

CHAPTER V

**THE (m, N) POLICY FOR $M^X/G/1$ QUEUEING SYSTEM WITH
OPTIONAL RE-SERVICES UNDER RANDOMIZED VACATION
POLICY DURING IDLE PERIOD AND OPTIONAL SINGLE
VACATION DURING BUSY PERIOD**

INTRODUCTION

The present chapter examines the (m, N) policy for a $M^X/G/1$ queueing system with feedback customers, in which the server operates randomized (p, J) vacation policy with at most J consecutive vacations during idle period and takes an optional single vacation between two services during busy period. Whenever the system is empty, the server immediately takes a vacation. But at the end of a vacation if the number of customers waiting for service is less than m , the server either joins the system and stays idle with probability $(1 - p)$ or leaves for another vacation with probability p . And this process continues until the queue length reaches at least m at a vacation termination point (or) the server takes at most J -consecutive vacations. Two more vacation control parameters are introduced, so that the choice of server vacation between services will depend on whether the customer leaves the system after completing the service (or) demand a re-service. Though, the (p, J) vacation policy coincides with J -vacation policy (of chapters II and III) when $p = 1$, the derivation of the PGF of the system size during idle period is more complicate than in previous models. Thus in the present chapter, only single service queueing system is considered for a reliable server. However the concept of feedback is introduced and it is assumed that, the customers if unsatisfied with the primary service, may demand for re-services any number of times.

5.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

5.1.1 Model Description

The customers arrive in batches according to the time-homogeneous Poisson process with group arrival rate λ as mentioned in Chapter II.

Idle Period

The server is deactivated whenever the system becomes empty and the deactivated server immediately takes a vacation of random length (VI). Upon returning from the first vacation, the server is activated for setup work if atleast m customers are waiting in the queue. On the other hand, if the number of customers waiting in the queue is less than m , then the server either joins the system and remains idle with probability $(1 - p)$ or leaves for another vacation with probability p . This pattern continues until the number of vacations reaches J . At the end of the J^{th} vacation if the queue length is still less than m , the server then necessarily joins the queue and stays idle until the queue length reaches atleast m to start the setup work. The process after setup period regarding (m, N) policy is as in Chapter II, i.e., the setup process starts when the queue length reaches atleast m and busy period initiates when atleast N customers accumulated in the system.

Busy Period

During busy period the server provides single service to the customers according to the FCFS queue discipline. At the completion of each service, the customer who is unsatisfied with the service, may demand a re-service with probability f or leave the system with probability $(1 - f)$. The service time S follows general distribution and the distribution function, density function and the first two moments of the service time are denoted by $S(x)$, $s(x)$, $E(S^k)$, $k = 1, 2$ respectively.

At the completion of each service the server may decide to take a vacation of random length or may continue to serve the next customer. The probability that the server takes vacation after each service depends on whether the customer leaves the system or demand re-service. It is assumed that if the customer leaves the system after the completion of service, then the server takes a single vacation of random length VB_1 with probability p_{b_1} , whereas the probability that the server takes vacation in the case of re-service is p_{b_2} and the vacation time is VB_2 . Thus the probabilities with which the server continues with the next service without taking vacations are $(1 - p_{b_1})$

or $(1 - p_{b_2})$ according as the customer leaves the system or demands re-service respectively. This type of service continues until the system becomes empty.

In the present model, the server operates, randomized vacation policy during idle period and takes optional single vacations during busy period. Thus the cycle is made up of server idle vacation, buildup period, setup period, busy period (including feedback service) and vacation period between services. Various stochastic processes involved in the queueing system are assumed to be independent of each other. The customers continue to arrive and join the system independent of the system states. The system is denoted by $M_{(m,N)}^X / G / 1 / V(p, J) / \text{feedback} / VB_i, i = 1, 2$ vacations.

Using supplementary variable technique the steady state system equations under the steady-state condition are analysed and the PGF of the system size is obtained so that various performance measures of the model can be derived from it. The notations $N_S(t), \lambda, X, g_k, g_n^{(i)}, X(z)$ are same as in Chapter II (Table 2.0).

At time t , let $Y(t) = 0, 1, 2, 3, 4, 5$ and 6 respectively denote that the server is on vacation (VI) during idle period, in buildup, setup, dormant, busy state, on vacations (VB_1) and (VB_2) during busy period. The notations of Random Variables (RV), Cumulative Distribution Functions (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments of the RV_S are listed below.

	RV	CDF	PDF	LST	k^{th} moments $k = 1, 2$
Setup time	D	$D(x)$	$d(x)$	$D^*(\theta)$	$E(D^k)$
Service time	S	$S(x)$	$s(x)$	$S^*(\theta)$	$E(S^k)$
Vacation time during idle period	VI	$VI(x)$	$vI(x)$	$VI^*(\theta)$	$E(VI^k)$
Vacation time between services when customers opting / not opting re-services	VB_i	$VB_i(x)$	$vB_i(x)$	$VB_i^*(\theta)$	$E(VB_i^k) \quad i = 1, 2$

Let $D^o(t), S^o(t), VI^o(t), VB_1^o(t), VB_2^o(t)$ denote the remaining times of random variables ; setup time, service time and vacation times at time t . Then

the state space $\{N_S(t), \delta(t)\}$, where $\delta(t) = (VI^o(t), 0, D^o(t), 0, S^o(t), VB_1^o(t), VB_2^o(t))$ according as $Y(t) = 0, 1, 2, 3, 4, 5$ and 6 respectively follow Markov process.

Let $Z(t) = j$ ($j = 1, 2, \dots, J$) denote that the server is on j^{th} vacation at time t counting from the idle period initiation point. Let

$$QI_{n,j}(x, t) dt = \Pr \{N_S(t) = n, x < VI^o(t) \leq x + dt, Y(t) = 0, Z(t) = j\}, n \geq 0,$$

$$1 \leq j \leq J \text{ (during idle vacation period)}$$

$$PI_n(t) = \Pr \{N_S(t) = n, Y(t) = 1\}, 0 \leq n \leq m-1 \text{ (buildup period)}$$

$$SE_n(x, t) dt = \Pr \{N_S(t) = n, x < D^o(t) \leq x + dt, Y(t) = 2\}, n \geq m \text{ (setup period)}$$

$$U_n(t) = \Pr \{N_S(t) = n, Y(t) = 3\}, m \leq n \leq N-1 \text{ (dormant period)}$$

$$P_n(x, t) dt = \Pr \{N_S(t) = n, x < S^o(t) \leq x + dt, Y(t) = 4\}, n \geq 1 \text{ (busy period)}$$

$$QB_{1,n}(x, t) dt = \Pr \{N_S(t) = n, x < VB_1^o(t) \leq x + dt, Y(t) = 5\}, n \geq 1 \text{ (vacation during busy period when the customer is not opting feedback)}$$

$$QB_{2,n}(x, t) dt = \Pr \{N_S(t) = n, x < VB_2^o(t) \leq x + dt, Y(t) = 6\}, n \geq 1 \text{ (vacation during busy period when the customer is opting feedback)}$$

5.1.2 The Steady State System Size Equations

The steady state system size equations for the queueing system, using supplementary variable technique are obtained and listed below :

Vacation State During Idle Period

$$-\frac{d}{dx} QI_{0,1}(x) = -\lambda QI_{0,1}(x) + P_1(0) (1-f) vI(x)$$

$$-\frac{d}{dx} QI_{0,j}(x) = -\lambda QI_{0,j}(x) + QI_{0,j-1}(0) p vI(x), \quad 2 \leq j \leq J$$

$$-\frac{d}{dx} QI_{n,j}(x) = -\lambda QI_{n,j}(x) + \lambda \sum_{k=1}^n QI_{n-k,j}(x) g_k + (1 - \delta_{1,j}) QI_{n,j-1}(0) p vI(x),$$

$$1 \leq n \leq m-1, 1 \leq j \leq J$$

$$-\frac{d}{dx} QI_{n,j}(x) = -\lambda QI_{n,j}(x) + \lambda \sum_{k=1}^n QI_{n-k,j}(x) g_k, \quad n \geq m, 2 \leq j \leq J$$

Buildup State

$$\lambda P I_n = Q I_{n,j}(0) + (1 - \delta_{0,n}) \lambda \sum_{k=1}^n P I_{n-k} g_k + \sum_{j=1}^{J-1} Q I_{n,j}(0)(1 - p)$$

$$0 \leq n \leq m-1$$

Setup State

$$-\frac{d}{dx} S E_n(x) = -\lambda S E_n(x) + (1 - \delta_{m,n}) \lambda \sum_{k=1}^{n-m} S E_{n-k}(x) g_k + \sum_{j=1}^J Q I_{n,j}(0) d(x)$$

$$+ \lambda \sum_{k=n-m+1}^n P I_{n-k} g_k d(x), \quad n \geq m$$

Dormant State

$$\lambda U_n = S E_n(0) + (1 - \delta_{m,n}) \lambda \sum_{k=1}^{n-m} U_{n-k} g_k, \quad m \leq n \leq N-1$$

