Single Server Non Markovian Models

By

M. Thangam

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE
AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE-641 043,
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CERTIFIED AS BONAFIDE RESEARCH WORK

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"Experience has shown that the power of Mathematical models in the study of real World has increased along with the complexity of the models and the level of abstraction of the Mathematics".

SYNOPSIS

SYNOPSIS

In the first chapter we discuss a non-markovian model GI/M/1 in which the inter arrival time has general distribution and service time is exponential. An imbedded Markovian chain approach is used to obtain the solutions. Since the inter arrival time has general distribution, the number of units present in the system at arrival epochs will form semi Markov chain. Here we have considered the system just prior to the arrival and derived the study state probability distribution at these points. Waiting time distribution of an arriving customer in the system has been obtained.

There are situations where the service provided is such that a group (or batch) of customers can be served simultaneously. Some examples are shuttle-bus service, freight trains express elevators and batch servicing in manufacturing processes. There are a number of rules according to which batches may formed for bulk service.

In second chapter we use general bulk service rule introduced by Neuts [16]. With this rule the steady state probability distributions at prearrival epochs and at arbitrary epochs have been derived in section 1 of this chapter. In section 2 the waiting time distribution of an arriving customer and his expected waiting time have also been calculated.

Recently, U.Chatterjee and C.P. Mukkerj have studied a GI/M/1 with exhaustive service and multiple vacations which are independently and identically distributed with a general distribution. In chapter III we discuss this model. The imbedded Markov chain technique is used to obtain the equilibrium probability distributions of system size at prearrival and at random epoch separately assuming that these probabilities exist. Finally for this model the distribution of waiting time of a customer in the queue (excluding service) has been derived.

Introduction

INTRODUCTION

A queueing system can be described by the flow of units for service, forming or joining the queue, if service is not immediately available, and leaving the system after being served (or sometimes without being served).

The basic features which characterise a system are;

1. Input process
2. Service mechanism
3. Queue discipline
4. Number of servers
5. System capacity

1. Input Process

The input process describes the manner in which units arrive and join the system. A unit may arrive either singly or in a group. The interval between two consecutive arrivals (or between two consecutive arrival groups, incase of bulk arrivals) is called the interarrival time or interval.

2. Service Mechanism

The service mechanism describes the manner in which service is rendered. A unit may be served either singly or in a batch. The time required for servicing a unit (or a group, in case of batch service) is called the service time.

3. Queue discipline

It is the principle by which the units form a queue and served. They are

a. First come first served (FCFS)

b. Last come first served (LCFS)

c. Service in random order (SIRO)

4. Number of Servers

A queueing system may contain a number of service channels (either parralel or series or a combination of both) to provide service. When the system has a number of parallel servers it is known as multi server model.

5. System capacity

The system capacity may be finite or infinite.

Kendall's notation

Kendall (1951) formulated the convenient symbolic form A/B/C/X/Y to specify any queueing model completely. The first and second symbols denote the type of distributions of interarrival times and of inter service times respectively. Third symbol specifies the number of servers, fourth the capacity of the system. The last symbol denote the queue discipline. But only the first three symbols are used commonly. Unless otherwise stated the queue discipline can be considered as FIFO and the system capacity can be considered infinite.

Some of the familiar notations are the following:-

M: Exponential (Markovian) distribution.

Gl: General input distribution.

Thus GI/M/1 defines a single server queueing system with general input and exponential service time distribution.

G/M a,b/1 denotes the queing model with single server in which the service time is exponential and the interarrival time has general distribution with mean $\frac{1}{\mu}$ and $\frac{1}{\lambda}$ respectively, following the bulk service rule.

Queueing models with vacations

In most queueing models, the server on completion of service to all the existing units continues to stay in the empty system, awaiting a new arrival. Service commences immediately upon a customer's arrival. But there are some physical systems in which an idle server will leave the system for some uninterruptable task (such as a tea break, a telephone call, a checkup operation in case the server is a machine etc.) In some cases the server may utilize his idle time in a useful and optimal way to perform additional jobs or for the preventive maintenance work in case the server is a machine idle. periods of the system during which these additional tasks will be performed will be referred to as 'vacation'.

There are several types of vacations namely.

1. Single vacation
2. Repeated vacation
3. Limited service vacation
4. Exhaustive service vacation
5. Exceptional first vacation
6. Gated vacation

Queueing models with inter-arrival time of customers and service time exponentially distributed are called Markovian queueing models.

Many authors have considered the models where the interarrival time is not exponential. In these models the process N(t) giving the system size at time t will not be Markovian. Using various techniques the non Markovian can be associated with Markovian. The imbedded Markov chain technique introduced by Kendall uses the concept of regeneration point (due to Palm) by suitable choice of regeneration points and extracts, from the process N(t) , Markov chains in discrete time at those points.

We can find the detailed study of G/M/1 model in Medhi[14], Gross & Harris [6], Saaty [17].

In most of the practical situations, the customers are served in batches. There are number of policies or rules according to which batches for bulk service may be formed. Neuts [16] have considered a general bulk service rule in which the batch takes a minimum of 'a' units and a maximum of 'b' units. Borthakur [1], by assuming that the service times were exponentially distributed and independent of the batch size, was able to obtain explicit results for the steady-state probabilities of queue length. Medhi [13] has considered the waiting-time (in queue) distribution for this M/M a,b/1 system and has given explicit results for the first two moments of this distribution. Easton and Chaudhry [3] have obtained steady-state results for the system $E_k/M(a,b)/1$ including the waiting time distribution.

Madill and Chaudhry [12] have discussed G/Ma,b/1 model under the general bulk service rule. They have derived the steady state probability distributions and the moments of the steady state waiting time for this model. There are some physical system in which an idle server will leave the system for some uninterruptable task [such as a tea break a telephone call, a checkup operation in case the server is a machine etc.]. Single server queues with vacation periods have later been studied by a number of authors, Notable among them are Miller [15], Cooper [4], Levy and Yachiali [11], Heyman and Sobel [9], Shantikumar [19], Scholl and Kleinrock [18], Lee [10], Keilson and Servi [7], Fuhrman [5].

Naishou TIAN, Daqing ZHANH and Chengxuan CAO (1989) [20] in their paper have discussed the GI/M/1 queue with exponential vations. They have expressed the transition matrix of the imbedded Markov chain as a black Jacobi form and obtained a matrix - geometric solution of the arrival point steady state queue size probabilities, using the approach developed by Neuts [16]. Recently U.Chatterjee and Mukerjee [2] have studied GI/M/1 queue in which vacations are independently and identically distributed with a general distribution.

In this dissertation we study in detail the paper "Waiting time moments in the queueing system GI/M a,b/1" presented by B.R.Madill & M.L.Chaudhry and U.Chatterjee & S.P.Mukerjee paper entitled by "GI/M/1 queue with server vacation".

We give some known results used in this dissertation.

1. **Rouches Theorem:**

If f(z) and g(z) are functions analytic inside and on a closed contour C and if $|g(z)| < |f(z)|$ on C then f(z) and f(z) + g(z) have the same number of zeros inside C.

2. If G(z) is the probability generating function of P_i's then $R(Z) = Z-G(z)$ has unique root inside $|z|=1$ If $G'(1) \quad 1$

3. If D denotes $\frac{d}{d\mu}$ then $e^{-(\mu w^b D)} \; [f(\mu)] = f(\mu(1-w^b))$

4. If $\int_0^\infty e^{-\mu t} \, dA(t) = \tilde{a}(\mu)$ then

$$L(A(t) = \int_0^\infty e^{-\mu t} A(t)dt = \frac{\tilde{a}\,(\mu)}{\mu}$$

5. If the random variable A_{a-j-1} represents the time for a-j-1 arrivals, $\tilde{a}(\mu)^{a-j-1} = \int_0^{\infty} e^{-\mu t} \, dA_{a-j-1}(t)$

6. Let $W_q(t)$, denote the waiting time distribution then the expected waiting time in the queue is

$$E(W_q) = \int_0^t t \; dw_q(t)$$

7. Imbedded Markov chain technique.

Non Markovian queues are reduced to Markovian by this technique which was introduced by Kendall [8].

SINGLE SERVER QUEUES WITH GENERAL INDEPENDENT INPUT AND EXPONENTIAL SERVICE

CHAPTER - I

SINGLE SERVER QUEUES WITH GENERAL INDEPENDENT INPUT AND EXPONENTIAL SERVICE

In this Chapter we study the non Markovian model GI|M|1 with single server where the service times are exponential and the interarrival time has general distribution with mean $\frac{1}{\mu}$ and $\frac{1}{\lambda}$ respectively. The imbedded Markov chain technique is used here to obtain our results.

