Some Interesting Results on Fuzzy Neighbourhood Spaces

By

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Introduction

INTRODUCTION

"THE ESSENCE OF MATHEMATICS IS ITS FREEDOM"

- George Cantor.

In 1968, Chang [5] introduced the concept of fuzzy topology. Since then a number of articles have been published on generalizations of topological concepts like Neighbourhoods, Continuity, Compactness, Connectedness etc., to Fuzzy Topological Spaces.

In this thesis, we attempt to give a brief survey of the fundamental results in fuzzy neighbourhood structures. Our study is based on the following articles:

1. Fuzzy Topologies Characterized By Neighbourhood Systems [18].
2. Characterization of Fuzzy Topologies from Neighbourhoods of Fuzzy Points[6].
3. Fuzzy Topology I, Neighbourhood Structure of a Fuzzy Point and Moore-Smith Convergence [17].
4. Lattice Properties of Neighbourhood Systems in Chang Fuzzy Topological Spaces [8].

This thesis is split into two chapters. In Section One of the First Chapter, we discuss a Characterization of fuzzy topological space in terms of fuzzy neighbourhood syustems. The results discussed here are contained in the paper "Fuzzy topologies characterized by Neighbourhood Systems"[18]. Given a fuzzy topology, we can associate for every x in X, a fuzzy neighbourhood system N_x. This neighbourhood suystem N_x satisfies the properties similar to a neighbourhood system of a point in a topological space. Conversely,given a neighbourhood system N_x for every x in X satisfying certain properties, we can construct a fuzzy topology.

The main theorem proved here is as follows:

"Let (X,τ) be a fuzzy topological space and let θ_τ be the function $\{ (x, N_x) : x \in X \}$ where N_x is the family of all neighbourhoods of x with respect to τ. Then θ_τ is a fuzzy neighbourhood system on X and $\tau(\theta_\tau) = \tau$."

In Section Two, similar characterization is obtained by considering fuzzy neighbourhood of fuzzy points.

In Section One of the Chapter Two, we define the Pu-neighbourhood of a fuzzy point and study the properties of Pu-neighbourhood system at a fuzzy point. We can characterize an open fuzzy set in terms of Pu-neighbourhoods. Further, we define adherence point and accumulation point of a fuzzy set. We can characterize closure of a fuzzy set in terms of accumulation points. The important results proved here are the following:

1. "Let $\mathcal{P}_{x_\lambda}(\tau)$ be the family of Pu-neighbourhoods of a fuzzy point x_λ in (X, τ).

Then we have,

(1) If $U \in \mathcal{P}_{x_\lambda}(\tau)$, then x_λ is quasi-coincident with U.

(2) If $U, V \in \mathcal{P}_{x_\lambda}(\tau)$, then $U \cap V \in \mathcal{P}_{x_\lambda}(\tau)$.

(3) If $U \in \mathcal{P}_{x_\lambda}(\tau)$ and $U \subseteq V$, then $V \in \mathcal{P}_{x_\lambda}(\tau)$

(4) If $U \in \mathcal{P}_{x_\lambda}(\tau)$, then there exists $V \in \mathcal{P}_{x_\lambda}(\tau)$, such that $V \subseteq U$ and

$V \in \mathcal{P}_{x_\mu}(\tau)$, for every fuzzy point x_μ which is quasi coincident with V.

Conversely, for each fuzzy point x_λ in x, if $\mathcal{P}_{x_\lambda}(\tau)$, is a family of fuzzy sets in X satisfying the above conditions (1) - (3), then the family τ of all the fuzzy sets U, such that $U \in \mathcal{P}_{x_\lambda}(\tau)$ satisfies the condition (4), mentioned above, then $\mathcal{P}_{x_\lambda}(\tau)$ is exactly the Pu - neighbourhood system of x_λ relative to the fuzzy topology τ".

2. "A fuzzy point $x_\lambda \in$ Cl-A if and only if each Pu-neighbourhood of x_λ is quasi-coincident with A"
3. "A fuzzy set A is closed if and only if A contains all the accumulation points of A".

In Section Two of Chapter Two, we define Kerre neighbourhood and Mashhour neighbourhood and study the lattice properties of the Pu-neighbourhood system, Kerre neighbourhood system and Mashhour neighbourhood system.

"A fuzzy set A in X is called a Kerre neighbourhood of a fuzzy singleton x_λ if there exists $O \in \tau$ such that $x_\lambda \subseteq O$ and $O \subseteq A$".

"A fuzzy set A in X is called a Mashhour neighbourhood of a fuzzy point x_λ if there exists $O \in \tau$ such that $x_\lambda \,\bar{\in}\, O$ and $O \subseteq A$".

The collection of Kerre neighbourhood of a fuzzy singleton x_λ is denoted by $\mathcal{K}_{x_\lambda}(\tau)$ where as the collection all Mashhour neighbourhoods of a fuzzy point x_λ is denoted by $\mathcal{M}_{x_\lambda}(\tau)$.

The important results proved here are the following :

1. "Let (X, τ) be a Chang fuzzy topological space and $x \in X$. The Kerre, Mashhour and Pu-neighbourhoods satisfy the following lattice completeness properties.

(1) $(\mathcal{M}_x(\tau), \subseteq)$ is a totally ordered sublattice of $(2^{I^X}, \subseteq)$ having the following sup-completeness property :

$\forall\ A \subseteq (0,1),\ \inf A > 0 \Rightarrow \mathcal{M}_{x_{\inf A}}(\tau) = \bigcup_{\alpha \in A} \mathcal{M}_{x_\alpha}(\tau).$

(2) $(\mathcal{K}_x(\tau), \subseteq)$ is a totally ordered sublattice of $(2^{I^X}, \subseteq)$ having the following inf-completeness property:

$\forall\ A \subseteq (0,1],\ \mathcal{K}_{x_{\sup A}}(\tau) = \bigcap_{\alpha \in A} \mathcal{K}_{x_\alpha}(\tau).$

(3) $(\mathcal{P}_x(\tau), \subseteq)$ is a totally ordered sublattice of $(2^{I^X}, \subseteq)$ having the following sup-completeness property:

$\forall\ A \subseteq (0,1],\ \mathcal{P}_{x_{\sup A}}(\tau) = \bigcup_{\alpha \in A} \mathcal{P}_{x_\alpha}(\tau).$

2. "Let (X,τ) be a Chang fuzzy topological space. The following are equivalent :

(1) τ is crisp

(2) $\forall\, x \in X,\ \forall\, A \subseteq (0,1] \ni \inf A > 0 \Rightarrow \mathcal{K}_{x_{\inf A}}(\tau) = \bigcup_{\alpha \in A} \mathcal{K}_{x_\alpha}(\tau).$

(3) $\forall\, x \in X,\ \forall\, A \subseteq (0,1) \ni \sup A > 0 \Rightarrow \mathcal{M}_{x_{\sup A}}(\tau) = \bigcap_{\alpha \in A} \mathcal{M}_{x_\alpha}(\tau).$

(4) $\forall\, x \in X,\ \forall\, A \subseteq (0,1] \ni \inf A > 0 \Rightarrow \mathcal{P}_{x_{\inf A}}(\tau) = \bigcap_{\alpha \in A} \mathcal{P}_{x_\alpha}(\tau)$."

Review of Literature

REVIEW OF LITERATURE

The fundamental concept of a fuzzy set was introduced by ***Zadeh*** in 1965 [19]. In the area of fuzzy topology, much research has been carried out since 1968. The notion of fuzzy neighbourhoods which enables us to introduce a local theory of fuzzy topologies was studied by ***R. Lowen*** in 1982 [10].

R.H. Warren, P.M. Pu and Y.M. Liu, C.De Mitri and E. Pascali, E. E. Kerre and P.L. Ottoy, Ahsanuallh, T. M. G., Mashinchi and Mashallah, Ganguly, Shobhakar, Sostak, A.P., Morsi, Nehad, N., Mashhour Ali, S. have published a number of articles in fuzzy neighbourhood system.

We have studied some of the contributions by ***R. H. Warren, C. De. Mitri and E. Pascali, P.M. Pu and Y. M. Liu and E. E. Kerre and P.L. Ottoy*** in this dissertation.

Now we present abstracts of some important papers on fuzzy neighbourhood spaces published by ***AHSANULLAH, T. M. G., MASHINCHI, MASHALLAH, GANGULY, SHOBHAKAR, SOSTAK, A.P., MORSI, NEHAD, N., MASHHOUR*** .

1. "***Some results on fuzzy neighbourhood spaces***" by ***AHSANULLAH, T.M.G.*** (1985) [1].

   The authors present some new results on Lowen fuzzy neighbourhood spaces. Various notions of closedness, the concepts of regular and Urysohn spaces are studied.

2. "***An intermediate value theorem in neighbourhood spaces***" by ***MASHINCHI, MASHALLAH*** (1986) [12].

   In this paper an intermediate value theorem for neighbourhood maps on connected neighbourhood spaces with values on a fixed neighbourhood space is given.

3. "***On fuzzy neighbourhood groups***" by ***AHSANULLAH, T.M.G.*** (1988) [2].

   Based on the notion of a fuzzy neighbourhood structure [10] the notion of a fuzzy neighbourhood group is introduced. Some basic results are established ; one of them is that every fuzzy neighbourhood group is uniformizable in Lowen's sense [9].

4. "***On fuzzy neighbourhood rings***" by ***AHSANULLAH, T.M.G., GANGULY, SHOBHAKAR*** (1990) [3].

   The authors introduce the concept of fuzzy neighbourhood rings with the help of the notion of fuzzy neighbourhood system as given by ***R. Lowen*** [10]. They give necessary and sufficient conditions for a ring structure and a fuzzy neighbourhood system to be compatible.

5. "***On the neighbourhood structure of fuzzy topological spaces***" by ***SOSTAK, A.P.*** (1990) [16].

   In this paper the notions of a neighbourhood structure and of a q-neighbourhood structure are introduced and studied. These structures are applied to obtain a local description of fuzzy topological spaces. As a special case the theory obtained contains in itself the local theory of Chang fuzzy topological spaces developed by ***P. M. Pu and Y.M. Liu*** [17].

6. "***The Urysohn lemma for fuzzy neighbourhood spaces***" by ***MORSI, NEHAD N.*** (1991) [13].

   The framework FNS of Lowen's fuzzy neighbourhood spaces is a special environment for the study of fuzzy topology. It encompasses its own metrics, interesting constructions. This is a fresh and fast growing field of research. It promises to develop quickly to a calculus which combines depth with applicability. Within this frame work, we establish the Ursohn lemma for the Mashhour-Morsi axiom of N - normality. The formulation employs the Mashhour-Morsi N. Euclidean spaces, as the suitable fuzzy analogue of the real line. The proof makes use of the Artico - Moresco fuzzy proximities, the Lowen fuzzy uniformities and the Mashhour-Morsi axiom of N - complete regularity.

7. "***Dual fuzzy neighbourhood spaces - I***" by ***MORSI, NEHAD N.*** (1991)[14].

   A fuzzy neighbourhood space (fns) is determined by means of a system of fuzzy neighbourhoods of the ordinary points of that space. We provide a general method for constructing bases for all fuzzy neighbourhood systems, in a natural way, from the fuzzy topologies spaces (fts's), which we call the fuzzy neighbourhood space. This construction is shown to be functional of those new fts's, the full subcategory of dual fuzzy neighbourhood spaces is shown to be isomorphic to the category of fns's and an fns will coincide with its dual fts if and only if it is topologically generated.

8. ***"On regularity axioms in fuzzy neighbourhood spaces"*** by ***MASHHOUR ALI. S, MORSI, NEHAD. N*** (1991) [11].

Two notions of regularity in fuzzy neighbourhood spaces (fns's) have been introduced, one by T.M.G., Ahsanullah and the other by us. We prove that those two notions coincide. We then characterize a regular fns as an fns whose α - level topologies, for all $0 < \alpha < 1$, are regular.

Chapter 1

CHAPTER I

FUNDAMENTALS OF FUZZY NEIGHBOURHOOD SPACES

In Section 1 of this Chapter we discuss a characterization of fuzzy topological space in terms of fuzzy neighbourhood system. The result discussed here is contained in the paper "Fuzzy Topologies Characterized by Neighbourhood Systems" [18].

Given a fuzzy topology, we can associate for every x in X, a fuzzy neighbourhood system N_x. This neighbourhood system N_x satisfies the properties similar to a neighbourhood system of a point in a topological space. Conversely, given for every x in X, a neighbourhood system N_x satisfying certain properties, we can construct a fuzzy topology.

In Section 2 of this Chapter, similar characterization is obtained by considering fuzzy neighbourhood of fuzzy points.

The results discussed above are contained in the papers "Fuzzy Topologies Characterized by Neighbourhood Systems" [18] and "Characterization of Fuzzy Topologies from Neighbourhoods of Fuzzy Points " [6].

SECTION 1.1

FUZZY TOPOLOGIES CHARACTERIZED BY NEIGHBOURHOOD SYSTEMS

Definition 1.1.1

Let X be a non empty set. A function A from X to [0,1] is called a fuzzy set in X.