Busy State

$$-\frac{d}{dx} P_n(x) = -\lambda P_n(x) + P_{n+1}(0) (1 - f) (1 - p_{b_1}) s(x)$$

$$+ (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} P_{n-k}(x) g_k + P_n(0) f (1 - p_{b_2}) s(x)$$

$$+ Q B_{1,n}(0) s(x) + Q B_{2,n}(0) s(x), \quad 1 \leq n \leq N-1$$

$$-\frac{d}{dx} P_n(x) = -\lambda P_n(x) + P_{n+1}(0) (1 - f) (1 - p_{b_1}) s(x) + \lambda \sum_{k=1}^{n-1} P_{n-k}(x) g_k$$

$$+ P_n(0) f (1 - p_{b_2}) s(x) + Q B_{1,n}(0) s(x) + Q B_{2,n}(0) s(x)$$

$$+ S E_n(0) s(x) + \lambda \sum_{k=n-N+1}^{n-m} U_{n-k} g_k s(x), \quad n \geq N$$

Vacation during busy period when the customer is not opting feedback

$$-\frac{d}{dx} Q B_{1,n}(x) = -\lambda Q B_{1,n}(x) + P_{n+1}(0) (1 - f) p_{b_1} v B_1(x)$$

$$+ (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} Q B_{1,n-k}(x) g_k, \quad n \geq 1$$

Vacation during busy period when the customer is opting feedback

$$-\frac{d}{dx} Q B_{2,n}(x) = -\lambda Q B_{2,n}(x) + P_n(0) f p_{b_2} v B_2(x) + (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} Q B_{2,n-k}(x) g_k,$$

$$n \geq 1$$

The LST for the steady-state equations are given by,

$$\theta QI_{0,1}^*(\theta) - QI_{0,1}(0) = \lambda QI_{0,1}^*(\theta) - P_1(0) (1 - f) VI^*(\theta) \quad (5.1)$$

$$\theta QI_{0,j}^*(\theta) - QI_{0,j}(0) = \lambda QI_{0,j}^*(\theta) - QI_{0,j-1}(0) p VI^*(\theta) \quad 2 \leq j \leq J \quad (5.2)$$

$$\begin{aligned} \theta QI_{n,j}^*(\theta) - QI_{n,j}(0) &= \lambda QI_{n,j}^*(\theta) - \lambda \sum_{k=1}^n QI_{n-k,j}^*(\theta) g_k \\ &\quad - (1 - \delta_{1,j}) QI_{n,j-1}(0) p VI^*(\theta) \quad 1 \leq n \leq m-1, 1 \leq j \leq J \end{aligned} \quad (5.3)$$

$$\theta QI_{n,j}^*(\theta) - QI_{n,j}(0) = \lambda QI_{n,j}^*(\theta) - \lambda \sum_{k=1}^n QI_{n-k,j}^*(\theta) g_k, \quad n \geq m, 2 \leq j \leq J \quad (5.4)$$

$$\begin{aligned} \lambda PI_n &= QI_{n,J}(0) + (1 - \delta_{0,n}) \lambda \sum_{k=1}^n PI_{n-k} g_k + \sum_{j=1}^{J-1} QI_{n,j}(0) (1 - p) \\ &\quad 0 \leq n \leq m-1 \end{aligned} \quad (5.5)$$

$$\begin{aligned} \theta SE_n^*(\theta) - SE_n(0) &= \lambda SE_n^*(\theta) - (1 - \delta_{m,n}) \lambda \sum_{k=1}^{n-m} SE_{n-k}^*(\theta) g_k \\ &\quad - \sum_{j=1}^J QI_{n,j}(0) D^*(\theta) - \lambda \sum_{k=n-m+1}^n PI_{n-k} g_k D^*(\theta), \\ &\quad n \geq m \end{aligned} \quad (5.6)$$

$$\lambda U_n = SE_n(0) + (1 - \delta_{m,n}) \lambda \sum_{k=1}^{n-m} U_{n-k} g_k, \quad m \leq n \leq N-1 \quad (5.7)$$

$$\begin{aligned} \theta P_n^*(\theta) - P_n(0) &= \lambda P_n^*(\theta) - P_{n+1}(0) (1 - f) (1 - p_{b_1}) S^*(\theta) \\ &\quad - (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} P_{n-k}^*(\theta) g_k - P_n(0) f (1 - p_{b_2}) S^*(\theta) \\ &\quad - QB_{1,n}(0) S^*(\theta) - QB_{2,n}(0) S^*(\theta) \quad 1 \leq n \leq N-1 \end{aligned} \quad (5.8)$$

$$\begin{aligned} \theta P_n^*(\theta) - P_n(0) &= \lambda P_n^*(\theta) - P_{n+1}(0) (1 - f) (1 - p_{b_1}) S^*(\theta) \\ &\quad - \lambda \sum_{k=1}^{n-1} P_{n-k}^*(\theta) g_k - P_n(0) f (1 - p_{b_2}) S^*(\theta) \\ &\quad - QB_{1,n}(0) S^*(\theta) - QB_{2,n}(0) S^*(\theta) - SE_n(0) S^*(\theta) \\ &\quad - \lambda \sum_{k=n-N+1}^{n-m} U_{n-k} g_k S^*(\theta), \quad n \geq N \end{aligned} \quad (5.9)$$

$$\begin{aligned} \theta QB_{1,n}^*(\theta) - QB_{1,n}(0) &= \lambda QB_{1,n}^*(\theta) - P_{n+1}(0) (1 - f) p_{b_1} VB_1^*(\theta) \\ &\quad - (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} QB_{1,n-k}^*(\theta) g_k, \quad n \geq 1 \end{aligned} \quad (5.10)$$

$$\begin{aligned} \theta QB_{2,n}^*(\theta) - QB_{2,n}(0) &= \lambda QB_{2,n}^*(\theta) - P_n(0) f p_{b_2} VB_2^*(\theta) \\ &\quad - (1 - \delta_{1,n}) \lambda \sum_{k=1}^{n-1} QB_{2,n-k}^*(\theta) g_k, \quad n \geq 1 \end{aligned} \quad (5.11)$$

5.1.3 Probability Generating Functions

The following partial PGFs are considered to analyse the model.

$$\begin{aligned} PI(z) &= \sum_{n=0}^{m-1} PI_n z^n & U(z) &= \sum_{n=m}^{N-1} U_n z^n \\ QI_j^*(z, \theta) &= \sum_{n=0}^{\infty} QI_{n,j}^*(\theta) z^n, & QI_j(z, 0) &= \sum_{n=0}^{\infty} QI_{n,j}(0) z^n \quad 1 \leq j \leq J \\ QB_i^*(z, \theta) &= \sum_{n=1}^{\infty} QB_{i,n}^*(\theta) z^n, & QB_i(z, 0) &= \sum_{n=1}^{\infty} QB_{i,n}(0) z^n \quad i = 1, 2 \\ SE^*(z, \theta) &= \sum_{n=m}^{\infty} SE_n^*(\theta) z^n, & SE(z, 0) &= \sum_{n=m}^{\infty} SE_n(0) z^n \\ P^*(z, \theta) &= \sum_{n=1}^{\infty} P_n^*(\theta) z^n, & P(z, 0) &= \sum_{n=1}^{\infty} P_n(0) z^n \end{aligned}$$

Multiplying the corresponding equations by suitable powers of z and adding the equations, the partial generating functions are derived through some algebraic manipulations.

Equations (5.1) and (5.3) at $j = 1$ imply,

$$(\theta - w_X(z)) QI_1^*(z, \theta) = QI_1(z, 0) - P_1(0) (1 - f) VI^*(\theta) \quad (5.12)$$

where $w_X(z) = \lambda (1 - X(z))$

$$\text{At } \theta = w_X(z), QI_1(z, 0) = P_1(0) (1 - f) VI^*(w_X(z)) \quad (5.13)$$

$$\text{and } QI_1^*(z, \theta) = \frac{P_1(0) (1 - f) (VI^*(w_X(z)) - VI^*(\theta))}{(\theta - w_X(z))} \quad (5.14)$$

Similarly equations (5.2), (5.3) and (5.4) for $2 \leq j \leq J$ lead to

$$QI_j(z, 0) = p VI^*(w_X(z)) \sum_{n=0}^{m-1} QI_{n,j-1}(0) z^n, \quad 2 \leq j \leq J \quad (5.15)$$

$$\text{and } QI_j^*(z, \theta) = \left(\frac{VI^*(w_X(z)) - VI^*(\theta)}{\theta - w_X(z)} \right) p \sum_{n=0}^{m-1} QI_{n,j-1}(0) z^n, \quad 2 \leq j \leq J \quad (5.16)$$

The sum of the equations (5.13) and (5.15) over $j = 1$ to J yields

$$QI(z, 0) = \sum_{j=1}^J QI_j(z, 0) = VI^*(w_X(z)) y_V(z) \quad (5.17)$$

$$\text{where } y_V(z) = p \sum_{j=1}^{J-1} \sum_{n=0}^{m-1} QI_{n,j}(0) z^n + P_1(0) (1-f) \quad (5.18)$$