Consider the system just prior to an arrival. Let $X_i, i=0,1,2, \ldots$ represents the number in the system when the i^{th} arrival comes inside the system. Then we have

$$X_{n+1} = X_n+1 - B_n \qquad [B_n \leqslant X_n+1 \ ; \ X_n \geqslant 0]$$

where B_n is the number of customers served during the interarrival time $T^{(n)}$ between the n^{th} and $(n+1)^{th}$ arrivals. The random variable $T^{(n)}$ can be denoted by T since the interarrival times are assumed independent. We denote its cumulative distribution function by A(t). Since the service is exponential the random variable B_n depends on only the length of the interval and not on the length of the service time of the present customer has already received.

$$\begin{aligned} \Pr\{B_n=b\} &= \Pr\{B=b\} \\ &= \int_0^\infty \Pr\{B=b / T=t\}\, dA(t) \\ &= \int_0^\infty \frac{e^{-\mu t}(\mu t)^b}{b!}\, dA(t) \end{aligned}$$

So that $P_{ij} \;=\; Pr\left\{X_{n+1}=j \;/\; X_n=i\right\}$

$$= P_r\left\{B = i+1-j\right\} \qquad \text{if } i+1 \geqslant j \geqslant 1$$

$$= \begin{cases} \dfrac{e^{-\ t}\ (\ \ t)^{i+1-j}}{i+1-j}\ dA(t) & \text{if } i+1 \geqslant j \geqslant 1 \\ 0 & \text{if } i+1 < j \end{cases} \qquad \ldots (1)$$

$$P_{i0} + \sum_{j=1}^{i+1} P_{ij} = 1$$

This implies $P_{i0} = 1 - \sum_{j=1}^{i+1} P_{ij}$ $\ldots (2)$

Hence we can readily see that the imbedded process is Markovian, since only the indices (i,j) are involved in (1) & (2), and further more since the state variable is discrete it is a Markov chain.

Let $b_n \;=\; Pr\left\{n \text{ services during an interarrival time}\right\}$

$$= Pr\ \left\{B=n\right\}$$

$$= \int_0^\infty \frac{e^{-\mu t}\ (\mu t)^n}{n!}\ dA(t)$$

Let the transition probability matrix of the imbedded stochastic process at the arrival point be P.

$P = (P_{ij})$ where $P_{ij} = \Pr\left\{X_{n+1}=j \,/\, X_n=i\right\}$

$$P = \begin{bmatrix} 1-b_0 & b_0 & 0 & 0 & 0 & \cdot\cdot\cdot\cdot \\ 1-\sum_{k=0}^{1} b_k & b_1 & b_0 & 0 & 0 & \cdot\cdot\cdot\cdot \\ 1-\sum_{k=0}^{2} b_k & b_2 & b_1 & b_0 & 0 & \cdot\cdot\cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot \end{bmatrix}$$

Assuming that the steady state-solution exists, the steady state probability vectar $q = q_n$ can be found as the solution to the stationary equation $qP = q$.

$$(q_0 q_1 q_2 \cdot\cdot q_n \cdot\cdot) \begin{bmatrix} 1-b_0 & b_0 & 0 & 0 & 0 & \cdot\cdot\cdot\cdot \\ 1-\sum_{k=0}^{1} b_k & b_1 & b_0 & 0 & 0 & \cdot\cdot\cdot\cdot \\ 1-\sum_{k=0}^{2} b_k & b_2 & b_1 & b_0 & 0 & \cdot\cdot\cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot \end{bmatrix} = (q_0 q_1 q_2 \cdots q_n \cdots)$$

from the above equation we get

$$q_0 = q_0(1-b_0)+q_1(1-\sum_{k=0}^{1} b_k)+q_2(1-\sum_{k=0}^{2} b_k) + \dots +q_i(1-\sum_{k=0}^{i} b_k)+ \dots$$

$$= [\sum_{\ell=0}^{\infty} q_\ell (1-\sum_{k=0}^{\ell} b_k)] \qquad \dots (3)$$

$$\text{and } q_i = q_0.0+q_i.0+\dots +q_{i-1}b_0+q_ib_1+ \dots +q_{i+K-1} b_k + \dots$$

$$= \sum_{k=0}^{\infty} q_{i+k-1}b_k \qquad \dots (4)$$

Let $Dq_i = q_{i+1}$ denote the shifting operator

Then $(D - \sum_{n=0}^{\infty} b_n D^n) q_{i-1} = 0$

$$(D - \sum_{n=0}^{\infty} b_n D^n) q_i = 0 \quad , \quad i \geq 0$$

Let $\beta(Z)$ denote the probability generating function of b_n. Clearly $\beta'(1) = \mu/\lambda$. If $\lambda/\mu<1$ then $Z = \beta(Z)$ has unique solution r_o in $|Z| = 1$

$$q_j = Cr_o^j \quad , \quad j \geq 0 \quad \text{and} \quad 0 < r_o < 1$$

As $_{j=o}\, q_j=1$

we have $C \sum_{i=0}^{\infty} r_o^i = 1$

$$C \frac{1}{1-r_o} = 1$$

hence $C = 1-r_o$

$$q_j = (1-r_o)r_o^j \qquad j \geq 0,\ 1 < r_o < 1$$

It is interesting to note that the steady state probability for M|M|1 model is $P_n = (1-\rho)\rho^n$ when $\rho = \frac{\lambda}{\mu}$. Hence all the expected value measures for G|M|1 model can be obtained by merely replacing ρ by r_o in M|M|1 model.

However, it must be pointed out that q_n is the steady - state probability of n in the system just prior to an arrival and not the general - time steady state probability P_n, So that the expected value measures apply only at arrival points. Unlike the M|G|1 model it is not true that $q_n = P_n$. Infact, it turns out that this is true if, and only if, the arrivals are poisson i.e $q_n = P_n$ for GI|M|1 if and, only if, GI=M. The waiting time distribution $W_q(t)$ can be obtained from M|M|1 with r_o replacing we get

$$W_q(t) = \begin{cases} 1 - r_o & t = 0 \\ 1-r_o\ e^{-\mu(1-r_o)t} & t > 0 \end{cases}$$

WAITING TIME MOMENTS IN THE QUEUEING SYSTEM GI/Ma,b/1

CHAPTER - II

WAITING TIME MOMENTS IN THE QUEUEING SYSTEM $GI|M_{a,b}|1$

In this model the interarrival times are assumed to be independent identically distributed random variables with distribution function A(t) and mean interarrival time $1/\lambda$.

The Service time is exponentially distributed with mean $\frac{1}{\mu}$ and distribution function B(t). The arrivals are served in batches according to the following general bulk service rule if a server immediately after his service finds.

1) greater than 'a' customers but less than 'b' customers in the queue, takes all the customers for service.
2) greater than b customers, takes only 'b' customers for service and leaving others to wait.
3) less than 'a' customers does not start service and waits for a minimum numbers of a customers.

§ 1 : Prearrival

Let t_n, n=1,2, ... (t_0=0) be the epoch at which the n^{th} customer arrives. We examine the system at $t_n - 0$. We define the two dimensional state space of the system as

$$\left\{ \begin{array}{ll} (i,k) : i=0 ; k=0,1,2 \ldots a-1 & \text{and} \\ \qquad\quad i=1 ; k \quad 0 & \end{array} \right\}$$

where i=0(1) corresponds to the server is idle [busy] and k is number of customers in the queue.

Clearly k is of the form rb+j , $0 \leq j \leq a-1$ and r=0,1,2 ...

If $\tau(t)$ denotes the system state observed at time t then the sequence of random variables $(i_n, j_n) = \tau(t_n - 0)$ defines a semi Markov Chain.

Let g_r denote the probability that r batches are served in an interarrival time A(t).

$$\text{Then} \quad g_r = \int_0^\infty \frac{e^{-\ x}(\ x)^r}{r!} \, dA(t)$$

$P_{ij} = \lim_{n \to \infty} \Pr\{i_n = i, j_n = j\}$ denotes the steady state probability that the server is idle (i=0) or busy (i=1) and there are j customers in the queue at prearrival epoch.

The steady state equations are:-

$$P_{10} = g_0 P_{0,a-1} + \sum_{r=1}^{\infty} g_r \sum_{j=a-1}^{b-1} P_{1,(r-1)b+j} \qquad \ldots (1)$$

$$P_{1\,j+1} = \sum_{r=0}^{\infty} g_r P_{1,rb+j} \qquad j \geq 0 \qquad \ldots (2)$$

$$P_{00} = \hat{g}_0 P_{0a-1} + \sum_{r=1}^{\infty} \hat{g}_r \sum_{j=a-1}^{b-1} P_{1,(r-1)b+j} \qquad \ldots (3)$$

$$P_{0j+1} = P_{0j} + \sum_{r=0}^{\infty} \hat{g}_r P_{1,rb+j} \qquad 0 \leq j \leq a-2$$

$$\text{where} \quad \hat{g}_r = 1 - \sum_{k=0}^{r} g_k \qquad \ldots (4)$$

Let E denote the shifting operator then the equation (2) can be written as

$$(E - \sum_{r=0}^{\infty} g_r \, E^{rb}) \, P_{i_j} = 0 \qquad j \geq 0 \qquad \ldots (5)$$

Let G(Z) denote the probability generating function of g_j with parameter Z^b. It can be shown that if G'(1)   1 then R(Z)=Z-G(z) has only one zero inside $|Z|=1$.

$$G'(Z) = \sum_{r=0}^{\infty} g_r \, (rb) Z^{rb-1}$$

$$\begin{aligned}
G'(1) &= b \sum_{r=0}^{\infty} r g_r \\
&= b \sum_{r=0}^{\infty} r \int_0^{\infty} \frac{e^{-\mu t} (\mu t)^r}{r!} \, dA(t) \\
&= b\mu \int_0^{\infty} t \; e^{-\mu t} \; e^{-\mu t} \, dA(t) \\
&= b\mu \int_0^{\infty} t \, dA(t) \\
&= b\mu . \frac{1}{\lambda} = \frac{1}{\rho}
\end{aligned}$$

Hence $G'(1) > 1$ implies $\frac{1}{\rho} > 1$

Thus when $\rho < 1$, Z-G(Z) has a unique root w inside $|Z|=1$.