For every $x \in A$, $A(x)$ is called the grade of membership of x in A. If A takes only the values 0,1 then A is called a crisp set in X. The crisp set which always takes the value 0 on X is denoted by 0_X and the crisp set which always takes the value 1 on X is denoted by 1_X.

The set $\{x \in X \,/\, A(x) > 0\}$ is called the support of A. The power class of X and the fuzzy power class of X are denoted by 2^X and 2^{I^X} respectively.

Definition 1.1.2

Let Λ be an indexed set and let $\mathscr{A} = \{A_\alpha / \alpha \in \Lambda\}$ be a family of fuzzy sets in X.

Then the union $\cup\mathscr{A}$ and the intersection $\cap\mathscr{A}$ are defined by

$(\cup\mathscr{A})(x) = \text{Sup}\ \{A_\alpha(x) / \alpha \in \Lambda\},\ x \in X$ and

$(\cap\mathscr{A})(x) = \text{Inf}\ \{A_\alpha(x) / \alpha \in \Lambda\},\ x \in X$ where Λ is any index set.

Definition 1.1.3

Let A be a fuzzy set. The complement of A, denoted by A' is defined by $A'(x) = 1 - A(x)$ for all $x \in X$.

Definition 1.1.4

Let A and B be two fuzzy sets. Then

(i) they are said to be equal if $A(x) = B(x)$ for every $x \in X$.

(ii) B includes A or A is contained in B, denoted as $A \subseteq B$ is $A(x) \leq B(x)$, $\forall\, x \in X$.

Definition 1.1.5

Let X be a non empty set and let τ be the family of fuzzy sets in X. Then τ is called the fuzzy topology on X if

(1) $0_X, 1_X \in \tau$

(2) if $\{A_i\}_{i = 1,2,\ldots n} \in \tau$, then $\bigcap_{i=1}^{n} A_i \in \tau$.

(3) if $\{ A_j \,/\, j \in \Lambda\} \in \tau$, then $\bigcup_{j\in\Lambda} A_j \in \tau$ where Λ is any index set.

The pair (X, τ) is called a fuzzy topological space (briefly, fts). Every member of τ is called a τ-open fuzzy set or simply open fuzzy set. The complement of a τ-open fuzzy set is called a τ-closed fuzzy set or simply a closed fuzzy set.

Definition 1.1.6

A fuzzy set A in a fuzzy topological space (X, τ) is a neighbourhood of a point $x \in X$ if, there exists $B \in \tau$ such that $B \subseteq A$ and $A(x) = B(x) > 0$. The family of all neighbourhoods of x with respect to the fuzzy topology τ on X is denoted by N_x.

Theorem 1.1.6

Let (X, τ) be fuzzy topological space. Then for each $x \in X$, N_x satisfies the following conditions :

(i) $1_X \in N_x$.

(ii) if $A \in N_x$, then $A(x) > 0$.

(iii) if $A \in N_x$, $A \subseteq B$ and $A(x) = B(x)$, then $B \in N_x$.

(iv) if $A_\alpha \in N_x$, $\alpha \in \Lambda$ then $\cup\{A_\alpha / \alpha \in \Lambda\} \in N_x$.

(v) if A, B $\in N_x$, then $A \cap B \in N_x$.

(vi) if $A \in N_x$, then there exists $B \in N_x$ such that $B \subseteq A$, $B(x) = A(x)$ and if $B(y) > 0$, then $B \in N_y$.

Proof:

Given (X, τ) is a fuzzy topological space.

(i) Then for each $x \in X$, $1_X \in N_x$ obviously.

(ii) Let $A \in N_x$.

To prove $A(x) > 0$.

$A \in N_x \Rightarrow$ A is a neighbourhood of X.

Therefore, by definition there exists $B \in \tau$ such that $B \subseteq A$ and $A(x) = B(x) > 0$.

Hence $A(x) > 0$.

(iii) Given $A \in N_x$, $A \subseteq B$ and $A(x) = B(x)$.

To prove $B \in N_x$.

$A \in N_x \quad \Rightarrow$ there exists $D \in \tau$ such that $D \subseteq A$ and $A(x) = D(x) > 0$.

Therefore, $D \subseteq A \subseteq B$ and $B(x) = A(x) = D(x) > 0$.

Hence, there exists $D \in \tau$ such that $D \subseteq B$ and $B(x) = D(x) > 0$.

$\Rightarrow \quad B \in N_x$.

(iv) Let $A_\alpha \in N_x$ for all $\alpha \in \Lambda$.

To prove $\cup \{A_\alpha / \alpha \in \Lambda\} \in N_x$.

$A_\alpha \in N_x \quad \Rightarrow$ there exists $B_\alpha \in \tau$ such that $B_\alpha \subseteq A_\alpha$ and $A_\alpha(x) = B_\alpha(x) > 0$.

Let $C = \cup\{ B_\alpha / \alpha \in \Lambda \}$ and $D = \cup\{ A_\alpha / \alpha \in \Lambda \}$

Then, $C \in \tau$ such that $C \subseteq D$ and $D(x) = C(x) > 0$

$\Rightarrow D \in N_x$.

That is, $\cup\{ A_\alpha / \alpha \in \Lambda \} \in N_x$.

(v) Let $A, B \in N_x$.

To prove $A \cap B \in N_x$.

$A, B \in N_x \quad \Rightarrow$ there exist $C, D \in \tau$ such that

$C \subseteq A,\ D \subseteq B$ and $A(x) = C(x) > 0,\ B(x) = D(x) > 0$.

$\Rightarrow C \cap D \subseteq A \cap B$ and $(A \cap B)(x) = (C \cap D)(x) > 0$

$\Rightarrow A \cap B \in N_x$

(vi) If we take B to be the open fuzzy set whose existence is specified in Definition 1.1.6 then the condition (vi) is obtained.

Definition 1.1.8

Let X be a set and θ be a function from X into the power class of $[0,1]^X$. Then θ is called a fuzzy neighbourhood system on X if θ satisfies the following conditions :

N1 : $1_X \in \theta(x)$ for each $x \in X$.

N2 : if $A \in \theta(x)$, then $A(x) > 0$.

N3 : if $A \in \theta(x)$, $A \subseteq B$ and $A(x) = B(x)$, then $B \in \theta(x)$.

N4 : if $A_\alpha \in \theta(x)$, $\alpha \in \Lambda$, then $\cup\{ A_\alpha / \alpha \in \Lambda\} \in \theta(x)$

N5 : if $A, B \in \theta(x)$, then $A \cap B \in \theta(x)$.

N6 : if $A \in \theta(x)$, then there exists $B \in \theta(x)$ such that $B \subseteq A$, $B(x) = A(x)$ and if $B(y) > 0$, then $B \in \theta(y)$.

Definition 1.1.9

If θ is a fuzzy neighbourhood system on X, then we define $\tau(\theta)$ as the family of all fuzzy sets A in X with the property that, if $A(x) > 0$, then $A \in \theta(x)$. Symbolically, $\tau(\theta) = \{A \in I^X \ / \text{ if } A(x) > 0, \text{ then } A \in \theta(x)\}$.

Theorem : 1.1.10

If θ is a fuzzy neighbourhood system on X, then $(X, \tau(\theta))$ is a fuzzy topological space.

Proof:

Clearly, 0_X and 1_X are in $\tau(\theta)$.

For each $\alpha \in \Lambda$, let $A_\alpha \in \tau(\theta)$.

Let $A = \cup\{ A_\alpha / \alpha \in \Lambda\}$

To prove $A \in \tau(\theta)$.

If $A(x) > 0$, then there exists a non empty $\Lambda_x \subset \Lambda$ such that $A_\alpha(x) > 0$ if and only if $\alpha \in \Lambda_x$.

By the definition of $\tau(\theta)$, if $\alpha \in \Lambda_x$, then $A_\alpha \in \theta(x)$

$\Rightarrow \cup\{A_\alpha / \alpha \in \Lambda_x\} \in \theta(x)$ by **N4**.

$\Rightarrow \cup\{A_\alpha / \alpha \in \Lambda\} \in \theta(x)$ by **N3**.

Hence, if $A(x) > 0$, then $A \in \theta(x)$.

$\Rightarrow A \in \tau(\theta)$.

Let $A,B \in \tau(\theta)$.

To prove $A \cap B \in \tau(\theta)$.

Suppose that $(A \cap B)(x) > 0$, then $A(x) > 0$ and $B(x) > 0$.

Hence, by the definition of $\tau(\theta)$, A and B are in $\theta(x)$.

$\Rightarrow A \cap B \in \theta(x)$.

Hence if $A \cap B > 0$, then $A \cap B \in \theta(x)$.

$\Rightarrow A \cap B \in \tau(\theta)$.

Hence $(X, \tau(\theta))$ is a fuzzy topological space.

Theorem 1.1.11

For every $x \in X$, the family $\mathcal{N}_x$ be the family of all neighbourhoods of x with respect to fuzzy topology $\tau(\theta)$ is exactly $\theta(x)$.

Proof:

Let $\mathcal{N}_x$ be the family of all neighbourhoods of x with respect to fuzzy topology $\tau(\theta)$.

To prove $\mathcal{N}_x = \theta(x)$.

If $A \in \mathcal{N}_x$, then there exists $B \in \tau(\theta)$ such that $B \subseteq A$ and $A(x) = B(x) > 0$.

By the definition $\tau(\theta)$, $B \in \theta(x)$.

Then it follows from **N3** that $A \in \theta(x)$.

Hence $\mathcal{N}_x \subseteq \theta(x)$.

If $A \in \theta(x)$, then by **N2**, $A(x) > 0$ and by **N6** there exists $B \in \theta(x)$ such that $B \subseteq A$ and $B(x) = A(x)$ and if $B(y) > 0$, then $B \in \theta(y)$.

It follows from the definition of $\tau(\theta)$ that $B \in \tau(\theta)$.

Consequently, $A \in \mathcal{N}_x$.

Hence, $\theta(x) \subseteq \mathcal{N}_x$.

Thus, $\mathcal{N}_x = \theta(x)$.

Theorem 1.1.12

Let (X, τ) be a fuzzy topological space and let θ_τ be the function $\{ (x, N_x) / x \in X \}$. Then θ_τ is a fuzzy neighbourhood system on X and $\tau(\theta_\tau) = \tau$.

Proof:

As a result of theorem 1.1.7, θ_τ satisfies conditions **N1** through **N6** and therefore θ_τ is a fuzzy neighbourhood system on X.

By theorem 1.1.10, $\tau(\theta_\tau)$ is a fuzzy topology on X. From theorem 1.1.11, we conclude that for each $x \in X$, the neighbourhoods of x with respect to $\tau(\theta_\tau)$ are exactly the members of N_x.

Since a fuzzy set is open if and only if it is a neighbourhood of each point at which it assumes a positive value, it follows that $\tau(\theta_\tau) = \tau$.

By the following example, we shall show that it is not possible to replace **N6** with the following axiom.

"If $A \in \theta(x)$ then there exists $B \in \theta(x)$ such that $B \subseteq A$ and $B(x) = A(x)$ and if $B(y) > 0$ then $A \in \theta(y)$."

Example:

Let X contain atleast two elements and choose $x \in X$. Define fuzzy sets A and B in X as follows :

$A(x) = B(x) = 3 / 4$.

$B(y) = 1 / 2$, if $y \in X - \{x\}$.

$A(y) = 3 / 4$, if $y \in X - \{x\}$.

Let the fuzzy topology on X be $\{ 0_X, 1_X, B \}$

Then $A \in N_x$, but $A \notin N_y$ for each $y \in X - \{x\}$.

SECTION 1.2

CHARACTERIZATION OF FUZZY TOPOLOGIES FROM NEIGHBOURHOOD OF FUZZY POINTS :

Definition 1.2.1

A fuzzy set in X is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is λ $(0 < \lambda \leq 1)$, we denote this fuzzy point by x_λ, where the point x is called its support.

Definition 1.2.2

The fuzzy point x_λ is said to be contained in a fuzzy set A, or to belong to A, denoted by $x_\lambda \in A$, or $x_\lambda \subseteq A$ if $\lambda \leq A(x)$.

Note:

Every fuzzy set A can be expressed as the union of all fuzzy points which belong to A.

Theorem 1.2.3:

Let A and B be fuzzy sets in X. The following statements are equivalent:

(a) $A \subseteq B$

(b) $x_\lambda \subseteq B$ for all $x_\lambda \subseteq A$

Equivalently,

(b_1) $x_\lambda \in B$ for all $x_\lambda \in A$.

Proof:

(a) $\Rightarrow$ (b) :

Assume $A \subseteq B$ and $x_\lambda \subseteq A$.

$\Rightarrow$ $A(x) \leq B(x)$ and $\lambda \leq A(x)$ for all x in X.

$\Rightarrow \lambda \leq B(x)$ for all x in X.

$\Rightarrow x_\lambda \subseteq B$ for all $x_\lambda \subseteq A$.

(b) $\Rightarrow$ (a) is obvious.