The sum of the equations (5.14) and (5.16) imply

$$QI^*(z, \theta) = \sum_{j=1}^J QI_j^*(z, \theta) = \frac{(VI^*(w_X(z)) - VI^*(\theta)) y_V(z)}{(\theta - w_X(z))} \quad (5.19)$$

Next to calculate the PGF corresponding to setup state, equations (5.6) and (5.5) are used, which in turn lead to

$$\begin{aligned} (\theta - w_X(z)) SE^*(z, \theta) &= SE(z, 0) + D^*(\theta) PI(z) w_X(z) - D^*(\theta) \left[\sum_{j=1}^J QI_j(z, 0) \right. \\ &\quad \left. - p \sum_{n=0}^{m-1} \sum_{j=1}^{J-1} QI_{n,j}(0) z^n \right] \end{aligned} \quad (5.20.0)$$

Substituting for $QI(z, 0)$ from (5.17) and using the definition of $y_V(z)$, the above equation at $\theta = w_X(z)$ gives,

$$SE(z, 0) = D^*(w_X(z)) [(VI^*(w_X(z)) - 1) y_V(z) + P_1(0) (1-f) - PI(z) w_X(z)] \quad (5.20)$$

Thus equation (5.20.0) implies,

$$SE^*(z, \theta) = \frac{(D^*(w_X(z)) - D^*(\theta))}{(\theta - w_X(z))} [(VI^*(w_X(z)) - 1) y_V(z) + P_1(0) (1-f) - PI(z) w_X(z)] \quad (5.21)$$

Next to evaluate $y_V(z)$, the following arguments are used. By conditioning on the arrival size during previous vacation

$$QI_{n,1}(0) = \alpha I_n (1-f) P_1(0), \quad n \geq 0 \quad (5.22)$$

$$\text{and } QI_{n,j}(0) = p \sum_{k=0}^n \alpha I_k QI_{n-k,j-1}(0), \quad 0 \leq n \leq m-1, \quad 2 \leq j \leq J \quad (5.23)$$

where αI_n denotes the probability that n customers arrive during a vacation period (VI).

Using equations (5.22), (5.23) recursively, it is found that

$$QI_{n,j}(0) = (1-f) P_1(0) p^{j-1} \alpha I_n^{(j)}, \quad 0 \leq n \leq m-1, \quad 1 \leq j \leq J \quad (5.23.1)$$

with the convolution product $\alpha I_n^{(j)} = \sum_{k=0}^n \alpha I_{n-k}^{(j-1)} \alpha I_k$.

Adding equations (5.22) and (5.23) over $1 \leq j \leq J$,

$$\sum_{j=1}^J QI_{n,j}(0) = \alpha I_n (1-f) P_1(0) + p \sum_{j=1}^{J-1} \sum_{k=0}^n \alpha I_k QI_{n-k,j}(0), \quad 0 \leq n \leq m-1 \quad (5.24)$$

$$\text{By letting } \delta_0 = (1-f) P_1(0) + p \sum_{j=1}^{J-1} QI_{0,j}(0) \text{ and} \quad (5.24.1)$$

$$\delta_n = p \sum_{j=1}^{J-1} QI_{n,j}(0), \quad 1 \leq n \leq m-1 \quad (5.24.2)$$

$y_V(z)$ of equation (5.18) can be written as

$$y_V(z) = \sum_{n=0}^{m-1} \delta_n z^n \quad (5.24.3)$$

Then equation (5.24) together with (5.23.1), at $n = 0$ lead to,

$$\delta_0 = (1-f) P_1(0) \frac{(1 - (\alpha I_0 p)^J)}{1 - \alpha I_0 p} \quad (5.25)$$

and for $1 \leq n \leq m-1$,

$$\delta_n = p \sum_{k=0}^n \alpha I_k \delta_{n-k} - p QI_{n,J}(0) \quad (5.25.1)$$

follows from (5.24) and (5.24.2)

For further simplification the following theorem is used.

Theorem : 5.1

$$\begin{aligned} \text{If } \beta_0 &= (1 - (\alpha I_0 p)^J), \\ \beta_n &= p \sum_{k=1}^n \frac{\alpha I_k \beta_{n-k}}{1 - \alpha I_0 p} - \alpha I_n^{(J)} p^J, \quad \text{for } 1 \leq n \leq m-1 \end{aligned} \quad (5.26.1)$$

$$\text{Then } \delta_n = P_1(0) (1-f) \frac{\beta_n}{1 - \alpha I_0 p}, \quad 0 \leq n \leq m-1 \quad (5.26.2)$$

Proof

Equation (5.25) shows that the equation (5.26.2) is true for $n = 0$.

Equation (5.25.1) implies,

$$\delta_n (1 - p \alpha I_0) = p \sum_{k=1}^n \alpha I_k \delta_{n-k} - p QI_{n,J}(0)$$

Thus by induction, substituting for δ_{n-k} and using equation (5.23.1)

$$(1 - \alpha I_0 p) \delta_n = P_1(0) (1 - f) \left[p \sum_{k=1}^n \frac{\beta_{n-k} \alpha I_k}{1 - \alpha I_0 p} - \alpha I_n^{(j)} p^j \right]$$

$$\text{i.e., } \delta_n = P_1(0) (1 - f) \frac{\beta_n}{(1 - \alpha I_0 p)}, \quad 1 \leq n \leq m-1 \quad (5.26.3)$$

$$\text{Thus, } y_V(z) = \sum_{n=0}^{m-1} \delta_n z^n = P_1(0) (1 - f) \beta(z) \quad (5.27)$$

$$\text{Where } \beta(z) = \sum_{n=0}^{m-1} \frac{\beta_n z^n}{1 - \alpha I_0 p} \quad (5.27.1)$$

Next to find the PGF of the number of customers in the system when the server is in buildup period, the following result is used.

Theorem : 5.2

$$\text{If } \pi_0 = 1, \quad \pi_n = \sum_{k=1}^n \pi_{n-k} g_k, \quad (5.28)$$

$$\psi_{n,1} = \sum_{k=0}^n \pi_{n-k} (\alpha I_k^{(j)}) p^{j-1}, \quad \psi_{n,2} = \left(\sum_{k=0}^n \frac{\pi_{n-k} \beta_k}{1 - \alpha I_0 p} - \pi_n \right), \quad 0 \leq n \leq m-1$$

$$\text{and } \psi_n = p \psi_{n,1} + (1 - p) \psi_{n,2} \quad (5.28.1)$$

$$\text{then } PI(z) = P_1(0) (1 - f) \psi(z), \quad \text{where } \psi(z) = \left(\frac{1}{\lambda p} \right) \sum_{n=0}^{m-1} \psi_n z^n \quad (5.28.2)$$

Proof

Equation (5.5) recursively implies,

$$\lambda PI_n = \sum_{k=0}^n \pi_{n-k} QI_{k,j}(0) + (1 - p) \sum_{k=0}^n \pi_{n-k} \sum_{j=1}^{j-1} QI_{k,j}(0)$$

Substituting for $QI_{k,j}(0)$ from (5.23.1) and using the definition for δ_n ($0 \leq n \leq m-1$),

$$\lambda PI_n = \sum_{k=0}^n \pi_{n-k} (\alpha I_k^{(j)}) p^{j-1} (1 - f) P_1(0) + \frac{1-p}{p} \quad (5.28.3)$$

Substituting for δ_n from (5.26.3),

$$\lambda PI_n = \frac{(1-f)P_1(0)}{p} [p \psi_{n,1} + (1-p) \psi_{n,2}] = \frac{(1-f)P_1(0)}{p} \psi_n$$

where ψ_n ($0 \leq n \leq m-1$) is as in equation (5.28.1)

$$\text{(i.e.) } PI(z) = P_1(0) (1 - f) \psi(z) \quad (5.28.4)$$

Corollary

For $0 \leq n \leq m-1$

$$(i) \quad \sum_{i=0}^{n-1} \psi_{i,1} g_{n-i} = \sum_{k=0}^{n-1} \alpha I_k^{(j)} p^{j-1} \pi_{n-k}$$

$$(ii) \quad \alpha I_n^{(j)} p^{j-1} + \sum_{i=0}^{n-1} \psi_{i,1} g_{n-i} = \psi_{n,1}$$

$$(iii) \quad \sum_{i=0}^{n-1} \psi_{i,2} g_{n-i} = \sum_{k=0}^{n-1} \frac{\beta_k \pi_{n-k}}{1 - \alpha I_0 p} - \pi_n$$

$$(iv) \quad \frac{\beta_n}{1 - \alpha I_0 p} + \sum_{i=0}^{n-1} \psi_{i,2} g_{n-i} = \psi_{n,2}$$

The proof of the corollary is analogous to the Corollary of Theorem 2.2 of Chapter II.

