CHAPTER III

Vertex corona product of Cycle and Path with Tadpole and Barbell graphs

In this Chapter, the b-chromatic number of vertex corona product of tadpole graph with path graph, tadpole graph with cycle graph, path graph with tadpole graph, barbell graph with cycle graph and barbell graph with path are obtained.

3.1 Introduction

Tadpole Graph [Kalpana, M et al., 2018]

The (m,n)-tadpole graph is a special type of graph consisting of a cycle graph on m (at least 3) vertices and a path graph on n vertices, connected with a bridge.

It is denoted by $T_{m,n}$ let 'm' denote the number of vertices of cycle graph C_m and n denote the number of vertices of path graph P_n .Tadpole graph is also called as dragon graph. Generally, In G, vertex set can be denoted as $V(G)$ and set of all edges represents $E(G)$ respectively. Also, the number of vertices $|V(G)| = m + n$ and number of edges $|E(G)| = m + n$.

Barbell Graph [Jonathan Gross, et al., 2004]

A $B(2, n)$ Barbell graph is a simple graph obtained by connecting two copies of a complete graph k_n by a bridge and denoted by $B(k_n, k_n)$.

A $B(3, n)$ Barbell graph is a simple graph obtained by connecting three copies of a complete graph k_n by a bridge and denoted by $B(k_n, k_n, k_n)$.

In general a $B(N, n)$ Barbell graph is a simple graph obtained by connecting N copies of a complete graph k_n by a bridge and it is denoted by $B(k_n, k_n, ..., N\, times \quad k_n)$.

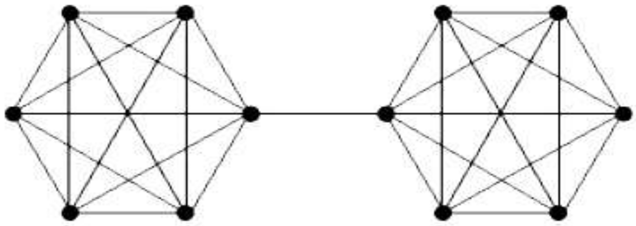

Fig 3.1: Barbell Graph B_6

Properties of barbell graph

- The barbell graph contains cycles in it.
- The barbell graph is connected every two nodes have a path between them.
- It has a bridge between 2 complete graphs.
- Bridge may and may not have nodes in it.
- Barbell Graphs are used in many areas particularly in networks.

3. 2: Vertex Corona Product of Cycle and Path with Tadpole

Theorem 3.2.1: The b-coloring of vertex corona product of $\varphi[T_{3,n} \circ P_n]$

Let $n \geq 1$, be a positive integer then, $\varphi[T_{3,n} \circ P_n] = n+2$

Proof:

Let $V(T_{3,n}) = \{x_i : 1 \leq i \leq n+2\}$ and $V(P_n) = \{p_i : 1 \leq i \leq n\}$

By the definition of corona graph, each vertex of $T_{3,n}$ is adjacent to every vertex of number of copies of P_n.

i.e., every vertex from the set $V(T_{3,n})$ is adjacent to every vertex from the set $V(P_n)$.

Then the vertex set of the corona product

$V(T_{3,n} \circ P_n) = \{x_i : 1 \leq i \leq n+2\} \cup \{p^i{}_j : 1 \leq i \leq n+2, 1 \leq j \leq n+2\}$.

Assign a proper coloring to $V(T_{3,n} \circ P_n)$ as follows

- For $1 \le i \le n+2$, color the vertices x_i with colors $c_1, c_2, \ldots, c_{n+2}$ respectively
- For $1 \le i \le n+2, 1 \le j \le n+2$, $1 \le j \le n+2$ color the vertices of $p^i{}_j$ with colors $c_1, c_2, \ldots, c_{n+2}$ respectively. By this coloring procedure, we have that $\varphi[T_{3,n} \circ P_n] \ge n+2$

To prove the lower bound, let us assume that, b-chromatic number of corona product of tadpole graph $T_{3,n}$ with path graph P_n is greater than $n+2$.

That is the b-chromatic number of $T_{3,n} \circ P_n = n+3$

We can assign n+3 colors is only possible, when the graph

$T_{3,n} \circ P_n$ contains n+3 vertices with degree n+2. But the graph

$T_{3,n} \circ P_n$ contains only n+2 vertices with maximum degree n+3 which makes the maximization of n+3 colors impossible.