Since $P_{ij} \leq 1$, solving the equation (5) we get

For $j \geq 0$, $P_{ij} = Aw^j$ for some constant A

Taking j=0 we have $P_{10} = A$

From (1)

$$P_{0\;a-1} = \frac{1}{g_0}\left[P_{10} - \sum_{r=1}^{\infty} g_r \sum_{j=a-1}^{b-1} Aw^{(r-1)b+j}\right]$$

$$= \frac{A}{g_0}\left[1- \sum_{r=1}^{\infty} g_r \; w^{(r-1)b} \sum_{j=a-1}^{b-1} w^j\right]$$

$$= \frac{A}{g_0}\left[1- \sum_{r=1}^{\infty} g_r \; w^{(r-1)b} \; \frac{w^{a-1}-w^b}{1-w}\right]$$

$$= \frac{A}{g_0}\left[1- \frac{w^{a-1}-w^b}{1-w} \cdot \frac{1}{w^b} \sum_{r=1}^{\infty} g_r \; w^{rb}\right]$$

$$= \frac{A}{g_0}\left[1- \frac{w^{a-1}-w^b}{w^b\,(1-w)}\,(w-g_0)\right]$$

$$= \frac{A}{w^b(1-w)g_0}\left[w^b-w^{b+1}-w^a+w^{a-1}g_0+w^{b+1}-w^b g_0\right]$$

$$= \frac{A}{w^b(1-w)g_0}\left[w^b-w^a+g_0(w^{a-1}-w^b)\right] .$$

$$= \frac{A}{1-w}\left[\frac{1-w^{a-b}}{g_0} + w^{a-b-1}-1\right] \qquad \ldots (6)$$

Substituting for P_{0a-1} and P_{ij} , $i \geqslant 0$ in equation (3),

We get

$$P_{00} = g_0\,\frac{A}{1-w}\left\{\frac{1-w^{a-b}}{g_0} + w^{a-b-1}-1\right\}$$

$$+ A \sum_{r=1}^{\infty} \hat{g}_r \sum_{j=a-1}^{b-1} w^{(r-1)b+j}$$

$$= \frac{A(1-g_0)}{1-w} \left\{ \frac{1-w^{a-b}}{g_0} + w^{a-b-1} - 1 \right\}$$

$$+ A \sum_{r=1}^{\infty} (1-\sum_{k=0}^{r} g_k)\, w^{(r-1)b} \left(\frac{w^{a-1}-w^{b}}{1-w} \right)$$

$$= \frac{A}{1-w}\Big[(1-g_0) \left\{ \frac{1-w^{a-b}}{g_0} + w^{a-b-1} - 1 \right\}$$

$$+ (w^{a-1}-w^{b}) \left\{ \frac{1}{1-w^{b}} - \sum_{r=1}^{\infty} w^{(r-1)b} \sum_{k=0}^{r} g_k \right\} \Big]$$

$$= \frac{A}{1-w} \Big\{ [(1-g_0) \, [\frac{1-w^{a-b}}{g_0} + w^{a-b-1} - 1]$$

$$+ \frac{w^{a-1} - w^{b}}{1-w^{b}} \, [1+ \frac{g_0(1-w^{b})}{w^{b}} - \frac{1}{w^{b-1}}] \Big\}$$

$$P_{00} = \frac{A}{1-w} \left[\frac{1-w^{a-b}}{g_0} + w^{a-b-1} - 1 + \frac{(1-w)\,(w^{a-1}-1)}{1-w^{b}} \right] \qquad \ldots (7)$$

Summing equation (4) for $j=0$ to $k-1$

$$P_{0k} = P_{00} + A \sum_{j=0}^{k-1} \sum_{r=0}^{\infty} \hat{g}_r \, w^{rb+j}$$

$$= P_{00} + A \sum_{r=0}^{\infty} (1- \sum_{i=0}^{r} g_i) w^{rb} \left(\frac{1-w^{k}}{1-w} \right)$$

$$= P_{00} + A \, \frac{1-w^{k}}{1-w} \left[\frac{1}{1-w^{b}} - \frac{w}{1-w^{b}} \right]$$

$$P_{0k} = P_{00} + A \, \frac{(1-w^{k})}{1-w^{b}} \quad , \quad 1 \leq k \leq a - 1 \qquad \ldots (8)$$

Next we shall calculate the constant A using the normalizing equation

$$\sum_{j=0}^{a-1} P_{oj} + \sum_{j=0}^{\infty} P_{1j} = 1$$

$$a\ P_{00} + A \sum_{j=1}^{a-1} \frac{1 - w^j}{1 - w^b} + \frac{A}{1-w} = 1$$

$$a\ [\frac{A}{1-w} \left\{ \frac{1-w^{a-b}}{g_0} + w^{a-1-1} - 1 + \frac{(1-w)(w^{a-1}-1)}{1-w^b} \right\}]$$

$$+ \frac{A}{1-w^b} \left\{ a-1- \frac{w-w^a}{1-w} \right\} + \frac{A}{1-w} = 1$$

$$\frac{A}{1-w} [a \left\{ \frac{1-w^{a-b}}{g_0} + w^{a-b-1} - 1 + \frac{(1-w)(w^{a-1}-1)}{1-w^b} \right\}$$

$$+1 + \frac{(a-1)\ (1-w)}{1-w^b} - \frac{w-w^a}{1-w^b}] = 1$$

$$\frac{A}{1-w} \quad \frac{1}{g_0(1-w^b)} \quad [a(1-w^{a-b})\ (1-w^b)+g_0 \left\{ a(w^{a-b-1}-1).(1-w^b) \right.$$

$$\left. + a(1-w)\ (w^{a-1}-1)+(1-w^b)+(a-1)(1-w)-(w-w^a) \right\}] = 1$$

$$\frac{A}{(1-w)g_0(1-w^b)} \quad [ac+g_0(w^a-w^b)] = 1$$

Where $c = (1-g_0)\ (1+w^a-w^b)-w^{a-b-1}(w-g_0)$

$$A = \frac{g_0(1-w)(1-w^b)}{ac+g_0(w^a-w^b)} \qquad \ldots (9)$$

Substituting the value of P_{00} in the right hand side of (8) we get,

$$P_{00} + \frac{A}{1-w^b} = \frac{A}{1-w} \left[\frac{1-w^{a-b}}{g_0} + w^{a-b-1} -1+ \frac{(1-w)(w^{a-1}-1)}{1-w^b} \right] + \frac{A}{1-w^b}$$

$$= \frac{A}{(1-w)(1-w^b)} [(1-w^b)(1-w^{a-b})+(1-w^b)(w^{a-b-1}-1)g_0$$

$$+ g_0(w^{a-1}-1)(1-w)+g_0(1-w)] \qquad \ldots (10)$$

Substituting the value of A in (10) we get

$$P_{00} + \frac{A}{1-w^b} = \frac{1}{ac+g_0(w^a-w^b)} [-g_0(1-w^b)+g_0 w^{a-b-1}-w^{a-b}$$

$$+1-w^b+w^a+g_0 \; [-w^{a-1}+w^{a-1}-1-w^a+w+1-w]$$

$$= \frac{1}{ac+g_0(w^a-w^b)} [(1-g_0) \; (1+w^a-w^b)-w^{a-b-1}(w-g_0)]$$

$$= \frac{c}{ac+g_0(w^a-w^b)}$$

$$= \frac{1}{a+g_0 \dfrac{(w^a-w^b)}{c}}$$

Accordingly (8) becomes

$$P_{0j} = \frac{1}{a+g_0(w^a-w^b)/c} - \frac{Aw^j}{1-w^b} \; , \; 1 \leq j \leq a-1$$

§ 2 : Random epoch

We shall consider the steady state probabilities at an arbitrary epoch. The equations are

$$P_{10}^{*} = g_0^{*} P_{0\,a-1} + \sum_{r=1}^{\infty} g_r^{*} \sum_{j=a-1}^{b-1} P_{1\,(r-1)b+j}$$

$$P_{1\,j+1}^{*} = \sum_{r=0}^{\infty} g_r^{*} P_{1rb+j} \qquad j \geq 0$$

$$P_{00}^{*} = \bar{g}_1^{*} P_{0a-1} + \sum_{r=1}^{\infty} \bar{g}_{r+1}^{*} \sum_{j=a-1}^{b-1} P_{1(r-1)b+j}$$

$$P_{0\,j+1}^{*} = P_{0j} + \sum_{r=0}^{\infty} \bar{g}_{r+1}^{*} P_{1rb+j} \qquad 0 \leq j \leq a-2$$