Now substituting for $y_v(z)$ and $PI(z)$ from (5.27) and (5.28.4) in equations (5.17), (5.19), (5.20) and (5.21), we have

$$QI(z, 0) = VI^*(w_X(z)) \beta(z) P_1(0) (1 - f) \quad (5.29)$$

$$QI^*(z, \theta) = \left(\frac{VI^*(w_X(z)) - VI^*(\theta)}{\theta - w_X(z)} \right) \beta(z) P_1(0) (1 - f) \quad (5.30)$$

$$SE(z, 0) = D^*(w_X(z)) ((VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z)) P_1(0) (1 - f) \quad (5.31)$$

$$SE^*(z, \theta) = \frac{(D^*(w_X(z)) - D^*(\theta))}{\theta - w_X(z)} ((VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z)) P_1(0) (1 - f) \quad (5.32)$$

where $\beta(z)$ is given by the equation (5.27.1).

Using equations (5.10) and (5.11), the PGFs corresponding to the vacation during busy period are obtained as,

$$QB_1(z, 0) = \frac{(1-f)}{z} p_{b_1} VB_1^*(w_X(z)) (P(z, 0) - P_1(0) z) \quad (5.33)$$

$$QB_1^*(z, \theta) = \frac{(1-f)}{z} p_{b_1} (P(z, 0) - P_1(0) z) \left(\frac{VB_1^*(w_X(z)) - VB_1^*(\theta)}{\theta - w_X(z)} \right) \quad (5.34)$$

$$QB_2(z, 0) = f p_{b_2} VB_2^*(w_X(z)) P(z, 0) \quad (5.35)$$

$$QB_2^*(z, \theta) = f p_{b_2} P(z, 0) \left(\frac{VB_2^*(w_X(z)) - VB_2^*(\theta)}{\theta - w_X(z)} \right) \quad (5.36)$$

The generating functions of the system size at service completion epoch and at arbitrary epoch, when the server is busy are derived as follows.

Equations (5.8) and (5.9) together with (5.33) and (5.35) lead to

$$\begin{aligned}
 (\theta - w_X(z)) P^*(z, \theta) = & P(z, 0) - \frac{S^*(\theta)}{z} [(1-f)(1-p_{b_1})(P(z, 0) - P_1(0)z) \\
 & + f(1-p_{b_2})P(z, 0) + z(QB_1(z, 0) + QB_2(z, 0))] \\
 & - S^*(\theta) \left(\sum_{n=N}^{\infty} SE_k(0)z^n + \lambda \sum_{n=N}^{\infty} z^n \sum_{k=n-N+1}^{n-m} U_{n-k} g_k \right) \quad (5.37)
 \end{aligned}$$

Summation of equation (5.7) over $n = m$ to $N - 1$ gives,

$$\lambda U(z) = \sum_{n=m}^{N-1} SE_n(0)z^n + \lambda \sum_{n=m+1}^{N-1} z^n \sum_{k=1}^{n-m} U_{n-k} g_k \quad (5.38)$$

Adding equation (5.37) with, equation (5.38) multiplied by $(-S^*(\theta))$, and substituting for $QB_i(z, 0)$ ($i = 1, 2$), it is found that

$$(\theta - w_X(z)) P^*(z, \theta) = \frac{P(z, 0)}{z} (z - S^*(\theta) \Phi(z)) - S^*(\theta) \mathcal{J}(z), \text{ where} \quad (5.39)$$

$$\Phi(z) = (1-f)((1-p_{b_1}) + p_{b_1} VB_1^*(w_X(z))) + f z ((1-p_{b_2}) + p_{b_2} VB_2^*(w_X(z)))$$

$$\text{and} \quad (5.39.1)$$

$$\mathcal{J}(z) = SE(z, 0) - U(z) w_X(z) - (1-f) P_1(0) (1 - p_{b_1} + p_{b_1} VB_1^*(w_X(z))) \quad (5.39.2)$$

At $\theta = w_X(z)$, equation (5.39) gives,

$$P(z, 0) = \frac{z S^*(w_X(z)) \mathcal{J}(z)}{z - S^*(w_X(z)) \Phi(z)} \quad (5.40)$$

Substituting for $P(z, 0)$ in equation (5.39),

$$P^*(z, \theta) = \frac{z (S^*(w_X(z)) - S^*(\theta)) \mathcal{J}(z)}{(\theta - w_X(z)) (z - S^*(w_X(z)) \Phi(z))} \quad (5.41)$$

Finally, the probability generating function of the system size, when the server is in dormant period is derived as follows :

Equation (5.7) recursively gives,

$$\lambda U_n = \sum_{k=m}^n SE_k(0) \pi_{n-k} \quad m \leq n \leq N-1 \quad (5.42)$$

where $SE_k(0)$ is the coefficient of z^k of $SE(z, 0)$

For further simplification, the following result is used.

Theorem : 5.3

$$(i) \quad (VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z) = \sum_{n=m}^{\infty} S_n z^n, \quad (5.43)$$

$$\text{where } S_n = \sum_{k=0}^{m-1} \left(\frac{\alpha I_{n-k} \beta_k}{1 - \alpha I_0 \rho} + \frac{\psi_k g_{n-k}}{\rho} \right)$$

and $\beta(z)$ is given by the equation (5.27.1).

$$(ii) \quad D^*(w_X(z)) ((VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z)) = \sum_{n=m}^{\infty} z^n \quad (5.44)$$

where h_k is the probability that k customers arrive during the setup period D .

Proof

$$\begin{aligned} & (VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z) \\ &= \left(\sum_{n=0}^{\infty} \alpha I_n z^n \right) \left(- \sum_{n=0}^{m-1} \frac{\beta_n z^n}{1 - \alpha I_0 \rho} + 1 - \sum_{n=0}^{m-1} \frac{\psi_n z^n}{\rho} + \left(\sum_{n=0}^{m-1} \frac{\psi_n z^n}{\rho} \right) \left(\sum_{k=1}^{\infty} g_k z^k \right) \right) \end{aligned}$$

The right hand side can be simplified using the corollary of Theorem 5.2. The procedure is analogous to Theorem 2.3.

After simplification

$$\begin{aligned} (VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z) &= \sum_{n=m}^{\infty} z^n \left(\sum_{k=0}^{m-1} \left(\frac{\beta_k \alpha I_{n-k}}{1 - \alpha I_0 \rho} + \frac{\psi_k g_{n-k}}{\rho} \right) \right) \\ &= \sum_{n=m}^{\infty} z^n S_n \end{aligned}$$

$$\text{where } S_n = \sum_{k=0}^{m-1} \left(\frac{\alpha I_{n-k} \beta_k}{1 - \alpha I_0 \rho} + \frac{\psi_k g_{n-k}}{\rho} \right)$$

Thus

$$\begin{aligned} D^*(w_X(z)) ((VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z)) &= \left(\sum_{k=0}^{\infty} h_k z^k \right) \left(\sum_{i=m}^{\infty} S_i z^i \right) \\ &= \sum_{n=m}^{\infty} z^n \sum_{i=m}^n S_i h_{n-i} \end{aligned}$$

For $k \geq m$, $SE_k(0) =$ the coefficient of z^k in $SE(z, 0)$ implies

$$SE_k(0) = P_1(0) (1 - f) (\text{coefficient of } z^k \text{ in } [D^*(w_X(z)) ((VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z))])$$

$$SE_k(0) = P_1(0) (1 - f) \sum_{i=m}^k S_i h_{k-i} \text{ follows from (5.44)}$$

Substituting for $SE_k(0)$ in equation (5.42) and rearranging the summation,

$$\lambda U_n = P_1(0) (1-f) \sum_{r=m}^n S_r \sum_{k=0}^{n-r} h_k \pi_{n-r-k}$$

$$\text{Thus } U(z) = P_1(0) (1-f) \sum_{n=m}^{N-1} \frac{\phi_n z^n}{\lambda} = P_1(0) (1-f) \phi(z) \quad (5.45)$$

$$\text{where, } \phi_n = \sum_{r=m}^n S_r \sum_{k=0}^{n-r} h_k \pi_{n-r-k} \text{ and } \phi(z) = \left(\frac{1}{\lambda} \right) \sum_{n=m}^{N-1} \phi_n z^n \quad (5.45.1)$$

Using equations (5.31) and (5.45), it is found that,

$$\begin{aligned} SE(z, 0) - U(z) w_X(z) - (1-f) P_1(0) ((1-p_{b_1}) + p_{b_1} VB_1^*(w_X(z))) \\ = \mathcal{J}(z) = -P_1(0) (1-f) w_X(z) I_{(m,N)}(z) \end{aligned} \quad (5.46)$$

$$\begin{aligned} \text{where } I_{(m,N)}(z) = \frac{(1-VI^*(w_X(z)))}{w_X(z)} \beta(z) D^*(w_X(z)) + \frac{1-D^*(w_X(z))}{w_X(z)} + \phi(z) \\ + \psi(z) D^*(w_X(z)) + p_{b_1} \frac{(VB_1^*(w_X(z))-1)}{w_X(z)} \end{aligned} \quad (5.47)$$

Substituting for $\mathcal{J}(z)$, equations (5.41) and (5.40) at $\theta = 0$ imply,

$$P^*(z, 0) = z (S^*(w_X(z)) - 1) \frac{P_1(0)(1-f) I_{(m,N)}(z)}{(z - S^*(w_X(z)) \Phi(z))} \quad (5.48)$$

$$P(z, 0) = \frac{z S^*(w_X(z)) P_1(0)(1-f) (-w_X(z)) I_{(m,N)}(z)}{(z - S^*(w_X(z)) \Phi(z))} \quad (5.48.1)$$

where $\Phi(z)$ is given by equation (5.39.1).

