CHAPTER - II

CHAPTER II

FUZZY ANALYTIC HIERARCHY PROCESS METHOD

Multiple criteria decision making (MCDM) refers to making decisions in the presence of multiple, usually conflicting, criteria. MCDM problems are common in everyday life. MCDM methods have been developed to support the decision makers in their unique and personal decision process.

Multiple Criteria Decision Making (MCDM) methods provide a stepping stones and techniques for finding a compromise solution. A large number of methods have been developed to solve multiple criteria problems. One of the most popular methods in MCDM is the Analytic Hierarchy Process (AHP).

The Analytic Hierarchy Process (AHP) method was developed by Professor Thomas Saaty 1980s [57]. This decision making method can help people set priorities and choose the best options by reducing complex decision problems to a system of hierarchies. Since its inception, it has evolved into several different variants and has been widely used to solve a broad range of multi criteria decision problems.

In spite of its popularity, the method is often criticized for its inability to adequately handle the inherent uncertainty and imprecision associated with the mapping of a decision-maker's perception to crisp numbers. Hence Fuzzy AHP (FAHP) method was developed to tackle this problem. There are many FAHP methods proposed by various authors. These methods are systematic approaches to the alternative selection and justification problems using the concepts of fuzzy set theory and hierarchical structure analysis.

SECTION 2.1

FUZZY AHP METHOD USING TRIANGULAR FUZZY NUMBERS

(Laarhoven and Pedrycz's Approach)

The Procedure of the fuzzy Analytic Hierarchy Process method using triangular fuzzy number is stated as follows:

Step 1: Construct a $n+1$ fuzzy reciprocal matrix that takes the following form as shown in (2)

$$\tilde{A} = \begin{pmatrix} (1,1,1) & \tilde{a}_{121} & \cdots & \tilde{a}_{1n1} \\ & \tilde{a}_{122} & & \tilde{a}_{1n2} \\ & \vdots & & \vdots \\ & \tilde{a}_{12P_{12}} & & \tilde{a}_{1nP_{1n}} \\ \tilde{a}_{211} & (1,1,1) & \cdots & \tilde{a}_{2n1} \\ \tilde{a}_{212} & & & \tilde{a}_{2n2} \\ \vdots & & & \vdots \\ \tilde{a}_{21P_{21}} & & & \tilde{a}_{2nP_{2n}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{a}_{n11} & \tilde{a}_{n21} & \cdots & (1,1,1) \\ \tilde{a}_{n12} & \tilde{a}_{n22} & & \\ \vdots & \vdots & & \\ \tilde{a}_{n1P_{n1}} & \tilde{a}_{n2P_{n2}} & & \end{pmatrix} \qquad (2)$$

where $\tilde{a}_{ijP_{ij}}$ are fuzzy numbers representing the relative importance of i^{th} factor to j^{th} one from the k^{th} expert's point of view. Note that P_{ij} may be zero when no expert expresses his/her comparison ratios or greater than 1 when more than one decision maker expresses his/her comparison ratios.

Step 2: Let $Z_i = (\, l_i,\ m_i,\ u_i\,)$. Solve the following linear equations:

$$l_i \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} \right] - \sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} u_j = \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{k=1}^{P_{ij}} \ln\, l_{ijk}\,, \qquad \forall\, i \tag{3}$$

$$m_i \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} \right] - \sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} m_j = \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{k=1}^{P_{ij}} \ln\, m_{ijk}\,, \qquad \forall\, i \tag{4}$$

$$u_i \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} \right] - \sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} l_j = \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{k=1}^{P_{ij}} \ln\, u_{ijk}\,, \qquad \forall\, i \tag{5}$$

where $\ln\, l_{ijk}$ and $\ln\, u_{ijk}$ are lower and upper values of $\ln a_{ijk} = -\ln\, a_{jik}$.