$\therefore \quad \varphi[T_{3,n} \circ P_n] \le n+2$

Hence $\quad \varphi[T_{3,n} \circ P_n] = n+2$

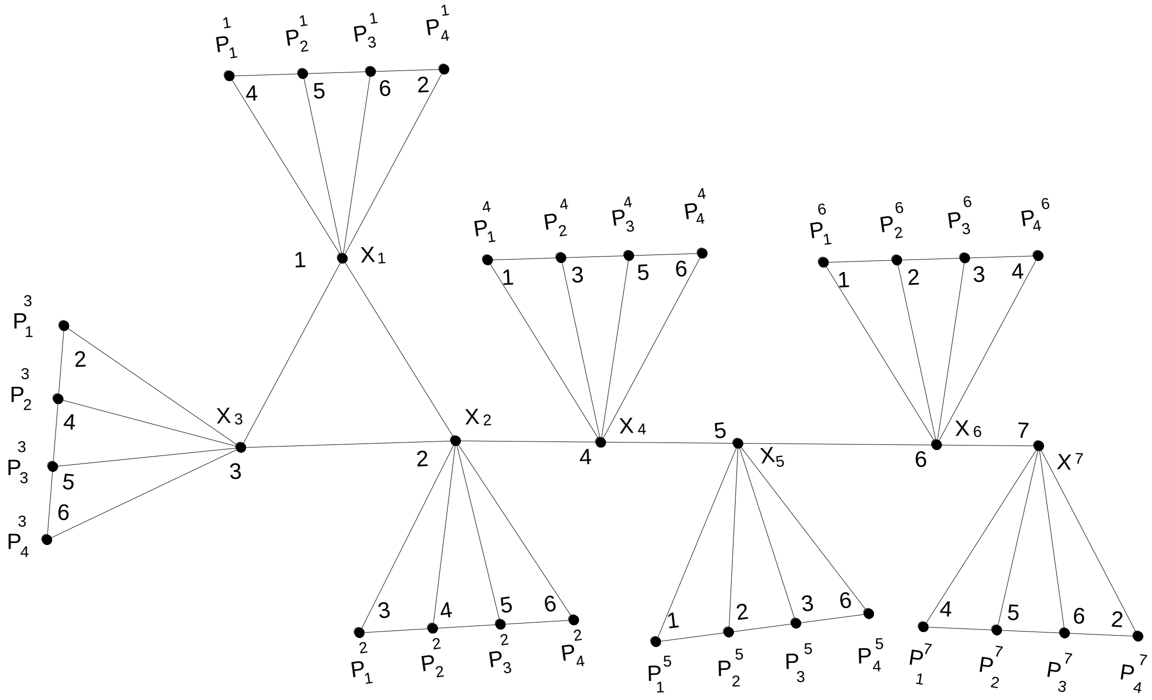

Fig: 3.2 $\varphi(T_{3,4} \circ P_4) = 6$

Theorem 3.2.2: The b-coloring of vertex corona product of $\varphi[\,T_{m,n} \circ P_n\,]$

Let $n \geq 1$, be a positive integer, the b-coloring of corona product of tadpole graph with path graph is $\varphi[\,T_{m,n} \circ P_n\,] = n+3$, m=4, 5 and n≥ 1

Proof:

Let the vertices of $T_{m,n}$ be $x_1, x_2, \ldots, x_{n+3}$. Let the vertices of path graph be $p_1, p_2, \ldots, p_{n+3}$.

$V(T_{m,n}) = \{x_j : 1 \leq j \leq m+n\}$ and $V(P_n) = \{p_i : 1 \leq i \leq n\}$

By the definition of corona graph, each vertex of $T_{m,n}$ is adjacent to every vertex of number of copies of P_n. i.e., every vertex from the set $V(T_{m,n})$ is adjacent to every vertex from the set $V(P_n)$.

Then the vertex set of the corona product

$V(T_{m,n} \circ P_n) = \{x_j : 1 \leq j \leq m+n\} \cup \{p^i{}_j : 1 \leq i \leq m+n, 1 \leq j \leq m+n\}$.

Assign a proper coloring to $V(T_{m,n} \circ P_n)$ as follows

$C(x_j) = i, 1 \leq i \leq n+3, 1 \leq j \leq m+n-1$

$C(x_j) = 1, \ j = m+n$

$C(p^i{}_j) = k, \ 1 \leq k \ \leq n+3$

An easy check shows that this coloring is a b-coloring.

The maximization of coloring depends the degree of the tadpole graph which is n+2. Then we can assign maximum number of n+3 colors to get b-coloring.

Hence $\varphi(T_{m,n} \circ P_n) = n+3$.