The partial PGFs of the system size of the model are listed below :

$$QI^*(z, 0) = \frac{P_1(0)(1-f)(1-VI^*(w_X(z)))}{w_X(z)} \beta(z) \quad (5.49.1)$$

$$SE^*(z, 0) = \frac{P_1(0)(1-f)(1-D^*(w_X(z)))}{w_X(z)} [(VI^*(w_X(z)) - 1) \beta(z) + 1 - \psi(z) w_X(z)] \quad (5.49.2)$$

$$U(z) = \frac{P_1(0)(1-f)}{\lambda} \sum_{n=m}^{N-1} \phi_n z^n = P_1(0) (1-f) \phi(z) \quad (5.49.3)$$

$$PI(z) = P_1(0) (1-f) \psi(z) \quad (5.49.4)$$

$$P^*(z, 0) = z (S^*(w_X(z)) - 1) \frac{(P_1(0)(1-f) I_{(m,N)}(z))}{(z - S^*(w_X(z)) \Phi(z))} \quad (5.49.5)$$

$$QB_1^*(z, 0) = \frac{p_{b_1} (1-f)(1-VB_1^*(w_X(z)))}{w_X(z)} \frac{(P(z, 0) - P_1(0)z)}{z} \quad (5.49.6)$$

$$QB_2^*(z, 0) = \frac{f p_{b_2} P(z, 0)(1-VB_2^*(w_X(z)))}{w_X(z)} \quad (5.49.7)$$

where $P(z, 0)$ is given by the equation (5.48.1).

Thus the partial generating functions are expressed in terms of the only one unknown $P_1(0)$.

To derive the total PGF of the system size distribution, the following generating functions are considered.

$$\begin{aligned} P_{Idle}(z) &= \text{Probability generating function of the system size when the} \\ &\quad \text{server is idle (vacation + buildup + setup + dormant)} \\ &= PI(z) + SE^*(z, 0) + U(z) + QI^*(z, 0) + QB_1^*(z, 0) + QB_2^*(z, 0) \\ &= P_1(0) (1-f) I_{(m,N)}(z) \frac{(z - S^*(w_X(z))(1-f + f z))}{(z - S^*(w_X(z)) \Phi(z))} \end{aligned} \quad (5.50)$$

$$\begin{aligned} P_{Busy}(z) &= \text{The PGF of the system size when server is busy} \\ &= P^*(z, 0) = \frac{z P_1(0)(1-f) I_{(m,N)}(z) (S^*(w_X(z)) - 1)}{z - S^*(w_X(z)) \Phi(z)} \end{aligned} \quad (5.51)$$

Thus the total PGF of the system size distribution is given by

$$P_{(m,N)}^{BVF}(z) = P_{Idle}(z) + P_{Busy}(z) = \frac{P_1(0)(1-f)^2 (z-1) S^*(w_X(z)) I_{(m,N)}(z)}{z - S^*(w_X(z)) \Phi(z)} \quad (5.52)$$

$P_1(0)$ can be calculated by using the normalizing condition.

If $D(z) = (z - S^*(w_X(z)) \Phi(z))$, then $D'(1) = (1-f)(1 - \rho_{BVF})$. Thus,

$$P_{(m,N)}^{BVF}(1) = 1 \text{ implies } P_1(0) = \frac{(1 - \rho_{BVF})}{(1-f) I_{(m,N)}(1)} \quad (5.53)$$

$$\text{where } \rho_{BVF} = \lambda E(X) E(H_{BVF}) \quad (5.53.1)$$

$$E(H_{BVF}) = \frac{1}{(1-f)} (E(S) + f p_{b_2} E(VB_2) + (1-f) p_{b_1} E(VB_1)) \text{ and} \quad (5.54)$$

$$I_{(m,N)}(1) = E(D) + \left(\frac{1}{\lambda}\right) \sum_{n=m}^{N-1} \phi_n + \left(\frac{1}{p\lambda}\right) \sum_{n=0}^{m-1} \psi_n + E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0 p} - p_{b_1} E(VB_1) \quad (5.55)$$

$$\text{Hence } P_{(m,N)}^{\text{BVF}}(z) = \frac{(1-f)(1-\rho_{\text{BVF}})(z-1)S^*(w_X(z))}{z - S^*(w_X(z))\Phi(z)} \frac{I_{(m,N)}(z)}{I_{(m,N)}(1)} \quad (5.56)$$

where $\Phi(z)$ is given by equation (5.39.1)

Using equation (5.50) $P_{(m,N)}^{\text{BVF}}(z)$ can be written as,

$$P_{(m,N)}^{\text{BVF}}(z) = \frac{(1-f)(1-\rho_S)(z-1)S^*(w_X(z))}{[z - S^*(w_X(z))(1-f+fz)]} \frac{P_{\text{Idle}}(z)}{P_{\text{Idle}}(1)} \quad (5.56.1)$$

where $\rho_S = \lambda E(X)E(S)$.

$\frac{(1-f)(1-\rho_S)(z-1)S^*(w_X(z))}{[z - S^*(w_X(z))(1-f+fz)]}$ gives the generating function of the system

size of $M^X/G/1$ queueing model with feedback and without threshold policies.

5.1.4 Decomposition Property

Theorem : 5.3

Under the steady-state condition $\rho_{\text{BVF}} < 1$, the PGF of the stationary system size of the (m, N) – policy queueing model under consideration is the product of PGF of the system size of $M^X/G/1$ feedback queue (without threshold policy and without vacation) and the PGF of the conditional system size during the server idle period.

5.1.5 Queue Size Distribution at Departure Epoch

Following the arguments of Lee et al. (1994a), the PGF $\pi^+(z)$ of the queue size distribution $\{\pi_n^+; n \geq 0\}$ at departure epoch is given by

$$\pi^+(z) = \sum_{n=0}^{\infty} \pi_n^+ z^n = \frac{D_1}{z} [P(z, 0)] \text{ where } D_1 \text{ is the normalizing constant.}$$

Substituting for $\frac{P(z, 0)}{z}$ from (5.48.1) and evaluating the constant D_1 ,

$$\pi^+(z) = \frac{(X(z)-1)}{(z-1)E(X)} P_{(m,N)}^{\text{BVF}}(z)$$

5.1.6 Performance Measures

In this section, the steady-state system size probabilities and the mean number of customers in the system at various states are calculated.

The Server in Idle State

Let P_{VI} , P_{Set} , $P_{Buildup}$, P_{Dor} , P_{VB_1} and P_{VB_2} denote the steady-state system size probabilities and L_{VI} , L_{Set} , L_{Build} , L_{Dor} , L_{VB_1} and L_{VB_2} denote the mean number of customers, when the system is on vacation (during idle period), in setup, buildup, dormant states and on vacation (during busy period). Then these measures can be calculated using equations (5.49.1) to (5.49.4), (5.49.6) and (5.49.7).

Thus,

$$P_{VI} = \lim_{z \rightarrow 1} QI^*(z, 0) = P_1(0) (1 - f) E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0 \rho} \quad (5.57)$$

$$\begin{aligned} L_{VI} &= \left[\frac{d}{dz} QI^*(z, 0) \right]_{z=1} \\ &= P_1(0) (1 - f) \end{aligned} \quad (5.57.1)$$

$$P_{Set} = \lim_{z \rightarrow 1} SE^*(z, 0) = P_1(0) (1 - f) E(D) \quad (5.58)$$

$$\begin{aligned} L_{Set} &= \left[\frac{d}{dz} SE^*(z, 0) \right]_{z=1} \\ &= P_1(0) (1 - f) \lambda E(X) \left[\frac{E(D^2)}{2} + E(D) \left(E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0 \rho} + \sum_{n=0}^{m-1} \frac{\psi_n}{\rho \lambda} \right) \right] \end{aligned} \quad (5.58.1)$$

$$P_{Buildup} = \lim_{z \rightarrow 1} PI(z) = P_1(0) (1 - f) \sum_{n=0}^{m-1} \frac{\psi_n}{\rho \lambda} \quad (5.59)$$

$$L_{Buildup} = \left[\frac{d}{dz} PI(z) \right]_{z=1} = P_1(0) (1 - f) \sum_{n=0}^{m-1} \frac{n \psi_n}{\rho \lambda} \quad (5.59.1)$$

$$P_{Dor} = \lim_{z \rightarrow 1} U(z) = P_1(0) (1 - f) \sum_{n=m}^{N-1} \frac{\phi_n}{\lambda} \quad (5.60)$$