Now we solve for the steady state probabilities

$$P_{10}^{*} = \frac{\rho\,(1-g_0)A}{1-w}\left[\frac{1-w^{a-b}}{g_0} + w^{a-b-1}-1\right] + \sum_{r=1}^{\infty} g_r \sum_{j=a-1}^{b-1} A\, w^{(r-1)b+j}$$

$$= \frac{\rho\,(1-g_0)A}{1-w}\left[\frac{1-w^{a-b}}{g_0} + w^{a-b-1}-1\right] + A\,\frac{(w^{a-1}-w^{b})}{(1-w)w^{b}}\left[\frac{\rho(1-w)}{1-w^{b}} - \rho(1-g_0)\right]$$

$$= \frac{\rho A}{1-w}\left\{(1-g_0)\left(\frac{1-w^{a-b}}{g_0} + w^{a-b-1}-1\right) + (w^{a-1-b}-1)\left[\frac{1-w}{1-w^{b}} - (1-g_0)\right]\right\}$$

$$= \frac{\rho A}{1-w}\left\{\frac{1-w^{a-b}}{g_0} -1-w^{a-b} + w^{a-b-1}-1\right.$$

$$\left. + g_0(1-w^{a-b-1}+w^{a-1-b}-1] - w^{a-b-1}+1+ \frac{(w^{a-b-1}-1)(1-w)}{1-w^b}\right\}$$

$$= \frac{\rho A}{1-w}\left\{\frac{1-w^{a-b}}{g_0} + \frac{w^b - w^{a-b-1}-1+w}{1-w^b} - 1\right\} \qquad \ldots (11)$$

$$P^*_{1\ j+1} = \sum_{r=0}^{\infty} g_r^* \, A w^{rb+j}$$

$$= A\, w^j \frac{(1-w)}{1-w^b} \qquad \left[\sum_{\gamma=0}^{\infty} g_r^* \, w^{rb} = \frac{\rho(1-w)}{1-w^b}\right] \qquad \ldots (12)$$

$$P^*_{00} = (1-g_0^*)\, \frac{A}{1-w}\left[\frac{1-w^{a-b}}{g_0} + w^{a-b-1}-1\right]$$

$$+ \sum_{r=1}^{\infty} \left(1-\sum_{k=0}^{r} g_k^*\right) \sum_{j=a-1}^{b-1} A\, w^{(r-1)b+j}$$

$$= \frac{[1-\rho(1-g_0)]A}{1-w}\left[\frac{1-w^{a-b}}{g_0} + w^{a-b-1}-1\right] + A\,\frac{[w^{a-1}-w^b]}{1-w} \cdot \frac{1}{1-w^b}$$

$$-A\, \frac{w^{a-1}-w^b}{1-w} \cdot \frac{1}{w^b(1-w^b)}\left\{\frac{\rho(1-w)}{1-w^b} - (1-g_0)(1-w^b)\right\} \ldots (13)$$

$$P^*_{0\ j+1} = P_{00} + \frac{A(1-w^j)}{1-w^b} + A\sum_{r=0}^{\infty}\left(1-\sum_{k=0}^{r} g_k^*\right) w^{rb+j}$$

$$= P_{00} + \frac{A(1-w^j)}{1-w^b} + \frac{Aw^j}{1-w^b} - Aw^j \rho \,\frac{(1-w)}{(1-w^b)^2}$$

$$= P_{00} + \frac{A}{1-w^b} - \rho\, \frac{Aw^{j-1}(1-w)}{(1-w^b)^2} \qquad \ldots (14)$$

§ 3 : Waiting time distribution

In this section we shall calculate the waiting time of a customer in the queue i.e the time between the arrival of a customer at a queueing system and his entry into the service. An arriving customer may find the system in any one of the following states:-

1. $(0,j)\ ,\ 0 \leq j \leq a-2$
2. $(1,hb+j)\ ,\ a-1 \leq j \leq b-1,\quad h=0,1,2\ldots$
3. $(1,hb+j)\ ,\ 0 \leq j \leq a-2,\quad h=0,1,2\ \ldots$

The arriving customer has to wait

In case 1, for a-1-j customers to arrive

In case 2, for h+1 service complition

In case 3, till either the services of (h+1) batches are completed or a-1-j units arrive which ever occurs latter.

Let W(t), w(t) denote the waiting time distribution and the probability density function of the waiting time respectively.

Let A_{a-j-1} represents the time for a-j-1 arrivals and B_{n+1} represents the time for (h+1) service completions.

$$\text{Then } w(t) = P_{0a-1}\,\delta(t)dt + \sum_{j=0}^{a-2} P_{0j}\,dA_{(a-j-1)}(t) + \sum_{h=0}^{\infty}\ \sum_{j=a-1}^{b-1} P_{1,hb+j}\,dB_{h+1}(t)$$

$$+ \sum_{h=0}^{\infty}\ \sum_{j=0}^{a-2} P_{1,hb+j}\quad dw^{hb+j}$$

Where

1. $dA_{a-j-1}(t)$ is the probability density function of the random variable A_{a-j-1}.

2. $B_{h+1}(t) = 1-\sum\limits_{i=0}^{h} (\mu t)^{i}\ e^{-\mu t}/i!$ $\qquad t > 0$

3. $dB_{h+1}(t) = \left[\mu^{h+1}\ t^{h}\ e\frac{-\mu t}{n!}\ \right] dt$

4. $w^{hb+j}(t) = \max\ [A_{a-j-1}(t),\ B_{h+1}(t)]$

5. $dw^{(hb+j)}(t) = A_{a-j-1}(t)\ dB_{h+1}(t) + dA_{a-j-1}(t)\ B_{h+1}(t)$

6. $\delta(t)$ is the dirac delta function.

The expected waiting time of the customer in the queue is

$$\begin{aligned} E(w(t)) = & \int_{0}^{\infty} t\ P_{0,a-1}\ (t)dt \\ & + \int_{0}^{\infty} t \sum_{j=0}^{a-2} P_{0j}\ dA_{a-j-1}(t) + \int_{0}^{\infty} t \sum_{h=0}^{\infty} \sum_{j=a-1}^{b-1} P_{1,hb+j}\ dB_{h+1}(t) \\ & + \int_{0}^{\infty} t \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} P_{1,hb+j}\ dw^{hb+j}(t) \qquad \ldots (15) \end{aligned}$$

Consider

$$\begin{aligned} \int_{0}^{\infty} t \sum_{j=0}^{a-2} P_{0j} dA_{a-j-1}(t) & = \sum_{j=0}^{a-2} P_{0j} \int_{0}^{\infty} t dA_{a-j-1}(t) \\ & = \sum_{j=0}^{a-2} P_{0j}\ E(A_{a-j-1}) \end{aligned}$$

$$= \sum_{j=0}^{a-2} \left\{ \left[\frac{c}{ac+g_0(w^a-w^b)} \right] - \frac{Aw^j}{1-w^b} \right\} E(A_{a-j-1})$$

$$= \sum_{j=0}^{a-2} \left\{ \frac{c\,E(A_{a-j-1})}{ac+g_0(w^a-w^b)} - \frac{AE(A_{a-j-1})w^j}{1-w^b} \right\} \qquad \ldots (16)$$

Third term of (15) becomes,

$$\sum_{h=0}^{\infty} \sum_{j=a-1}^{b-1} Aw^{hb+j} \int_0^t t \, dB_{h+1}(t)$$

$$= \sum_{h=0}^{\infty} \sum_{j=a-1}^{b-1} Aw^{hb+j} \int_0^t t\left(\frac{\mu^{h+1} t^h e^{-\mu t}}{h!} \right) dt$$

$$= \sum_{h=0}^{\infty} \sum_{j=a-1}^{b-1} Aw^{hb+j} \frac{\mu^{h+1}}{h!} \cdot \frac{(h+1)!}{h+1+1}$$

$$= \sum_{h=0}^{\infty} w^{hb} A \left(\frac{w^{a-1}-w^b}{1-w} \right) \frac{h+1}{\mu}$$

$$= A \left(\frac{w^{a-1}-w^b}{1-w} \right) \frac{1}{\mu} \cdot \frac{1}{(1-w^b)^2} \qquad \ldots (17)$$

Consider the last term of (15)

$$\sum_{h=0}^{\infty} \sum_{j=0}^{a-2} w^{hb+j} \int_0^{\infty} t \left[A_{a-j-1}(t)\, dB_{h+1}(t) + B_{h+1}(t)\, dA_{a-j-1}(t) \right] \qquad \ldots (18)$$

Substituting $B_{h+1}(t)$ and $dB_{h+1}(t)$ we have the expression.