Step 3: The right sides of the equations above are operated using logarithmic operations. Then we obtain the fuzzy weight in Equation (6)

$$w_i = (\, \lambda_1 \exp(l_i),\ \lambda_2 \exp(m_i),\ \lambda_3 \exp(u_i)\,) \tag{6}$$

where,

$$\lambda_1 = \left[\sum_{i=1}^{n} \exp(u_i) \right]^{-1}$$

$$\lambda_2 = \left[\sum_{i=1}^{n} \exp(m_i) \right]^{-1}$$

$$\lambda_3 = \left[\sum_{i=1}^{n} \exp(l_i\,) \right]^{-1}$$

Equation (6) can also be used to determine r_{ij} , the performance score of the i^{th} alternative under the j^{th} criteria.

Step 4: Steps *1-3* are repeated several times until all reciprocal matrices are solved. With the fuzzy weights and performance scores, we can calculate the fuzzy utility for each alternative A_i as

$$u_i = \sum_{j=1}^{n} w_j r_{ij} \tag{7}$$

These fuzzy utilities can be ranked by any appropriate fuzzy ranking method. Thus the alternative with highest fuzzy utility is selected as best alternative.

SECTION 2.2

FUZZY AHP METHOD USING TRIANGULAR FUZZY NUMBERS
(Debmallya Chatterjee and Bani Mukherjee's Approach)

Definition: 2.2.1 [12]

The geometric mean of a data set $\{a_1, a_2 \dots a_n\}$ is given by

$$g = \left(\prod_{i=1}^{n} a_i\right)^{1/n}$$

Then the ***Normalization of Geometric Mean*** is given by

$$n = g/\sum_{i=1}^{n} a_i$$

In this procedure of the Fuzzy AHP method uses the triangular fuzzy number in the pairwise comparison matrix and Normalization of Geometric Mean (NGM) method is used to compute weights from pairwise comparison matrix.

The algorithm of Fuzzy AHP method using triangular fuzzy number with normalization of geometric mean is given as follows:

Step 1: **Conceptual Hierarchy of Fuzzy –AHP Model**

Analytical Hierarchy Process starts by laying out the overall hierarchy of the decision making problem. The hierarchy is structured from the top (the overall goal of the problem) through the intermediate levels (criteria and sub-criteria on which subsequent levels depend) to the bottom level (the list of alternatives). Each criterion in the lower level of hierarchy is compared with respect to the criteria in the upper level of hierarchy. The criteria in the same level are compared using pair wise comparison. Fig 3 describes the hierarchy of a decision making problem.

