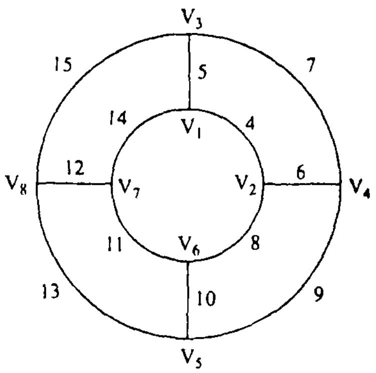
Solution: Every vertex, except root, is the child of an internal vertex. Since each of the i internal vertices has m children, there are mi vertices in the tree other than the root Therefore The tree containts $n = mi + 1$ vertices.

Problem 14 Draw a binary tree whose level order indices are $\{1, 2, 4, 5, 8, 10, 11\}$

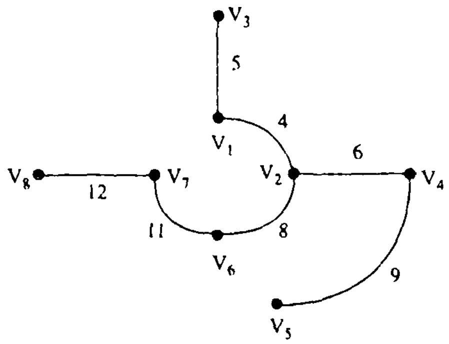
Solution:



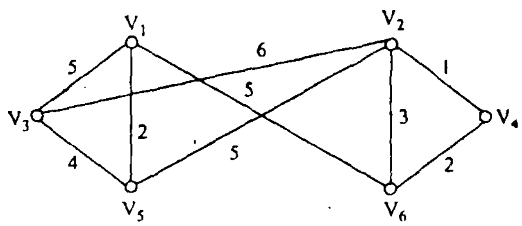
Problem 15 Find a minimal spanning tree of the weighted graph G



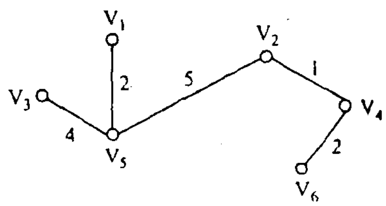
Solution: By Kruskal's algorithm



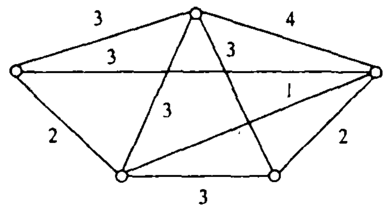
'roblem 16 Find a MST of the weighted graph G



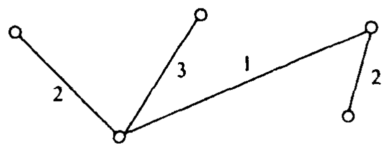
Solution:



Problem 17 Find a MST of the weighted graph G (by Kruskal's algorithm)



Solution:



Problem 18 Evaluate the expression which is given in postfix notation

x 2 - 3 + 2 3 y + - w 3 - x ÷

where x is 7, y is 2 and w is 1

Solution: $\frac{8}{6}$

Problem 19 Show that the maximum number of vertices in a binary tree of height *r* is $2^{n+1} - 1$

Solution: The maximum number of vertices on level k is 2^k. Hence the maximun number of vertices is $1 + 2 + 2^2 + \ldots\ldots + 2^n$ (or) $2^{n+1} - 1$

Problem 20 Let T be a tree. Suppose that T has x vertices and e edges. Find thi formula relating x to e

Solution: Each vertex except the root has in degree 1

Thus $e = x - 1$.

Problem 21 Show that the number of vertices in a binary tree is always odd.

Solution: Consider a binary tree on n vertices. Since it contains exactly one verte: of degree two and other vertices are of degree one or three, it follows that there are $n -$ odd degree vertices in the graph. But the number of odd degree vertices in a graph i even, it follows that $n - 1$ is even and hence n is odd.

Problem 22 Count the number of vertices of degree three in a binary tree on n vertice having k number of pendant vertices.

Solution: Since the binary tree contains k number of pendant vertices and one vert of degree two, we have total number of remaining vertices which are of degree three i $n - k - 1$.

Problem 23 In any binary tree T on n vertices, show that the number of pendan vertices (edges) is equal to $\frac{n+1}{2}$.

Solution: Let the number of pendant edges in a binary tree on n vertices be k. Then we have $n-k-1$ vertices of degree three, one vertex of degree two, k vertices of degree one and $n-1$ edges. Therefore

$$\text{sum of degrees of vertices} = 2 \times \text{number of edges}$$
$$(n-k-1)\times 3 + 1 \times 2 + k \times 1 = 2(n-1) \Rightarrow k = \frac{n+1}{2}.$$

Problem 24 What is the maximum possible number of vertices in any k-level tree?

Solution: The level of a root is zero and it is the only one vertex at level zero. There are two vertices at a level 1 that are adjacent to the root vertex. From these vertices we can find maximum four vertices at level 2. So on to get a minimum height tree we have to keep the vertex at higher level only after filling all the vertices in its lower level. Thus maximum number of vertices possible for such a k level tree is

$$\therefore \quad n \le 2^0 + 2^1 + 2^2 + \cdots + 2^k = \frac{1(1-2^{k+1})}{1-2} = 2^{k+1} - 1.$$

Problem 25 What is the minimum possible height of a binary tree on $2n - 1 (n \ge 1)$ vertices?

Solution: Let k be the minimum height of a binary tree on $2n-1$ vertices. For minimum height we have to keep maximum number of vertices in the previous level before placing any vertex in the next level. Thus it follows from the above problem that k should satisfy the inequality

$$2n - 1 \le 2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$
$$\text{i.e.} \quad 2n - 1 \le 2^{k+1} - 1 \Rightarrow 2n \le 2^{k+1} \Rightarrow n \le 2^k.$$

Now taking logarithm on both sides we get

$$\log_2^n \le k \Rightarrow k \ge \log_2^n.$$

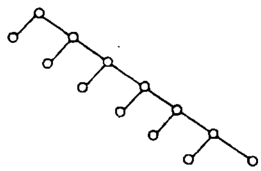
Since k is an integer, this implies that the minimum value of $k = \lceil \log_2^n \rceil$.

Problem 26 What is the maximum possible level (height) of a binary tree on $2n +$ $(n \ge 0)$ vertices.

Solution: Let k be the height of a binary tree on $2n+1$ vertices. To get a vertex in maximum level we must keep exactly two (minimum) vertices in each level except the root vertex. That is out of $2n+1$ vertices one is a root and the remaining $2n$ vertices can keep in exactly n levels. Thus the maximum height of a tree is n. Hence maximum possible value of k is n.

This can also be shown as $2 + 2 \cdots + 2(k \text{ times}) \ge 2n \Rightarrow 2k \ge 2n \Rightarrow k \ge n \Rightarrow$ minimum value of k is n).

Problem 27 Find the maximum possible height of a binary tree with 13 vertices and draw graph of the tree.



Solution: we have $n = 13$, maximum possible height of the binary tree $\frac{n-1}{2} = 6$.

Problem 28 Find the minimum height of the tree with 9 vertices .

Solution: $n = 9$.