$$\int_0^t t \left\{ A_{a-j-1}(t)\, dB_{h+1}(t) + B_{h+1}\, dA_{a-j-1}(t) \right\}$$

$$= \int_0^t t \, (1- \sum_{i=0}^{\infty} \frac{(\mu t)^i \, e^{-\mu t}}{\underline{|i}}$$

$$+ \int_0^t t \, [\frac{\mu^{h+1} \, t^h \, e^{-\mu t}}{h!}] \, dA_{a-j-1}(t) \, dt$$

$$= \int_0^t t dA_{a-j-1}(t) - \sum_{i=0}^{h} \frac{\mu^i}{i!} \int_0^\infty t \, t^i \, e^{-\mu t} \, dA_{a-j-1}(t)$$

$$+ \frac{\mu^{h+1}}{h!} \int_0^\infty e^{-\mu t} t_t^{h+1} \, A_{a-j-1}(t) \, dt$$

$$[\int_0^\infty e^{-\mu t} dA_{a-j-1}(t) = (\tilde{a}(\mu))^{a-j-1}$$

and $\int_0^\infty e^{-\mu t} A_{a-j-1}(t) dt = \int_0^\infty e^{-\mu t} [\int_0^t dA_{a-j-1}(x)]$

$$= L \, [\int_0^t dA_{a-j-1}(x)]_{s=\mu}$$

$$= \int_0^\infty e^{-\mu t} \, dA_{a-j-1}(t)/\mu$$

$$= \{\tilde{a}(\mu)\}^{a-j-1}/\mu \,]$$

The expression (18) simplified as

$$\sum_{h=0}^{\infty} \sum_{j=0}^{a-2} Aw^{hb+j} \, [\int_0^t tdA_{a-j-1}(t) - \sum_{i=0}^{h} \frac{\mu^i}{i!} (-D)^{i+1} \, (\tilde{a}(\mu))^{a-j-1}$$

$$+ \frac{\mu^{h+1}}{h!} (-D)^{h+1} \, \frac{(\tilde{a}(\mu))^{a-j-1}}{\mu} \,]$$

$$= \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} Aw^{hb+j} \, E \, (A_{a-j-1})$$

$$+ \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} Aw^{hb+j} \left\{ - \sum_{i=0}^{h} \frac{\mu^i (-D)^{i+1}}{i!} \, q_j + \frac{\mu^{h+1}}{h!} (-D)^{h+1} \, \frac{q_j}{\mu} \, \right]$$

where $q_j = \{\tilde{a}(\mu)\}^{a-j-1}$

$$= \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} Aw^{hb+j} E(A_{a-j-1})$$

$$+ A \sum_{j=0}^{a-2} w^{j} \sum_{h=0}^{\infty} w^{hb} \left\{ - \sum_{i=0}^{h} \frac{\mu^{i}(-D)^{i+1}}{i!} q_j + \frac{{}^{h+1}}{h!} (-D)^{h+1} \frac{q_j}{\mu} \right\}$$

$$= \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} Aw^{hb+j} E(A_{a-j-1})$$

$$+ A \sum_{j=0}^{a-2} w^{j} [\frac{1}{1-w^{b}} \sum_{i=0}^{\infty} \frac{(-\mu D w^{b})^{i}}{i!} D]q_j + A \sum_{j=0}^{a-2} w^{j} \sum_{h=0}^{\infty} w^{hb}$$

$$\frac{\mu^{h+1}}{h!} (-D)^{h+1} \frac{q_j}{\mu}$$

$$= \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} Aw_{hb+j} E\ (A_{a-j-1}) - \frac{A}{1-w^{b}} \sum_{j=0}^{a-2} w^{j} \left\{ e^{-\mu D w^{b}} Dq_j \right\}$$

$$+ A \sum_{j=0}^{a-2} w^{j} \sum_{h=0}^{\infty} w^{hb} \frac{\mu^{h+1}}{h!} (-D)^{h+1} \frac{q_j}{\mu}$$

$$= \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} Aw^{hb+j} E(A_{a-j-1}) +$$

$$+ A \sum_{j=0}^{a-2} w^{j} \left\{ \frac{1}{1-w^{b}} e^{-\mu w^{b} D} (Dq_j) - \mu e^{-\mu w^{b} D} \frac{(\mu D q_j - q_j)}{\mu^{2}} \right.$$

$$= \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} Aw^{hb+j} E(A_{a-j-1})$$

$$+ A \sum_{j=0}^{a-2} w^{j} \left(\frac{\mu}{\mu^{2}(1-w^{b})^{2}} \cdot e^{-\mu w^{b} D} \right) (\tilde{a}(\mu)^{a-j-1} \qquad \ldots (19)$$

Substituting (16) (17) (19) in (15) we get

$$E(w(t)) = \sum_{j=0}^{a-2} \left\{ \frac{CE(A_{a-j-1})}{a+g_0(w^a-w^b)} -A \frac{E(A_{a-j-1})}{1-w} w^j \right\}$$

$$+ A\left(\frac{w^{a-1}-w^b}{1-w}\right)\frac{1}{\mu}\cdot\frac{1}{(1-w^b)^2}$$

$$+ \sum_{h=0}^{\infty}\sum_{j=0}^{a-2} Aw^{hb+j}\, E(A_{a-j-1})$$

$$+ A\sum_{j=0}^{a-2} w^j \left\{\frac{e^{-\mu w^b D}}{(1-w^b)^2}\right\}(\tilde{a}(\mu))^{a-j-1}$$

$$= \sum_{j=0}^{a-2}\frac{CE(A_{a-j-1})}{ac+g_0(w^a-w^b)} + A\frac{(w^{a-1}-w^b)}{(1-w)\,\mu\,(1-w^b)^2}$$

$$+ A\sum_{j=0}^{a-2} w^j\left\{\frac{e^{-\mu w^b D}}{(1-w^b)^2}\right\}(\tilde{a}(\mu))^{a-j-1}$$

Using the fact that $e^{-\mu w^b D}(\tilde{a}(\mu))^{a-j-1} = (\tilde{a}(\mu(1-w^b)^{a-j-1}$

$$= w^{a-j-1}$$

and $E(A_{a-j-1}) = (-D)\left\{\tilde{a}(\mu)\right\}^{a-j-1\,\prime}\Big]_{\mu=0}$

We get

$$W(t) = \sum_{j=0}^{a-2}\frac{CE(A_{a-j-1})}{ac+g_0(w^a-w^b)} + \frac{A(w^{a-1}-w^b)}{\mu(1-\mu)(1-w^b)^2} + A\sum_{j=0}^{a-2}\frac{w^j}{\mu(1-w^b)^2}w^{a-j-1}$$

$$= \sum_{j=0}^{a-2}\frac{C(-D)\,\tilde{a}(\mu)^{a-j-1}\Big]_{\mu=0}}{ac+g_0(w^a-w^b)} + \frac{A(w^{a-1}-w^b)}{\mu(1-w)(1-w^b)^2} + \frac{A}{\mu}\,\frac{(a-1)w^{a-1}}{(1-w^b)^2}$$

$$= \sum_{j=0}^{a-2}\frac{C(-D)\tilde{a}(\mu)^{a-j-1}\Big]_{\mu=0}}{ac+g_0(w^a-w^b)} + \frac{A}{\mu(1-w^b)^2}\left[aw^{a-1}-w^{a-1}+\frac{w^{a-1}-w^b}{1-w}\right]$$

$$= \frac{c}{ac+g_0(w^a-w^b)} \left[\sum_{k=1}^{a-1} \frac{a-k}{\lambda} \right] + \frac{A}{\mu(1-w^b)^2} \left\{ aw^{a-1} + \frac{w^a-w^b}{1-w} \right\}$$

$$= \frac{c}{ac+g_0(w^a-w^b)} \left[\frac{1}{\lambda} \frac{(a-1)(a-1+1)}{2} \right] + \frac{A}{\mu(1-w^b)^2} \left\{ aw^{a-1} + \frac{w^a-w^b}{1-w} \right\}$$

$$= \frac{c\,(a-1)a}{2\lambda\,(ac+g_0(w^a-w^b)} + \frac{\rho b}{\lambda(1-w^b)^2} A \left[aw^{a-1} + \frac{w^a-w^b}{1-w} \right]$$

$$\left[\text{Since } \rho = \frac{\lambda}{b\mu} \Rightarrow \frac{1}{\mu} = \frac{\rho b}{\lambda}\right]$$

$$= \frac{1}{\lambda} \left[\frac{c(a-1)a}{2(ac+g_0(w^a-w^b)} + \frac{A}{(1-w^b)^2} \left[aw^{a-1} + \frac{w^a-w^b}{1-w} \right] \right]$$

GI/M/1 QUEUE WITH SERVER VACATION

CHAPTER - III

GI / M / I QUEUE WITH SERVER VACATION

In this Chapter GI/M/I Queue in which vacations are independently and identically distributed with a general distribution are considered. The equilibrium Probability distributions at pre-arrival epoch and at random epoch for this model are derived. The distribution of waiting time of a customer in the queue is also obtained.

Consider the GI/M/1 queue with multiple servers vacation in which the service rule is exhaustive. We assume that the service time distribution is exponential with mean $\frac{1}{\mu}$ the interarrival time is a random variable U, having a general distribution with mean $\frac{1}{\lambda}$ and successive vacations are independent and identically distributed random variable V_1, V_2 ... each having a general distribution with mean $\frac{1}{\nu}$. Vacations, Service times and interarrival times are mutually independent random variable.

We adopt the following notations;

$\hat{V}$ = residual life time of a vacation of duration V measured from an arrival epoch

Z_1^+ = excess time of U and V

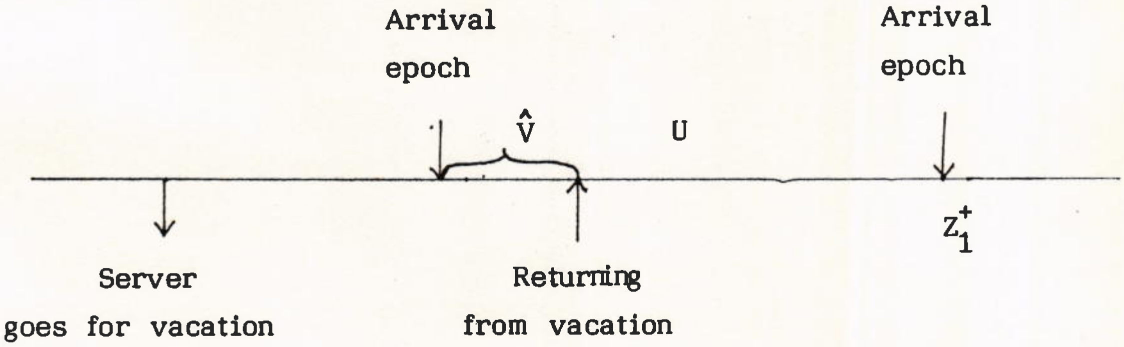
$S_j (j \geq 1)$ = sum of j consecutive service time

$F_x(.)$ = The distribution function of random variable X. $[f_x(.) = 0 \text{ for } X \leq 0]$

$f_x(.)$ = The density function of the random variable x.