Theorem 1.2.4

Let x_λ be a fuzzy point and A and B be fuzzy sets in X. Then

(a) $x_\lambda \in A \cup B \Leftrightarrow x_\lambda \in A$ or $x_\lambda \in B$.

(b) $x_\lambda \in A \cap B \Leftrightarrow x_\lambda \in A$ or $x_\lambda \in B$.

More generaally, let $\{A_i\}$ be a family of fuzzy sets in X, then

(a′) $x_\lambda \in \bigcup_\alpha A_\alpha \Leftrightarrow x_\lambda \in A_\alpha$ for some α.

(b′) $x_\lambda \in \bigcap_\alpha A_\alpha \Rightarrow x_\lambda \in A_\alpha$ for all α.

(c′) $x_\lambda \subseteq A_\alpha$ for all $\alpha \Rightarrow x_\lambda \subseteq \bigcap_\alpha A_\alpha$

Proof:

(a) $x_\lambda \in A \cup B$

$\Leftrightarrow \lambda \leq \text{Sup}\{ A(x), B(x) \}$ for $x \in X$.

$\Leftrightarrow \lambda \leq A(x)$ or $\lambda \leq B(x)$.

$\Leftrightarrow x_\lambda \in A$ or $x_\lambda \in B$.

(b) $x_\lambda \in A \cap B$.

$\Leftrightarrow \lambda \leq \text{Inf}\{ A(x), B(x) \}$ for $x \in X$.

$\Leftrightarrow \lambda \leq A(x)$ and $\lambda \leq B(x)$.

$\Leftrightarrow x_\lambda \in A$ and $x_\lambda \in B$.

(a′) Let $x_\lambda \in \bigcup_\alpha A_\alpha$.

$\Leftrightarrow \lambda \leq \sup_{\alpha \in \Lambda} \{ A_\alpha(x) \}$.

$\Leftrightarrow \lambda \leq A_\alpha(x)$ for some α.

$\Leftrightarrow x_\lambda \in A_\alpha$ for some α.

(b′) $x_\lambda \in \bigcap_{\alpha} A_\alpha$

$\Rightarrow \lambda \leq \inf_{\alpha \in \Lambda} \{ A_\alpha(x) \}$.

$\Rightarrow \lambda \leq A_\alpha(x)$ for all α.

$\Rightarrow x_\lambda \in A_\alpha$ for all α.

(c′) $x_\lambda \subseteq A_\alpha$ for all α.

$\Rightarrow \lambda \leq A_\alpha(x)$ for all $x \in X$, for all α.

$\Rightarrow x_\lambda \subseteq \bigcap_{\alpha} A_\alpha$

Definition 1.2.5

In a fuzzy topological space (X,τ), A is a neighbourhood of a fuzzy set M if there exists an open fuzzy set B such that $M \subseteq B \subseteq A$.

Theorem 1.2.6

A fuzzy set A is open if and only if for each fuzzy set B contained in A, A is a neighbourhood of B.

Proof:

Let A be an open fuzzy set.

To prove for each fuzzy set B contained in A, A is a neighbourhood of B.

Since A is an open fuzzy set, the result is obvious.

Assume that the converse part is true.

To prove A is open.

Since $A \subseteq A$, there exists an open fuzzy set O such that $A \subseteq O \subseteq A$.

Hence $A = O$ and A is open.

Theorem 1.2.7

If U is the neighbourhood system of a fuzzy set, the finite intersection of members of U belong to U and each fuzzy set which contains a member of U belongs to U.

Proof:

If R and S are neighbourhoods of a fuzzy set A, there are open neighbourhoods R_0 and S_0 contained in R and S respectively.

Then $R \cap S$ contains the open neighbourhood $R_0 \cap S_0$ and hence a neighbourhood of A. Thus the intersection of two members of U is a member of U. Hence, intersection of any finite number of members of U is a member of U. Hence if a fuzzy set R contains a neighbourhood of A, it contains an open neighbourhood of A and consequently, itself is a neighbourhood.

Definition 1.2.8

In a fuzzy topological space (X, τ) a fuzzy set A is called a neighbourhood (briefly, nhbd) of a fuzzy point x_λ if there exists an open fuzzy set B such that $x_\lambda \subseteq B \subseteq A$.

The neighbourhood system of a fuzzy point is the family of all the neighbourhood of the fuzzy point, denoted by $\mathcal{U}_{x_\lambda}$.

Theorem 1.2.9

Let A be a fuzzy set in a fuzzy topological space (X, τ). The following statements are equivalent :

(a) A is open.

(b) for each fuzzy point $x_\lambda \subseteq A$, A is a neighbourhood of x_λ

Equivalently, for each fuzzy point $x_\lambda \in A$, A is a neighbourhood of x_λ.

Proof :

(a) $\Rightarrow$ (b) is obvious.

To prove (b) $\Rightarrow$ (a): .

Assume (b) is true.

To prove A is open.

The assumption ensures that for each $x_\lambda \subseteq A$, there exists an open fuzzy set A_{x_λ} such that $x_\lambda \subseteq A_{x_\lambda} \subseteq A$.

$$\Rightarrow \bigcup_{x_\lambda \in A} x_\lambda \subseteq \bigcup_{x_\lambda \in A} A_{x_\lambda} \subseteq A$$

$$\Rightarrow A \subseteq \bigcup_{x_\lambda \in A} A_{x_\lambda} \subseteq A$$

$$\Rightarrow A = \bigcup_{x_\lambda \in A} A_{x_\lambda}$$

$\Rightarrow$ A is open.

Theorem 1.2.10

$\mathcal{U}_{x_\lambda}$, the family of neighbourhoods of fuzzy points x_λ satisfies the following properties :

P1 : If $U \in \mathcal{U}_{x_\lambda}$ and $U \subseteq V$, then $V \in \mathcal{U}_{x_\lambda}$ and consequently, $X \in \mathcal{U}_{x_\lambda}$ for each fuzzy point x_λ in X.

P2 : Every finite intersection of members of $\mathcal{U}_{x_\lambda}$ belonge to $\mathcal{U}_{x_\lambda}$.

P3 : If $U \in \mathcal{U}_{x_\lambda}$, then $x_\lambda \subseteq U$.

P4 : If $U \in \mathcal{U}_{x_\lambda}$, then a fuzzy set $V \in \mathcal{U}_{x_\lambda}$ exists such that $U \in \mathcal{U}_{x_\lambda}$ for each fuzzy point $x_\mu \subseteq V$.

Proof :

To prove **P1 :**

Given $U \in \mathcal{U}_{x_\lambda}$ and $U \subseteq V$.

To prove $V \in \mathcal{U}_{x_\lambda}$.

$U \in \mathcal{U}_{x_\lambda} \Rightarrow$ U is a neighbourhood of x_λ.

$\Rightarrow$ there exists an open fuzzy set A such that $x_\lambda \subseteq A \subseteq U$.

Since $U \subseteq V$, we get $x_\lambda \subseteq A \subseteq U \subseteq V$. That is, $x_\lambda \subseteq A \subseteq V$.

$\Rightarrow$ V is a neighbourhood of x_λ.

$\Rightarrow V \in \mathcal{U}_{x_\lambda}$.

Also $X \in \mathcal{U}_{x_\lambda}$ for each fuzzy point x_λ in X.

To prove **P2 :**

Let $U_1, U_2, \ldots\ldots\ldots\ldots U_n \in \mathcal{U}_{x_\lambda}$.

To prove $\bigcap_{i=1}^{n} U_i \in \mathcal{U}_{x_\lambda}$

$U_1, U_2, \ldots\ldots\ldots\ldots U_n \in \mathcal{U}_{x_\lambda}$.

$\Rightarrow$ there exists open fuzzy sets $A_1, A_2, \ldots\ldots\ldots\ldots\ldots A_n$ such that

$x_\lambda \subseteq A_i \subseteq U_i$ for $i = 1, 2, \ldots.n$.

$\Rightarrow x_\lambda \subseteq \bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{n} U_i$.

Since $A_1, A_2, \ldots\ldots\ldots.A_n$ are open fuzzy sets, $\bigcap_{i=1}^{n} A_i$ is an open fuzzy set.

This implies that $\bigcap_{i=1}^{n} U_i \in \mathcal{U}_{x_\lambda}$.

To prove **P3** :

Let $U \in \mathcal{U}_{x_\lambda}$.

Then by the definition of neighbourhood of a fuzzy point, we get $x_\lambda \subseteq U$.

To prove **P4** :

Let $U \in \mathcal{U}_{x_\lambda}$.

Then there exists an open fuzzy set A such that $x_\lambda \subseteq A \subseteq U$.

$\Rightarrow A \in \mathcal{U}_{x_\lambda}$.

Choose $A = V$.

Then $V \in \mathcal{U}_{x_\lambda}$.

Let $x_\mu \subseteq V$. Then $x_\mu \subseteq V \subseteq U$.

$\Rightarrow$ U is a neighbourhood of x_μ.

$\Rightarrow U \in \mathcal{U}_{x_\lambda}$.

The following theorem is a consequence of properties **P3** and **P4** :

Theorem 1.2.11

If $U \in \mathcal{U}_{x_\lambda}$ and $x_\mu \subseteq x_\lambda$, then $U \in \mathcal{U}_{x_\lambda}$. Conversely, if $U \in \mathcal{U}_{x_\lambda}$ for each fuzzy point $x_\mu \subseteq x_\lambda$, then $U \in \mathcal{U}_{x_\lambda}$.

Proof :

Let $U \in \mathcal{U}_{x_\lambda}$.

Then by **P4** there exists a fuzzy set $V \in \mathcal{U}_{x_\lambda}$ such that $U \in \mathcal{U}_{x_\lambda}$ for each fuzzy point $x_\mu \in V$.

This implies that $x_\lambda \subseteq V$, by **P3**.

Thus, in particular that $U \in \mathcal{U}_{x_\lambda}$ for each fuzzy point $x_\mu \subseteq x_\lambda$.

To prove the converse :

Given $U \in \mathcal{U}_{x_\lambda}$ for each fuzzy point $x_\mu \subseteq x_\lambda$.

To prove $U \in \mathcal{U}_{x_\lambda}$.

$U \in \mathcal{U}_{x_\lambda}$

$\Rightarrow$ for each $x_\mu \subseteq x_\lambda$ there exists an open fuzzy set A_{x_μ} such that $x_\mu \subseteq A_{x_\mu} \subseteq U$.

$$\Rightarrow \bigcup_{x_\mu \subseteq x_\lambda} x_\mu \subseteq \bigcup_{x_\mu \subseteq x_\lambda} A_{x_\mu} \subseteq U.$$

$$\Rightarrow x_\lambda \subseteq \bigcup_{x_\mu \subseteq x_\lambda} A_{x_\mu} \subseteq U.$$

Since A_{x_μ} is open for each x_μ, $\bigcup_{x_\mu \subseteq x_\lambda} A_{x_\mu}$ is open.

Therefore, $U \in \mathcal{U}_{x_\lambda}$.

Note :

The converse part in the above theorem is treated as the fifth property **P5** of the neighbourhood system $\mathcal{U}_{x_\lambda}$.

An important consequence of **P5** is expressed as follows:

Theorem 1.2.12

If $U \in \mathcal{U}_{x_\lambda}$ for each fuzzy point $x_\lambda \in U$, then $U \in \mathcal{U}_{x_\lambda}$ for each fuzzy point $x_\lambda \subseteq U$.

Proof :

Let $x_\lambda \subseteq U$ be a fuzzy point.

By assumption, it follows in particular that $U \in \mathcal{U}_{x_\lambda}$ for each $x_\mu \subseteq x_\lambda$ and by **P5** we have that $U \in \mathcal{U}_{x_\lambda}$, for each fuzzy point $x_\lambda \subseteq U$.

Note :

Let X be the real line R and $\mathcal{F}(X)$ denote the totality of fuzzy sets in X. Symbolically, $\mathcal{F}(X) = \{A : R \to [0, 1]\}$.

Now we shall establish the following characterization theorem.

Theorem 1.2.13

If with each fuzzy point x_λ in a set X is associated a family $\mathcal{U}_{x_\lambda}$ of fuzzy sets of X in such a way that the properties **P1 - P5** in theorem 1.2.10 and in theorem 1.2.11 are satisfied, then one and only one fuzzy topology for X exists in which, for each fuzzy point x_λ of X, $\mathcal{U}_{x_\lambda}$ is the neighbourhood system of x_λ.

Proof :

$$\text{Let } \tau = \left\{A \in F(X) \ / \ A \in \mathcal{U}_{x_\lambda} \ \text{ for each } x_\lambda \subseteq A\right\}.$$

Now we will prove that τ is a fuzzy topology for X.

Obviously, the empty fuzzy set $\phi \in \tau$. As a consequence of **P1** of theorem 1.2.10 it follows that $X \in \mathcal{U}_{x_\lambda}$ for all $x_\lambda \subseteq X$.

$\Rightarrow X \in \tau$.

Let $\{A_i\}_i$ be a family of members of τ.

To prove $\bigcup\limits_i A_i \in \tau$.

If $x_\lambda \in \bigcup\limits_i A_i$, then there exists one i_0 such that $x_\lambda \in A_{i_0}$ by theorem 1.2.4(a′).