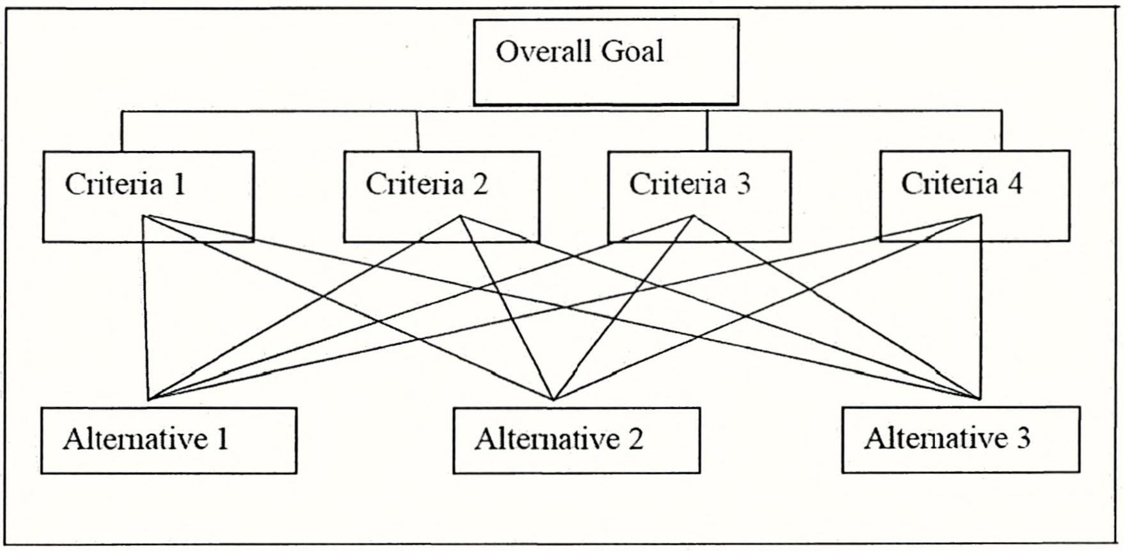

Fig 3. Hierarchy of the Decision Making Problem

Step 2: **Recruiting Pairwise Comparisons Among Decision Elements**

Once the hierarchy is established, the pair wise comparison evaluation takes place. All the criteria on the same level of the hierarchy are compared to each of the criterion of the preceding (upper) level. A pair wise comparison is performed by using Fuzzy linguistic terms in the scale of 0 – 10 described by the Triangular Fuzzy Numbers in the Table 1.

Verbal judgment	Explanation	Fuzzy number
Extremely Un-important	A criterion is strongly inferior to another	(0, 1, 2)
Un-important	A criterion is slightly inferior to another	(1, 2.5, 4)
Equally Important	Two criteria contribute equally to the object	(3, 5, 7)
Moderately Important	Judgment slightly favor one criterion over another	(6, 7.5, 9)
Extremely Important	Judgment strongly favor one criterion over another	(8, 9, 10)

Table 1. Fuzzy Important scale with Triangular Fuzzy Number

To reflect pessimistic, most likely and optimistic decision making environment, triangular fuzzy numbers with minimum value, most plausible value and maximum value are considered.

The fuzzy comparison matrix is defined as,

$$\tilde{A} = \begin{bmatrix} 1 & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & 1 & \cdots & \tilde{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & 1 \end{bmatrix}$$

where $\tilde{a}_{ij} = (a_{ij}^{L}, a_{ij}^{M}, a_{ij}^{U})$ is the relative importance of each criteria in pair wise comparison and number $a_{ij}^{L}, a_{ij}^{M}, a_{ij}^{U}$ are the are the minimum value, most plausible value and maximum value of the triangular fuzzy number.

Step 3: Generation of Criteria and Sub-Criteria Weight

The Normalization of the Geometric Mean (NGM) method is applied to compute weights from the fuzzy pair wise comparison matrices which is given by

$$w_i = a_i \Big/ \sum_{i=1}^{n} a_i$$

where,

$$a_i = \Big(\prod_{j=1}^{n} a_{ij}\Big)^{1/n}$$

In the above equations, a_i is geometric mean of i^{th} criteria. a_{ij} is the comparison value of i^th criteria to the j^th criteria. w_i is the i^th criterion's weight, where $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$; $1 \le i \le n$.

For group evaluation, it is required to aggregate evaluator's opinions into one. Considering the evaluation given by expert $E_i = (\, a_L^{(i)},\ a_M^{(i)},\ a_U^{(i)}\,)$, the aggregate of all experts' judgments can be calculated using average means.

$$\tilde{A} = \left(\frac{1}{n}\sum_{i=1}^{n} a_L^{(i)}, \frac{1}{n}\sum_{i=1}^{n} a_M^{(i)}, \frac{1}{n}\sum_{i=1}^{n} a_U^{(i)} \right)$$

The final weight vector is generated by defuzzyfying the average

$$w_i = \left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n} a_L^{(i)} + 2\left\{ \frac{1}{n}\sum_{i=1}^{n} a_M^{(i)} \right\} + \frac{1}{n}\sum_{i=1}^{n} a_U^{(i)} \right)}{4} \right]$$

The weight of i^{th} sub criteria under k^{th} main criteria is obtained by

$$(w_k \times s_{ki})$$

where w_k is the k^{th} main criteria weight and s_{ki} is the weight of i^{th} sub criteria with respect to k^{th} main criteria.

Step 4: Calculation of Overall Score for Alternatives

Once the weight of criteria, sub criteria are evaluated and are multiplied to obtain global weight of sub criteria, it is required to calculate the overall score of alternatives for their evaluation.