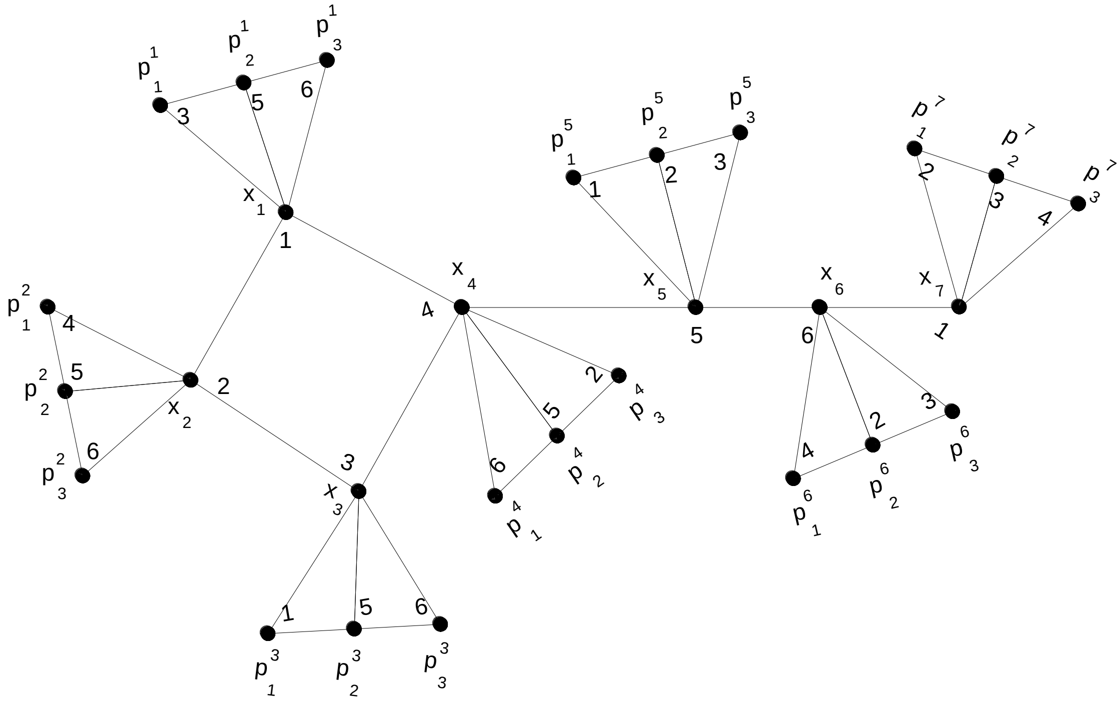

Fig: 3.3 $\varphi(T_{4,3} \circ P_3) = 6$

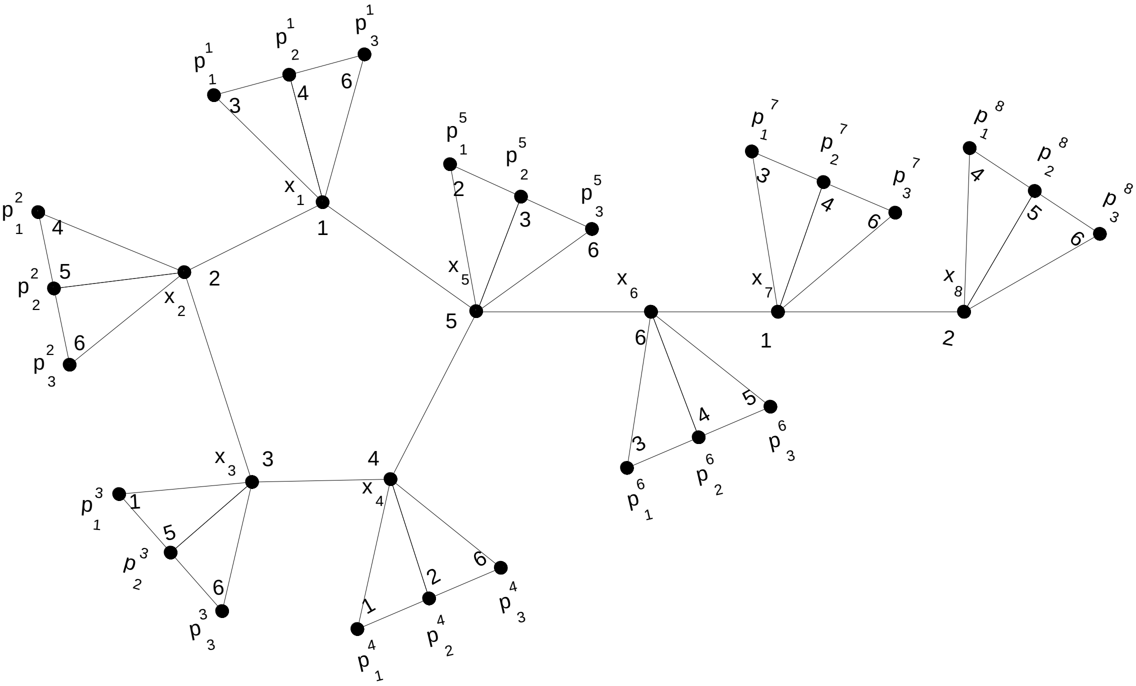

Fig: 3.4 $\varphi(T_{5,3} \circ P_3) = 6$

Theorem 3.2.3: The b-coloring of vertex corona product of $\varphi[P_n \circ T_{3,n}]$

For any positive number n, the b-chromatic number of the corona graph of path graph with tadpole graph is

$$\varphi[P_n \circ T_{3,n}] = \begin{cases} n+2, & n \le 3 \\ n+1, & n = 4 \\ n, & n \ge 5 \end{cases}$$

Proof:

Let $V(P_n) = \{p_i : 1 \le i \le n\}$ and $V(T_{3,n}) = \{x_i : 1 \le i \le n\}$

By the definition of corona graph, each vertex of P_n is adjacent to every vertex of number of copies of $T_{3,n}$. i.e., every vertex $pi \in V(P_n)$ is adjacent to every vertex from the set $\{x^i{}_j : 1 \le i \le n, 1 \le j \le n+3\} \in V(F_{3,n})$

Let $V(P_n \circ T_{3,n}) = \{p_i : 1 \le i \le n\} \cup \{x^i{}_j : 1 \le i \le n, 1 \le j \le n+3\}$

We prove the results by the following three cases.

Case 1: For $n \le 3$,

Assign the colors as follows

- For $p_i : 1 \le i \le n$, assign the color c_i
- For $x^i{}_j : 1 \le i \le n$, $1 \le j \le n+3$, assign the color c_k, $1 \le k \le n+2$

By the proper coloring procedure for the graph $P_n \circ T_{3,n}$, the maximization of b-coloring depends the vertex set of $T_{3,n}$. But $T_{3,n}$ has $n+2$ vertices of degree n+1, then we can assign n+2 colors to get b-coloring.