Since $A_{i_0} \in \tau$, we have $A_{i_0} \in \mathcal{U}_{x_\lambda}$.

Hence by property **P1** in the theorem 1.2.10, we get $\bigcup\limits_i A_i \in \mathcal{U}_{x_\lambda}$.

Since this is true for every $x_\lambda \in \bigcup\limits_i A_i$ it follows that $\bigcup\limits_i A_i \in \tau$.

Consider a family $A_1, A_2, \ldots\ldots A_n$ of τ.

Suppose $x_\lambda \in A_1 \cap A_2 \cap \ldots\ldots\ldots\ldots \cap A_n$.

Beacuse of therorem 1.2.4 (b), we have that $x_\lambda \subseteq A_i$ for each $i = 1, 2, \ldots n$.

$\Rightarrow A_1 \cap A_2 \cap \ldots\ldots \cap A_n \in \mathcal{U}_{x_\lambda}$.

Since this is true for every $x_\lambda \subseteq A_1 \cap A_2 \cap \ldots\ldots \cap A_n$, we get $\bigcap\limits_{i=1}^{n} A_i \in \tau$.

Thus the family τ defines in X is a fuzzy topology.

In this τ, we shall now prove that the member of $\mathcal{U}_{x_\lambda}$ are the neighbourhoods of the fuzzy point x_λ.

Let U be a neighbourhood of x_λ in the above mentioned fuzzy topology τ.

Then there exists an open fuzzy set V such that $x_\lambda \subseteq V \subseteq U$.

Since $V \in \tau$, we have that $V \in \mathcal{U}_{x_\lambda}$ and thus by **P1** of the theorem 1.2.10, we get

$U \in \mathcal{U}_{x_\lambda}$.

Conversely, let $U \in \mathcal{U}_{x_\lambda}$.

We shall now prove that U is a neighbourhood of x_λ in the specified fuzzy topology.

Set $\Omega = \left\{x_\mu \;/\; x_\mu \text{ is a fuzzy point in X such that } U \in \mathcal{U}_{x_\lambda}\right\}$.

Since $x_\lambda \in \Omega$, we have that Ω is non empty.

Let $V = \bigcup\limits_{x_\mu \in \Omega} x_\mu$.

Then $x_\lambda \subseteq V$.

From the fact that $U \in \mathcal{U}_{x_\lambda}$ for all $x_\mu \in \Omega$, it follows that by **P3** of theorem 1.2.10

that $x_\mu \subseteq U$ for all $x_\mu \in \Omega$ and hence $V = \bigcup\limits_{x_\mu \in \Omega} x_\mu \subseteq U$.

Let us now prove that V is open, which means that $V \in \mathcal{U}_{\bar{x}_{\bar{\lambda}}}$ for all $\bar{x}_{\bar{\lambda}} \in V$.

Let the fuzzy point $\bar{x}_{\bar{\lambda}} \in V = \bigcup\limits_{x_\mu \in \Omega} x_\mu$ be fixed.

We have that $\bar{\lambda} < \operatorname{Sup}\limits_{x_\mu \in \Omega} \bar{x}_{\bar{\mu}}$ and hence there exists a fuzzy point $\bar{x}_{\bar{\mu}} \in \Omega$ such that

$\bar{\lambda} < \bar{x}_{\bar{\mu}} \;\Rightarrow\; \bar{x}_{\bar{\lambda}} \in \bar{x}_{\bar{\mu}}$.

Since $U \in \mathcal{U}_{\overline{x}_{\overline{\mu}}}$, by the definition of Ω, **P4** of theorem 1.2.10 implies that a fuzzy set

$W \in \mathcal{U}_{\overline{x}_{\overline{\mu}}}$ exists such that $U \in \mathcal{U}_{x_\gamma}$ for each fuzzy point $x_\gamma \in W$.

That is, $x_\gamma \in \Omega$ for each $x_\gamma \in W$.

It follows that $\bigcup_{x_\gamma \in W} x_\gamma \subseteq \bigcup_{x_\mu \in \Omega} x_\mu$. That is, $W \subseteq V$.

Thus **P1** of theorem 1.2.10 ensures that $W \in \mathcal{U}_{\overline{x}_{\overline{\mu}}}$ and from **P3** and **P4** of theorem 1.2.10 we have that $V \in \mathcal{U}_{\overline{x}_{\overline{\lambda}}}$.

We have thus proved that V is open.

Therefore U which contains V, is an neighbourhood of x_λ in the fuzzy topology.

Finally it is clear by the characterization of the open fuzzy sets given in theorem 1.2.9 that, if τ' is a fuzzy topolgy for X, where $\mathcal{U}_{x_\lambda}$ is again the neighbourhood system of x_λ for each fuzzy point x_λ in X, then τ' coincides with τ.

Hence the proof.

BASES AND FUNDAMENTAL NEIGHBOURHOOD SYSTEMS

Definition 1.2.14

A family $\{A_i\}$ of fuzzy sets in X is called a cover of a fuzzy set B if $B \subseteq \bigcup_i A_i$.

Remark 1.2.15

If $\{A_i\}$ is a family of fuzzy sets in X and B is a fuzzy set in X, saying that $B \subseteq \bigcup_i A_i$ is the same as saying that for each $x_\lambda \in B$, there is an i such that $x_\lambda \in A_i$.

Definition 1.2.16

If (X, τ) is a fuzzy topological space, an arbitrary family $\mathcal{B}$ of nonempty fuzzy sets in X is a base for τ if each member of τ can be expressed as the union of members of $\mathcal{B}$.

We can also say that for each open fuzzy set A of τ and for each fuzzy point $x_\lambda \in A$, there exist $B \in \mathcal{B}$ such that $x_\lambda \in B \subseteq A$.

In particular, the empty fuzzy set is the union of the empty family of elements $\mathcal{B}$.

Theorem 1.2.17

Let $\mathcal{B}$ be a base for the fuzzy topological space (X, τ), then the following properties are satisfied:

(a) For each fuzzy point $x_\lambda \in X$, there exists an element $B \in \mathcal{B}$ such that $x_\lambda \in B$.

(b) If B_1 and B_2 are two elements of $\mathcal{B}$ such that $B_1 \cap B_2 \neq \phi$ and $x_\lambda \in B_1 \cap B_2$, then there exists an element $B \in \mathcal{B}$ such that $x_\lambda \in B \subseteq B_1 \cap B_2$.

Remark 1.2.18

The following is equivalent to (a).

(a′) $X = \bigcup_{B \in \mathcal{B}} B$ that is, the family $\mathcal{B}$ is a cover of X.

Note :

The properties (a) and (b) of theorem 1.2.17 can be considered as axioms for a fuzzy topology in the sense specified as follows :

Theorem 1.2.19

Let $\mathcal{B}$ be a family of non empty fuzzy sets in X for which properties (a) and (b) of theorem 1.2.17 hold. One and only one fuzzy topology exists in which $\mathcal{B}$ is a base.

Proof :

It is easy to prove that $\tau = \{A \in \mathcal{F}(X) \,/\, A \text{ is the union of members of } \mathcal{B}\}$ is the required fuzzy topology for X.

Definition 1.2.20

Let (X, τ) be an fuzzy topological space and x_λ is a fuzzy point in X. A family $\mathcal{G}(x_\lambda)$ of neighbourhoods of x_λ is called a fundamental neighbourhood system of x_λ if for each neighbourhood U of x_λ, there is a member V of $\mathcal{G}(x_\lambda)$ such that $V \subseteq U$.

Theorem 1.2.21

Let $\mathcal{B}$ be a family of nonempty open fuzzy sets in an fuzzy topological space (X, τ). If for each fuzzy point $x_\lambda \in X$, the family of the members of $\mathcal{B}$ that contain x_λ is a fundamental neighbourhood system of x_λ, then $\mathcal{B}$ is a base for the fuzzy topology τ.

Note :

In an ordinary topology, the converse of the previous theorem is also true.

Theorem 1.2.22

In a fuzzy topological space (X, τ) however, the members of a fundamental neighbourhood system $\mathcal{G}\,(x_\lambda)$ of a fuzzy point x_λ in X satisfy the following properties:

(a1) Every finite intersection of members of $\mathcal{G}\,(x_\lambda)$ contains a member of $\mathcal{G}\,(x_\lambda)$.

(b1) If $U \in \mathcal{G}\,(x_\lambda)$ then $x_\lambda \subseteq U$.

(c1) If $U \in \mathcal{G}\,(x_\lambda)$, then there exists $V \in \mathcal{G}\,(x_\lambda)$ such that for each $x_\mu \in V$ there is a $W \in \mathcal{G}\,(x_\mu)$ such that $W \subseteq U$.

(d1) Given a fuzzy set U in X, if for each $x_\mu \subseteq x_\lambda$ there exists $V_{x_\mu} \in \mathcal{G}\,(x_\mu)$ such that $V_{x_\mu} \subseteq U$, then there exists $V \in \mathcal{G}\,(x_\lambda)$ such that $V \subseteq U$.

Properties (a1) - (d1) derive one by one from properties **P2 - P5** of the neighbourhoods, respectively and they can be considered as axioms for a fuzzy topology as stated as follows :

Theorem 1.2.23

If with each fuzzy point x_λ in a set X is associated a family $\mathcal{G}\,(x_\lambda)$ of fuzzy sets in X in such a way that conditions (a1) - (d1) are satisfied, then there is one and only one fuzzy topology for X in which $\mathcal{G}(x_\lambda)$ is a fundamental neighbourhood system of x_λ, for each fixed fuzzy point x_λ in X.

Note :

We note explicitly that the converse of the theorem 1.2.23 is not true in general. In order to show this, we shall use the following example :

Let X be the set R of real numbers. Consider, for each fuzzy point x_λ in X, the family of all the open fuzzy intervals containing x_λ, that is, the set $\mathcal{G}(x_\lambda) = \{ [a, b]_\lambda \text{ such that } x_\lambda \subseteq [a, b]_\lambda \}$.

We can show that the family $\mathcal{G}(x_\lambda)$ doesnot satisfy property (d1) of the fundamental neighbourhood system and on the otherhand, we can easily see that the family of all the open fuzzy intervals fulfills conditions (a) and (b) of the theorem 1.2.19 and thus it is a base for a fuzzy topology in X.

Clearly, in this fuzzy topology the family $\mathcal{G}(x_\lambda)$ cannot be an fundamental neighbourhood system of x_λ.

Chapter II

CHAPTER II

LATTICE PROPERTIES OF FUZZY NEIGHBOURHOOD SPACES

In the First Section, we define the Pu - neighbourhood of a fuzzy point and study the properties of neighbourhood system at a fuzzy point. We can characterize an open fuzzy set in terms of Pu - neighbourhoods. Further, we define adherence point and accumulation point of a fuzzy set. We can characterize closure of a fuzzy set in terms of accumulation points.

In the Second Section, Lattice properties of fuzzy neighbourhood spaces are obtained.

SECTION 2.1

PU - NEIGHBOURHOOD STRUCTURE OF A FUZZY POINT

Definition 2.1.1

A fuzzy point x_λ is called quasi coinicdent with a fuzzy set A if $\lambda + A(x) > 1$ or $\lambda > A'(x)$. This relation is denoted by $x_\lambda \; q \; A$.

Definition 2.1.2

Let (X, τ) be a fuzzy topological space. A fuzzy set A in X is called a Pu - neighbourhood of a fuzzy point x_λ if there exists $B \in \tau$ such that $x_\lambda \; q \; B$ and $B \subseteq A$.

Note :

The collection of Pu-neighbourhoods of a fuzzy point x_λ is denoted by $\mathcal{P}_{x_\lambda}(\tau)$.

Theorem 2.1.3

Let A and B be two fuzzy sets. Then $A \subseteq B$ if and only if A and B' are not quasi coincident. Particularly, $x_\lambda \in A$ if and only if x_λ is not quasi coincident with A'.

Proof :

Let A and B be two fuzzy sets.

Since $A \subseteq B \Leftrightarrow A(x) \leq B(x)\ \forall\ x$ in X,

$$A(x) + B'(x) \leq A(x) + 1 - B(x) \leq 1.$$

Hence $A \subseteq B$ if and only if $A \not{q} B'$

Hence the result.

Theorem 2.1.4

Let $\mathcal{P}_{x_\lambda}(\tau)$ be the family of Pu - neighbourhoods of a fuzzy point x_λ in (X, τ).

Then we have ,

(1) If $U \in \mathcal{P}_{x_\lambda}(\tau)$, then x_λ is quasi coincident with U.

(2) If U, $V \in \mathcal{P}_{x_\lambda}(\tau)$, then $U \cap V \in \mathcal{P}_{x_\lambda}(\tau)$.

(3) If $U \in \mathcal{P}_{x_\lambda}(\tau)$ and $U \subseteq V$, then $V \in \mathcal{P}_{x_\lambda}(\tau)$.

(4) If $U \in \mathcal{P}_{x_\lambda}(\tau)$, then there exists $V \in \mathcal{P}_{x_\lambda}(\tau)$ such that $V \subseteq U$ and $V \in \mathcal{P}_{x_\mu}(\tau)$ for every fuzzy point x_μ which is quasi coincident with V.