The overall score of m^{th} alternative is obtained by

$$A_m = \sum_{l=1}^{N} s_l \times a_{ml}$$

where s_l is the weight of l^{th} sub criteria and $a_{ml}$is the weight of m^{th} alternative with respect to l^{th} sub criteria.

Finally the alternative with highest weight is selected as the best alternative.

SECTION 2.3

FUZZY AHP METHOD USING TRAPEZOIDAL FUZZY NUMBERS (Buckley's Approach)

The Fuzzy AHP method using trapezoidal fuzzy numbers is illustrated as follows:

Step 1: Consult the decision maker, and obtain the comparison matrix A whose elements are $\tilde{t}_{ij} = (a_{ij},\ b_{ij},\ c_{ij},\ d_{ij})$, where all i and j are trapezoidal fuzzy numbers.

Step 2: The fuzzy weights w_i can be calculated as follows.

The geometric mean for each row is determined as:

$$\tilde{z}_i = \left[\sum_{j=1}^{n} \widetilde{t_{ij}} \right]^{1/n} \qquad \forall\, i$$

The fuzzy weight w_i is given as

$$w_i = \tilde{z}_i \oplus \left[\sum_{j=1}^{n} \tilde{z}_j \right]^{-1}$$

In the following discussion, we will detail the derivation of fuzzy weight w_i .

In order to calculate the membership functions and its graph and also for the fuzzy weights the following values are calculated

$$f_i(\alpha) = \left[\prod_{j=1}^{n} \left((b_{ij} - a_{ij})\alpha + a_{ij} \right) \right]^{1/n} , \qquad \alpha \in [0,1]$$

$$g_i(\alpha) = \left[\prod_{j=1}^{n} \left((c_{ij} - d_{ij})\alpha + b_{ij} \right) \right]^{1/n} , \qquad \alpha \in [0,1]$$

$$f(\alpha) = \sum_{i=1}^{m} f_i(\alpha)$$

$$g(\alpha) = \sum_{i=1}^{m} g_i(\alpha)$$

Furthermore,

$$a_i = \left[\prod_{j=1}^{n} a_{ij}\right]^{1/n}$$

and

$$a = \sum_{i=1}^{m} a_i$$

Similarly,

$$b_i = \left[\prod_{j=1}^{n} b_{ij}\right]^{1/n}$$

and

$$b = \sum_{i=1}^{m} b_i$$

$$c_i = \left[\prod_{j=1}^{n} c_{ij}\right]^{1/n}$$

and

$$c = \sum_{i=1}^{m} c_i$$

$$d_i = \left[\prod_{j=1}^{n} d_{ij}\right]^{1/n}$$

and

$$d = \sum_{i=1}^{m} d_i$$

The final fuzzy weights are defined as

$$w_i = \left(\frac{a_i}{a}, \frac{b_i}{b}, \frac{c_i}{c}, \frac{d_i}{d}\right), \qquad \forall\, i$$

The membership function $\mu_{w_i}(x)$ is defined as follows: Let x be a real number on the horizontal axis. The $\mu_{w_i}(x)$ can be summarized as in Table 2.

x	$\mu_{w_i}(x)$
$\leq (a_i/d)$	0
$\geq (a_i/d)$	0
$[b_i/c, c_i/b]$	1
$[a_i/d, b_i/c]$	$\alpha \in [0,1]$
$[c_i/b, d_i/a]$	$\alpha \in [0,1]$

Table 2. Interpretation of Entities in a Pair-wise Comparison Matrix

When $x \in [a_i/d, b_i/c]$ or $x \in [c_i/b, d_i/a]$, then x can be calculated as

$$x = \begin{cases} f_i(\alpha)/g(\alpha) & if\ x \in [a_i/d, b_i/c] \\ g_i(\alpha)/f(\alpha) & if\ x \in [c_i/b, d_i/a] \end{cases}$$

Thus the graph of membership function is zero to the left of a_i/d, on the interval $[a_i/d, b_i/c]$ is defined by $x = f_i(\alpha)/g(\alpha)$ and is a horizontal line at the value of 1 on the interval $[b_i/c, c_i/b]$ and then on the interval $[c_i/b, d_i/a]$, it is defined by $x = g_i(\alpha)/f(\alpha)$.