$\varphi[P_n \circ T_{3,n}] = n+2, n \le 3.$

Case 2: For n=4

Assign the following colors as b-chromatic number for $P_n \circ T_{3,n}$

- For p_1, p_2, p_3, p_4 assign the colors c_1, c_2, c_3, c_4 respectively
- For $x^i{}_j : 1 \le i \le 4$, $1 \le j \le 3, i \ne 3$, assign the color c_i

- For $x^i{}_j : 1 \le i \le 4\ , 4 \le j \le 7$, assign the color $c_{i\text{-}4}$
- For $x^i{}_j : i = 3\ , 1 \le j \le 7$, assign the color c_5 .
- By considering the above coloring procedure,
  we get that $\varphi[\,P_4 \circ T_{3,4}\,] = 5$

Case 3: For $n \ge 5$

Assign the following n colors as b-chromatic number for $P_n \circ T_{3,n}$

- For $p_i : 1 \le i \le n$, assign the color c_i
- For $x^i{}_j : 1 \le i \le n\ , 1 \le j \le n+3$, assign the color $c_k , 1 \le k \le n$

By this coloring procedure, we get that

$\varphi[\,P_n \circ T_{3,n}\,] \ge n.$

To prove the lower bound, consider that $\varphi[\,P_n \circ T_{3,n}\,] \ge n.$

i.e., $\varphi[\,P_n\ o T_{3,n}\,] = n+1.$

To assign n+1 colors, the graph should have n+1 distinct vertices with degree n. But the graph $P_n \circ T_{3,n}$ have only n vertices with maximum degree $\ge n+1$. which is the contradiction.