Conversely, for each fuzzy point x_λ in X, if $\mathcal{P}_{x_\lambda}(\tau)$ is a family of fuzzy sets in X satisfying the above conditions (1) - (3), then the family τ of all the fuzzy sets U, such that $U \in \mathcal{P}_{x_\lambda}(\tau)$ when ever x_λ q U is a fuzzy topology for X. If, in addition, $\mathcal{P}_{x_\lambda}(\tau)$ satisfies the condition (4) metioned above, then $\mathcal{P}_{x_\lambda}(\tau)$ is exactly the Pu- neighbourhood system of x_λ relative to the fuzzy topology τ.

Proof :

Let $\mathcal{P}_{x_\lambda}(\tau)$ be the family of Pu - neighbourhoods of a fuzzy point x_λ in (X, τ).

(1) To prove :

x_λ is quasi coincident with U if $U \in \mathcal{P}_{x_\lambda}(\tau)$.

$U \in \mathcal{P}_{x_\lambda}(\tau)$

$\Rightarrow$ there exists $B \in \tau$ such that x_λ q B and $B \subseteq U$.

Therefore x_λ q U.

(2) Let $U, V \in \mathcal{P}_{x_\lambda}(\tau)$

To prove :

$U \cap V \in \mathcal{P}_{x_\lambda}(\tau)$.

$U \in \mathcal{P}_{x_\lambda}(\tau)$

$\Rightarrow$ there exists $B_1 \in \tau$ such that x_λ q B_1 and $B_1 \subseteq U$.

$V \in \mathcal{P}_{x_\lambda}(\tau)$

$\Rightarrow$ there exists $B_2 \in \tau$ such that x_λ q B_2 and $B_2 \subseteq V$.

$\Rightarrow B_1 \cap B_2 \in \tau$ such that $x_\lambda \; q \, B_1 \cap B_2$ and $B_1 \cap B_2 \subseteq U \cap V$.

$\Rightarrow U \cap V \in \mathcal{P}_{x_\lambda}(\tau)$.

(3) Let $U \in \mathcal{P}_{x_\lambda}(\tau)$ and $U \subseteq V$.

To prove $V \in \mathcal{P}_{x_\lambda}(\tau)$.

$U \in \mathcal{P}_{x_\lambda}(\tau)$

$\Rightarrow$ there exists $B \in \tau$ such that $x_\lambda \; q \, B$ and $B \subseteq U$.

But $U \subseteq V$. Therefore there exists $B \in \tau$ such that $x_\lambda \; q \, B$ and $B \subseteq V$

$\Rightarrow V \in \mathcal{P}_{x_\lambda}(\tau)$.

(4) Let $U \in \mathcal{P}_{x_\lambda}(\tau)$

To prove (4):

$U \in \mathcal{P}_{x_\lambda}(\tau)$

$\Rightarrow$ there exists $V \in \tau$ such that $x_\lambda \; q \, V$ and $V \subseteq U$

For a fuzzy point x_μ, $x_\mu \; q \, V \Rightarrow V \in \mathcal{P}_{x_\mu}(\tau)$ since $V \in \tau$.

Conversely we assume that, for each fuzzy point $x_\lambda \in X$, $\mathcal{P}_{x_\lambda}(\tau)$ is a collection of fuzzy sets satisfying (1), (2), (3).

Let $\tau = \left\{U \in I^X \;/\; U \in \mathcal{P}_{x_\lambda}(\tau) \text{ whenever } x_\lambda \; q \, U\right\}$

(i) Obvioulsy 0_X, $1_X \in \tau$

(ii) Let $U, V \in \tau$.

To prove $U \cap V \in \tau$.

$U, V \in \tau$

$\Rightarrow U, V \in \mathcal{P}_{x_\lambda}(\tau)$ whenever $x_\lambda \, q \, U$ and $x_\lambda \, q \, V$.

$\Rightarrow U \cap V \in \mathcal{P}_{x_\lambda}(\tau)$ whenever $x_\lambda \, q \, (U \cap V)$.

$\Rightarrow U \cap V \in \tau$.

(iii) Let $\{U_\alpha\}_{\alpha \in \Lambda} \in \tau$

To prove $\underset{\alpha \in \Lambda}{\cup} (U_\alpha) \in \tau$

$\{U_\alpha\}_{\alpha \in \Lambda} \in \tau$

$\Rightarrow U_\alpha \in \mathcal{P}_{x_\lambda}(\tau)$ and $x_\lambda \, q \, U_\alpha$ for all $\alpha \in \Lambda$.

$\Rightarrow \underset{\alpha \in \Lambda}{\cup} (U_\alpha) \in \mathcal{P}_{x_\lambda}(\tau)$ and $x_\lambda \, q \, \underset{\alpha \in \Lambda}{\cup} (U_\alpha)$

$\Rightarrow \underset{\alpha \in \Lambda}{\cup} (U_\alpha) \in \mathcal{P}_{x_\lambda}(\tau)$.

If $\mathcal{P}_{x_\lambda}(\tau)$ satisfies the condition (4), then $\mathcal{P}_{x_\lambda}(\tau)$ is exactly the Pu - neighbourhood system of x_λ relative to the fuzzy topology τ.

Theorem 2.1.5

Let $\{A_\alpha\}$ be a family of fuzzy sets in X. Then a fuzzy point x_λ is quasi coincident with $\cup\mathcal{A}$ if and only if there exists some $A_\alpha \in \mathcal{A}$ such that $x_\lambda \, q \, A_\alpha$.

Proof :

If $x_\lambda \, q \, A_\alpha$, then $x_\lambda \, q \cup A_\alpha$ is true. The necessary condition is obvious.

Theorem 2.1.6

A subfamily $\mathcal{B}$ of a fuzzy topology for X is a base for τ if and only if for each fuzzy point x_λ in (X, τ) and for each open Pu - neighbourhood A of x_λ, there exists a member $B \in \mathcal{B}$ such that $x_\lambda \, q \, B \subseteq A$.

Proof :

The necessary condition follows directly from the definition of a base and the necessary condition of theorem 2.1.5. We shall now show the sufficient part.

If $\mathcal{B}$ is not a base for τ, then there exists a member, $A \in \tau$, such that $G = \{B \in \mathcal{B} \,/\, B \subseteq A\} \neq A$ and hence there is an x such that $G(x) < A(x)$.

Let $\lambda = 1 - G(x)$, which is obviously positive. We obtain a fuzzy point x_λ. Since $A(x) + \lambda > G(x) + \lambda = 1$, we get $x_\lambda \, q \, A$.

But since any member $B \in \mathcal{B}$ whcih is contained in A is cotained in G, we have $B(x) + \lambda \leq G(x) + \lambda = 1$. That is, x_λ is not quasi coincident with B.

This contradicts the assumption.

Definition 2.1.7

Let A be a fuzzy set in (X, τ). The union of all the τ - open sets contained in A is called the interior of A, denoted by A^o or by $\mathrm{Int}_\tau A$.

Note :

(i) A^o is the largest open set contained in A and $\left(A^o\right)^o = A^o$.

(ii) The fuzzy set A is open if and only if $A = A^{o}$.

Definition 2.1.8

The intersection of all the τ-closed sets containing A is called the closure of A, denoted by $Cl_{\tau}A$ or simply Cl-A.

Note :

(i) Obviously Cl-A is the smallest τ-closed set containing A and

(ii) A fuzzy set A is closed if and only if A = Cl-A.

Theorem 2.1.9

A fuzzy point $x_{\lambda} \in A^{o}$ if and only if x_{λ} has a neighbourhood contained in A.

Theorem 2.1.10

A fuzzy point $x_{\lambda} \in$ Cl-A if and only if each Pu - neighbourhood of x_{λ} is quasi coincident with A.

Proof :

$x_{\lambda} \in$ Cl-A if and only if for every closed set $F \supset A$, $x_{\lambda} \in F$ or $F(x) \geq \lambda$.

By taking complement, this fact can be stated as follows :

$x_{\lambda} \in$ Cl-A if and only if, for every open set $B \subseteq A'$, $B(x) \leq 1 - \lambda$.

In otherwords, for every open set B satisfying $B(x) > 1 - \lambda$, B is not contained in A'.

From the theorem 2.1.3 B is not contained in A' if and only if B is quasi coincident with $(A')' = A$.

We have thus proved that $x_\lambda \in$ Cl-A if and only if every open Pu - neighbourhood B of x_λ is quasi coincident with A.

Definition 2.1.11

A fuzzy point x_λ is called an adherence point of a fuzzy set A if every Pu - neighbourhood of x_λ is quasi coincident with A.

Theorem 2.1.12

A° is the union of all the adherence points of A.

Definition 2.1.13

A fuzzy point x_λ is called a boundary point of a fuzzy set A if $x_\lambda \in$ Cl-A $\cap$ Cl-A′. The union of all the boundary points of A is called a boundary of A denoted by b(A).

Note :

It is clear that b(A) = Cl-A $\cap$ Cl-A′.

Theorem 2.1.14

Cl-A $\supset$ A $\cup$ b(A), where the inclusion symbol cannot be replaced by an equlaity.

It should be noticed that in general topology, we have Cl-A = A $\cup$ b(A). This is not true in the case of fuzzy topology as shown by the following example.

Example :

Let $x \in X$, $\tau = \{X, \phi, x_{1/2}\}$ and $A = x_{2/3}$ and $x_\lambda = x_{3/4}$.

Then the Pu - neighbourhood of x_λ in (X, τ) are X and $x_{1/2}$, which are all quasi coincident with A.

Hence by theorem 2.1.10 $x_\lambda \in$ Cl-A. On the otherhand $x_\lambda \notin A$ and the Pu - neighbourhood of $\{x_{1/2}\}$ is not quasi coincident with A'.

That is, $x_\lambda \notin b(A)$ and hence $x_\lambda \notin A \cup b(A)$.

Definition 2.1.15

A fuzzy point x_λ is called an accumulation point of a fuzzy set A if

(i) x_λ is an adherence point of A.

(ii) every Pu - neighbourhood of x_λ and A are quasi coincident at some point different from supp (x_λ), whenever $x_\lambda \in A$.

The union of all the accumulation points of A is called the derived set of A, denoted by A^d. It is evident that $A^d \subseteq$ Cl-A.

Theorem 2.1.16

Cl-A = $A \cup A^d$, where A^d is the dervied set of A.

Proof :

Let $\Omega = \{ x_\lambda \, / \, x_\lambda$ is an adherence point of A $\}$.

Then, from theorem 2.1.10 Cl-A $= \cup\, \Omega$.

On the otherhand, $x_\lambda \in \Omega$ is either $x_\lambda \in A$ or $x_\lambda \notin A$.

For the latter case, by definition of accumulation point, we get $x_\lambda \in A^d$.

Hence Cl-A $= \cup\, \Omega \subseteq A \cup A^d$.

The inverse inclusion relation is obvious.

Theorem 2.1.17

A fuzzy set A is closed if and only if A contains all the accumulation points of A.

Proof:

We know A is closed if and only if A = Cl-A. Thus, by the above theorem, we obtain the proof.

Theorem 2.1.18

In (X,τ), let $A = x_\lambda$, then

(1) for $y \neq x$, $(Cl\text{-}A)(y) = A^d(y)$.

(2) if $(Cl\text{-}A)(x) > \lambda$, $(Cl\text{-}A)(x) = A^d(x)$.

(3) $(Cl\text{-}A)(x) = \lambda$ if and only if $A^d(x) = 0$.

Proof :

Since $Cl\text{-}A = A \cup A^d$, the conclusions (1) and (2) and the sufficient part of (3) are true.

To prove the necessary part of (3).

Let $A(x) = \lambda$. We claim that any fuzzy point x_μ is not an accumulation point of A and hence $A^d(x) = 0$.

In fact, when $\mu > \lambda$, $x_\mu \notin A$ and hence $x_\mu \notin A^d$.

When $\mu \leq \lambda$, $x_\mu \in A$.

But then and Pu - neighbourhood of x_μ and A cannot be quasi coincident at a point different from x. Therefore, x_μ is not an accumulation point of A.

Hence the proof.

Theorem 2.1.19

In (X, τ), let $A = \{x_\lambda\}$, then

(1) When $A^d(x) > 0$, $A^d = A$ is closed.

(2) When $A^d(x) = 0$, A^d is closed if and only if there exists an open set B^* such that $B^*(x) = 1$ and for $y \neq x$, $B^*(y) = (Cl\text{-}A)'(y) = \left(A^d\right)'(y)$.

(3) $A^d(x) = 0$ if and only if there exists an open set B such that $B(x) = 1 - \lambda$.

Proof :

(1) It is obvious, by theorem 2.1.18

(2) A^d is closed if and only if $\left(A^d\right)'$ is open. When $A^d(x) = 0$, in view of (1) and (3) of theorem 2.1.18 it follows that $\left(A^d\right)'$ is the required B^*. The proof is thus completed.