$$L_{Dor} = \left[\frac{d}{dz} U(z) \right]_{z=1} = P_1(0) (1 - f) \sum_{n=m}^{N-1} \frac{n \phi_n}{\lambda} \quad (5.60.1)$$

$$P_{VB_1} = \lim_{z \rightarrow 1} QB_1^*(z, 0) = p_{b_1} E(VB_1) \left[\lambda E(X) - \frac{(1 - \rho_{BVF})}{I_{(m,N)}(1)} \right] \quad (5.61)$$

$$\begin{aligned}
L_{VB_1} &= \left[\frac{d}{dz} QB_1^*(z, 0) \right]_{z=1} \\
&= (1-f) p_{b_1} \left[FVB_i - \frac{\lambda E(X)(1-\rho_{BVF}) E(VB_1^2)}{I_{(m,N)}(1)(1-f)} \frac{E(VB_1^2)}{2} \right] \quad (5.61.1)
\end{aligned}$$

where

$$\begin{aligned}
FVB_i &= \frac{\lambda E(X(X-1))E(VB_i) + (\lambda E(X))^2 E(VB_i^2)}{2(1-f)} + \frac{\lambda E(X)E(VB_i)I'_{(m,N)}(1)}{(1-f)I_{(m,N)}(1)} \\
&+ \frac{\lambda E(X)E(VB_i)(-D''(1))}{2(1-\rho_{BVF})(1-f)^2} + \frac{(\lambda E(X))^2 E(S)E(VB_i)}{(1-f)}, \quad i = 1, 2 \quad (5.61.2)
\end{aligned}$$

$$P_{VB_2} = \lim_{z \rightarrow 1} QB_2^*(z, 0) = f p_{b_2} E(VB_2) \frac{\lambda E(X)}{1-f} \quad (5.62)$$

$$L_{VB_2} = \left[\frac{d}{dz} QB_2^*(z, 0) \right]_{z=1} = f p_{b_2} \left(FVB_2 + \frac{\lambda E(X)E(VB_2)}{1-f} \right) \quad (5.62.1)$$

where

$$\begin{aligned}
(-D''(1)) &= \lambda E(X(X-1)) [E(S) + (1-f) p_{b_1} E(VB_1) + f p_{b_2} E(VB_2)] \\
&+ (\lambda E(X))^2 \{E(S^2) + (1-f) p_{b_1} E(VB_1^2) + f p_{b_2} E(VB_2^2) \\
&+ 2 E(S) [f p_{b_2} E(VB_2) + (1-f) p_{b_1} E(VB_1)]\} \\
&+ 2\lambda E(X) f [p_{b_2} E(VB_2) + E(S)] \quad (5.63)
\end{aligned}$$

$$\begin{aligned}
I'_{m,N}(1) &= \lambda E(X) \left[E(D)E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1-\alpha I_0 p} + \sum_{n=0}^{m-1} \frac{\beta_n}{1-\alpha I_0 p} \frac{E(VI^2)}{2} + \frac{E(D^2)}{2} \right. \\
&\quad \left. + E(D) \sum_{n=0}^{m-1} \frac{\psi_n}{p \lambda} - p_{b_1} \frac{E(VB_1^2)}{2} \right] \\
&+ E(VI) \sum_{n=0}^{m-1} \frac{n \beta_n}{1-\alpha I_0 p} + \sum_{n=m}^{N-1} \frac{n \phi_n}{\lambda} + \sum_{n=0}^{m-1} \frac{n \psi_n}{p \lambda} \quad (5.64)
\end{aligned}$$

The Server in Busy State

The probability that the server is busy with the service (P_{Busy}) and the expected number of customers in the system L_{Busy} are calculated using the equation (5.49.5). One can note that,

$$\begin{aligned}
P_{Busy} &= \text{Probability that the server is busy} \\
&= \lim_{z \rightarrow 1} P^*(z, 0) = \frac{\lambda E(X)E(S)}{(1-f)} = \frac{\rho_S}{1-f} \quad (5.65)
\end{aligned}$$

$$L_{\text{Busy}} = \left[\frac{d}{dz} P^*(z, 0) \right]_{z=1} = F_{\text{BVF}} + \frac{\lambda E(X)E(S)}{1-f}$$

where

$$F_{\text{BVF}} = \frac{\lambda E(X(X-1))E(S) + (\lambda E(X))^2 E(S^2)}{2(1-f)} + \frac{\lambda E(X)E(S)(-D''(1))}{2(1-f)^2(1-\rho_{\text{BVF}})} + \frac{\lambda E(X)E(S)}{(1-f)} \frac{I'_{(m,N)}(1)}{I_{(m,N)}(1)} \quad (5.65.1)$$

Let $E(\text{Cycle})$ denote the expected length of the cycle, then

$$E(\text{Cycle}) = \frac{(1-f)I_{(m,N)}(1)}{1-\rho_{\text{BVF}}}$$

Mean System Size

The expected system size $L_{(m,N)}^{\text{BVF}}$ of the model is given by

$$L_{(m,N)}^{\text{BVF}} = \left[\frac{d}{dz} P_{(m,N)}^{\text{BVF}}(z) \right]_{z=1} = L^{\text{BVF}} + \frac{I'_{(m,N)}(1)}{I_{(m,N)}(1)}$$

$$\text{where } L^{\text{BVF}} = \lambda E(X)E(S) + \frac{(-D''(1))}{2(1-f)(1-\rho_{\text{BVF}})} \quad (5.65.2)$$

Note :

It is also verified that

$$L_{(m,N)}^{\text{BVF}} = L^{\text{BF}} + \frac{P'_{\text{idle}}(1)}{P_{\text{idle}}(1)}$$

$$\text{where } L^{\text{BF}} = \rho_S + \frac{\lambda E(X(X-1))E(S) + (\lambda E(X))^2 E(S^2) + 2f \lambda E(X)E(S)}{2(1-f)(1-\rho_S)}$$

5.2 OPTIMAL MANAGEMENT POLICY

By following the procedure given in Chapter II, the optimal threshold values (m^*, N^*) for the present model can be obtained.

Recalling the cost structure (C_y (startup cost per cycle), C_{set} (setup cost), C_{build} (buildup cost), C_{dor} (standby cost), C_{br} (breakdown cost), C_{busy} (operating cost), C_h (holding cost per customer), C_{vl} (reward per unit time due to idle vacation), C_{VB_1} (cost due to VB_1) and C_{VB_2} (cost due to VB_2), the total average cost per unit time of the system is given by

$$T_C^{BVF}(m, N) = \frac{C_y}{E_{\text{cycle}}} + C_{\text{set}} P_{\text{set}} + C_{\text{dor}} P_{\text{dor}} + C_{\text{busy}} P_{\text{busy}} + C_{\text{build}} P_{\text{build}} \\ + C_{VB_1} P_{VB_1} + C_{VB_2} P_{VB_2} + C_h L_{(m,N)}^{BVF} - C_{VI} P_{VI}$$

By substituting the corresponding measures, the above equation can be re-written as

$$T_C^{BVF}(m, N) = \bar{A} + \frac{1}{I_{(m,N)}(1)} [A_1 + Z(m) + C_{\text{dor}} (1 - \rho_{BVF}) \sum_{n=m}^{N-1} \frac{\phi_n}{\lambda} + \frac{C_h}{\lambda} \sum_{n=m}^{N-1} n \phi_n]$$

where

$$\bar{A} = C_h L^{BVF} + C_{\text{busy}} P_{\text{busy}} + \lambda E(X) E(VB_1) p_{b_1} C_{VB_1} + \lambda E(X) E(VB_2) p_{b_2} C_{VB_2} \frac{f}{1-f}$$

L^{BVF} is given by equation (5.65.2).

$$A_1 = (1 - \rho_{BVF}) \left[\frac{C_y}{1-f} + C_{\text{set}} E(D) - C_{VB_1} p_{b_1} E(VB_1) \right] \\ + C_h \left[\lambda E(X) \left(\frac{E(D^2)}{2} - p_{b_1} \frac{E(VB_1^2)}{2} \right) \right]$$

$$Z(m) = C_h \left\{ \lambda E(X) \left[\left(\frac{E(VI^2)}{2} + E(D) E(VI) \right) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0 \rho} + \sum_{n=0}^{m-1} \frac{\psi_n}{\rho \lambda} E(D) \right] \right. \\ \left. + E(VI) \sum_{n=0}^{m-1} \frac{n \beta_n}{1 - \alpha I_0 \rho} + \sum_{n=0}^{m-1} \frac{n \psi_n}{\rho \lambda} \right\} \\ + C_{\text{build}} (1 - \rho_{BVF}) \sum_{n=0}^{m-1} \frac{\psi_n}{\rho \lambda} - C_{VI} (1 - \rho_{BVF}) E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0 \rho}$$

By calculation,

$$T_C^{BVF}(m, k+1) - T_C^{BVF}(m, k) = \frac{\phi_k}{\lambda I_{(m, k+1)}(1) I_{(m, k)}(1)} H^{BVF}(m, k) \text{ where}$$

$$H^{BVF}(m, k) = C_{\text{dor}} (1 - \rho_{BVF}) d_m^{BV_1} + C_h \left(k d_m^{BV_1} + \sum_{n=m}^{k-1} \frac{(k-n)}{\lambda} \phi_n \right) - (A_1 + Z(m))$$

$$d_m^{BV_1} = E(D) + \sum_{n=0}^{m-1} \frac{\psi_n}{\rho \lambda} + E(VI) \sum_{n=0}^{m-1} \frac{\beta_n}{1 - \alpha I_0 \rho} - p_{b_1} E(VB_1)$$

By proceeding as in Chapter II, it is found that the condition under which $T_C^{BVF}(m, k+1) - T_C^{BVF}(m, k) > 0$ for the first value of k is given by $N^*(m) = \min \{k / (H^{BVF}(m, k) > 0)\}$. The optimal threshold values $(m^*, N^*(m))$ can be obtained by following the algorithm given in Chapter II.