Thus $\varphi[\,P_n\ o T_{3,n}\,] \le n.$

Hence $\varphi[\,P_n\ o T_{3,n}\,] = n.$

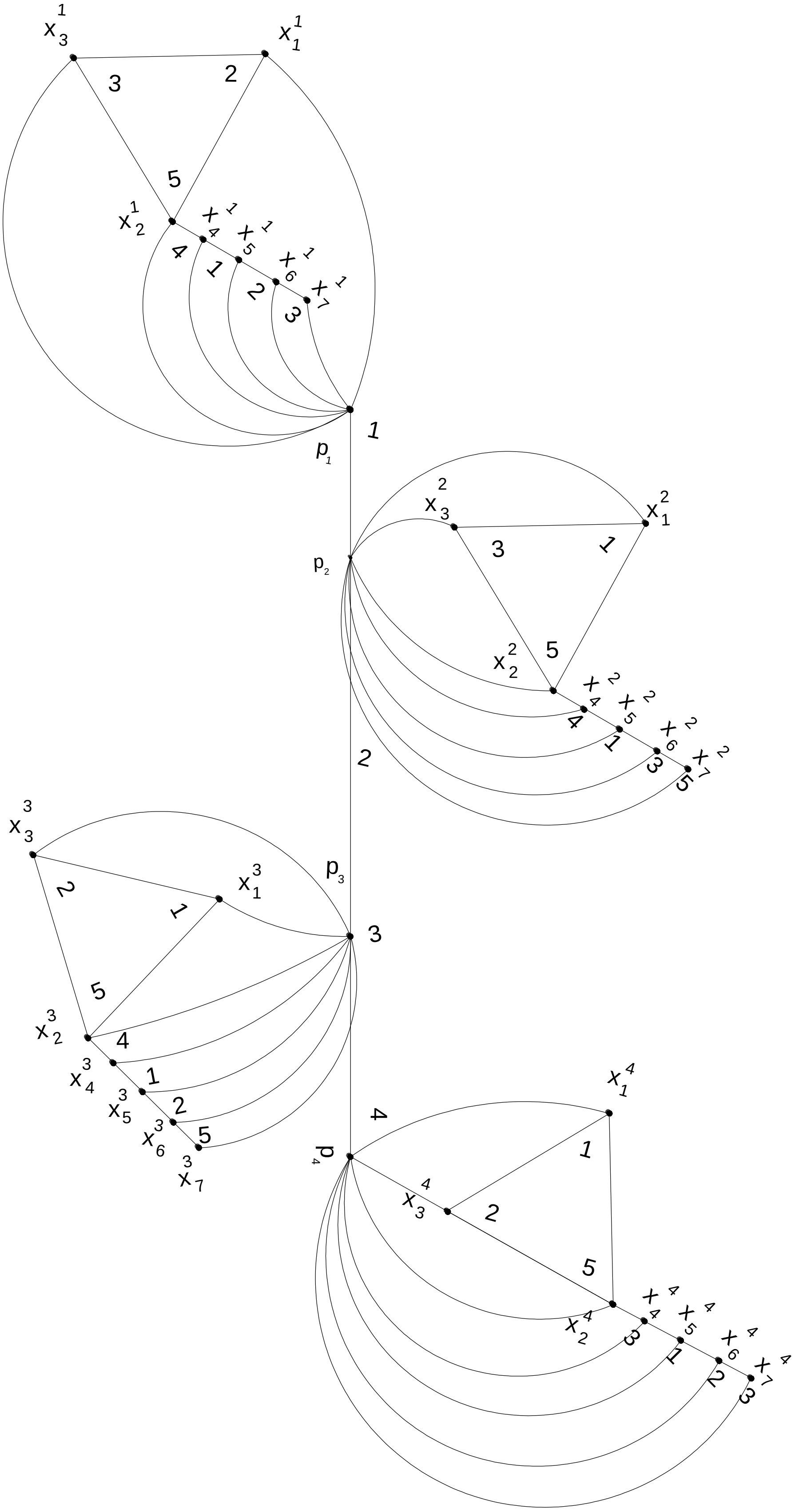

Fig: 3.5 $\varphi[\, P_4 \circ T_{3,4} \,] = 5$

3.2.4: The b-coloring of vertex corona product of $\varphi[T_{3,n} \circ C_n]$

Algorithm:

Input: The number “n” of $T_{3,n} \circ C_n$

Output: Assigning b-coloring to the vertices of $T_{3,n} \circ C_n$

```
begin
for $i = 1$ to n+2
{
    $V_1 = \{x_i\};$
    $C(x_i) = i;$
for $i = 1$ to n
{
    $V_2 = \{x_{n+3}\};$
    $C(x_{n+3}) = 1;$
}
for $i = 1$ to $n+3$, $j = 1$ to $n+3$, $k = 1$ to $n+2$
{
    $V_3 = \{y^i{}_j\};$
    $C(y^i{}_j) = c_k;$
}
$V = V_1 \cup V_2 \cup V_3;$
end.
```

Theorem 3.2.4: For any tadpole graph $T_{3,n}$ and a cycle graph C_n ,the b-chromatic number of the corona product $T_{3,n} \circ C_n$ is given by $\varphi[T_{3,n} \circ C_n] = n + 2, n \geq 1.$

Proof:

Let the vertex set of tadpole graph be $V(T_{3,n}) = \{x_i : 1 \leq i \leq n + 3\}$

Let the vertex set of cycle graph be $V(C_n) = \{y_i : 1 \leq i \leq n\}$

By the definition of corona product, each vertex of $T_{3,n}$ is adjacent to every vertex of number of copies of C_n.

i.e., every vertex $x_i \in V(T_{3,n})$ is adjacent to every vertex from the set

$\{y^i{}_j : 1 \leq i \leq n+3, 1 \leq j \leq n+3\}$

Then the vertex set $T_{3,n} \circ C_n$ is

$V(T_{3,n} \circ C_n) = \{x_i : 1 \leq i \leq n+3\} \cup \{y^i{}_j : 1 \leq i \leq n+3, 1 \leq j \leq n+3\}$

Assign a proper coloring to $T_{3,n} \circ C_n$ by using the above algorithm.

Thus we have, $\varphi[T_{3,n} \circ C_n] \geq n + 2$

To maximize the color as $n+3$, the color class c_{n+3} should adjacent with all other colors. Here this condition is not satisfied. Thus getting b-coloring is impossible.