The minimum height of the binary tree

$$\lceil \log_2(n-1)+1 \rceil = \lceil \log_2(9-1)+1 \rceil = 3.$$

Quiz Questions

1. A tree is a Connected, Undirected, Simple ___________ graph.
   (Ans: acyclic)
2. The level of a vertex v is a tree is the ___________ of the simple path from the root.
   (Ans: length)
3. It T is tree $\Longleftrightarrow$ T has no ___________ and
   (Ans: cycles, $|E| = |V| - 1$)
4. A Simple undirected graph G is a tree $\Longleftrightarrow$ if G is ___________ and contains ___________
   (Ans : Connected, no cycles)
5. The circuit rank of G with $|V|$ vertices and |E| edges is ___________
   (Ans: $|E| - |V| + 1$)
6. An undirected graph G has a spanning tree $\Longleftrightarrow$ G is ___________
   (Ans: Connected)
7. The value of prefix expression
   $+ - \times 235/ \uparrow 234$?
   (Ans: 3)
8. The value of postfix expression
   $723 \times -4 \uparrow 93/+$?
   (Ans: 4)
9. A connected graph has a spanning tree with ___________ edges
   (Ans: $|V| - 1$)
10. Is the sequence (1, 1, 2, 2, 3, 3, 3, 3,) the degree sequence of a tree
    (Ans: No)
11. A forest is a simple graph with no ___________
    (Ans: circuits or cycles)
12. If G is a tree then $\sum_{\text{all} v \in V(G)} deg(V) =$ ___________
    (Ans: $2|V| - 2$)
13. Is there any tree with degree sequence (1, 1, 2, 2, 3, 3)
    (Ans: No)
14. In a connected plane (Simple) graph G with $|E| > 1$ then $|E| \leq$ ___________
    (Ans: $3|V| - 6$)
15. $\chi(K_{n,n}) =$ ___________
    (Ans: n + 1)
16. What is the value of the following prefix expression
    $\times + 3 + 3 \uparrow 3 + 3\,3\,3$
    (Ans: 2205)
17. The number of spanning Trees of the simple graph K_3 is ___________
    (Ans: 3)

18. The number of spanning Trees of the simple graph K_4 is ___________
(Ans: 4)
19. Any Tree can be coloured with ___________ colours
(Ans: Two)
20. A graph is bipartite $\Longleftrightarrow$ it can be coloured with ___________ colours
(Ans: Two)

6. Graph theoretic Algorithms

6.0 Introduction

In this chapter, we consider different types of computer inputs and outputs for a given graph, other than the matrix from that we are studied in the previous chapters. Alsc present some algorithms to find the spanning tree, shortest path, connectivity, planarity and Euler cycle etc., of the graphs.

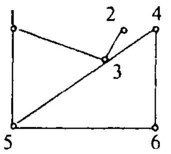
6.1 Computer representation of a graph (Input)

Generally, A graph is stored in a digital computer in one of the following five forms. Each has advantages and disadvantages. The choice depends on the graph, type of a problem. type of machine, type of (computer) language in which the algorithm is to be executed, any modification in the graph during the process.

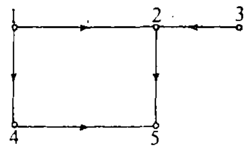
1. *Adjacency matrix:* We already defined adjacency matrix of a graph in the Graphs chapter and observed its properties. As the adjacency matrix is symmetrical, input of all the entries of the matrix to the computer is not necessary, instead we can input only upper triangle entries of the adjacency matrix. The advantage of this matrix is that the input entries are binary numbers. Disadvantage is it requires more complex operations, requires more computational time and requires more memory storage.
2. *Incidence matrix:* The input of the graph in incidence matrix form usually requires more memory locations than that of adjacency matrix because the number of vertices in a graph is usually more than the number of vertices in it provided the graph is not acyclic. Incidence matrices are particularly favored for the graphs of electrical, and switching networks where the loops (circuits) are avoided.
3. *Edge listing:* Let G be a graph of order n whose vertices are labeled by the integers 1 to n. Representation of the edges of G as a two element subsets of elements of the vertex set (ordered or unordered) is called edge listing. This list may be a multiset thus parallel edges and self loops can also be represented in this method. Edge listing is convenient to enter the data into the computer but it requires lot of manipulations.
4. *Two linear arrays:* In this method, edges of the graph is represented by two linear array $F = (f_1, f_2, \ldots f_n)$ and $H = (h_1, h_2, \ldots, h_n)$. Each entry in F and H are vertex label, written in such a way that i the edge e_i of the graph is the edge $f_i h_i$

5. *Successor Listing:* If the ratio $\frac{|E|}{|V|}$ of a graph is not large, then graph can be written after assigning all of its vertices by the numbers $1, 2, \ldots n$. Each vertex label by K is a linear array, whose first element is K and whose remaining elements are the vertices that are immediate successors of K, that is they are the labels of the vertices which have the path of length one from v (in the case of undirected graphs these are the adjacent /neighboring vertices).

Example: Consider the following graphs



undirected graph G_1



directed graph G_2

Input graph	Edges listing	Two linear arrays	Successor listing
G_1	$E =$ (1,3) (1,5) (2,3) (3,2) (3,4) (3,5) (4,3) (4,6) (5,1) (5,5) (5,6) (6,4) (6,5)	$F = \{1, 1, 2, 3, 3, 3, 4, 4, 5, 5, 5, \}$ $H = \{3, 5, 3, 2, 4, 5, 3, 6, 1, 3, 6\}$	1 : 3,5 2 : 3 3 : 2, 4, 5 4 : 3, 6 5 : 1, 3, 6 6 : 4, 5
G_2	$E =$ (1,2), (1, 4), (2, 5) (3,2), (4, 5), (5, 5)	$F = \{1, 1, 2, 3, 4, 5\}$ $H = \{2, 4, 5, 2, 5, 5\}$	1 : 2,4 2 : 5 3 : 2 4 : 5 5 : 5

The output of the graph varies with the problem. If the output consists of subgraphs, we may make the problem to print the appropriate adjacency matrix.

5.2 Warshall's algorithm (to find the path matrix from A(G))

Given the adjacency matrix A(G) of a simple digraph, then the following steps produce the pathmatrix P

Step1: $P^{[0]} = A$

Step2: $K = 1$

Step3: $i = 1$

Step4: $P_{ij}^{[K]} = P_{ij}^{[K-1]} \vee \left(P_{iK}^{[K-1]} \wedge P_{Kj}^{[K-1]}\right) \quad \forall j = 1 \text{to} n$

Step5: i = i + 1 if $i \leq n$ go to step4

Step6: $K = K + 1$ If $K \leq n$ go to step3
otherewise stop.

Note:

If $P_{iK}^{[K-1]} = 1$

then step4: $P_{ij}^{[K]} = P_{ij}^{[K-1]} \vee P_{Kj}^{[K-1]} \quad \forall j = 1 \text{ to } n$

Note: Warshall's algorithm can be used to get transitive closure of a relation (Refer se theory)

Input: The adjacency matrix $A(G)$ of a simple digraph G with vertices $v_1, v_2, \ldots v_n$.

output: The path matrix P

Quick computation of path matrix by Warshall's algorithm

We start with $A(G) \left(= P^{[0]}\right)$ the adjacency matrix of the given graph G, anc then successively construct the matrices $P^{[1]}, P^{[2]}, \ldots P^{[n]}$ where n is the number of vertices. Moreover for each $K \geq 1$, we construct $P^{[K]}$ interms of the previously constructed $P^{[K-1]}$ we construct $P^{[K-1]}$ by following steps

Step1: First transfer all 1's in $P^{[K-1]}$ to $P^{[K]}$

Step2: List all positions $s_1, s_2, \ldots$ in *column* K of $P^{[K-1]}$
where the entry is 1 and the positions $t_1, t_2 \ldots$
in *row* K of $P^{[K-1]}$ where the entry is 1

Step3: Put 1 in each position $s_i t_j$ of $P^{[K]}$

Example: Find the path matrix of the graph G whose adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$P^{[0]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding $P^{[1]}$: Consider column 1 of $P^{[0]}$. The second position of the column 1 has '1' and inrow 1 of $P^{[0]}$, second position has '1'. Therefore put '1' in the position (2, 2) of $P^{[0]}$ to get the matrix $P^{[1]}$

$$\therefore \quad P^{[1]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding $P^{[2]}$: Consider column 2 of $P^{[1]}$, the positions 1 and 2 have 1's, in row 2, positions 1, 2 and 3 have 1's.

Therefore put '1' in the positions (1,1), (1,2), (1,3), (2,1), (2,2) and (2,3) of $P^{[1]}$ to get $P^{[2]}$

$$\therefore \quad P^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding $P^{[3]}$: Consider column 3 of $P^{[2]}$, the positions 1 and 2 have 1's, in row 3 position 4 has a 1 Therefore put 1 in the positions (1, 4) and (2, 4) of $P^{[2]}$ to get $P^{[3]}$

$$\therefore \quad P^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding $P^{[4]}$: Consider column 4 of $P^{[3]}$, the positions 1,2 and 3 have 1's, in row 4, there *is no 1*.

Therefore *no new 1's* are added to $P^{[3]}$ to get $P^{[4]}$

$$\therefore \quad P^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As n = number of vertices = 4. STOP.