(3) From (3) of theorem 2.1.18 $A^d(x) = 0$ if and only if $(Cl\text{-}A)(x) = \lambda = A(x)$. This implies that there exists a closed F such that $F(x) = \lambda$, or equivalently, there exists an open B such that $B(x) = 1\text{-}\lambda$.

Hence the proof.

Theorem 2.1.20

The derived set of each fuzzy set is closed if and only if the derived set of each point is closed.

SECTION 2.2

LATTICE PROPERTIES OF FUZZY NEIGHBOURHOOD SPACES

In this section, fuzzy set x_λ where $0 < \lambda \leq 1$ is referred to as fuzzy singleton. The fuzzy set x_λ is called fuzzy point if $0 < \lambda < 1$.

The set of all fuzzy singletons on X and set of all fuzzy points in X are denoted by $\zeta(x)$ and $\mathcal{P}(x)$ respectively.

Definition 2.2.1

A fuzzy point $x_\zeta \,\bar{\in}\, A$ if and only $\zeta < A(x)$, where A is a fuzzy set in X.

Definition 2.2.2

Let (X, τ) be a fuzzy topological space. A fuzzy set A in X is called a Kerre neighbourhood of a fuzzy singleton x_λ if and only if there exists $O \in \tau$ such that $x_\lambda \subseteq O$ and $O \subseteq A$.

Definition 2.2.3

Let (X, τ) be a fuzzy topological space. A fuzzy set A in X is called a Mashhour neighbourhood of a fuzzy point x_λ if and only if there exists a $O \in \tau$ such that $x_\lambda \,\bar{\in}\, O$ and $O \subseteq A$.

Note :

The collection of Kerre neighbourhoods of a fuzzy singleton x_λ is denoted as $\mathcal{K}_{x_\lambda}(\tau)$.

The collection of Mashhour neighbourhoods of a fuzzy point x_λ is denoted as $\mathcal{M}_{x_\lambda}(\tau)$.

As outlined in the article published by ***E.E. Kerre and P.L. Ottoy***, the approaches of ***Mashhour and Pu*** lead to complete formal parallelism with ordinary topology, That is, the characteristic properties of the neighbourhood concept hold and moreover they are strong enough to characterize a fuzzy Chang topology. In the approach of Kerre, the neighbourhood system satisfies the characteristic properties: However, these properties only define a base for a fuzzy Chang topology. Further on, we will make use of the following sets. Let (X, τ) be a Chang fuzzy topological space and x ∈ X, then we put,

(1) $\mathcal{K}_x(\tau)=\left\{\mathcal{K}_{x_\lambda}(\tau)\,/\,x_\lambda\in\zeta(x) \text{ and } \operatorname{supp} x_\lambda=\{x\}\right\}$
$=\left\{\mathcal{K}_{x_\varepsilon}(\tau)\,/\,\varepsilon\in(0,1]\right\}$

(2) $\mathcal{M}_x(\tau)=\left\{\mathcal{M}_{x_\lambda}(\tau)\,/\,x_\lambda\in\mathcal{P}(x) \text{ and } \operatorname{supp} x_\lambda=\{x\}\right\}$
$=\left\{\mathcal{M}_{x_\varepsilon}(\tau)\,/\,\varepsilon\in(0,1)\right\}$

(3) $\mathcal{P}_x(\tau)=\left\{\mathcal{P}_{x_\lambda}(\tau)\,/\,x_\lambda\in\zeta(x) \text{ and } \operatorname{supp} x_\lambda=\{x\}\right\}$
$=\left\{\mathcal{P}_{x_\varepsilon}(\tau)\,/\,\varepsilon\in(0,1]\right\}$

Next we shall give some definitions from lattice theory :

Definition 2.2.4

Let P be a non empty set. A partially order relation in P is a relation symbolized by $\leq$ and have the following properties :

1. $x\leq x, \quad \forall x$ (Reflexivity)
2. $x\leq y$ and $y\leq x \Rightarrow x=y$ (Antisymmtery)
3. $x\leq y$ and $y\leq z \Rightarrow x\leq z$ (Transtivity)

Definition 2.2.5

A partially order relation having the following property that "Any two elements are comparable" is called a totally order relation.

Definition 2.2.6

A nonempty set P in which there is defined a partial order relation is called a partially ordered set.

Definition 2.2.7

A lattice is a partially ordered set L in which each pair of elements has a greatest lower bound and least upper bound.

Definition 2.2.8

If x and y are two elements in a lattice $(L, \leq)$, we denote their greatest lower bound by $(x \wedge y)$ and is called a meet and least upper bound by $(x \vee y)$ and is called a join.

Definition 2.2.9

A subset $K \subset L$ is called a sublattice if $(K, \subseteq)$ itself forms a lattice.

Definition 2.2.10

A lattice is called a complete lattice if every non-empty subset has a greatest lower bound (glb) and a least upper bound (lub).

Definition 2.2.11

A lattice $(L, \leq)$ is called conditionally complete if and only if every nonempty and bounded subset of $(L,\leq)$ has a greatest lower bound (glb) and a least upper bound (lub).

Definition 2.2.12

Let $(L, \leq)$ be a complete lattice and $(K, \leq)$ be a sublattice of $(L, \leq)$. $(K, \leq)$ is called a conditionally closed sublattice of $(L, \leq)$ if for every nonempty subset H of K that is bounded in $(K, \leq)$, $\text{Inf}_L H$, $\text{Sup}_L H$ belong to K.

In this case $(K, \leq)$ constitute a conditionally complete lattice.

Definition 2.2.13

$(K, \leq)$ is called a conditionally infimum (supremum) closed sublattice of $(L, \leq)$ if for every nonempty subset H of K that is lower(upper) bounded in $(K, \leq)$, $\text{Inf}_L H$ $(\text{Sup}_L H)$ belongs to K.

Note :

1. If $(K, \leq)$ is a conditionally infimum (supremum) closed sublattice of $(L, \leq)$ then the lattice $(K, \leq)$ is conditionally complete.
2. The concepts conditionally infimum closed and conditionally supremum closed are not equivalent.

LATTICE PROPERTIES OF KERRE, MASHHOUR AND PU NEIGHBOURHOODS

Theorem 2.2.14

Let (X, τ) be a Chang fuzzy topological space and $x \in X$. The Mashhour, Kerre and Pu- neighbourhoods satisfy the following lattice completeness properties :

1. $(\mathcal{M}_x(\tau), \subseteq)$ is a totally ordered sublattice of $(2^{I^X}, \subseteq)$ having the following sup-completeness property:

   $\forall A \subseteq (0,1),\ \inf A > 0 \ \Rightarrow\ \mathcal{M}x_{\inf A}(\tau) = \underset{\alpha \in A}{\cup} \mathcal{M}_{x_\alpha}(\tau).$

2. $(\mathcal{K}_x(\tau), \subseteq)$ is a totally ordered sublattice of $(2^{I^X}, \subseteq)$ having the following inf-completeness property:

   $\forall A \subseteq (0, 1], \quad \mathcal{K}_{x_{\sup A}}(\tau) = \bigcap_{\alpha \in A} \mathcal{K}_{x_\alpha}(\tau)$

3. $(\mathcal{P}_x(\tau), \subseteq)$ is a totally ordered sublattice of $(2^{I^X}, \subseteq)$ having the following sup-completeness property:

   $\forall A \subseteq (0, 1], \quad \mathcal{P}_{x_{\sup A}}(\tau) = \bigcup_{\alpha \in A} \mathcal{P}_{x_\alpha}(\tau)$

Proof :

Proof of (1) :

Obviously, we get $(\mathcal{M}_x(\tau), \subseteq)$ is a sublattice of the lattice $(2^{I^X}, \subseteq)$.

Claim 1

To prove $(\mathcal{M}_x(\tau), \subseteq)$ is a totally ordered sublattice of $(2^{I^X}, \subseteq)$.

Let $\mathcal{M}_{x_\varepsilon}(\tau)$, $\mathcal{M}_{x_\sigma}(\tau)$ be two elements of $(\mathcal{M}_x(\tau), \subseteq)$.

That is, $(\varepsilon, \sigma) \in (0, 1)^2$.

Case 1 :

If $\varepsilon = \sigma$, then $\mathcal{M}_{x_\varepsilon}(\tau) = \mathcal{M}_{x_\sigma}(\tau)$.

Case 2 :

If $\varepsilon < \sigma$, then consider $N \in \mathcal{M}_{x_\sigma}(\tau)$.

Then there exist $O \in \tau$ such that $x_\sigma \,\tilde{\in}\, O$ and $O \subseteq N$.

$\Rightarrow$ there exist $O \in \tau$ such that $\sigma < O(x)$ and $O \subseteq N$.

$\Rightarrow$ there exist $O \in \tau$ such that $\varepsilon < O(x)$ and $O \subseteq N$.

$\Rightarrow$ there exist $O \in \tau$ such that $x_\varepsilon \,\vec{\in}\, O$ and $O \subseteq N$.

$\Rightarrow N \in \mathcal{M}_{x_\varepsilon}(\tau)$.

Hence if $\varepsilon < \sigma$, $\mathcal{M}_{x_\sigma}(\tau) \subseteq \mathcal{M}_{x_\varepsilon}(\tau)$.

Case 3:

If $\sigma < \varepsilon$, consider $N \in \mathcal{M}_{x_\varepsilon}(\tau)$.

$\Rightarrow$ there exist $O \in \tau$ such that $x_\epsilon \,\vec{\in}\, O$ and $O \subseteq N$.

$\Rightarrow$ there exist $O \in \tau$ such that $\varepsilon < O(x)$ and $O \subseteq N$.

$\Rightarrow$ there exist $O \in \tau$ such that $\sigma < O(x)$ and $O \subseteq N$.

$\Rightarrow$ there exist $O \in \tau$ such that $x_\sigma \,\vec{\in}\, O$ and $O \subseteq N$.

$\Rightarrow N \in \mathcal{M}_{x_\sigma}(\tau)$.

Hence if $\sigma < \varepsilon$, then $\mathcal{M}_{x_\varepsilon}(\tau) \subseteq \mathcal{M}_{x_\sigma}(\tau)$.

Hence we have proved that $(\mathcal{M}_x(\tau), \subseteq)$ is a descending chain, which implies

$(\mathcal{M}_x(\tau), \subseteq)$ is a totally ordered sublattice of $(2^{I^X}, \subseteq)$.

Claim 2 :

$\forall A \subseteq (0, 1)$ and $\inf A > 0 \quad \Rightarrow \mathcal{M}_{x_{\inf A}}(\tau) = \bigcup_{\sigma \in A} \mathcal{M}_{x_\sigma}(\tau)$.

$A \subseteq (0, 1)$ and $\inf A > 0$ so that $\mathcal{M}_{x_{\inf A}}(\tau)$ makes sense.

Since for every $\varepsilon \in A$ and $\varepsilon \geq \inf A$, we get

$\forall \varepsilon \in A$, $\mathcal{M}_{x_\varepsilon}(\tau) \subseteq \mathcal{M}_{x_{\inf A}}(\tau)$

$$\Rightarrow \bigcup_{\varepsilon \in A} \mathcal{M}_{x_\varepsilon}(\tau) \subseteq \mathcal{M}_{x_{\inf A}}(\tau) \qquad (*)$$

To prove $\mathcal{M}_{x_{\inf A}}(\tau) \subseteq \bigcup_{\varepsilon \in A} \mathcal{M}_{x_\varepsilon}(\tau)$

Consider $V \in \mathcal{M}_{x_{\inf A}}$.

$\Rightarrow$ there exist $O \in \tau$ such that $O \subseteq V$ and $\inf A \,\vec{\in}\, O$.

$\Rightarrow$ there exist $O \in \tau$ such that $O \subseteq V$ and $\inf A < O(x)$.

By infimum property, there exists one $\varepsilon \in A$ such that $\varepsilon < O(x)$.

$\Rightarrow V \in \mathcal{M}_{x_\varepsilon}(\tau)$.

$$\Rightarrow \mathcal{M}_{x_{\inf A}}(\tau) \subseteq \bigcup_{\varepsilon \in A} \mathcal{M}_{x_\varepsilon}(\tau). \qquad (**)$$

From (*) and (**) we get,

$$\forall A \subseteq (0, 1), \inf A > 0, \quad \Rightarrow \mathcal{M}_{x_{\inf A}}(\tau) = \bigcup_{\varepsilon \in A} \mathcal{M}_{x_\varepsilon}(\tau).$$

Similarly (2) and (3) can be proved.

The following two theorems give the link between Kerre, Mashhour and Pu neighbourhoods.

Theorem 2.2.15

Let (X,τ) be a Chang fuzzy topological space and $x \in X$. The Pu-neighbourhoods of the (crisp) singleton $\{ x \}$ are completely determined by the Pu - neighbourhoods of every fuzzy point with support $\{ x \}$, for $\mathcal{P}_{x_1}(\tau) = \bigcup_{\varepsilon \in (0,1)} \mathcal{P}_{x_\varepsilon}(\tau)$.