Step 2 is repeated for all the fuzzy performance score, r_{ij}.

Step 3: The fuzzy weights and fuzzy performance scores are aggregated. The fuzzy utilities $U_i, \forall\, i$ are obtained based on

$$U_i = \sum_{j=1}^{n} w_i\, r_{ij}\,, \qquad \forall\, i$$

The fuzzy utilities are ranked by using a suitable fuzzy ranking method. Thus the alternative with highest fuzzy utility is selected as best alternative.

SECTION 2.4

FUZZY AHP METHOD USING COMBINATION OF TRIANGULAR FUZZY NUMBERS AND TRAPEZOIDAL FUZZY NUMBERS

(Pan Nang-Fei's Approach)

In this method of Fuzzy Analytic Hierarchy Process a combination of triangular fuzzy numbers and trapezoidal fuzzy numbers are utilized. Furthermore, the α- cut concept is applied to describe specific levels of uncertainty associated with the decision environment.

Definition 2.4.1 [51]

The ***Center-of-Gravity (COG) defuzzification technique*** is given by the formula

$$z^* = \frac{\int \mu(z).zdz}{\int \mu(z)\, dz}$$

where $\mu(z)$ is the membership value; z^* is the weighted average and $\int$ denotes an algebraic integration.

Definition 2.4.2 [51]

For the fuzzy sets $A_i : i = 1, \dots n$; the ***Fuzzy max-min aggregation technique*** is given by the formula

$$\mu_A(x) = \max\{\min\{\mu_1(x), \dots \mu_n(x)\}\}$$

where $\mu_A(x)$ is the membership value of x in the aggregated subset A and $\mu_i(x)$ is the membership grades of the fuzzy subset A_i.

The proposed Fuzzy AHP method is summarized as follows:

Step 1: **Construction of Hierarchy**

The typical Fuzzy AHP decision problem consists of

1. a number of alternatives, M_i $(i = 1, \dots m)$,
2. a set of evaluation criteria, C_j $(j = 1 \dots n)$,
3. a linguistic judgment r_{ij} representing the relative importance of each pair criteria, and
4. a weighting vector, $W = (w_1, \dots, w_n)$.

The first step of the proposed model is to determine all the important criteria and their relationship of the decision problem in the form of a hierarchy. This step is crucial because the selected criteria can influence the final choice. The hierarchy is structured from the top (the overall goal of the problem) through the intermediate levels (criteria and sub-criteria on which subsequent levels depend) to the bottom level (the list of alternatives).

Step 2: **Evaluation of Fuzzy Pairwise Comparison**

Once the hierarchy is established, the pairwise comparison evaluation takes place. All the criteria on the same level of the hierarchy are compared to each of the criterion of the preceding (upper) level. A pairwise comparison is performed by using linguistic terms. The five linguistic terms, "Very Unimportant"(VU),"Less Important" (LI), "Equally Important" (EI), "More Important" (MI) and "Very Important"(VI) ranging 0–10 are used to develop fuzzy comparison matrices. These five linguistic variables are described by fuzzy numbers as denoted in Table 3 or by membership functions as illustrated in Fig. 4.

Verbal judgment	Explanation	Fuzzy number
Very Unimportant (VU)	A criterion is strongly inferior to another	(0, 0, 1, 2)
Less Important (LI)	A criterion is slightly inferior to another	(1, 2.5, 4)
Equally Important (EI)	Two criteria contribute equally to the object	(3, 5, 7)
More Important (MI)	Judgment slightly favor one criterion over another	(6, 7.5, 9)
Very Important (VI)	Judgment strongly favor one criterion over another	(8, 9, 10, 10)