Note: If $P_{ij}^{[K-1]} = 1$ then $P_{ij}^{[K]} = 1$—that is every '1' in $P^{[K-1]}$ remains as '1' in $P^{[K]}$.

Moreover If $P_{ij}^{[K-1]} = 0$ then we get a *new 1* in position (i,j) of $P^{[K]}$ only if there were ones in positions (i,k) and (k,j) of $P^{[K-1]}$ – ––that is

$$P_{ij}^{[K]} = 1 \text{ if } P_{iK}^{[K-1]} = P_{Kj}^{[K-1]} = 1.$$

Thus if $P_{ij}^{[K-1]} = 0$ we need only examine column K and row K of $P^{[K-1]}$ and if there is a 1 in position i of column K and a 1 in position j of row K, a 1 will be entered in position (i j) of $P^{[K]}$.

Warshall's algorithm can be modified further to obtain a matrix which given the lengths of shortest paths between the vertices (path that uses the least number of edges: shortest path).

For this purpose. Let A be the *adjacency matrix* of the graph. Replace all those elements of A which are zero by ∞, which shows that there is no edge between the vertices in question. The following algorithm produces the required matrix which shows the lengths of minimum paths.

6.3 Minima Algorithm (shortest pathmatrix for a unweighted graph)

Start with the adjacency matrix $A(G)$. Replace the zero elements in the adjacency matrix A by infinity (∞) or by some very large number, denote this matrix by M. The matrix C produced by the following steps shows the minimum lengths of paths between the vertices.

Step 1: $C^{[0]} = M$

Step 2: $K = 1$

Step 3: $i = 1$

Step 4: $C_{ij}^{[K]} = \text{Min}\left\{C_{ij}^{[K-1]}, C_{iK}^{[K-1]} + C_{Kj}^{[K-1]}\right\} \forall j = 1 \text{ to } n$

Step 5: $i = i + 1$ If $i \leq n$ go to step 4

Step 6: $K = K + 1$ If $K \leq n$ go to step 3

otherwise STOP

Note: $+$ in step 4 means the ordinary adding of integers.

6.4 Floyd–Warshall algorithm (shortest distance algorithm)

If our goal is to find the shortest path between every pair of vertices in a weighted graph with n vertices we can employ Dijkstra's algorithm, but the complexity is more. Apart from Dijkstra's algorithm, there is an algorithm due to R.W. Floyd and S. warshall named 'Floyd-warshall algorithm', which determines the *shortest distances* between all pairs of vertices in a graph, this algorithm is popular because it is so easy to describe.

To final the *shortest distances* between all pairs of vertices in a weighted graph where the vertices are $v_1, v_2 \ldots v_n$, carry out the following procedure

Step 1: $i = 1$ to n, set $d(i, i) = 0$

For $i \neq j$ if $v_i v_j$ is an edge, Let $d(i, j)$ be

the weight of this edge. Set $d(i, j) = \infty$, otherwise

Step 2: For $k = 1$ to n

For $i, j = 1$ to n

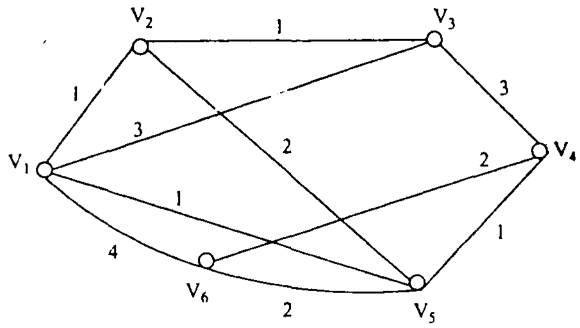
Let $d(i, j) = \min\{d(i, j), d(i, k) + d(k, J)\}$

The final value of $d(i, j)$ is the shortest distance from v_i to v_j.

Initially the algorithm sets the shortest distance from v_i to v_j to be the length of edge $v_i v_j$, if this is an edge. After the first iteration of step 2 ($k = 1$), this shortest distance has been replaced by the length of the path $v_i v_1 v_j$, if this is a path. Ingeneral after stage k, the algorithm has determined the shortest distance from v_i to v_j via the vertices $v_1, v_2 \dots v_k$. This distance is the true shortest distance after $k = n$.

The Floyd–Warshall algorithm is very efficient from the point of view of storage since it can be implemented by just updating the matrix of distance with each change in k; there is no need to store different matrices.

Example: Consider the graph.



Initial values of $d(i, j)$

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	3	∞	1	4
v_2	1	0	1	∞	2	∞
v_3	3	1	0	3	∞	∞
v_4	∞	∞	3	0	1	2
v_5	1	2	∞	1	0	2
v_6	4	∞	∞	2	2	0

when $K = 1$

$$\text{Consider } d(3,6) = \min\{d(3,6), d(3.1) + (1,6)\}$$
$$= \min\{\infty, 3 + 4\}$$
$$= 7$$
$$d(2,6) = \min\{d(2,6), d(2,1) + d(1,6)\}$$
$$= \min\{\infty, 5\}$$
$$= 5$$

After $K = 1$

$$\begin{bmatrix} 0 & 1 & 3 & \infty & 1 & 4 \\ 1 & 0 & 1 & \infty & 2 & 5 \\ 3 & 1 & 0 & 3 & 4 & 7 \\ \infty & \infty & 3 & 0 & 1 & 2 \\ 1 & 2 & 4 & 1 & 0 & 2 \\ 4 & 5 & 7 & 2 & 2 & 0 \end{bmatrix} \rightarrow$$

After $K = 2$

$$\begin{bmatrix} 0 & 1 & 2 & \infty & 1 & 4 \\ 1 & 0 & 1 & \infty & 2 & 5 \\ 2 & 1 & 0 & 3 & 3 & 6 \\ \infty & \infty & 3 & 0 & 1 & 2 \\ 1 & 2 & 3 & 1 & 0 & 2 \\ 4 & 5 & 6 & 2 & 2 & 0 \end{bmatrix}$$

After $K = 3$

$$\rightarrow \begin{bmatrix} 0 & 1 & 2 & 5 & 1 & 4 \\ 1 & 0 & 1 & 4 & 2 & 5 \\ 2 & 1 & 0 & 3 & 3 & 6 \\ 5 & 4 & 3 & 0 & 1 & 2 \\ 1 & 2 & 3 & 1 & 0 & 2 \\ 4 & 5 & 6 & 2 & 2 & 0 \end{bmatrix} \rightarrow$$

After $K = 4$

$$\begin{bmatrix} 0 & 1 & 2 & 5 & 1 & 4 \\ 1 & 0 & 1 & 4 & 2 & 5 \\ 2 & 1 & 0 & 3 & 3 & 5 \\ 5 & 4 & 3 & 0 & 1 & 2 \\ 1 & 2 & 3 & 1 & 0 & 2 \\ 4 & 5 & 5 & 2 & 2 & 0 \end{bmatrix}$$

After $K = 5$

$$\begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 3 \\ 1 & 0 & 1 & 3 & 2 & 4 \\ 2 & 1 & 0 & 3 & 3 & 5 \\ 2 & 3 & 3 & 0 & 1 & 2 \\ 1 & 2 & 3 & 1 & 0 & 2 \\ 3 & 4 & 5 & 2 & 2 & 0 \end{bmatrix} \rightarrow$$

After $K = 6\ (n)$

$$\begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 3 \\ 1 & 0 & 1 & 3 & 2 & 4 \\ 2 & 1 & 0 & 3 & 3 & 5 \\ 2 & 3 & 3 & 0 & 1 & 2 \\ 3 & 4 & 5 & 2 & 2 & 0 \end{bmatrix}$$

The final matrix is the shortest distance matrix.

Shortest path between a given pair of vertices

Recall that a weighted graph is a graph in which values are assigned to the edges and that the length of a path in a weighted graph is the sum of the weights of the edges in the path. We let $w(i, j)$ denote the weight of edge (i.j).

In weighted graphs, we often want to find a *shortest path* (a path having minimum length) between two given vertices. Dijkstra's algorithm efficiently solves this problem. Through this discussion G denotes a connected weighted graph we want to find a stortest path from vertex 'a' to vertex 'z' (The assumption G is connected can be dropped).