Proof :

Using theorem 2.2.14(3) we may write,

$$\mathcal{P}_{x_1}(\tau) = \bigcup_{\varepsilon \in (0,1]} \mathcal{P}_{x_\varepsilon}(\tau) \supset \bigcup_{\varepsilon \in (0,1)} \mathcal{P}_{x_\varepsilon}(\tau).$$

Since the converse is obvious, we get, $\mathcal{P}_{x_1}(\tau) = \bigcup_{\varepsilon \in (0,1)} \mathcal{P}_{x_\varepsilon}(\tau)$.

Hence the proof.

Theorem 2.2.16

Let (X,τ) be a Chang fuzzy topological space and $x \in X$. The following links between Kerre, Mashhour and Pu - neighbourhoods hold :

1. $\forall\ \varepsilon \in (0,1]$, $\mathcal{K}_{x_\varepsilon}(\tau) = \bigcap_{\varepsilon' \in (0,\varepsilon)} \mathcal{M}_{x_{\varepsilon'}}(\tau) = \bigcap_{\varepsilon' \in (1-\varepsilon,1]} \mathcal{P}_{x_{\varepsilon'}}(\tau)$.

2. $\forall\ \varepsilon \in (0,1)$, $\mathcal{M}_{x_\varepsilon}(\tau) = \mathcal{P}_{x_{1-\varepsilon}}(\tau) = \bigcup_{\varepsilon' \in (\varepsilon,1)} \mathcal{K}_{x_{\varepsilon'}}(\tau)$.

3. $\forall\ \varepsilon \in (0,1]$, $\mathcal{P}_{x_\varepsilon}(\tau) = \bigcup_{\varepsilon' \in (1-\varepsilon,1]} \mathcal{K}_{x_{\varepsilon'}}(\tau)$.

3′. $\forall\ \varepsilon \in (0,1]$, $\mathcal{P}_{x_\varepsilon}(\tau) = \mathcal{M}_{x'_{1-\varepsilon}}(\tau)$ and $\mathcal{P}_{x_1}(\tau) = \bigcup_{\varepsilon \in (1-\varepsilon,1]} \mathcal{M}_{x_\varepsilon}(\tau)$.

Proof :

(1) To show the first equality in (1), it suffices to remark that using theorem (2.2.14(2)),

$$\bigcap_{\varepsilon' \in (0,\varepsilon)} \mathcal{M}_{x_{\varepsilon'}}(\tau) \subseteq \bigcap_{\varepsilon' \in (0,\varepsilon)} \mathcal{K}_{x_{\varepsilon'}}(\tau) = \mathcal{K}_{x_\varepsilon}(\tau).$$

(2) To prove the second equality in (2) we start from $A \in \mathcal{M}_{x_\varepsilon}(\tau)$. Then from theorem 2.2.14(1) we obtain,

$$A \in \bigcup_{\varepsilon' \in (\varepsilon,1)} \mathcal{M}_{x'_\varepsilon}(\tau) \subseteq \bigcup_{\varepsilon' - (\varepsilon,1]} \mathcal{K}_{x_\varepsilon'}(\tau)$$

Conversely from $A \in \bigcup_{\varepsilon' - (\varepsilon,1]} \mathcal{K}_{x_\varepsilon'}(\tau)$;

we infer that there exists an $\varepsilon' \leq O(x)$, $O \in \tau$ and $O \subseteq A$ for some $\varepsilon' > \varepsilon$, and hence $A \in \mathcal{M}_{x_\varepsilon} \in \tau$.

The remaining equalities can be proved similarly.

Theorem 2.2.17

Let (X, τ) be a Chang fuzzy topological space and $x \in X$.

(1) $(\mathcal{M}_x(\tau), \subseteq)$ is a conditionally supremum closed sublattice of $(2^{I^X}, \subseteq)$.

(2) $(\mathcal{K}_x(\tau), \subseteq)$ is a conditionally infimum closed sublattice of $(2^{I^X}, \subseteq)$.

(3) $(\mathcal{P}_x(\tau), \subseteq)$ is a conditionally supremum closed sublattice of $(2^{I^X}, \subseteq)$.

Proof :

Proof of (1) :

Consider a non empty subset M of $\mathcal{M}_x(\tau)$. M is bounded above by $\mathcal{M}_{x_m}(\tau)$ where $m \in (0m\ 1)$.

Let $E = \{\varepsilon \,/\, \varepsilon \in (0, 1) \text{ and } \mathcal{M}_{x_\varepsilon}(\tau) \in M\}$.

To prove $\cup M = \bigcup_{\varepsilon \in E} \mathcal{M}_{x_\varepsilon}(\tau) \in \mathcal{M}_x(\tau)$

Case (1) :

If $m \in E$, then $\cup M = \mathcal{M}_{x_m}(\tau)$

Case (2) :

If $m \notin E$, then $\inf E > 0$.

Hence by theorem 2.2.14(1) we get,

$$\mathcal{M}_{x_{\inf E}}(\tau) = \bigcup_{\varepsilon \in E} \mathcal{M}_{x_\varepsilon}(\tau) \in \mathcal{M}_x(\tau)$$
$$\cup M = \bigcup_{\varepsilon \in E} \mathcal{M}_{x_\varepsilon}(\tau) \in \mathcal{M}_x(\tau)$$

Hence $(\mathcal{M}_x(\tau), \subseteq)$ is a conditionally supremum closed sublattice of $(2^{I^X}, \subseteq)$.

Proof of (2) :

Consider a non empty subset K of $\mathcal{K}_x(\tau)$. K is bounded by $\mathcal{K}_{x_m}(\tau)$ where

$m \in (0,1)$.

Let $E = \{\varepsilon \,/\, \varepsilon \in (0,1) \text{ and } \mathcal{K}_{x_\varepsilon}(\tau) \in K\}$.

To prove :

$$\cup K = \bigcap_{\varepsilon \in E} \mathcal{K}_{x_\varepsilon}(\tau) \in \mathcal{K}_x(\tau).$$

Case (1) :

If $m \in E$, then $\cup K = \mathcal{K}_{x_m}(\tau)$.

Case (2) :

If $m \notin E$, then $\sup E > 0$.

Then by theorem 2.2.14(2) we get,

$$\mathcal{K}_{x_{\sup E}}(\tau) = \bigcap_{\varepsilon \in E} \mathcal{K}_{x_\varepsilon}(\tau)$$
$$\Rightarrow \quad \cup K = \bigcap_{\varepsilon \in E} \mathcal{K}_{x_\varepsilon}(\tau) \in \mathcal{K}_x(\tau).$$

Hence $(\mathcal{K}_x(\tau), \subseteq)$ is a conditionally infimum closed sublattice of $(2^{I^X}, \subseteq)$.

Proof of (3) :

Consider a non empty subset P of $\mathcal{P}_x(\tau)$. P is bounded above by $\mathcal{P}_{x_m}(\tau)$ where $m \in (0,1)$.

Let $E = \{\varepsilon \in (0, 1) \text{ and } \mathcal{P}_{x_\varepsilon}(\tau) \in P\}$.

To prove :

$$\cup P = \bigcup_{\varepsilon \in E} \mathcal{P}_{x_\varepsilon}(\tau) \in \mathcal{P}_x(\tau).$$

Case (1) :

If $m \in E$, then $\cup P = \mathcal{P}_{x_m}(\tau)$.

Case (2) :

If $m \notin E$, then $\inf E > 0$.

Then by theorem 2.2.14(3) we get,

$$\mathcal{P}_{x_{\sup E}}(\tau) = \bigcup_{\varepsilon \in E} \mathcal{P}_{x_\varepsilon}(\tau) \in \mathcal{P}_x(\tau).$$

$$\cup P = \bigcup_{\varepsilon \in E} \mathcal{P}_{x_\varepsilon}(\tau) \in \mathcal{P}_x(\tau).$$

Hence $(\mathcal{P}_x(\tau), \subseteq)$ is a conditionally supremum closed sublattice of $(2^{I^X}, \subseteq)$.

Next we see the negations of the dual properties of the theorem 2.2.17. For that we need the following definition.

Definition 2.2.18

Let X contains only two elements and let x_λ be the fuzzy singleton on X. Then

$\tau_{x_\lambda} = \{O \,/\, O \in I^X \text{ and } x_\lambda \subseteq O\} \cup \{\phi\}$ is called the fuzzy sierpinski space.

Theorem 2.2.19

Let (X, τ) be a Chang fuzzy topological space. Generally there exists some x in X such that :

(1) $(\mathcal{M}_x(\tau), \subseteq)$ is not a conditionally infimum closed sublattice of $(2^{I^X}, \subseteq)$.

(2) $(\mathcal{K}_x(\tau), \subseteq)$ is not a conditionally supremum closed sublattice of $(2^{I^X}, \subseteq)$.

(3) $(\mathcal{P}_x(\tau), \subseteq)$ is not a conditionally infimum closed sublattice of $(2^{I^X}, \subseteq)$.

Proof :

We shall prove (2) by using the fuzzy sierpinski space as a counter example. Consider $X = \{x, y\}$ and let x_λ be the fuzzy singleton with support $\{x\}$ and value $\lambda \in (0, 1]$. The corresponding fuzzy sierpinski space is given by

$$\tau_{x_\lambda} = \{O \,/\, O \in I^X \text{ and } x_\lambda \subseteq O\} \cup \{\phi\}.$$

Then the neighbourhood systems for an arbitrary fuzzy singleton x_δ and an arbitrary fuzzy point x_σ are given by :

$$\mathcal{K}_{x_\delta}(\tau_{x_\lambda}) = \{O \,/\, O \in \tau_{x_\lambda} \text{ and } x_\delta \subseteq O\}$$

and

$$\mathcal{M}_{x_\sigma}(\tau_{x_\lambda}) = \{O \,/\, O \in \tau_{x_\lambda} \text{ and } x_\sigma \subseteq O\}.$$

As a consequence it follows that

$$\forall \mu \in (0, 1],\ \forall \gamma \in (0, 1) \qquad \mathcal{K}_{y_\mu}(\tau_{x_\lambda}) \neq \mathcal{M}_{y_\gamma}(\tau_{x_\lambda})$$

Indeed, if $\mu \leq \gamma$ then it suffices to put $O(x) = \lambda$ and $O(y) = \mu$ to have $O \in \mathcal{K}_{y_\mu}(\tau_{x_\lambda})$ and $O \notin \mathcal{M}_{y_\gamma}(\tau_{x_\lambda})$.

If $\gamma < \mu$ put $O(x) = \lambda$ and $O(y) = \frac{1}{2}(\mu + \gamma)$, then $O \in \mathcal{M}_{y_\gamma}(\tau_{x_\lambda})$ and $O \notin \mathcal{K}_{y_\mu}(\tau_{x_\lambda})$.

Let $\varepsilon \in (0, 1)$ and consider the set $K = \left\{\mathcal{K}_{y_{\varepsilon'}}(\tau_{x_\lambda}) \ / \ \varepsilon' \in (\varepsilon, 1]\right\}$.

This set is upper bounded by $\mathcal{K}_{y_\varepsilon}(\tau_{x_\lambda})$ in the lattice $(\mathcal{K}_y(\tau_{x_\lambda}), \subseteq)$ and by theorem 2.2.16 (2) it follows that

$$\begin{aligned} \cup K &= \bigcup_{\varepsilon' \in (\varepsilon, 1]} \mathcal{K}_{y_{\varepsilon'}}(\tau_{x_\lambda}) \\ &= \mathcal{M}_{y_\varepsilon}(\tau_{x_\lambda}) \notin \mathcal{K}_y(\tau_{x_\lambda}). \end{aligned}$$

This completes the proof.

Similarly we can prove (1) and (3).

The following theorem gives the negations of the dual properties of the theorem 2.2.14.

Theorem 2.2.20

There exists a Chang fuzzy topological space (X, τ) such that,

(1) there exist $x \in X$, there exist $A \subseteq (0, 1)$ such that $\inf A > 0$ and

$$\mathcal{K}_{x_{\inf A}}(\tau) \neq \bigcup_{\alpha \in A} \mathcal{K}_{x_\alpha}(\tau).$$

(2) there exist $x \in X$, there exist $A \subseteq (0, 1)$ such that $\sup A < 1$ and

$$\mathcal{M}_{x_{\sup A}}(\tau) \neq \bigcap_{\alpha \in A} \mathcal{M}_{x_\alpha}(\tau).$$

(3) there exist $x \in X$, there exist $A \subseteq (0, 1]$ such that $\inf A > 0$ and

$$\mathcal{P}_{x_{\inf A}}(\tau) \neq \bigcap_{\alpha \in A} \mathcal{P}_{x_\alpha}(\tau).$$

It is a quite interesting to look for those Chang fuzzy topological space (X, τ) that satisfy the negations of the formulas in the theorem 2.2.14. That is, for instance those fuzzy topological space for which the Kerre neighbourhoods satisfy the inf-completeness property of theorem 2.2.14 as well as its dual sup-completness property. As a remarkable result we find that these spaces precisely constitute the crisp topological spaces, hence we have the following characterizations of crisp topological spaces.

Theorem 2.2.21

Let (X, τ) be a Chang fuzzy topological spaces. The following are equivalent :

(1) τ is crisp.