$\overline{F}_x(.)$ = Laplace transform of $f_x(.)$

The expressions U, V, $\hat{V}$ and Z_1^+ are explained in the following figure.



$$\text{Now } F_{\hat{V}}(x) = \Pr(V \leq X)$$

$$= \gamma \int_0^x (1-F_V(Y)]dy$$

[From the renewal theory]

$$F_{Z_1^+}(X) = \Pr[Z_1^+ \leq X] \text{ Provided } Z_1^+ \text{ is positive}$$

$$= \frac{\Pr\ [Z_1^+ \leq X]}{\Pr[Z_1^+ > 0]}$$

$$\frac{\Pr\ [U-V \leq X]}{\Pr[U > \hat{V}]}$$

$$\Pr\ [U - \hat{V} \leq X] = \int_0^\infty F_U(Y + X)\ f_{\hat{V}}(Y)\ dy$$

[by convolution theorem]

$$\Pr\,[Z_1^{+} \leq X] \;=\; \int_0^{\infty} [\; \int_0^{X} f_U(Y+S)\; f_{\hat{V}}(Y)\; d\,S)\; dY$$

$$F_{S_j}(X) \;=\; \Gamma_X(\mu, j)$$

$$f_{S_j}(X) \;=\; \frac{\mu^j\, X^{j-1}\, e^{-\mu X}}{(j-1)!}$$ where $\Gamma_X(\mu, j)$ is the incomplete gamma function defined by

$$\Gamma_X(\mu, j) \;=\; \int_0^{X} f_{S_j}(Y)\,dY$$

Next we shall derive the limitting probabilities using the imbedded Markov Chain technique.

Pre arrival epoch

Let t_n, $n=1,2\ \ldots$ $(t_0=0)$ be the epoch at which the n^{th} customer arrives. We examine the system at t_n-0. We define the two dimensional state space of the system as

$[(i,j)\ ;\ i=0,\ j=0,\ 1,2\ \ldots]$ where $i=0$ corresponds to the server on vacation and $i=1$ corresponds to the server in the system; j is the number of customers in the system in any of these cases.

Let g_r denote the probability that r customers are served in an interarrival time and h_r the probability that r customers are served in the duration Z_1^{+}.

Then $$g_r = \int_0^{\infty} \frac{e^{-\mu x}(\mu x)^r}{r!}\, dF_U(x)$$ and

$$h_r = \frac{e^{-\mu x}\,(\mu x)^r}{r!}\, dF_{Z_1^{+}}(x)\ ,\quad r \geqslant 0$$

Let $\hat{g}_j = 1 - \sum_{k=0}^{j} g_k$, $\hat{h}_j = 1 - \sum_{k=0}^{j} h_k$ and $\theta = Pr[\hat{V} > U]$

$$\text{Then } \theta = P_r\,[U < \hat{V}]$$

$$= \int_0^\infty F_U(X)\, f_{\hat{V}}(X)\,dX$$

If $\tau(t)$ denote the system state observed at time t, then the sequence of random variables $(i_n, j_n) = \tau(t-0)$ defines a semi-Markov Chain with transition that eventually leads to the following equations involving the limiting probabilities.

$$\bar{P}_{ij} = \lim_{n \to \infty} Pr\,[i_n = i, j_n = j] \quad [i=0,1, j=0,1,2\ldots] \qquad \ldots (1)$$

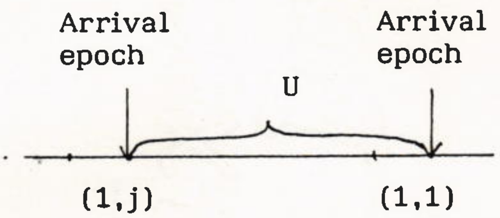
[Assuming that these exists]

$\bar{P}_{ij}$ = Probability that there are j customers in the system just before an arrival epoch and the server in on Vacation or in the system according as i=0 or1

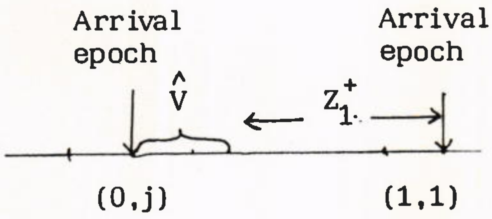
Then $\bar{P}_{1,1} = \sum_{r=1}^{\infty} \bar{P}_{1,r}$ Pr [r customers are served in the inter-arrival time] + $\sum_{r=0}^{\infty} \bar{P}_{0,r}$ Pr [r customers are served in the duration Z_1^+] Pr $[\hat{V} \leq U]$

$$= \sum_{r=1}^{\infty} \bar{P}_{1,r}\, g_r + (1-\theta) \sum_{r=0}^{\infty} \bar{P}_{0,r}\, h_r \qquad \ldots (2)$$

These arguments are justified from the following figures.



j customers are served in time U



$\dot{J}$ customers are served in Z_1^+ time.

Similarly for $j \geqslant 2$

$$\bar{P}_{1,j} = \sum_{r=0}^{\infty} \bar{P}_{1,j+r-1}\, g_r + (1-\theta) \sum_{r=0}^{\infty} \bar{P}_{0,j+r-1}\, h_r$$

i.e $$\bar{P}_{1,j+1} = \sum_{r=0}^{\infty} \bar{P}_{1,j+r}\, g_r + (1-\theta) \sum_{r=0}^{\infty} \bar{P}_{0,j+r}\, h_r, \quad j \geqslant 1 \qquad \ldots (3)$$

and

$$\bar{P}_{0,0} = \sum_{j=1}^{\infty} \bar{P}_{1j}\ \text{Pr [j+1 are served in the interarrival time]}$$
$$+ \sum_{j=0}^{\infty} \bar{P}_{0,j}\ \text{Pr [j+1 are served in } Z_1^{+} \text{ time]}\ \text{Pr}\,[\hat{V} < U]$$

$$= \sum_{j=1}^{\infty} \bar{P}_{1,j}\, \hat{g}_i + (1-\theta) \sum_{j=0}^{\infty} \bar{P}_{0,j}\, \hat{h}_j \qquad \ldots (4)$$

$$\left[\text{where } g_i = 1 - \sum_{k=0}^{j} g_k \ ;\ h_j = 1 - \sum_{k=0}^{j} h_k\right]$$

The transition form state (o,j-1) to state (0,j) is possible only when the Vacation duration is longer than the interarrival time.

Hence $\bar{P}_{0,j} = \bar{P}_{0,j-1} \; P_r[\hat{V} > U] \, , \; j \geqslant 1$ ... (5)

$= \theta \, \bar{P}_{0,j-1} \; , \quad j \geqslant 1$

This implies $\bar{P}_{0j} = \theta^j \, \bar{P}_{0,0} \, , \; j=0,1,2\ldots$ ... (6)

Denoting the forward displacement operator by E

i.e $[E(P_{1,j}) = P_{1,j+1}]$ [from (3)]

$$E \, \bar{P}_{1,j} = \sum_{r=0}^{\infty} E^r \bar{P}_{1,j} \, g_r + (1-\theta) \sum_{r=0}^{\infty} \theta^{j+r} \, \bar{P}_{0,0} \, h_r$$

[Using (6)]

$$\left[E - \sum_{r=0}^{\infty} E^r g_r\right] \bar{P}_{1,j} = (1-\theta) \, \theta^j \sum_{r=0}^{\infty} \theta^r \, h_r \; \bar{P}_{0,0}$$

$$[E - G(E)] \, \bar{P}_{1,j} = (1-\theta) \, \theta^j \, H(\theta) \, \bar{P}_{0,0} \, , \; j \geqslant 1 \qquad \ldots (7)$$

Where G(Z) and H(Z) are the probability generating functions of g_j and h_j respectively.