Dijkstra's algorithm involves assigning labels to vertices. We let $L(v)$ denote the label of vertex v. At any point some vertices have temporary Labels and the rest have permanent labels. we let T denote the set of vertices having temporary labels. We will circle vertices having permanent labels. We will show later that it $L(v)$ is the permanent label of vertex v, then $L(v)$ is the length of a shortest path from a to v. Initially all vertices label of vertex v, then $L(v)$ is the length of a shortest path from a to v. Initially all vertices have temporary labels. Each iteration of the algorithm changes the states of one label from temporary to permanent, thus we may terminate the algorithm when z receives a permanent label. At this point $L(z)$ gives the length of a shortest path from a to z.

6.5 Dijkstra's shortest path algorithm

Input: A connected, weighted graph in which all weights are positive; vertices a and z.

Output: $L(z)$, the length of a shortest path from a to z.

Step 1: Let $P = \phi$, where P is the set of those vertices which have permanent labels and T = all vertices of G.

Step 2: Let $P = \{a\}$, i.e., label the starting vertex 'a' permanently with label 0 ie., $L(a) = 0$ (initially), set $L(x) = \infty$ for $x \neq a.T = V - \{a\}$

Step 3: Label another vertex permanently according to the following rules:

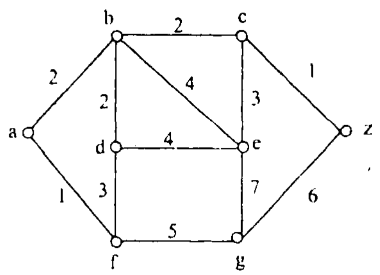
1. Every vertex 'j' (adjacent to previously permanent labeled vertex i) that is not yet permanently labeled gets the temporary label:
   $L(j) = \min\{\text{old}L(j), \text{old}L(i) + w(ij)\}$ where i is the latest vertex permanently labeled and $w(ij)$ is the weight of the edge (ij), it there is no edge (i j) then $w(ij) = \infty$
2. The smallest value among all the temporary labels becomes the permanent label of the corresponding vertex (in case of tie select any one for permanent labeling)

Step 3: Repeat step 2 until z gets a permanent label.

Note: This algorithm doesnot actually gives the shortest path, it gives only the shortest distance. The shortest path can easily be constructed by working backwards from the terminal vertex.

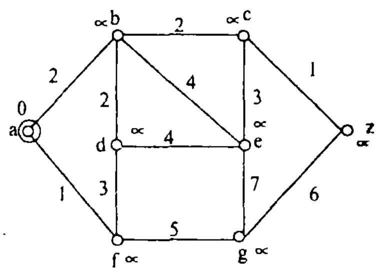
Note: If the digraph is not a weighted one the weights of all the edges are taken to be 1.

Example: Find a shortest path from a to z in following graph



Solution: $P = \phi, T$ = Set of all vertices $= \{a, b, c, d, e, f, g, z\}$. Let label of a i.e $L(a) = 0$ and $L(x) = \infty \forall x \neq a$

Initialization in Dijkstra's shortest path algorithm:

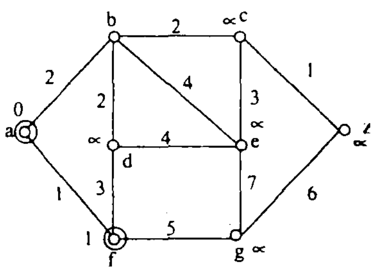


The uncircled vertices adjacent to a are b and f

∴ Then the new labels

$$L(b) = min\{\text{old } L(b), L(a) + w(a, b)\}$$
$$= \min\{\infty, 0 + 2\} = 2$$
$$L(f) = min\{\text{old } L(f), L(a) + w(a, f)\}$$
$$= min\{\infty, 0 + 1\} = 1$$

vertex f is with the smallest label, and circle it. Now



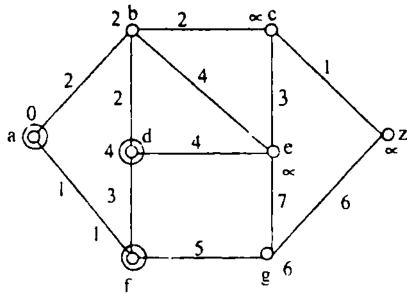
The uncircled vertices adjacent to f are d and g

$$L(d) = min\{\text{old } L(d), L(f) + w(f, d)\}$$
$$= min\{\infty, 1 + 3\} = 4$$
$$L(g) = \min\{\text{old} L(g), L(f) + w(f, g)\}$$
$$= min\{\infty, 1 + 5\} = 6$$

vertex d is with smallest label and circle it

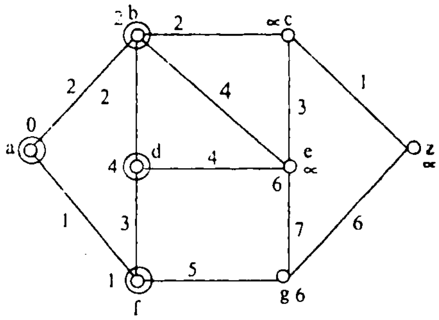
The uncircled vertices adjacent to d are b and e

$$L(b) = min\{\text{old } L(b), L(d) + w(d, b)\}$$
$$= min\{2, 4 + 2\}$$
$$= 2$$



$$
\begin{aligned}
L(e) &= min\{\text{old } L(e), L(d) + w(d, e)\} \\
&= min\{\infty, 4 + 4\} \\
&= \infty
\end{aligned}
$$

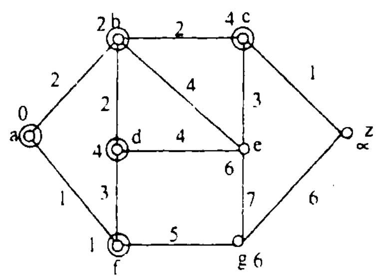
vertex b is with smallest label and circle it



The uncircled vertices adjacent to b are c and e

$$
\begin{aligned}
L(c) &= \min\{\text{old } L(c), L(b) + w(b, c)\} \\
&= \min\{\infty, 2 + 2\} = 4 \\
L(e) &= \{\text{old } L(e), L(b) + w(b, e)\} \\
&= \min\{\infty, 2 + 4\} = 6
\end{aligned}
$$

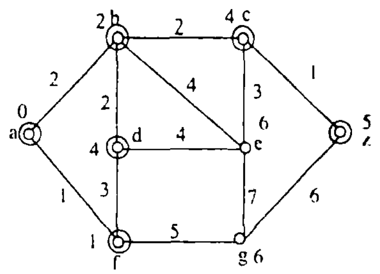
vertex c is with smallest label and circle it



Uncircled vertices adjacent to c are e and z

$$L(e) = min\{\text{old}L(e), L(c) + w(e, c)\}$$
$$= min\{6, 4 + 3\} = 6$$
$$L(z) = min\{\text{old } L(z), L(c) + w(c, z)\}$$
$$= min\{\infty, 4 + 1\} = 5$$

Vertex z is the smallest Label and circle it, and as it is the permanent label for z, stop.
$\therefore$ as z is labeled 5, indicating that the length of a shortest path from a to z is 5. A shortest path is given by $a - b - c - z$



6.6 Algorithm for connectedness (Fusion algorithm).

The Fusion algorithm is the most efficient algorithm for connectedness. This method also tells show many components the graph has.

Before going to the a actual algorithm we require the following steps:

Algorithm 1: To find new adjacency matrix after fusion.

Step 1: Change u's row to the sum of u's row with v's row and (symmetrically) change u's column to the sum of u's column with v's column.

Step 2: Delete the row and column corresponding to v. The resulting matrix is the adjacency matrix of the new graph G.

Algorithm 2 (Fusion algorithm)

Let G be a connected or disconnected graph having self loop and parallel edges. In finding the connectedness removal of self loop and parallel edges does not effect.

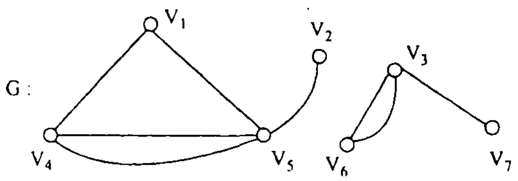
Step 1: Remove all the self loops and parallel edges Denote this new graph by G_1.

Step 2: Fuse vertex v_1 to the first of the vertices $v_2, \ldots v_n$ with which it is adjacent to give a new graph, denoted by G_2 in which the new vertex is also denoted by v_1.