Hence $\varphi[T_{3,n} \circ C_n] \leq n + 2,$

$\therefore \varphi[T_{3,n} \circ C_n] = n + 2.$

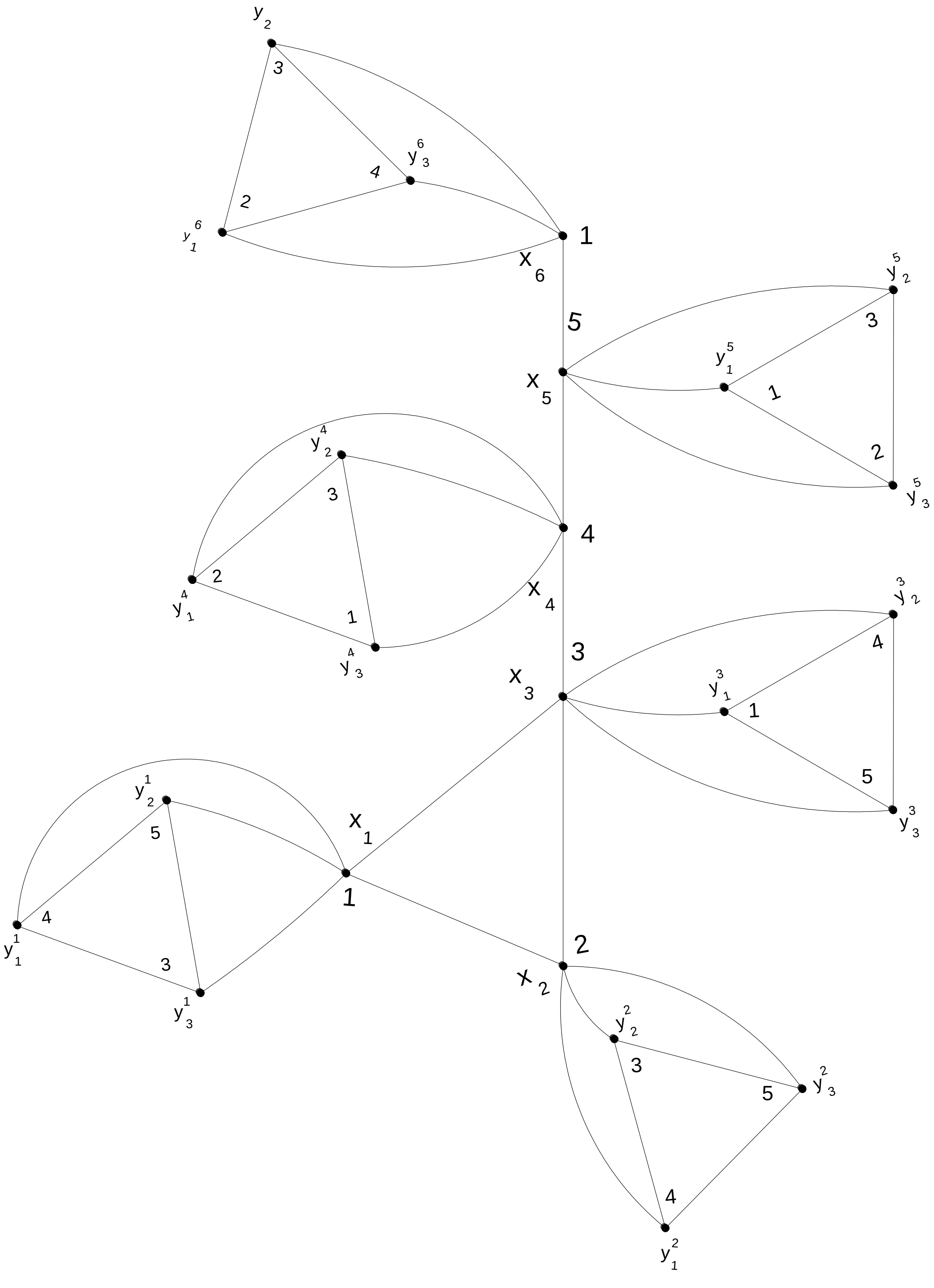

Fig 3.6: $\varphi[\,T_{3,3} \circ C_3] = 5$

3. 3: Vertex Corona Product of Cycle and Path with Barbell Graphs

Algorithm 3.3.1: The b-coloring of vertex corona product of $\varphi[B(K_n, K_n)^{\circ}C_n]$

Input: $B(K_n, K_n)^{\circ}C_n$, $n \geq 3$.

$V\{x_1, x_2, x_3, ..., x_n, y_1, y_2, y_3 .... y_n, p^1{}_1, p^1{}_2, ..., p^n{}_n, q^1{}_1, q^1{}_2, ... q^n{}_n.\}$.

```
For $i = 1$ to n
    $x_i \leftarrow i$;
end for
for $i = 1$ to n
    $y_i \leftarrow n + i$;
end for
for $i = 1$ to n, $j = 1$ to n
    $p^i{}_j \leftarrow n + j$;
end for
for $i = 1$ to n, $j = 1$ to n
    $q^i{}_j \leftarrow j$;
end for
end procedure
```

output: vertex colored $B(K_n, K_n)^{\circ}C_n$.

Theorem 3.3.1:

For any barbell graph $B(K_n, K_n)$ and cycle graph C_n, the b-chromatic number of corona graph of $B(K_n, K_n)°C_n$ is $\varphi[B(K_n, K_n)°C_n] = 2n, n \geq 3$.

Proof:

Let $V[B(K_n, K_n)] = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and
$V(C_n) = \{p_i : 1 \leq i \leq n\}$

By the definition of corona product, each vertex of $B(K_n ° K_n)$ is adjacent to every vertex of number of copies of C_n.

i.e., $\{x_i, y_i : 1 \leq i \leq n\} \in V(B(K_n, K_n))$, is adjacent to every vertex of

$\{p^i{}_j, q^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\} \in V(Cn)$

Then the vertex set of the corona product $B(K_n, K_n) oC_n$ is

$$V[B(K_n, K_n)°C_n] = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$$
$$\cup \{p^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{q^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\}$$

Assign the coloring as per the algorithm .

By this coloring procedure, we get that

$$\varphi[B(K_n, K_n)°C_n] \geq 2n$$

To prove the lower bound, let us assume that, the b-chromatic number of corona product of barbell graph $B(K_n, K_n)$ with cycle graph C_n is greater than 2n.

That is the b-chromatic number of $B(K_n, K_n)°C_n$ is equal to $2n + 1$.

Assigning $2n + 1$ colors, the graph $B(K_n, K_n)°C_n$ should have $2n + 1$ vertices with 2n degree along with distinct colors. But the graph $B(K_n, K_n)°C_n$ having only 2n vertices with maximum degree $2n - 1$, which is the contradiction. Therefore assigning $2n + 1$ colors is impossible.

$\therefore \quad \varphi[B(K_n, K_n)°C_n] \leq 2n$. Hence $\varphi[B(K_n, K_n)°C_n] = 2n$.