Table 3. Fuzzy importance scale

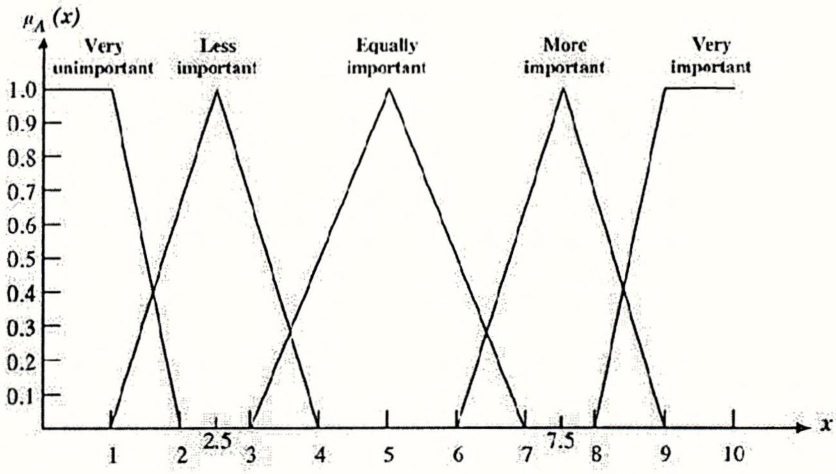

Fig. 4. Membership functions for linguistic values.

It can be found in the figure that "Very Unimportant" and "Very Important" are represented by half trapezoidal membership functions; whereas the remaining levels are characterized by symmetric triangular membership functions.

Fuzzy comparison matrix, $\tilde{A}$ representing fuzzy relative importance of each pair elements is given by

$$\tilde{A} = \begin{bmatrix} 1 & \tilde{r}_{12} & \tilde{r}_{13} & \cdots & \tilde{r}_{1n} \\ \tilde{r}_{21} & 1 & \tilde{r}_{23} & \cdots & \tilde{r}_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \tilde{r}_{n1} & \tilde{r}_{n2} & \tilde{r}_{n3} & \cdots & 1 \end{bmatrix}$$

where $\tilde{r}_{ij}$ is the relative importance of i^{th} criteria to the j^{th} criteria.

To reflect particular degrees of uncertainty regarding the decision making process, the α- cut concept is applied. The value of α is between 0 and 1. $\alpha = 0$ and $\alpha = 1$, signify the degree of uncertainty is greatest and least, respectively. In practical applications, $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 1$ are used to indicate the decision-making condition that has pessimistic, moderate, and optimistic view, respectively. Fig. 5 shows that a triangular fuzzy number regarding a given value can be denoted by $(X_{\alpha,L}, X_{\alpha M}, X_{\alpha R})$. $X_{\alpha,L}, X_{\alpha M}, X_{\alpha R}$ represents the most-likely value, minimum value, and maximum value of the fuzzy number, respectively.

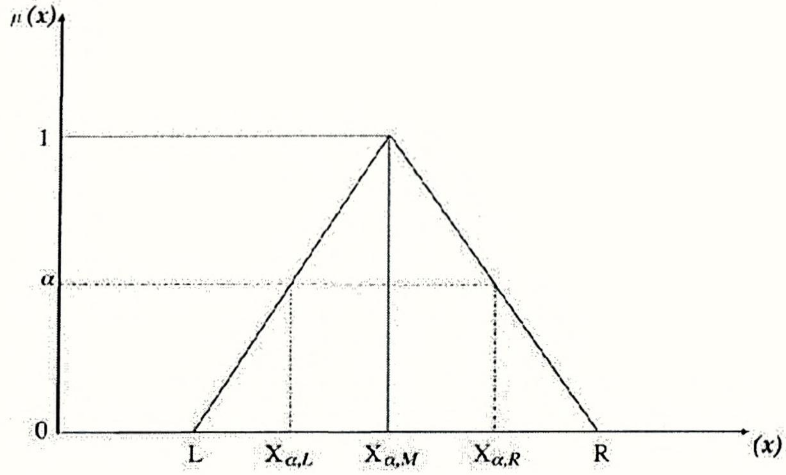

Fig. 5. Triangular fuzzy intervals under α- cut

The five membership functions shown in Fig. 4 can also be mathematically expressed through Equations. (8) – (12).