Step 3: The above two steps (algorithm 1) gives the adjacency matrix $A(G_2)$.
Step 4: Repeat steps 1 and 2 with until v_1 is not adjacent to any of the other vertices.
Step 5: Repeat step 2 and 4 on the vertex v_2 of the last graph and then on all remaining vertices of the resulting graphs.

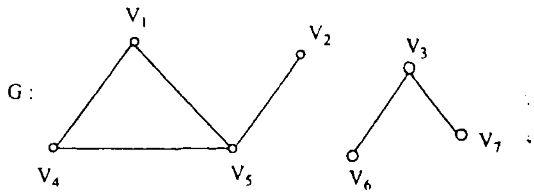
The final graph is empty and the number of its (isolated) vertices is the number of connected components of the initial graph G

Example: Given below is the adjacency matrix of graph G with seven vertices listed as $v_1 \dots v_n$. Use fusion algorithm to check the connectedness.

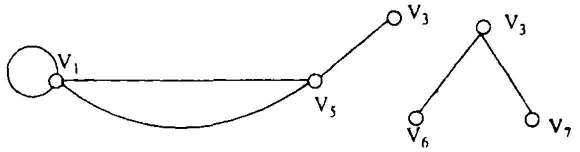


$$A(G) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: Step 1: Removal of parallel edges and self loops therefore the new graph is

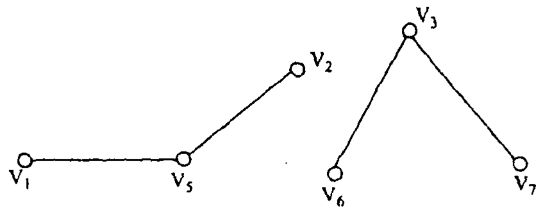


Step 2: Fuse v_1 to the first vertex v_4

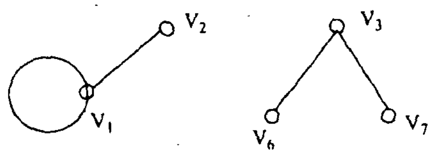


Removing self loop and parallel edge

$$\text{Then } A(G_2) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

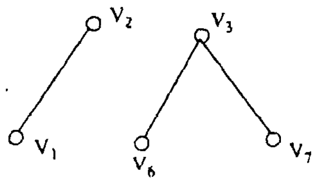


Fusing v_1 with v_5



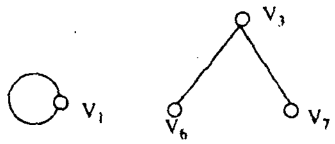
removing self loop

G3



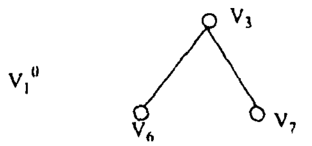
$$A(G_3) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fusing v_1 and v_2



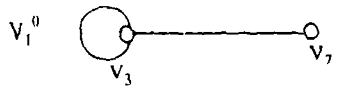
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

removing self loop



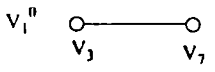
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Fusing v_3 with v_6



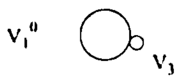
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

removing self loop



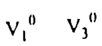
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Fusing v_3 with v_6



$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

removing self loop



$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since the final adjacency matrix is a 2×2 null matrix, we conclude that the original graph G has 2 connected components.

6.7 Planarity detecting algorithm

Step (i) A disconnected graph is planar $\Longleftrightarrow$ each of its components is planar

Step (ii) A separable graph is planar $\Longleftrightarrow$ each of its block is planar
Let $G = \{G_1, \ldots G_K\}$, where each G_i is a non separable block of G. test each G_i for planarity.

Step (iii) Since the addition or removal of a self loop does not affect planarity, remove all self loops for finding the planarity.

Step (iv) Since parallel edges also do not affect planarity eliminate edges in parallel while detecting the planarity.

Step (v) Elimination of a vertex of degree two by merging two edges in series, does not affect planarity. Therefore eliminate all edges in series.
Therefore repeat the application of steps(iv) and (v) till the graph can't be reduced any further

Step (vi) The reduced graph after the repeated application of steps (iii) and (iv) tested for planarity by using the following results:

(a) A single edge graph is a planar graph
(b) A complete graph of 4 vertices (K_4) is a planar graph
(c) If the graph is with $|V| \geq 5$ land $|E| \geq 7$, use the result

"Let G be a planar graph with $|V| \geq 3$ and $|E|$ edges then $|E| \leq 3|V| - 6$"
or
Use Kuratowski's theorem.

Example: Show that K_5 is not planar

Solution: If K_5 is planar, then

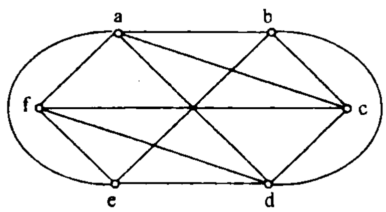
$$|E| \leq 3|V| - 6$$
$$\Rightarrow 10 \leq 3.5 - 6 = 9$$

a contradiction

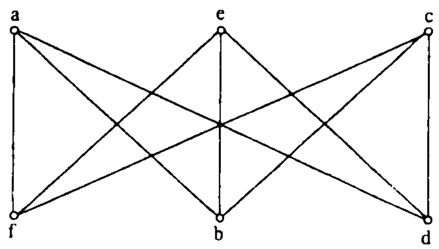
$\therefore$ K_5 is not planar

Example: Show that the following graph is not planar by finding a subgraph homomorphic to either K_5 or $K_{3,3}$

Solution: Given graph



Delete the edges (a, e), (b, d), (a, c) and (f, d) then the graph becomes



which is $K_{3,3}$.

$\therefore$ The given graph is not planar.

6.8 Fleury's algorithm (constructing an euler circuit)

Fleury's algorithm is useful in constructing an euler circuit in a connected graph G.

Input: A connected graph G with every vertex is of even degree

Output: Euler circuit in G.

Step 1: Choose a starting vertex u

Step 2: Traverse any available edge, choosing an edge that will disconnect the remaining graph only if there is no alternative

Step 3: After traversing each edge, remove it (together with any vertices of degree 0 with result)

Step 4: If no edges remain, stop. Otherwise, choose another available edge and go back to step 2.

Presenting the Fleury's algorithm using graph theory terminology i.e; without pure english sentences:

Step 1: Select any vertex v as the beginning vertex of the circuit $C : v$

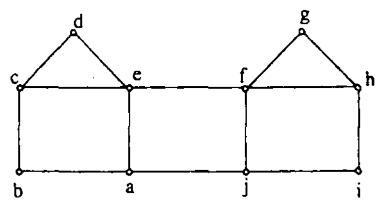
Step 2: Suppose $C : v, u, \ldots w$ has been constructed so far

(i) If there is only one edge (w, z) at w extend C to $C : v, u, \ldots w, z$. Delate (w, z) from E and w from V

(ii) If there are several edges at w, choose the one that is not a bridge [cut edge] to the remaining graph, say (w, z) extend C to $C : v, u, \ldots w, z$ and delete (w, z) from E and w from V

Step 3: Repeat Step 2 until no edges remain in E End.

Example: Use Fleury's algorithm on the below graph to find an Euler circuit



Solution: Choose the vertex a. The result of applying Step 2 are listed in the following table.

Current path	Next edge	Reasoning
$C : a$	aj	No edge from a is a bridge, choose any one
$C : aj$	jf	No edge from j is a bridge, choose any one
$C : ajf$	fg	fe is a bridge and fg is not a bridge
$C : ajfg$	gh	No edge from g is a bridge
$C : ajfgh$	hi	No bridge from
$C : ajfghi$	ij	- do
$C : ajfghij$	jh	- do -
$C : ajfghijh$	hf	- do -
$C : ajfghijhf$	fe	fe is the only edge no alternative

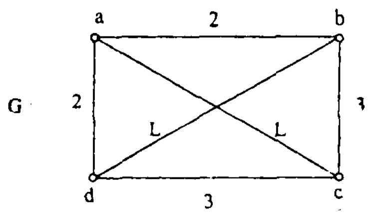
$C : ajfghijhfe$	ed	No bridge
$C : ajfghijhfed$	dc	- do -
$C : ajfghijhfedac$	ce	- do -
$C : ajfghijhfedace$	ea	- do -
$C : ajfghijhfedacea$	No edge remaining in E	End

Algorithm for TSP

The traveling salesperson problem is related to the problem of finding a hamiltonian cycle in a graph.