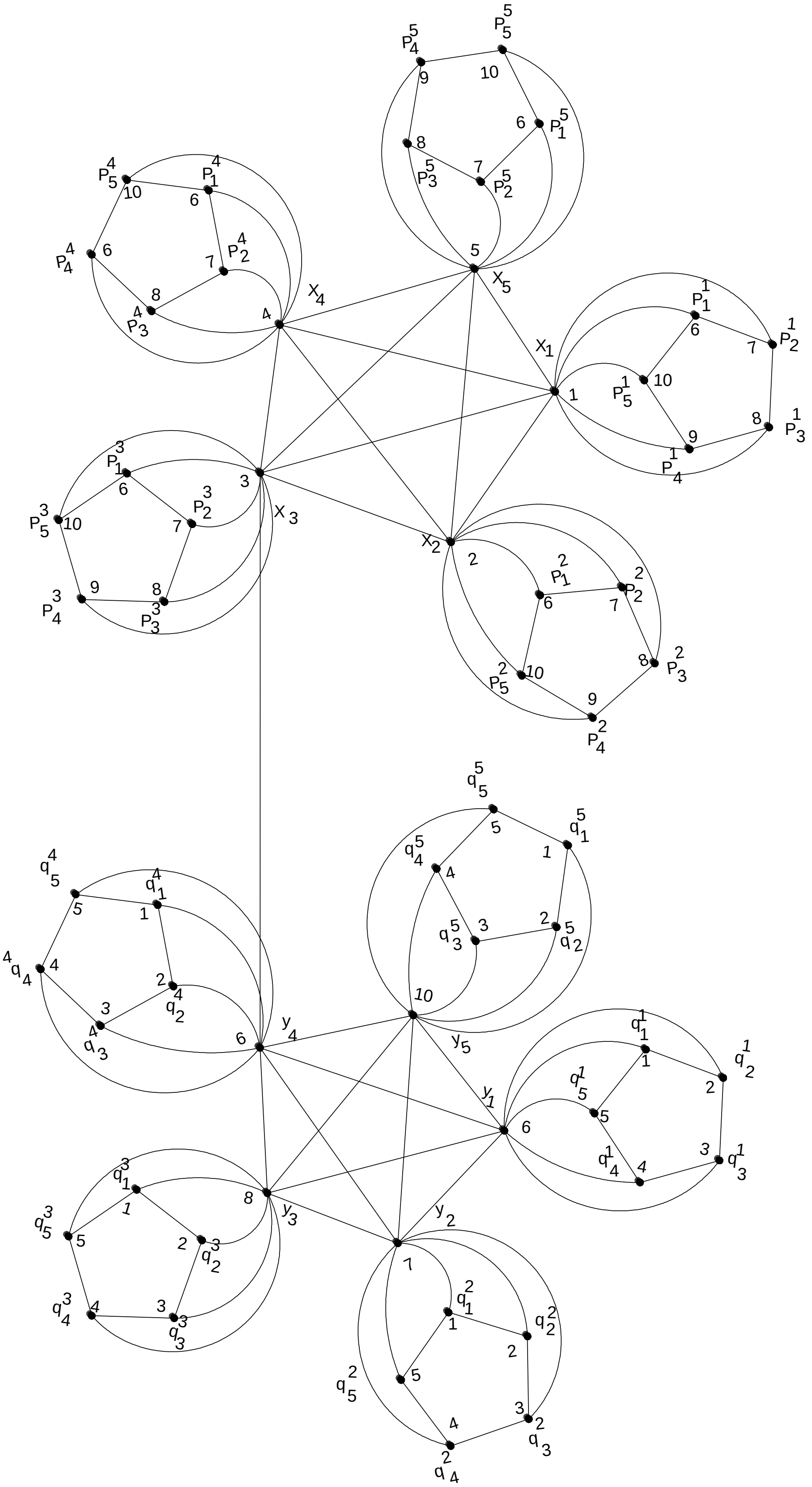

Fig 3.7: $\varphi[\mathrm{B}(\mathrm{K}_5, \mathrm{K}_5)^{\circ}\mathrm{C}_5] = 10$

Algorithm 3.3.2: The b-coloring of vertex corona product of $\varphi[B(K_n, K_n)°P_n]$

Input: $B(K_n, K_n)°P_n$, $n \geq 3$.

```
$V \leftarrow \{x_1, x_2, x_3, ..., x_n, y_1, y_2, y_3, ..., y_n, p^1{}_1, p^1{}_2, ..., p^n{}_n, q^1{}_1, q^1{}_2, ..., q^n{}_n\}$.
For $i = 1$ to n
$x_i \leftarrow i$;
end for
for $i = 1$ to n
$y_i \leftarrow n + i$;
end for
for $i = 1$ to n, $j = 1$ to n
 $p^i{}_j \leftarrow n + j$;
end for
for $i = 1$ to n, $j = 1$ to n
$q^i{}_j \leftarrow j$;
end for
end procedure
```

Output: vertex colored $B(K_n, K_n)°P_n$.

Theorem 3.3.2:

For any barbell graph $B(K_n, K_n)$ and path graph P_n, the b-chromatic number of corona graph of $B(K_n, K_n)°P_n$ is $[B(K_n, K_n)°P_n] = 2n, n \geq 3$.

Proof:

Let $V[B(K_n, K_n)] = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and $V(P_n) = \{p_i : 1 \leq i \leq n\}$.