But $$G(Z) = \sum_{r=0}^{\infty} Z^r \, g_r$$

$$= \sum_{r=0}^{\infty} \int_0^{\infty} e^{-\mu X} \, \frac{(\mu X)^r}{r!} \, Z^r \, d \, F_U(X)$$

$$= \int_0^\infty e^{-\mu X(1-Z)} \, d \, F_U(X)$$

$$= \bar{F}_U \, (\mu(1-Z))$$

Similarly $H(Z) = \bar{F}_{Z_1^+} \, (\mu(1-Z)]$

It can easily be shown that when

$$G'(1) = -\mu \bar{F}_U^{1}(0) = \mu/\lambda > 1$$

i.e $\lambda/\mu < 1$

$R(Z) = Z-G(Z)$ has only one Zero inside $|Z|=1$ we assume that $\rho = \lambda/\mu < 1$ and α is the root of $R(Z)=0$ with $|\alpha| < 1$

Then using the fact that $\sum_{j=1}^{\infty} \bar{P}_{ij} < 1$

We have from equation (7)

$$\bar{P}_{1j} = [A\alpha^j + \frac{\theta^j(1-\theta)H(\theta)}{\theta-G(\theta)}] \, \bar{P}_{0,0}$$

where A is a constant

$$A = \frac{(1-\theta) \, H(\theta)}{G(\theta)-\theta}$$

Thus $\bar{P} = A \, (\alpha^j - \theta^j) \, \bar{P}_{0,0} \quad , \ j \geq 1$

Now the only remaining unknown $\bar{P}_{00}$ is determined from the normalizing equation

$$\sum_{j=0}^{\infty} \bar{P}_{0j} + \sum_{j=1}^{\infty} \bar{P}_{1j} = 1$$

as $$(\bar{P}_{00})^{-1} = A\left(\frac{\alpha}{1-\alpha} - \frac{\theta}{1-\theta}\right) + \frac{1}{1-\theta}$$

The mean system size observed at a prearrival epoch is

$$E(\bar{N}) = \sum_{j=0}^{\infty} j\,\bar{P}_{0j} + \sum_{j=1}^{\infty} j\,\bar{P}_{1j}$$

$$= \left\{A\left[\frac{\alpha}{(1-\alpha)^2} - \frac{\theta}{(1-\theta)^2}\right] + \frac{\theta}{(1-\theta)^2}\right\} \bar{P}_{00}$$

Random epoch

Let us obtain the limiting prohabilities of the system size at a random epoch. In addition to the notations adopted earlier, we assume

T = Period of time between a random epoch and preceding arrival epoch.

Z_2^+ = Excess time of $U(T)$ over $\hat{V}$

θ^* = $\Pr\{\hat{V} > T\}$

$$F_T(X) = \lambda_0 \int_0^X (1-F_u(Y))\,dY$$

$$F_{Z_2^+}(x) = \frac{\int_0^x \int_0^\infty f_{\hat{V}}(Y)\, f_T(Y+S)\,dY\,dS}{\Pr(\hat{V} < T)}$$

We examine the system at some time t preceding an arrival epoch. Then using the relations between the two sequences of random variables.

$$(i_1, j_1) = \tau(t) \quad (t_n < t < t_{n+1}) \text{ and } (i_n, j_n) \quad (n=0,1,2,\ldots)$$

Let P_{ij} denote the probability that there are j customers in the system at random spoch and the server is on vacation or in the system according as i = 0 or 1.

Then the equations satisfied by

$$P_{ij} = \operatorname*{Lt}_{n \to \infty} \Pr[i_1 = i, j_1 = j] \text{ are :}$$

$$P_{1,1} = \sum_{r=1}^{\infty} \bar{P}_{1,r} g_r^* + (1-\theta^*) \sum_{r=o}^{\infty} \bar{P}_{o,r} h_r^*$$

$$P_{1,j} = \sum_{r=o}^{\infty} \bar{P}_{1,j+r-1} g_r^* f (1-\theta^*) \sum_{r=o}^{\infty} \bar{P}_{o,j+r-1} h_r^* \quad (j \geq 2)$$

$$P_{oo} = \sum_{j=1}^{\infty} \bar{P}_{1,j} g_j^* + (1-\theta^*) \sum_{j=o}^{\infty} \bar{P}_{o,j} h_j^*$$

$$P_{oj} = \bar{P}_{o,j-1} \theta^*$$

$$P_{oj} = \theta^* \theta^j \bar{P}_{oo}$$

To obtain the solution, we first find the relation between $g_j^*(h_j^*)$ and $g_j(h_j)$

$$g_r^* = \int_0^{\infty} \frac{e^{-\mu x} (\mu x)^r}{r!} \, d F_T(x)$$

$$dF_T(X) = \lambda [1-F_0(X)] dx$$

$$g_r^* = \int_0^\infty \frac{e^{-\mu x}(\mu x)^r}{r!} [1-F_U(x)]\, dx$$

$$= \frac{\lambda}{\mu} - \lambda \frac{\mu^r}{r!} (-D)^r \left[\frac{\tilde{F}_U(\mu)}{\mu} \right]$$

Where $\tilde{F}_U(\mu) = \int_0^\infty e^{-\mu x}\, d\, F_U(x)$

$$\left[L[F_U(x)] = \int_0^\infty e^{-\mu x} F_U(x) dx = \frac{\tilde{F}_U(\mu)}{\mu} \right]$$

$$= \frac{\lambda}{\mu} - \frac{\lambda}{r!} \mu^r (-1)^r \sum_{k=0}^{r} (-1)^k \frac{k!}{k+1}\, r\, c_k [D^{r-K} F_U(\mu)]$$

$$= \frac{\lambda}{\mu} \left[1 - \sum_{k=0}^{r} \frac{\mu^{r-k}(-D)^{r-k}}{(r-k)\,!} \tilde{F}_U(\mu) \right]$$

$$= \frac{\lambda}{\mu} \left[1 - \sum_{k=0}^{r} g_{r-k}\right]$$

$$= \rho \left[1 - \sum_{k=0}^{r} g_k \right]$$

$$h_r^* = \int_0^\infty \frac{e^{-\mu x}(\mu x)^r}{r!}\, d\, F^+_{Z_2}(x)$$

$$= \int_0^\infty \frac{e^{-\mu x}(\mu x)^r}{r!} \int_0^\infty \frac{f_{\hat{V}}(Y)\, f_T(Y+x)\, d\, dx}{(1-\theta^*)}$$

$$= \int_0^\infty \frac{e^{-\mu x}(\mu x)^r}{r!} \int_0^\infty \frac{f_{\hat{V}}(Y)\ \lambda(1-F_U(y+x)}{1-\theta^*}\ dydx$$

$$= \frac{\lambda}{1-\theta^*}\ [\ \frac{1}{\mu} - \int_0^\infty \frac{e^{-\mu x}(\mu x)^r}{r!}\ (1-\theta)F_{Z^+}(x)dx\ ]$$

$$= \frac{\lambda}{\mu(1-\theta^*)}\ [1 - (1-\theta) \sum_{k=o}^{r} h_k\ ]$$

$$= \frac{\rho}{1-\theta^*}\ [1-(1-\theta) \sum_{k=o}^{r} h_k\ ]$$

$$P_{ij} = \sum_{r=o}^{\infty} A\ (\alpha^{j+r-1} - \theta^{j+r-1})\ \bar{P}_{oo}\ g_r^*$$

$$+ (1-\theta^*) \sum_{r=o}^{\infty} \theta^{j+r-1})\ \bar{P}_{oo}$$

$$\sum_{r=o}^{\infty} \alpha^r g_r^* = \rho \sum_{r=o}^{\infty} \alpha^r\ (1- \sum_{k=o}^{r} g_k\ )$$

$$= \rho\ [\ \frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\ ]$$

$$= \rho$$

$$\sum_{r=o}^{\infty} \theta^r g_r^* = P\ [\ \frac{1}{1-\theta} - \frac{G(\theta)}{1-\theta}\ ]$$

$$(1-\theta^*) \sum_{r=o}^{\infty} \theta^r h_r^* = \rho\ [\ \frac{1}{1-\theta} - \sum_{r=o}^{\infty} (1-\theta)\ \theta^r \sum_{k=o}^{r} h_k\ ]$$

$$= \rho\ [\ \frac{1}{1-\theta} - (1-\theta)\ \frac{H\ (\theta)}{1-\theta}\ ]$$

$$= \rho\ [\ \frac{1}{1-\theta} - H(\theta)\ ]$$

$$P_{ij} = \bar{P}_{00}[A \alpha^{j-1} \rho - A\ \theta^{j-1} \rho\ \frac{(1-G(\theta)}{1-\theta} + \theta^{j-1} \rho\ [\frac{1}{1-\theta} - H(\theta)]$$

$$= \bar{P}_{00}\ \rho\ [A \alpha^{j-1} - \frac{A\theta^{j-1}}{1-\theta} + \frac{\theta^{j-1}}{1-\theta}\ [(1-\theta)H(\theta) + A\theta]$$

$$+ \theta^{j-1}\ [\frac{1}{1-\theta} - H\ (\theta)]$$

$$P_{ij} = \bar{P}_{00}\ \rho\ [A(\alpha^{i-1} - \theta^{j-1}) + \frac{\theta^{j-1}}{1-\theta}\ ]$$

$$P_{00} = [\sum_{j=1}^{\infty} A(\alpha^{j}-\theta^{j})\ (1- \sum_{k=0}^{j} g_k^{*}) +$$

$$+ (1-\theta^{*}) \sum_{j=0}^{\infty} \theta^{j}\ (1- \sum_{k=0}^{j} h_k^{*})]\ \bar{P}_{00}$$

$$= \{A\ [\frac{\alpha}{1-\alpha} - \frac{\theta}{1-\theta}\ ] - A\ [\ \frac{g_0^{*}\alpha}{1-\alpha} + \frac{1}{1-\alpha}\ [\sum_{k=0}^{\infty} g_k^{*} \alpha^{k} - g_0^{*}]$$

$$+ A[\ \frac{g_0^{*}\theta}{1-\theta} + \frac{1}{1-\theta}\ [\sum_{h=0}^{\infty} g_k^{*}\ \theta^{k} - g_0^{*}\ ]$$