The problem is: Given a weighted graph G, find a minimum length hamiltonian cycle in G. If we think of the vertices in a weighted graph as cities and the edge weights as distances, the traveling salesperson problem is to find a shortest route in which the salesperson can visit each city one time, starting and ending at the same city.

Example:



The cycle C: $abcda$ is a hamiltonian cycle for the graph G Replacing any of the edges in C by either of the edges labeled L would increase the length of C; thus C is a minimum length hamiltonian cycle for G. Thus C solves the traveling salesperson problem for G.

Although there are algorithms for finding an euler cycle. No efficient algorithm is available to find a hamiltonian cycle of a graph. But the following algorithm is the best possible one, but there may exist some minimum hamiltonian cycle.

6.9 TSP algorithm (to find a Hamiltonian cycle)

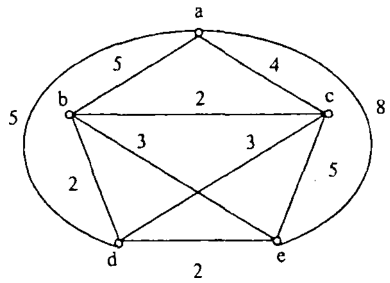
Nearest neighbourhood method

Step 1: Start with an arbitrary chosen vertex v, and scan all the vertices adjacent to v. Find the vertex that is closest to this vertex (- that is the end vertex of an edge incident from v having minimum weight) Take a path P by joining v and u.

Step 2: Choose a vertex, which is not already chosen closest to the last vertex chosen by scanning all the vertices adjacent to it as in step 1. Extend the path P to this vertex

Step 3: If no vertex is left to scan, obtain a circuit by joining the vertex v and the last vertex chosen.
Otherwise return to step 2.

Example: Find a minimum hamiltonian cycle of the following network using nearest neighbourhood method

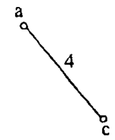


Solution: Start with an arbitrary chosen vertex a.

* The vertices adjacent to a are b, c, d and e

$$w(a, b) = 5, w(a, c) = 4, w(a, d) = 5$$

$$w(a, e) = 8 \quad \therefore \quad \min\{4, 5, 5, 8\} = 5 \text{ for the edge } ac$$

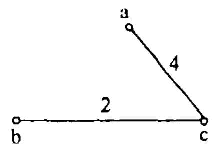


hence take the path: $a - c$

* The vertices adjacent to 'c' are b, d and e

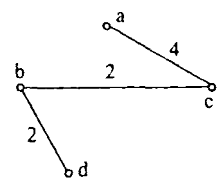
$$w(c, b) = 2, w(c, d) = 3, w(c, e) = 5$$

$$\min(2, 3, 5) = 2 \quad \text{for the edge } cd$$



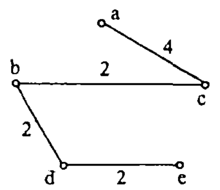
Hence take the path: $a - c - b$

* The vertices adjacent to b are d and $e w(b, d) = 2$, $w(b, e) = 3$, $\min(2, 3) = 2$ and that is for the edge bd

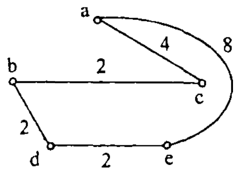


Hence the path: $a - c - b - d$

* The vertex adjacent to 'd' is $e w(d, e) = 2$
Hence the path: $a - c - b - d - e$

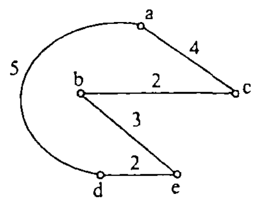


* If no vertex adjacent to e is left to scan. Join e and a. The minimum hamiltonian circuit H is shown below



The weight of $HC : 4 + 2 + 2 + 2 + 8 = 18$

Note: As we mentioned earlier, the hamiltonian circuit obtained by this method is one of the best possible one. However, there may exist a hamiltonian circuit with lesser weight. For the above graph the circuit mentioned below is also a HC. The weight of the circuit is 16, which is less than that of the circuit obtained above.



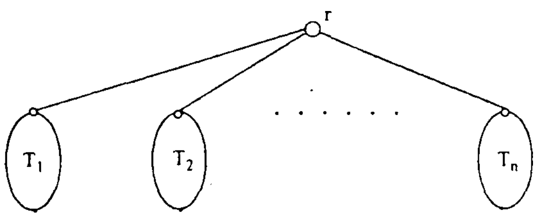
□

6.10 Tree Traversal Algorithms

A traversal of a tree is a process that enumerates each of the vertices in the tree exactly one, when a vertex is encountered in the order of enumeration specified by a particular process, we say that we *visit* the given vertex.

Apart from $BFS\&DFS$, we describe here three principal ways that may be used to traverse a given tree, later inparticular traversing a binary tree.

Consider the following tree with root 'r' and the subtrees $T_1, T_2, \ldots T_n$ (from left to right)



Preorder traversal:

Step 1: Visit root

Step 2: Visit subtrees left to right in *pre order*

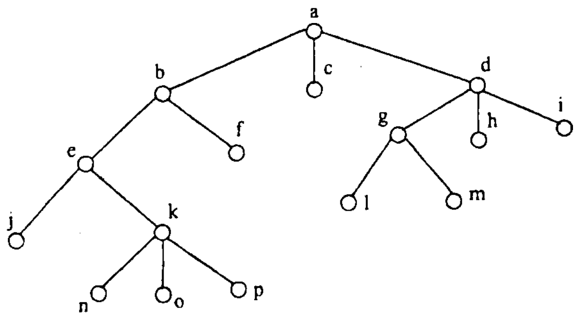
In order traversal:

Step 1: Visit left most subtree in inorder
Step 2: Visit root
Step 3: Visit other subtrees left to right in inorder

Postorder traversal:

Step 1: Visit subtrees left to right in postorder
Step 2: Visit root

Example: Consider the following tree



Pre order traversal: *abcd*

abefcdghi

abejkfcdglmhi

| *abejknopfcdglmhi* |

In order traversal: *bacd*

ebfacgdhi

jekbfaclgmdhi

| *jenkopbfaclgmdhi* |

Post order traversal: *bcda*

efbcghida

jkefbclmghida

| *jnopkefbclmghida* |

□

Now considering traversal algorithms especially for *binary trees:* each scheme will be defined by specifying the order for processing the 3 entities: the root (N), the left subtree (L) and the right subtree (R).

Preorder traversal

Step 1: Visit the root
Step 2: Visit the left subtree from the root in post order (is subtree exists)
Step 3: Visit the right subtree from the root in the post order.

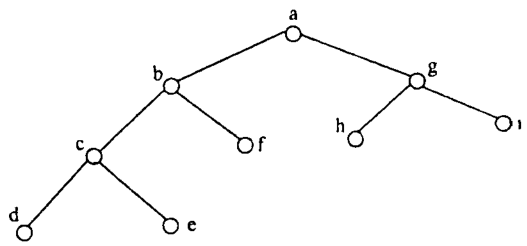
Inorder traversal

Step 1: Visit the left subtree from the root in inorder (if it exists)
Step 2: Visit the root
Step 3: Visit the right subtree from the root in inorder

Post order traversal

Step 1: Visit the left subtree from the root in postorder
Step 2: Visit the rightsubtree from the root in postorder
Step 3: Visit the root.

Example:



Preorder: *abcdefghi*
Inorder: *dcebfahgi*
Postorder: *decfbhiga*.

□

6.11 Cyclic exchange algorithm (finding all spanning trees)

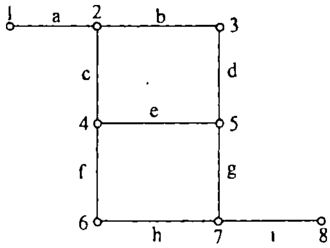
Addition of a chord to a tree always produces a circuit. Moreover, deletion of any edge from this circuit again gives a tree. The following "cyclic exchange algorithm" is based on this strategy.