By the definition of corona graph, each vertex of $B(K_n, K_n)$ is adjacent to every vertex of number of copies of P_n.

i.e., $\{x_i, y_i : 1 \leq i \leq n\} \in V[B(K_n, K_n)]$ is adjacent to every vertex of

$\{p^i{}_j, q^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\} \in V(Pn).$

Then the vertex set of the corona product $B(K_n, K_n)^\circ P_n$ is

$$\begin{aligned} V[B(K_n, K_n)^\circ P_n] = {} & \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \\ & \cup \{p^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{q^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\} \end{aligned}$$

Assign the coloring as per the algorithm. By this coloring procedure, we get that

i.e., $[B(K_n, K_n)^\circ P_n] \geq 2n$

To prove the lower bound, let us assume that, b-chromatic number of corona product of barbell graph $B(K_n, K_n)$ with path graph P_n is greater than 2n.

That is the b-chromatic number of $B(K_n, K_n)^\circ P_n$ is equal to $2n + 1$.

Assigning 2n+1 colors, the graph $B(K_n, K_n)^\circ P_n$ should have $2n + 1$ vertices with 2n degree along with distinct colors. But the graph $B(K_n, K_n)^\circ P_n$ having only 2n vertices with maximum degree $2n - 1$, which is the contradiction.

Therefore assigning $2n + 1$ colors is impossible.

Thus, $\varphi[B(K_n, K_n)^\circ P_n] \leq 2n.$

Hence $\varphi[B(K_n, K_n)^\circ P_n] = 2n.$

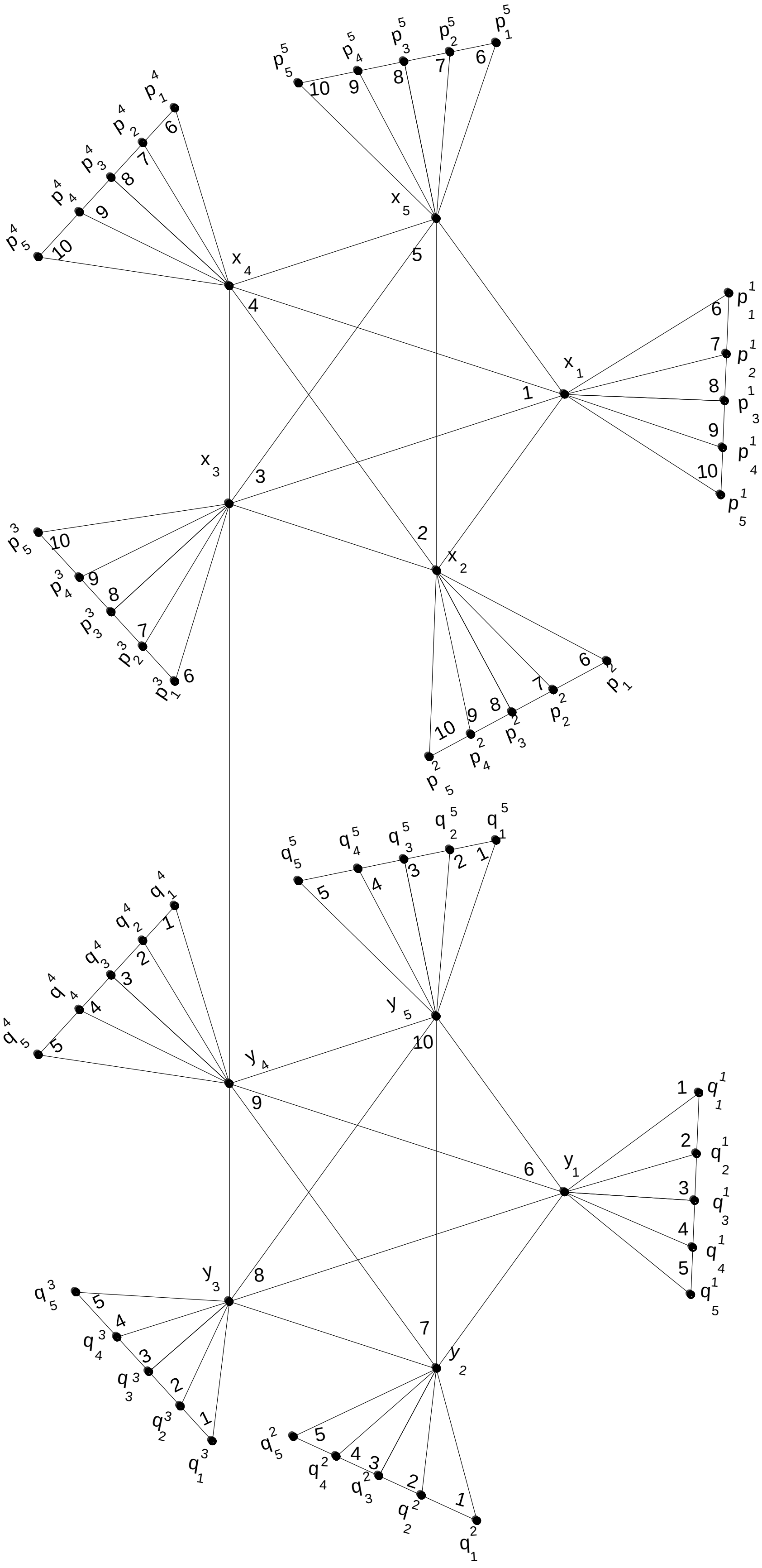

Fig 3.8: $\varphi[\,B(K_5, K_5)\circ P_5\,] = 10$