5.3 PARTICULAR CASES

In this section, the PGF of the system size of some queueing models are derived as special cases and presented.

Case 1

The J-vacation policy of Chapter IV and (p, J) vacation policy of Chapter V are the same when $p = 1$. It is verified that $\beta(z)$, $\psi(z)$ and $\phi(z)$ of Chapter II and V coincide with each other when $p = 1$.

If $p_{b_1} = 0$ then $I_{(m,N)}(z)$ of Chapter V coincides with that of Chapter II. In addition if $p_{b_2} = 0$ and $f = 0$ the total PGFs of the system size of Chapter V coincides with that of Chapter II with, the selection of $r = 0$ and $a = a_i = 0$ for $i = 1$ to C in Chapter II.

Case 2

Ke and Huang (2010) analysed $M^X/G/1$ queueing model with delayed repair under $\langle p, J \rangle$ vacation for the classical 1-policy.

If $m = N = 1$; $p_{b_1} = p_{b_2} = 0$ $\Pr(D = 0) = 1$ and $f = 0$; then equation (5.47) is reduced to

$$I_{(1,1)}(z) = \frac{1 - VI^*(w_X(z))}{w_X(z)} \frac{\beta_0}{1 - \alpha I_0 p} + \frac{p \psi_{0,1} + (1-p) \psi_{0,2}}{p \lambda}. \text{ After simplification}$$

$$\frac{I_{(1,1)}(z)}{I_{(1,1)}(1)} = \frac{[(1 - (p \alpha I_0)^J)(1 - VI^*(w_X(z))) + \alpha I_0 (1 - X(z))(\alpha I_0^{J-1} p^J (1 - \alpha I_0) + 1 - p)]}{(1 - X(z))[\lambda E(VI)(1 - (\alpha I_0 p)^J) + \alpha I_0 (p^J \alpha I_0^{J-1} (1 - \alpha I_0) + 1 - p)]}$$

Thus the PGF of the present chapter leads to

$$P(z) = \frac{(1-p)(z-1)S^*(w_X(z))}{z - S^*(w_X(z))} \frac{I_{(1,1)}(z)}{I_{(1,1)}(1)} \text{ where } \rho = \lambda E(X) E(S). \text{ This}$$

coincides with the corresponding result of Ke and Huang (2010) for a reliable server model.

Case 3

The results of the present chapter can be compared with that of Chapter IV with the following identifications :

Let $J = 1$; $p_{b_1} = p_{b_2} = p$; $f = 0$; $VB_1 = VB_2 = VB$ in (5.56 and 5.47) of Chapter V and $r = 0$, $p' = 1$; $a_1 = a_2 = 0$; $M = 1$ in (4.54 and 4.47) of Chapter IV. Then the PGF of the system size for both the models will be

$$P(z) = \frac{(1-\rho)(z-1)S^*(w_X(z))}{z - S^*(w_X(z))(1-p + pVB^*(w_X(z)))} \frac{I_{(m,N)}(z)}{I_{(m,N)}(1)}$$

$$\text{and } I_{(m,N)}(z) = \frac{1 - VI^*(w_X(z))D^*(w_X(z))}{w_X(z)} + D^*(w_X(z))\psi(z) + \phi(z) \\ + \frac{p(VB^*(w_X(z)) - 1)}{w_X(z)}$$

5.4 NUMERICAL ANALYSIS

In order to study the effects of the probability (f) with which the customers opt feedback service, arrival rate (λ) and Bernoulli vacation probabilities p_{b_1} , p_{b_2} and p , on the optimum policies, numerical values are obtained in Tables 5.1 to 5.5. The different distributions used for numerical computations are presented in the following table.

Random variables (Y)		Distribution F(Y)	Mean E(Y)	Second order moments E(Y ²)
Service (S)		Two-stage hyper-exponential	$\frac{b_1}{\mu_1} + \frac{1-b_1}{\mu_2}$ $0 \leq b_1 \leq 1$	$2 \left(\frac{b_1}{\mu_1^2} + \frac{1-b_1}{\mu_2^2} \right)$
Setup time (D)		Erlang-3 type	$\frac{1}{\gamma}$	$\frac{4}{3\gamma^2}$
Vacation	VI	Erlang -3 type	$\frac{1}{\eta}$	$\frac{4}{3\eta^2}$
	VB; $i = 1, 2$	Erlang-2 type	$\frac{1}{\eta_i}$	$\frac{3}{2\eta_i^2}$
Batch size (X)		Geometric (Geo(p_1))	$\frac{1}{1-p_1}$	$\frac{p_1 + 1}{(1-p_1)^2}$
		Binomial B(2, p_1)	$2p_1$	$2p_1(1+p_1)$

The expected cost values of $T_C(m, N)$ for different values of m and N are listed in Table 5.1 and plotted in Figure 5.1. The optimal threshold values (m^*, N^*) and $T_C(m^*, N^*(m))$ are respectively obtained at (14, 14) and 502.201. This justifies the solution procedure in the theory.

Table 5.1 The Expected Cost $T_C(m, N)$ for Different Values m and N

$(C_h, C_{build}, C_{set}, C_{dor}, C_y, C_{busy}, C_{VI}, C_{VB_1}, C_{VB_2}) = (30, 8, 1000, 60, 10000, 10, 8, 20, 5)$

$(\rho, \gamma, \eta, p_1, \lambda, f, J) = (0.05, 1.0, 0.5, 0.3, 0.6, 0.03, 13)$

$(b_1, \mu_1, \mu_2, \eta_1, \eta_2, p_{b_1}, p_{b_2}) = (0.5, 1.5, 2, 0.3, 0.2, 0.01, 0.2)$

$m \backslash N$	1	4	7	10	13	14	16	20	22
1	1458.599	871.830	626.555	543.961	518.015	515.888	517.861	538.208	553.668
3		842.374	621.753	540.886	515.698	513.736	515.964	536.641	552.212
5			614.904	537.262	512.944	511.175	513.707	534.788	550.499
7			590.437	533.681	510.284	508.702	511.526	532.996	548.842
9				528.523	507.774	506.368	509.467	531.301	547.274
11					505.394	504.190	507.536	529.707	545.798
13					503.030	502.402	505.766	528.214	544.413
14						502.201	505.053	527.506	543.754
15							504.867	526.824	543.119

Figure 5.1. The Expected Cost $T_C(m, N)$ for Different Values of m and N

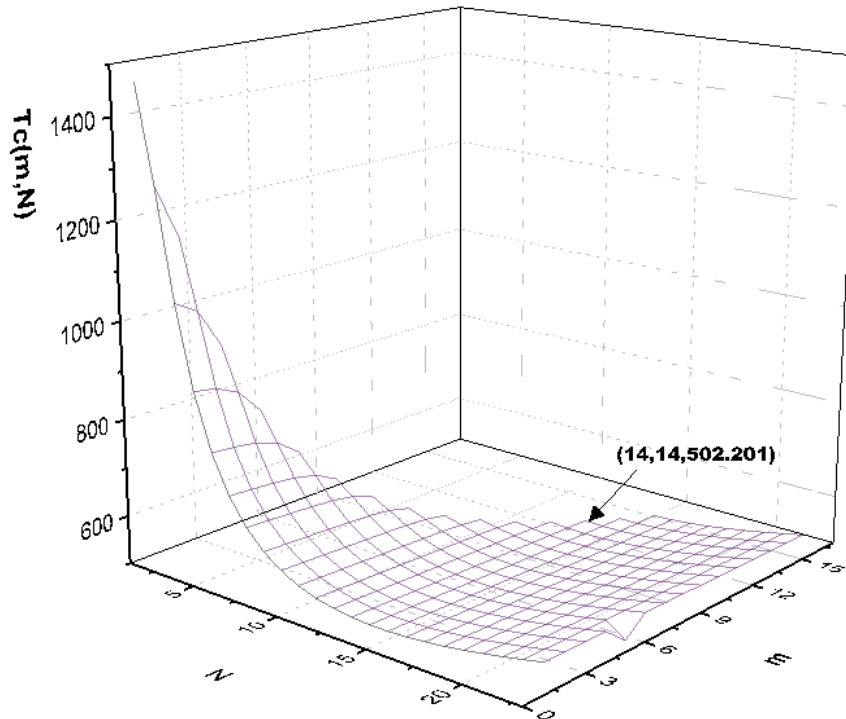


Table 5.2 gives the optimal values of both the (m, N) policy and (N, N) policy models as the standby cost C_{dor} changes. The table values show that as C_{dor} increases, the optimal values of (m, N) -policy approach the optimal values of (N, N) -policy. As in previous chapters (II to IV) (m, N) -policy provides lower average cost than the single threshold (N, N) -policy. The parameters used to create Table 5.2 to 5.5 are the same as in Table 5.1. The variant parameters are represented in the corresponding tables.