$$X(\alpha)_{Very\ Unimportant} = \begin{cases} X_{\alpha,L} = 0 \\ X_{\alpha M} = \dfrac{0.5 + (X_{\alpha,L} - 1)[(X_{\alpha,L} - 1)(0.33 + 0.17\alpha) + 1]}{1 + (0.5X_{\alpha,L} - 0.5)(1 + \alpha)} \\ X_{\alpha U} = 2 - \alpha \end{cases} \tag{8}$$

$$X(\alpha)_{Less\ Important} = \begin{cases} X_{\alpha,L} = 1 + 1.5\alpha \\ X_{\alpha M} = 2.5 \\ X_{\alpha U} = 4 - 1.5\alpha \end{cases} \tag{9}$$

$$X(\alpha)_{Equally\ Important} = \begin{cases} X_{\alpha,L} = 3 + 2\alpha \\ X_{\alpha M} = 5 \\ X_{\alpha U} = 7 - 2\alpha \end{cases} \qquad (10)$$

$$X(\alpha)_{More\ Important} = \begin{cases} X_{\alpha,L} = 6 + 1.5\alpha \\ X_{\alpha M} = 7.5 \\ X_{\alpha U} = 9 - 1.5\alpha \end{cases} \qquad (11)$$

$$X(\alpha)_{Very\ Important} = \begin{cases} X_{\alpha,L} = 8 + \alpha \\ X_{\alpha M} = \dfrac{8 + (9 - X_{\alpha,L})[(9 - X_{\alpha,L})(0.67 + 0.17\alpha) + 0.5]}{1 + (4.5 - 0.5X_{\alpha,L})(1 + \alpha)} \\ X_{\alpha U} = 10 \end{cases} \qquad (12)$$

Accordingly a fuzzy comparison matrix can be defined as follows:

$$\tilde{A} = \begin{pmatrix} 1 & (X_{12,L}, X_{12M}, X_{12R}) & \cdots & (X_{1n,L}, X_{1nM}, X_{1nR}) \\ (X_{\alpha 21L}, X_{21M}, X_{21R}) & 1 & \cdots & (X_{2nL}, X_{2nM}, X_{2nR}) \\ \cdots & \cdots & \cdots & \cdots \\ (X_{n1L}, X_{n1M}, X_{n1R}) & (X_{n2L}, X_{n2M}, X_{n2R}) & \cdots & (X_{nnL}, X_{nnM}, X_{nnR}) \end{pmatrix}$$

Step 3: **Calculation of Element Weight**

The Normalization of the Geometric Mean (NGM) method used in Buckley's [12] model is applied to compute local weights and given by

$$w_i = {}^{a_i}\!/_{\sum_{i=1}^{n} a_i}$$

where

$$a_i = \left(\prod_{j=1}^{n} r_{ij} \right)^{1/n}$$

In the above equations, a_i is geometric mean of criterion i. r_{ij} is the comparison value of criterion i to criterion j. w_i is the i^{th} criterion's weight, where $w_i > 0$ and $\sum_{1=1}^{n} w_i = 1\,,\, 1 \leq i \leq n$.

For group evaluation, it is required to aggregate manifold evaluators' opinions into one. The aggregate of multiple experts' evaluations encompasses a range of membership values that must be defuzzified in order to resolve a single representative value. This is done using fuzzy max–min operator and Center-of-Gravity (COG) techniques because of their simplicity and efficiency.

Accordingly, the overall weight of the l^{th} sub-criterion, S_l can be computed by

$$S_l = \sum_{i=1}^{L} w_k \times S_{lk}$$

where w_k is the weight of the k^{th} main-criterion; S_{lk} is the local weight of the l^{th} sub-criterion with respect to the k^{th} main-criterion.

Similarly, the overall weight of the m^{th} alternative regarding the l^{th} sub-criterion, R_m, is given by

$$R_m = \sum_{m=1}^{M} S_l \times R_{ml}$$

Ultimately the alternative with highest overall weight is selected as the best alternative.