Step 1: Find a spanning tree T_1 for the given graph (by already mentioned procedures). Add a chord e_1 to the tree T_1. This produces a fundamental circuit

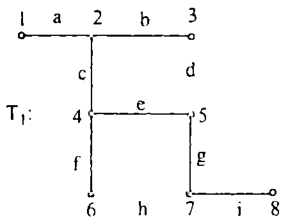
Step 2: Scan all branches of T_1 that are also edges of the fundamental circuit obtained in step 1. Choose one branch and delete it from the fundamental circuit. We get a tree, call it $T_{1,2}$. Add it again to the tree and choose another branch (if it exist). Repeat the process of addition and deletion obtain the trees $T_{1,3}$, $T_{1,4}$, ... for all possible branches.

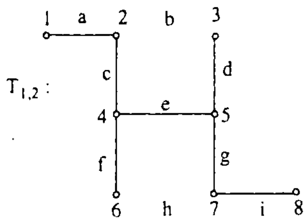
Step 3: Scan all chords of T_1, choose a chord e_i ($i > 1$ if it exist) of T_1, which is not chosen earlier return to step 1.

Example: Find all spanning trees of the following graph

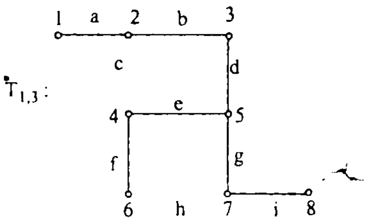


Solution: One of the spanning tree





Addition of 'd' and deletion of 'b'

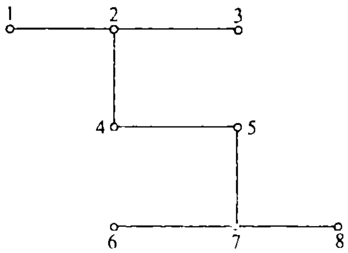


Addition of 'd' and deletion of 'c'

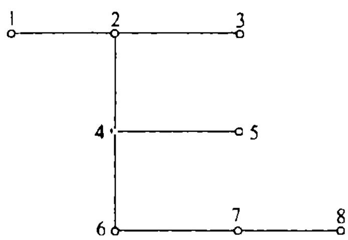
The chords of the graph with respect to T_1 are d and h. The branches of T are a, b, c, e, f, g and i

Addition of the chord d yields a fundamental circuit 3 – 2 – 4 – 5 – 3. The branches of the tree on this circuit are b, c and e. And addtion of the chord h yields a fundamental circuit 6 – 7 – 5 – 8 – 6.

The branches of the tree on this circuit are e, f and g. Thus addition of a chord to the tree T and deletion of one branch at a time with replacement gives the following spanning Trees $T_{1,2}$ and $T_{1,3}$ etc;



Addition of 'h' and deletion of 'f'



Addition of 'h' and deletion of 'g'

□

Minimal Spanning Tree Algorithms

Spanning Tree: A spanning tree of a connected graph G is a subgraph which is a tree and which includes every vertex of G.

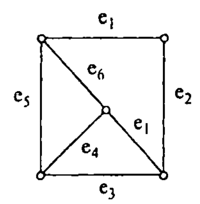
Minimal Spanning Tree: A minimal Spanning Tree of a connected weighted graph is a spanning tree of least weight, that is, a spanning tree for which the sum of the weights of all its edges is least among all spanning trees.

The concept of spanning tree exists only for a connected graph G and if G has n vertices, any spanning tree must necessarily contain $n - 1$ edges.

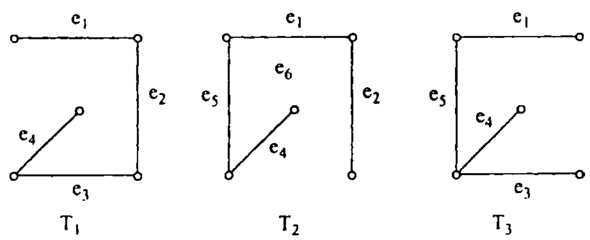
Algorithm to find a spanning tree in a connected graph G:

Step 1: If G has no cycles, then it is already a tree, $S.G$ is a spanning tree for G.
Step 2: If G contains a cycle, then delete an edge which breakes the cycle without disturbing the connectedness of the graph G
Identify the other remaining cycles in the graph G and
Step 3: Repeat the step 2 until a connected subgraph without cycles *containing all the vertices* of G will be obtained, that is a spanning tree,
Step 4: Also ensure $|E| = |V| - 1$

Example:



Deleting the edges which are forming cycles yields a spanning tree



6.12 Kruskals algorithm

Input: A connected weighted graph G with n vertices
Output: A minimal spanning tree for G

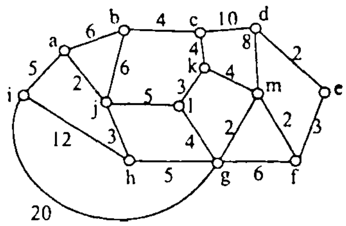
Step 1: Arrange the edges in order of increasing weights

Step 2: Select the edge with minimum weight (If there is more than one edge with minimum weight arbitrarly choose one of these edges). This is first edge of T.

Step 3: Proceed sequentially adding each edge of (next) minimum weight to T which does not result in a cycle until $n - 1$ edges are included

Step 4: Exit with minimal spanning Tree T

Example:



Kruskal's algorithm initially picks an edge of lowest (least) weight There are several edges of weight 2, any of these may be selected. Suppose begin with edge $e_1 = aj$. There still remain edges of weight 2, so choose any other say $e_2 = gm$. Among those edges which do not form a cycle with e_1 and e_2, choose one of least weight, say $de = e_3$. Continuing, we may have edges $aj, gm, de, mf, fe, lk, hj, km, ck, bc$. At this stage, there remains an edge (gl) of weight 4, but this edge would complete a cycle with lk, km, gm already selected, and so choose an edge of weight 5 say jl and another ia. Now any of the remaining edges will form a cycle also $|E| = 12 = |V| - 1 = 13 - 1$. So must stop.

The algorithm assures us we have the desired minimum spanning tree. It has weight 39. □

6.13 Prim's algorithm

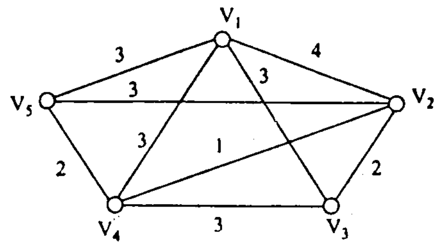
Input: A connected weighted graph G with n vertices
Output: A minimal spanning tree T

Step 1: Select an arbitrary vertex v_1 and an edge e_1 with minimum weight incident with vertex v_1. This forms initial MST T

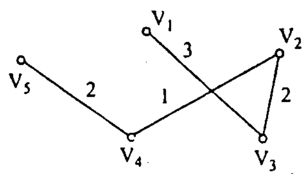
Step 2: If edges $e_1, e_2, \ldots e_i$ have been chosen involving end points $v_1, v_2, \ldots v_{i+1}$. Choose and edge $e_{i+1} = v_j v_k$ with $v_j \in T$ and $v_k \in T$ such that e_{i+1} has smallest weight among the edges of G with precisely one end in $\{v_1, \ldots v_{i+1}\}$

Step 3: Stop after $n - 1$ edges have been chosen. Otherwise go to step 3

Example: Find the minimal spanning tree of the weighted graph using Prim's algorithm



According to step 1, choose vertex v_1, Now edge with smallest weight incident on v_1 is $e = v_1 v_3$ or $v_1 v_5$, choose $e = v_1 v_3$ similarly choose the edges $v_3 v_2$, $v_2 v_4$, $v_4 v_5$

The minimal spanning tree is 

□

6.14 Breadth First Search Algorithm

The idea of BFS is to visit all vertices sequentially on a given level before going onto the next level.
Input: A connected graph G with vertices labelled $v_1, v_2 \ldots v_n$
Output: A spanning tree T of G

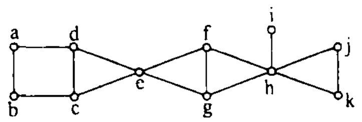
Step 1: Let v_1 be the root of G. Form the set $V = \{v_1\}$

Step 2: Adding the new edges: Consider the vertices of the graph in order, consistent with the orginal labelling, then for each vertex $x \in V$ add the new edge $\{x, v_k\}$ to T, where k is the minimum index such that adding the edge $\{x, v_k\}$ does not produce any cycle.
If no edge can be added then STOP. Then T is a spanning tree for G
After all the vertices of V have been considered in order go to step 3

Step 3: Replace V by all the children v in T of the vertices x of v where the edges $\{x, v\}$ were added in step 2.
Go back and repeat step 2 for the new set V

Example: Consider the ordering of the vertices $abcdefghijk$ Then select a as the first vertex chosen in the spanning tree T and designate it as the root of T. Thus at this stage, T consists of the single vertex a. Add to T all $\{a, x\}$ as x runs in order from b to k, that do not produce a cycle in T.