$$+ (1-\theta^{*})\ [\ \frac{1}{1-\theta} - \sum_{k=0}^{\infty} \frac{h_k\ \theta^{k}}{1-\theta}\ ]\}\ \bar{P}_{00}$$

$$= \{A\ [\frac{\alpha}{1-\alpha} - \frac{\theta}{1-\theta}\ ] - A\ [\ \frac{g_0^{*}}{-1}\ \ \frac{1}{1-\alpha} \rho\ ]$$

$$+ A\ [\ \frac{g_0^{*}}{-1} + \frac{1}{1-\theta}\ \ \frac{\rho\ (1-G(\theta)}{1-\theta}$$

$$+ \frac{1-\theta}{1-\theta} - \frac{1}{1-\theta}\ (\rho\ (\frac{1}{1-\theta} - H\ (\theta))]\}\ \bar{P}_{00}$$

$$= \left\{ A \left[\frac{1}{1-\alpha} - \frac{1}{1-\theta}\right] - A \left[\frac{\rho}{1-\alpha}\right] + \frac{A}{1-\theta} \left[\frac{\rho}{1-\theta}\right]\right.$$

$$\left. - \frac{\rho}{(1-\theta)^2} [A\theta + (1-\theta) H(\theta) + \frac{1-\theta^*}{1-\theta} - \frac{1}{1-\theta} [\rho(\frac{1}{1-\theta} - H(\theta))] \right\} \overline{P}_{oo}$$

$$[A\ G(\theta) = A\theta + (1-\theta)\ H(\theta)]$$

$$= P_{oo} \left\{ A \left[\frac{1}{1-\alpha} - \frac{1}{1-\theta} \right] - A \rho \left[\frac{1}{1-\alpha} - \frac{1}{1-\theta}\right] + \frac{1-\theta^*}{1-\theta} - \frac{\rho}{(1-\theta)^2} \right\}$$

$$P_{oo} = \left\{ A(1-\rho) \left[\frac{1}{1-\alpha} - \frac{1}{1-\theta} \right] + \frac{1}{1-\theta} \left[(1-\theta^*) - \frac{\rho}{1-\theta}\right] \right\} P_{oo}$$

Expected Waiting time

We shall derive the distribution of waiting time of a customer in the queue [excluding service].

Let W be the random waiting time of a customer in the queue. An arbitrary customer on arrival has to wait for a time $\hat{V}$ or $\hat{V} + S_j$ or $S' + S_{j-1} (S_o=0)$. If the system is in the state (0,0) or (0,j), $j \geq 0$ (or) (1,j), $(j \geq 1)$ where S' is the remaining service time of the customer, already in service. Then the distribution function of W is

$$F_W(t) = \Pr(V \quad t)\ \overline{P}_{oo} + \underset{j=1}{} \Pr(V + S_j \quad t)\ \overline{P}_{o,j}$$

$$+ \underset{j=1}{} \Pr(S' + S_{j-1} \quad t)\ \overline{P}_{1j} \qquad \ldots(8)$$

$$F_W(t) = F_{\hat{V}}(t)\bar{P}_{0,0} + \sum_{j=1}^{\infty} \theta^j \Pr(\hat{V} + S_j \leq t)\, \bar{P}_{0,0}$$

$$+ \sum_{j=1}^{\infty} \Pr(S' + S_{j-1} \leq t)\, A\, (\alpha^j - \theta^j)\, \bar{P}_{0,0}$$

Since the service time distribution is exponential, $S' + S_j$ is a gamma variates with parameter μ and j+1

Consider,

$$\Pr(\hat{V}+S_j \leq t) = \int_0^t f_{\hat{V}}(x)\, F_{S_j}(t-x)dx$$

$$= \int_0^t \Big[\int_0^{t-x} \frac{u^j y^{j-1} e^{-\mu y}}{(j-1)!}\, dy \Big]\, f_{\hat{V}}(x)dx$$

(Using the definition of S_j and $\hat{V}$]

$$\sum_{j=1}^{\infty} \theta^j \Pr[\hat{V} + S_j \leq t] = \int_0^t \Big[\int_0^{t-x} \mu\, e^{-\mu y(1-\theta)} dy \Big]\, f_{\hat{V}}(x)dx$$

$$= \int_0^t \left[\frac{\mu\theta(e^{-\mu y(1-\theta)}}{-\mu(1-\theta)} \right]_0^{t-x} f_{\hat{V}}(x)dx$$

$$= -\frac{\theta}{1-\theta} \int_0^t e^{-\mu(1-\theta)(t-x)} f_{\hat{V}}(x)dx + \frac{\theta}{1-\theta} \int_0^t f_{\hat{V}}(x)dx$$

$$= \frac{\theta}{1-\theta} F_{\hat{V}}(t) - \frac{\theta}{1-\theta}\, e^{-\mu(1-\theta)t} \int_0^t e^{-\mu(1-\theta)x} f_{\hat{V}}(x)dx$$

$$= \frac{\theta}{1-\theta} F_{\hat{V}}(t) - \frac{\theta}{1-\theta}\, e^{-\mu(1-\theta)t}\, \xi(t)$$

Where $\xi(t) = \int_0^t e^{-\mu(-\theta)x} f_{\hat{V}}(x)dx$

and $$\sum_{j=1}^{\infty} A\,\alpha^j \Pr[S'+S_{j-1} \leq t] = A \sum_{j=1}^{\infty} \alpha^j \int_0^t \frac{\mu^j x^{j-1} e^{-\mu x}}{(j-1)!} dx$$

$$= A\mu\alpha \int_0^t e^{-\mu(1-\alpha)x} dx$$

Similarly,

$$\sum_{j=1}^{\infty} A\,\theta^j \Pr(S'+S_{j-1} \leq t) = A \frac{\theta}{1-\theta} (1-e^{-\mu(1-\theta)t})$$

Substituting these in (8) and simplifying

$$F_W(t) = \overline{P}_{00} [\frac{1}{1-\theta} [F_{\hat{V}}(t) - \theta e^{-\mu(1-\theta)t} \xi(t)]$$

$$+ A [\frac{\alpha}{1-\alpha} (1-e^{-\mu(1-\alpha)t}) - \frac{\theta}{1-\theta} (1-e^{-\mu(1-\theta)t}]$$

Let $f_W(t) = \frac{d}{dt} F_W(t)$ be the density function of the waiting time. Then differentiating the above equation we get,

$$f_W(t) = \overline{P}_{00} \frac{1}{1-\theta} f_{\hat{V}}(t) - \frac{\theta}{1-\theta} (-\mu(1-\theta) e^{-\mu(1-\theta)t} \xi(t)$$

$$+ e^{-\mu(1-\theta)t} e^{\mu(1-\theta)t} f_{\hat{V}}(t)$$

$$+ A [\frac{\alpha}{1-\alpha} \mu(1-\alpha) e^{-\mu(1-\theta)t} - \frac{\theta}{1-\theta} \mu(1-\theta) e^{-\mu(1-\theta)t}]$$

$$= \overline{P}_{00} [f_{\hat{V}}(t) + \mu\theta e^{-\mu(1-\theta)t} \xi(t)$$

$$+ \mu A [\alpha e^{-\mu(1-\alpha)t} - \theta e^{-\mu(1-\theta)t}]$$

The expected waiting time in the queue is

$$E(W) = \int_0^\infty t\, f_W(t)\, dt$$

$$= \bar{P}_{00} \int_0^t t \quad f_{\hat{V}}(t)dt + \int_0^\infty t\mu\theta\, e^{-\mu(1-\theta)t}\, \xi(t)dt$$

$$+ A\, [\int_0^\infty t\alpha e^{-\mu(1-\alpha)t}dt - \int_0^\infty t\, \theta\, e^{-\mu(1-\theta)t}dt]$$

Consider,

$$\mu A \alpha \int_0^\infty t\, e^{-\mu(1-\alpha)t}dt = \mu A \alpha\, [\frac{t\, e^{-\mu(1-\alpha)t}}{-\mu(1-\alpha)} \Big]_0^\infty$$

$$+ \frac{e^{-\mu(1-\alpha)t}}{\mu(1-\alpha)}\, dt\,]$$

$$= \mu A \frac{\alpha}{[\mu(1-\alpha)]^2},$$

Similarly, $\mu A\theta \int_0^\infty t\, e^{-\mu(1-\theta)t}dt = \mu A \frac{\theta}{[\mu(1-\theta)]^2}$

Hence We get,

$$E(W) = \Big\{ E(\hat{V}) + \frac{\theta}{(\mu(1-\theta))^2} \int_0^\infty f_{\hat{V}}(t)dt + \frac{\theta}{1-\theta} E(\hat{V})$$

$$+ \frac{A}{\mu}\, [\frac{\alpha}{(1-\alpha)^2} - \frac{\theta}{(1-\theta)^2}] \Big\} \bar{P}_{00}$$

$$= \frac{E(\hat{V})}{1-\theta} \bar{P}_{00} + \frac{1}{\mu}\, [\frac{\theta}{(1-\theta)^2} + A \frac{\alpha}{(1-\alpha)^2} \quad \frac{\theta}{(1-\theta)^2}\,]\, \bar{P}_{00}$$

$$= \frac{E(\hat{V})}{1-\theta} \bar{P}_{00} + \frac{1}{\mu} E\, (\bar{N})$$

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