Appendices

APPENDIX – 1

The following are the identities and the results used in the analytical solutions. Let m and N be non negative integers.

IDENTITIES:

1. $$\left(\sum_{k=1}^{\infty} g_k z^k\right)\left(\sum_{n=m}^{\infty} D_n z^n\right) = \sum_{n=m+1}^{\infty} z^n \left(\sum_{k=1}^{n-m} D_{n-k}\, g_k\right)$$

2. $$\left(\sum_{k=0}^{\infty} h_k z^k\right)\left(\sum_{n=m}^{\infty} \xi_n z^n\right) = \sum_{n=m}^{\infty} z^n \left(\sum_{i=m}^{n} \xi_i\, h_{n-i}\right)$$

3. $$\left(\sum_{k=1}^{\infty} g_k z^k\right)\left(\sum_{n=m}^{N-1} U_n z^n\right) = \sum_{n=m+1}^{N-1} z^n \left(\sum_{k=1}^{n-m} U_{n-k}\, g_k\right) + \sum_{n=N}^{\infty} z^n \left(\sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k\right)$$
for $m \geq 0$

4. $$\left(\sum_{n=0}^{\infty} \alpha_n z^n\right)\left(\sum_{n=0}^{m-1} \delta_n z^n\right) = \sum_{n=0}^{m-1} z^n \left(\sum_{i=0}^{n} \alpha_i\, \delta_{n-i}\right) + \sum_{n=m}^{\infty} z^n \left(\sum_{i=0}^{m-1} \delta_i\, \alpha_{n-i}\right)$$

5. $$\sum_{r=m}^{n} S_r \left(\sum_{k=0}^{n-r} h_k\, \pi_{n-r-k}\right) = \sum_{k=m}^{n} \pi_{n-k} \left(\sum_{i=m}^{k} \delta_i\, h_{k-i}\right)$$

6. $$\sum_{k=1}^{n} g_k \left(\sum_{i=0}^{n-k} \alpha_i\, \pi_{n-k-i}\right) = \sum_{k=0}^{n-1} \alpha_k \left(\sum_{i=1}^{n-k} g_i\, \pi_{n-k-i}\right)$$

7. $$\sum_{r=0}^{\infty} x^{rb} \left(\sum_{i=0}^{r} c_i\right) = \frac{\sum_{k=0}^{\infty} c_k x^{kb}}{1-x^b}$$

II Results using L'Hospital Rule:

1. If f(1)=g(1)=0, then $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right)_{z=1} = \frac{g'(1)f''(1)-f'(1)g''(1)}{2(g'(1))^2}$ where the dashes represent the derivatives of the functions.

2. If $w_X(z)=\lambda(1-X(z))$, then $\frac{d}{dz}\left(\frac{1-V^*(w_X(z))}{w_X(z)}\right)=\lambda E(X)E\left(\frac{V^2}{2}\right)$ where $V^*(\theta)$ denotes the LST of V(t).

3. $$\frac{d}{dz}\left(\frac{1-V^*(w_X(z))D^*(w_X(z))}{w_X(z)}\right)=\frac{\lambda E(X)}{2}(E(D^2)+E(V^2)+2E(V)E(D))$$

4. $$\frac{d}{dz}\left(\frac{z-1}{z-S^*(w_X(z))}\right)=\frac{\lambda E(X(X-1))E(S)+(\lambda E(X))^2E(S^2)}{2(1-\rho)^2}$$

   where $\rho=(\frac{d}{dz}S^*(w_X(z)))_{z=1}=\lambda E(X)E(S)$

5. $$\frac{d^2}{dz^2}S^*(w_X(z))=(\lambda E(X))^2E(S^2)+\lambda E(X(X-1))E(S)$$

6 If $h_{ai}(w_X(z))=w_X(z)+a_i(1-B_i^{*1}(w_X(z)))$, then

   $$\frac{d}{dz}h_{ai}(w_X(z))=-\lambda E(X)(1+a_iE(B_i)) \quad \text{and}$$

   $$\frac{d^2}{dz^2}h_{ai}(w_X(z))=-[\lambda E(X(X-1))(1+a_iE(B_i))+a_i(\lambda E(X))^2E(B_i^2)]$$

7. If $H_i^*(w_X(z))=S_i^*(h_{ai}(w_X(z)))$, then

   $$\frac{d}{dz}H_i(w_X(z))=\lambda E(X)(1+a_iE(B_i))E(S_i)$$

   and $$\frac{d^2}{dz^2}H_i(w_X(z))=(\lambda E(X))^2((1+a_iE(B_i))^2E(S_i^2)+a_iE(B_i^2)E(S_i))+$$

   $$\lambda E(X(X-1))(1+a_iE(B_i))E(S_i)$$

APPENDIX – 2

Various distributions used in numerical analysis:

I. Discrete distributions:

1. Geometric distribution (X~Geo(p))

   The probability distribution is $Pr(X=k)=(1-p)p^k$.

   mean $E(X)=\frac{1}{1-p}$ and second moment $E(X(X-1))=\frac{2p}{(1-p)^2}$

2. Binomial distribution (X~B(n, p))

   The probability distribution is $p_k = nC_k p^k (1-p)^{n-k}$.

   mean $E(X)=np$ and second moment $E(X^2)=np(q+np)$

II. Continuous distributions:

1. Exponential distribution X:

   The density function $f(x)=\lambda e^{-\lambda x}$, mean $E(X)=\frac{1}{\lambda}$ and second moment $E(X^2)=\frac{2}{\lambda^2}$

2. Uniform distribution X:

   The density function $f(x) = 1 \quad 0 \le x \le 1$,

   $= 0$ Otherwise

   mean $E(X)=\frac{1}{2}$ and second moment $E(X^2)=\frac{1}{3}$

3. Gamma distribution X:

   The density function $f(x)=\frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma k}, \quad x>0$

   $=0 \quad , x<0$

   mean $E(X)=\frac{k}{\lambda}$ and second moment $E(X^2)=\frac{k(k+1)}{\lambda^2}$

4. k-Erlang distribution X:

The density function $f_{\lambda k,k}(x) = \frac{(\lambda k)^k x^{k-1} e^{-\lambda kx}}{\Gamma k}$, $x > 0$,

mean $E(X) = \frac{1}{\lambda}$ and

second moment $E(X^2) = \frac{1}{k\lambda^2} + \frac{1}{\lambda^2}$ $\lambda > 0$, k= 1,2,3....

5. k-stage Hyper exponential distribution X:

It is a mixture of k independent exponential distribution having probability density function $f(x) = \sum_{i=1}^{k} a_i \mu_i e^{-\mu_i x}$, $0 \le a_i \le 1$

and $\sum_{i=1}^{\infty} a_i = 1$, mean $E(X) = \sum_{i=1}^{k} \frac{a_i}{\mu_i}$ and second moment $E(X^2) = 2\sum_{i=1}^{k} \frac{a_i}{\mu_i^2}$.

APPENDIX - 3

Application – An example

The following quality control problem may be considered as one example of an application, fitting our models (Chapters II to V).

A manufacturing plant produces certain items that occasionally are defective in batches. The good items are marketed while the defective ones are kept in storage until they can be reworked to meet the specifications. Assume that one of the machines in the plant may be converted when needed (at some cost) from a production mode to repair – mode in order to perform this rework. An appropriate cut off number **m** may be chosen, such that, if the number of defective items is at least **m,** then the special machine will be converted from the production mode to the repair mode at the next opportunity. The time required for conversion can be considered as setup time. During the setup period more defective items may be accumulated and the management may raise the cut off number to N ($\geq$m) to start the rework of the defective items. And this will increase the cycle length. Because of the cost involved in switching model, after conversion into repair mode, the machine will rework all of the defective items (including the new arrivals) exhaustively, and then switch back to the productive mode when there are no defective items left.

To fit the quality- control problem in our queueing models, the defective items are interpreted as the customers and the special machine is considered as the server. The server is available only when the machine is in the repair mode. The service time required to rework a defective item to meet specifications.

The server who is released from duty whenever the system becomes empty may spend his time productively else where and then can be recalled to duty at the end of random (i.i.d) time intervals. Thus when the machine is in production mode, the server is assumed to be on a series of vacations and the time required to produce each set of items may be considered as a vacation time. Some time vacations are interpreted as preventive maintenance performed on the server (not breakdowns) rather than productive work done else where, in which case, the server takes only one vacation at a time. This gives rise to single-vacation. In certain situations, the server needs random amount of time between any two consecutive reworks which corresponds to Bernoulli vacation.

If any item is found to be defective at the end of the first rework it may be allowed to undergo a second re-work process of SOS facility considered in Chapters III to V. In some cases, different types of rework process may be facilitated so that, the defective items may under go the required rework process. And this corresponds to the service discussed in Chapter IV. And the machine may breakdown while doing the rework process and can be sent for repair immediately.

The example is originally suggested by Kella (1989) to suit his model and it is modified here..