Thus we add $\{a, b\}$ and $\{a, d\}$. These edges are now called tree edges for the BFS tree.



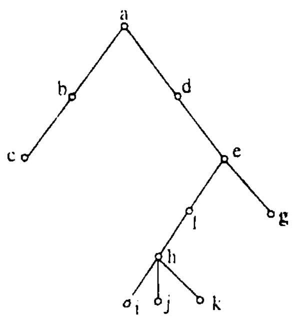
Now repeat the process for all vertices on level one from the root by examining each vertex in the designated order. Thus since b and d are at levelone, we first examine b.

For b, we include the edge $\{b, c\}$ as a tree edge. Then for d, we reject the edge $\{d, c\}$ since its inclusion would produce a cycle in T. But we include $\{d, e\}$. Next we consider the vertices at level two. Reject the edge $\{c, e\}$; include $\{e, f\}$ and $\{e, g\}$

Then repeat the procedure again for vertices on level three. Reject $\{f, g\}$, but include $\{f, h\}$. At g, reject $\{f, g\}$ and $\{g, h\}$.

On level four, include $\{h, i\}$, $\{h, j\}$ and $\{h, k\}$

Next we attempt to apply the procedure on level five at i, j and k, but no edge can be added at these vertices so the procedure ends. The spanning tree T therefore includes the vertices $a, b, c, d, e, f, g, h, i, j$ and k and the edges $\{a, b\}$, $\{a, d\}$, $\{b, c\}$, $\{d, e\}$, $\{e, f\}$, $\{e, g\}$, $\{f, h\}$, $\{h, i\}$, $\{h, j\}$ and $\{h, k\}$



□

6.15 Depth First search Algorithm

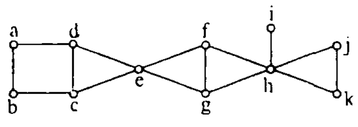
The idea of DFS is proceeding to higher levels successively in the first opportunity Later we back track and add the vertices which are not visited.
Input: A connected graph G with vertices labeled $v_1, v_2 \ldots v_n$
Output: A spanning tree T for G

Step 1: (Visit a vertex) Let v_1 be the root of T and set $L = v_1$ (L stands for the vertex last visited)

Step 2: (Find an unexamined edge and an unvisited vertex adjacent to L) For all vertices adjacent to L choose the edge $\{L, v_k\}$, where k is the minimum index such that adding $\{L, v_k\}$ to T does not create a cycle. If no such edge exists go to step 3; otherwise add edge $\{L, v_k\}$ to T and set $L = v_k$; repeat step 2 at the new value for L

Step 3: (Back track or terminate) If x is the parent of L in T set $L = x$ and apply step 2 at the new value of L. If on the otherhand, L has no parent in T (so that $L = v_1$) then the DFS terminates and T is a spanning tree for G.

Example:



Choose an ordering of the vertices, say $abcdefghijk$ select a as the root of T. The vertex a is said to be visited. Next, we select the edge $\{a, x\}$ where x is the first label in the designated order that doesnot form a cycle with those edges already chosen in T. In this case we add the edge $\{a, b\}$. The edge $\{a, b\}$ is now said to be examined and becomes a tree edge. In this context, a is the parent of b and b is the child of a. In general, while at some vertex x, two situations arise.

Situation 1: If there are some unexamined edges incident on x, then we consider the edge $\{x, y\}$, where y is the first vertex in the designated ordering on the vertices for which $\{x, y\}$ is unexamined.

In this situation, two cases present themselves

Case 1: If y has not been previously visited, visit y, select $\{x, y\}$ as a tree edge, and continue the search from y. In this case, x is the parent of y

Case 2: If y has been visited previously, then reject the edge $\{x, y\}$ consider it examined and proceed to select another unexamined edge $\{x, z\}$ incident on x where is the first vertex for which $\{x, z\}$ is an unexamined edge. Each such rejected edge in the context of DFS is called a back edge.

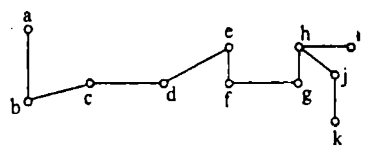
In the example at hand, we would select the edge $\{b, c\}$ continue the search at c, and select $\{c, d\}$. Then we would continue the search at d and first reject the edge $\{d, a\}$ and then select the edge $\{d, e\}$. At e, we reject the edge $\{e, c\}$, select $\{e, f\}$. Continuing in this manner, we select $\{f, g\}$, reject $\{g, e\}$ select $\{g, h\}$ reject $\{h, f\}$ and select $\{h, i\}$

At this point, we are presented with a second general situation.

Situation 2: If all the edges incident on x have already been examined, then we return to the parent of x and continue the search from the parent of x. The vertex x is now said to be *completely* scanned Moreover, the process of returning to the parent of x is called *back tracking*.

Thus, in the example that we are considering, since there are no unexamined edges at i, we must backtrack to h and continue the search from h. Then we select $\{h, j\}$ and $\{j, k\}$ and finally reject $\{k, h\}$.

Actually we are through, because there are no more unexamined edges. But if we had limited version (as a computer may have), we may be aware only that there are no unexamined edges at k. Therefore, we backtrack, according to situation 2 to j. But then we must backtrack to h, etc. Eventually we must backtrack all the way back to the root a.



Comparison between BFS and DFS

1. The idea of BFS is to visit all vertices sequentially on a given level before going onto the next level.
   The idea of DFS is proceeding successively to higher levels at the first opportunity.
2. The edges that were rejected in BFS are called cross edges. Clearly BFS partitions the edges of the graph G into the two sets of tree edges and cross edges
3. DFS terminates when the search returns to the root and all vertices have been visited.

□

6.16 Welch–Powell Algorithm for a coloring of a graph G

Step 1: Order the vertical in G according to decreasing degree

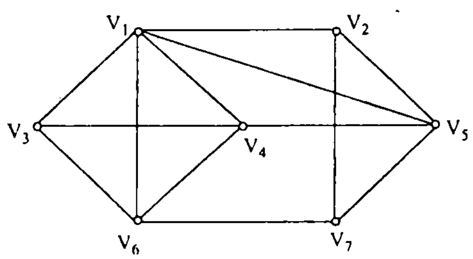
Step 2: Assign the first color C_1 to the first vertex and then in sequential order assign C_1 to each vertex which is not adjacent to a previous vertex which was assigned the color C_1

Step 3: Repeat the step 2 with a second color C_2 and the subsequence of non colored vertices

Step 4: Repeat the step 3, with a third color C_3 then a fourth color C_4 and so on until all the vertices are colored.

Note: Welch–Powell algorithm does not always yield a minimum coloring of a graph G, it gives only an upper bound for $\chi(G)$.

Example: Find $\chi(G)$ for the following graph using welch powell algorithm



Solution:

Vertex:	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree:	5	4	4	4	3	3	3
Color:	C_1	C_2	C_3	C_3	C_2	C_4	C_1

Color v_1 with C_1 and since v_7 is not adjacent with v_1 color v_7 also with C_1.

Color v_4 with C_2, v_2 and v_7 are not adjacent with v_4. Since v_7 is already colored. Color v_2 with C_2.

Color v_5 with C_3, since v_6 is not adjacent with v_5 Color v_6 with C_3, v_3 is also not adjacent with v_5 but it is adjacent with v_6. Hence v_3 can't be colored with C_3.

Color v_3 with C_4

All the vertices are colored and 4 colors are used. Hence $\chi(G) \leq 4$

But the vertices v_1, v_3, v_4 and v_6 are adjacent to each other. Hence at least 4 colors are required to color these vertices. Thus $\chi(G) = 4$.

□