Table 5.2 Optimal Policy as Dormant Cost Changes

$(C_h, C_{dor}, C_{busy}, C_y, C_{V1}, C_{build}, C_{set}, C_{VB_1}, C_{VB_2}) = (10, 25, 10, 1000, 18, 8, 150, 20, 5)$; $J =$

15

C_{dor}	(m^*, N^*)	$T_C(m^*, N^*)$	$L_{(m^*, N^*)}^J$	$E(\text{Cycle})$	N^*	$T_C(N^*, N^*)$	$L_{(m^*, N^*)}^J$	$E(\text{Cycle})$
10	(1, 8)	105.527	5.640	27.689	7	106.888	5.685	26.619
12	(2, 8)	106.121	5.661	27.719	7	106.888	5.685	26.619
15	(6, 8)	106.590	5.812	28.046	7	106.888	5.685	26.619
20	(7, 8)	106.848	5.920	28.529	7	106.888	5.685	26.619
25	(7, 7)	106.888	5.685	26.619	7	106.888	5.685	26.619

In Table 5.3 the influence of the group arrival rate λ , feedback probability f and the probability with which the server takes vacation between two consecutive services (p_{b_1}) on queue size (L) are analysed. It is found that the increasing values of λ or f or p_{b_1} cause, an increase in the mean system size. For the computation of Tables 5.3 to 5.5, $m = 5$ and $N = 10$.

Table 5.3 Mean System Size Vs. Feedback Probability f and Arrival Rate λ

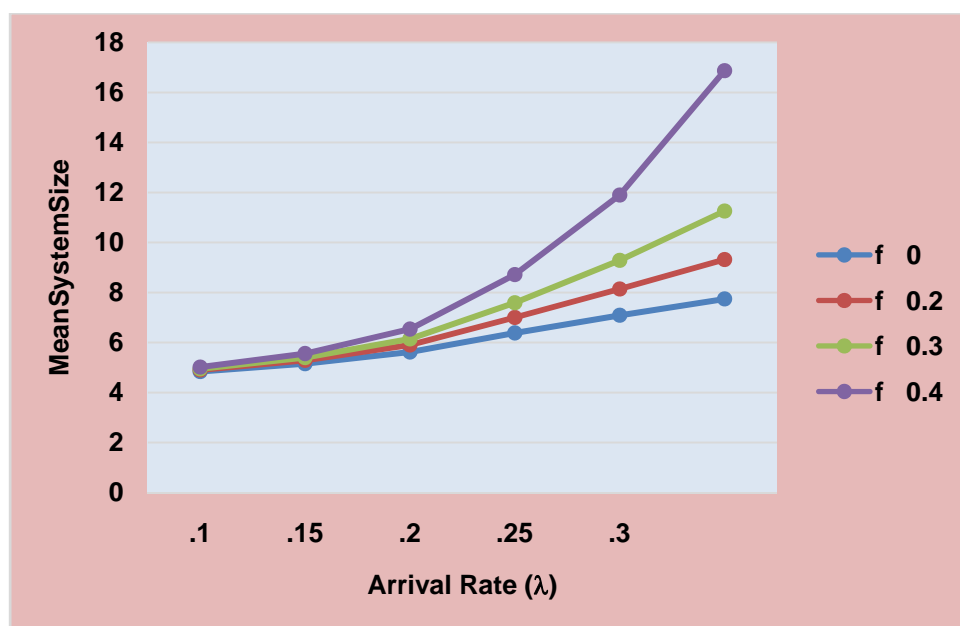
λ	$f = 0$	0.2	0.3	0.4
0.10	4.837	4.902	4.950	5.020
	4.689	4.742	4.783	4.839
0.15	5.148	5.286	5.396	5.559
	4.831	4.928	5.005	5.116
0.20	5.620	5.904	6.147	6.539
	5.004	5.166	5.298	5.500
0.25	6.382	6.999	7.593	8.719
	5.220	5.477	5.698	6.056
0.3	7.741	9.318	11.258	16.871
	5.493	5.895	6.264	6.918

■ $(p_{b_1}, p_{b_2}) = (0.3, 0.03)$

■ $(p_{b_1}, p_{b_2}) = (0.1, 0.03)$

Figure 5.3 Expected System Size Vs Arrival Rate λ for Different Values of (f)

with $(p_{b_1}, p_{b_2}) = (0.3, 0.03)$



The effects of batch size $E(X)$ and feedback probability (f) on the system size are studied for two different distributions of batch size in Table 5.4. The table values show that for both Binomial distribution and Geometric distribution, the queue becomes congested as $E(X)$ or f increases. The parametric values other than $\lambda = 0.2$, $m = 5$ and $N = 10$, $p = 0.7$ are same as in Table 5.1.

Table 5.4 Mean System Size Vs $E(X)$ and Feedback Probability (f) for Geometric and Binomial Distribution

f	$E(X) = 1$	$E(X) = 2$	$E(X) = 3$	$E(X) = 4$
0.01	3.477	4.202	5.629	8.087
	3.513	4.032	4.808	6.037
0.05	3.502	4.338	6.082	9.325
	3.543	4.143	5.107	6.779
0.10	3.539	4.542	6.822	11.622
	3.586	4.310	5.592	8.156
0.15	3.581	4.799	7.864	15.612
	3.636	4.520	6.273	10.548
0.2	3.631	5.130	9.440	24.256
	3.695	4.790	7.297	15.729

■ Geo (p_1)

■ B (2, p_1)

The optimal cost values $T_C(m^*, N^*)$ corresponding to different values of (p , J) are listed in Table 5.5, where J denotes the maximum number of consecutive vacations that server can take and p denotes the probability with which the server takes the multiple vacations. The parametric values are same as in Table 5.1.

Table 5.5 $T_C(m^*, N^*)$ for Different Values of p and J

p	$J = 1$	5	10	23
0.1	510.980	511.545	511.545	511.545
0.2	531.230	532.327	532.327	532.327
0.3	555.244	556.757	556.758	556.758
0.4	584.125	585.769	585.765	585.765
0.5	619.954	620.930	620.881	620.881

Figure 5.4 gives the mean system size for different values of feedback probability (f) and batch size $E(X)$ in the case of geometric distribution of Table 5.4. Figure 5.5 gives the graphical representation of Table 5.5.

Figure 5.4 Expected System Size Vs. f and $E(X)$

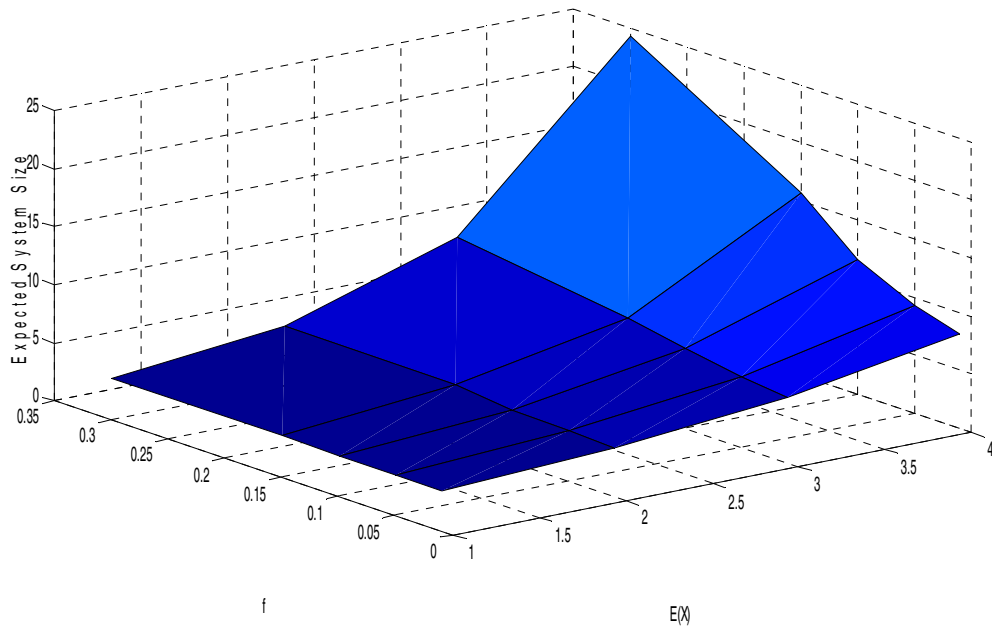


Figure 5.5 $T_C(m^*, N^*)$ Vs. J and p