(2) $\forall\ x \in X,\ \forall\ A \subseteq (0, 1]$ such that $\inf A > 0 \Rightarrow \mathcal{K}_{x_{\inf A}}(\tau) = \bigcup_{\alpha \in A} \mathcal{K}_{x_\alpha}(\tau).$

(3) $\forall\ x \in X,\ \forall\ A \subseteq (0, 1)$ such that $\sup A < 1 \Rightarrow \mathcal{M}_{x_{\sup A}}(\tau) = \bigcup_{\alpha \in A} \mathcal{M}_{x_\alpha}(\tau).$

(4) $\forall\ x \in X,\ \forall\ A \subseteq (0, 1]$ such that $\inf A > 0 \Rightarrow \mathcal{P}_{x_{\inf A}}(\tau) = \bigcap_{\alpha \in A} \mathcal{P}_{x_\alpha}(\tau).$

Proof :

(1) $\Rightarrow$ (2) is obvious.

To prove (2) $\Rightarrow$ (1) :

Let $O \in \tau$, $x \in X$ and $O(x) = \varepsilon > 0$.

We have to show that $O(x) = 1$.

Suppose assume the contrary that $\varepsilon < 1$. Then from our assumption that,

$\forall\, x \in X,\ \forall\, A \subseteq (0, 1],\ \inf A > 0.$

$\Rightarrow \mathcal{K}_{x_{\sup A}}(\tau) = \underset{\alpha \in A}{\cup} \mathcal{K}_{x_\alpha}(\tau)$ and from theorem 2.2.16(2) we get,

$$\mathcal{K}_{x_\varepsilon}(\tau) = \underset{\varepsilon' \in (\varepsilon, 1]}{\cup} \mathcal{K}_{x_{\varepsilon'}}(\tau) = \mathcal{M}_{x_\varepsilon}(\tau).$$

We know $O \in \mathcal{K}_{x_\varepsilon}(\tau)$ or equivalently $O \in \mathcal{M}_{x_\varepsilon}(\tau)$.

Hence a contradiction to the definition of Mashhour neighbourhood that needs $\varepsilon < O(x)$.

Definition 2.2.22

A Chang fuzzy topological space (X, τ) is called a Lowen fuzzy topological space if $\forall\, \varepsilon \in (0, 1)$, there exists $O \in \tau$ $\forall\, x \in X$ such that $O(x) = \varepsilon$.

That is every constant map on X constitutes an open fuzzy set. This supplementary condition states that every point by the same open fuzzy set.

Definition 2.2.23

A Chang fuzzy topological space (X, τ) is called a surjective fuzzy topological space if, $\forall\, \varepsilon \in (0, 1)$ and $\forall\, x \in X$, there exist $O \in \tau$ such that $O(x) = \varepsilon$.

That is, every ε-level can be attained at every point by a open fuzzy set (not necessarily the same for every point).

Note :

On a purely logical base we obtain (X, τ) is a Lowen fuzzy topological space.

$\Rightarrow (X, \tau)$ is a surjective fuzzy topological space.

$\Rightarrow (X, \tau)$ is a Chang fuzzy topological space.

The reverse implications however are not valid. Indeed every crisp topological space is a Chang fuzzy topological space but not a surjective space.

Example :

Consider an uncountable set X. The fuzzy co-countable topology on X is defined as,

$$\tau = \left\{ O \ / \ O \in I^X \text{ and } \operatorname{supp}(\text{co } O) \text{ is countable} \right\} \cup \{\phi\}.$$

Thus (X, τ) is surjective but not Lowen fuzzy topological spaces.

Next we want focus on those Chang fuzzy topological space (X, τ) for which the sets $\mathcal{K}_x(\tau)$, $\mathcal{M}_x(\tau)$ and $\mathcal{P}_x(\tau)$ derived from the Kerre, Mashhour and Pu - neighbourhood systems respectively, constitute conditionally closed sublattices of $(2^{I^X}, \subseteq)$ for every x in X.

Definition 2.2.24

For Kerre neighbourhoods we want the following to hold :

$\forall K \subseteq \mathcal{K}_x(\tau)$, $K \neq \phi$ and K is bounded in $\mathcal{K}_x(\tau) \Rightarrow \cup K \in \mathcal{K}_x(\tau)$ and $\cap K \in \mathcal{K}_x(\tau)$

or, taking into account the theorem 2.2.15, $\forall K \subseteq \mathcal{K}_x(\tau)$, $K \neq \phi$ and K is bounded in above $\mathcal{K}_x(\tau) \Rightarrow \cup K \in \mathcal{K}_x(\tau)$.

Such Chang fuzzy topological space will be called conditionally closed Kerre fuzzy topological space. Similar definitions for conditionally closed Mashhour fuzzy topological space and conditionally closed Pu - fuzzy topological space can be considered.

Note :

(1) Evidently every crisp topology is a conditionally closed Kerre topology from theorem 2.2.21.

(2) We can see from the following counter example that "Not every conditionally closed Kerre fuzzy topological space is a crisp topological space".

Counter Example :

Let X = I and consider the class τ of fuzzy sets on I defined by $\tau = \{\phi, I, D\}$ where D denotes the diagonal of I x I. That is the mapping $x \rightarrow x, \forall x \in I$.

For an obvious reason this space will be described as the Z-space. We can easily obtain the following neighbourhood system.

$$\mathcal{K}_x(\tau) = \begin{cases} \{I\}, & \text{if } \varepsilon \in (x, 1] \\ \{N \,/\, N \in I^I \text{ and } \forall y \in I,\ y \leq N(y)\} & \text{if } \varepsilon \in (0, x]. \end{cases}$$

$$\mathcal{M}_x(\tau) = \begin{cases} \{I\}, & \text{if } \varepsilon \in [x, 1) \\ \{N \,/\, N \in I^I \text{ and } \forall y \in I,\ y \leq N(y)\} & \text{if } \varepsilon \in (0, x). \end{cases}$$

$$\mathcal{P}_x(\tau) = \begin{cases} \{I\}, & \text{if } \varepsilon \in (0, 1-x] \\ \{N \,/\, N \in I^I \text{ and } \forall y \in I,\ y \leq N(y)\} & \text{if } \varepsilon \in (1-x, 1]. \end{cases}$$

Hence putting $A = \{N \,/\, N \in I^I \text{ and } \forall y \in I,\ y \leq N(y)\}$

we may write,

$$\mathcal{K}_x(\tau) = \mathcal{M}_x(\tau) = \mathcal{P}_x(\tau) = \{\{I\}, A\}\ \forall x \in (0, 1).$$

$$\mathcal{K}_0(\tau) = \mathcal{M}_0(\tau) = \mathcal{P}_0(\tau) = \{\{I\}\} \text{ and}$$

$$\mathcal{K}_1(\tau) = \mathcal{M}_1(\tau) = \mathcal{P}_1(\tau) = \{A\}.$$

Obviously $(\mathcal{K}_x(\tau), \subseteq)$, $(\mathcal{M}_x(\tau), \subseteq)$, $(\mathcal{P}_x(\tau), \subseteq)$ constitute conditionally closed sublattices of $(2^{I^X}, \subseteq)$, $\forall\, x \in I$. However since τ is not crisp, these structures do not satisfy the conditions of theorem 2.2.21. This can also easily be seen from the following example.

Consider for instance $x = a \in (0, 1)$ and put $M = \left\{\mathcal{M}_{a_\varepsilon}(\tau) \;/\; \varepsilon \in (0, a)\right\} = \{A\}$ then of course we have $\cap M = A$ but $\cap M = \bigcap_{\varepsilon \in (0,a)} \mathcal{M}_{a_\varepsilon}(\tau) \neq \mathcal{M}_{a_a}(\tau) = \{I\}$.

Similarly we obtain,

$$\bigcap_{\varepsilon \in (0,a]} \mathcal{P}_{a_\varepsilon}(\tau) \neq \mathcal{P}_{a_a}(\tau) \text{ and}$$

$$\bigcup_{\varepsilon \in (0,a]} \mathcal{K}_{a_\varepsilon}(\tau) \neq \mathcal{K}_{a_a}(\tau).$$

Finally let us determine the position of the conditionally closed fuzzy topological space with respect to the surjective fuzzy topological space. We will show:

A surjetive fuzzy topological space need to be a conditionally closed Kerre (Mashhour, Pu) fuzzy topological space.

Indeed, let $x \in X$ and $(\mathcal{K}_x(\tau), \subseteq)$ be a conditionally supremum closed sublattice of $(2^{I^X}, \subseteq)$. From theorem 2.2.16 and the hypothesis we obtain for $\varepsilon \in (0, 1)$,

$$\mathcal{M}_{x_\varepsilon}(\tau) = \bigcup_{\varepsilon' \in (\varepsilon,1)} \mathcal{K}_{x_{\varepsilon'}}(\tau) = \mathcal{K}_{x_\sigma}(\tau), \text{ for some } \sigma \in (0, 1].$$

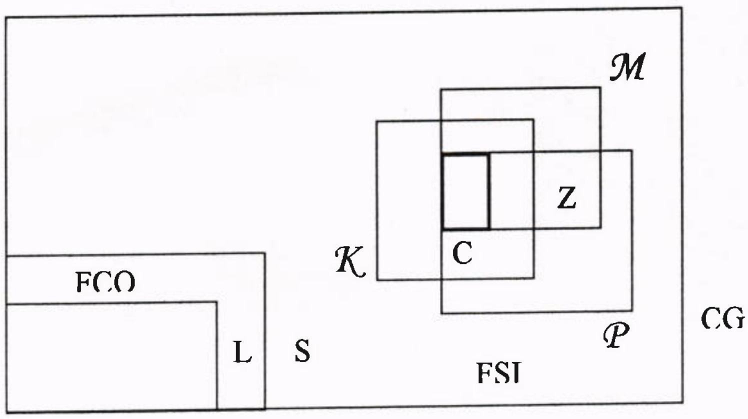
Hence $\forall\, \varepsilon \in [0, 1)$, there exist $\sigma \in (0, 1]$ such that $\mathcal{M}_{x_\varepsilon}(\tau) = \mathcal{K}_{x_\sigma}(\tau)$.

Firstly suppose $\varepsilon = \sigma$. The existence of $O \in \tau$ such that $O(x) = \frac{1}{2}(\varepsilon + \sigma)$ would imply $O \in \mathcal{M}_{x_\varepsilon}(\tau)$ and $O \notin \mathcal{K}_{x_\sigma}(\tau)$ and hence a contradiction again. So we obtain there exist $\varepsilon' \in (0, 1)\ \forall\ O \in \tau$ such that $O(x) \neq \varepsilon$.

The remaining case $\varepsilon > \sigma$ leads to the same conclusion. Hence by the method of cases we can conclude,

$\forall\ \varepsilon \in (0, 1),\ \forall\ x \in X,$ there exist $O \in \tau$ such that $O(x) = \varepsilon$

That is , (X, τ) is not surjective.



Remark that the reverse implications to those mentioned above do not hold; indeed, a straightforward calculation shows that the fuzzy sierpinski space [7] constitutes a Chang fuzzy topological space that is neither surjective nor a conditionally closed sublattice space in some sense (Kerre, Mashhour, Pu).

The above considerations lead to the diagram to figure, with

CG	=	Chang Goguen fuzzy topological spaces.
L	=	Lowen fuzzy topological space.
S	=	Surjective fuzzy topological space.

$\mathcal{K}$	=	Conditionally closed Kerre spaces.
$\mathcal{M}$	=	Conditionally closed Mashhour spaces.
$\mathcal{P}$	=	Conditionally closed Pu - spaces.
C	=	Crisp spaces.
FCO	=	Fuzzy co-countable topological space .
FSI	=	Fuzzy sierpinski space .
Z	=	Z-space .

Summary and Conclusion

SUMMARY AND CONCLUSION

In this thesis we have discussed the following articles in fuzzy neighbourhood spaces.

1. ***R.H. WARREN*** "Fuzzy topologies characterized by neighbourhood systems" [18].
2. ***C. DE MITRI and E. PASCALI*** "Characterization of fuzzy topologies from neighbourhoods of fuzzy points" [6].
3. ***P.M. PU and Y.M. LIU***, "Fuzzy topology I neighbourhood structure of a fuzzy point and Moore-Smith convergence" [17].
4. ***E.E. KERRE and P.L. OTTOY,*** "Lattice properties of neighbourhood systems in Chang fuzzy topological spaces" [8].

The main result of paper one is a characterization of fuzzy topology in terms of fuzzy neighbourhood systems. In the second paper, a similar characterization is obtained by considering fuzzy neighbourhoods of a fuzzy point. In the third paper, the fuzzy neighbourhood is defined by using the concept of quasi-coincidence. Here the authors have defined and studied adherence points and accumulation points of a fuzzy set. They have characterized closure of a set in terms of fuzzy neighbourhood. In the last paper lattice properties of neighbourhoods due to Kerre, Mashhour and Pu Pao-Ming are studied.

For further study on fuzzy neighbourhood spaces one can examine proximity, uniformity etc., in terms of fuzzy neighbourhoods.

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