Chapter 5

J-Continuous Functions in Topological Spaces

§ 5.1. Introduction

Continuous functions perform very essential aspect in general topology. The stronger and weaker forms of continuity have been brought out and analysed by many topologists. In this Chapter, another concept of continuous functions utilizing J-closed sets is established. The dependency and independency of J-continuous functions with other existing continuous functions are examined. Different kinds of continuities in particular, quasi J-continuity, totally J-continuity, strongly J-continuity, contra J-continuity and quasi totally J-continuity are introduced and characterized. Many interrelating properties on them are obtained.

§ 5.2. J-Continuous Functions

The concept of J-continuous functions in topological spaces are introduced and some of their related properties are analysed here.

Definition 5.2.1. A function f : (Y,ζ)→ (Z,σ) is said to be **J-continuous** if the inverse image of every closed set in (Z,σ) is J-closed in (Y,ζ).

Example 5.2.2. Let f : (Y,ζ)→ (Z,σ) be the many-one into function defined by f(p) = r, f(q) = q, f(r) = r. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z, ϕ, {p}, {q}, {p,q}}, σ^c= {Z, ϕ, {r}, {q,r}, {p,r}}.Then f is J-continuous as JC(Y,ζ)= P(Y) – {p}.

Proposition 5.2.3. A continuous function f :(Y,ζ) →(Z,σ) is a J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a continuous function. Let U be any closed set in (Z,σ).Since f is continous, f^{-1}(U) is closed in (Y,ζ).By **Proposition 2.3.2.**, f^{-1}(U) is J-closed in (Y,ζ).Hence f is J-continuous.

Counter Example 5.2.4. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p,q}} and σ = {Z, ϕ, {p}, {q}, {p,q}}.Here σ^c = {Z, ϕ, {r}, {q,r}, {p,r}} and ζ^c= {Y, ϕ, {r}}.Then f is J-continuous as JC(Y,ζ)= P(Y) but not continuous. Because for the closed sets {q,r} and{p,r} in (Z,σ),the inverse images are not closed in (Y,ζ).

Proposition 5.2.5. A δg*-continuous function f :(Y,ζ) →(Z,σ) is a J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a δg*-continuous function. Let U be any closed set in (Z,σ).Since f is δg*-continuous, f^{-1}(U) is δg*-closed in (Y,ζ).By **Proposition 2.3.6.**, f^{-1}(U) is J-closed in (Y,ζ).Hence f is J-continuous.

Counter Example 5.2.6. Let f : (Y,ζ)→ (Z,σ) be the many-one into function defined by f(p) = q, f(q) = q, f(r) = r. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z, ϕ,{p,q}}.Here σ^c={Z,ϕ,{r}} and δg*C(Y,ζ)={Y,ϕ,{q,r}}.Then f is J-continuous as JC(Y,ζ)= P(Y)−{p}but not δg*-continuous. Because for the closed set {r} in (Z,σ), f^{-1}({r}) = {r} is not a δg*-closed set in (Y,ζ).

Proposition 5.2.7. A δg-continuous function f :(Y,ζ) →(Z,σ) is a J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a δg-continuous function. Let U be any closed set in (Z,σ).Since f is δg-continuous, f^{-1}(U) is δg-closed in (Y,ζ).By **Proposition 2.3.8.**, f^{-1}(U) is J-closed in (Y,ζ).Hence f is J-continuous.

Counter Example 5.2.8. Let f : (Y,ζ)→ (Z,σ) be the bijective function defined by f(p) = p, f(q) = r, f(r) = q. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p},{p,q}} and σ = {Z, ϕ,{p,q}}.Here σ^c={Z,ϕ,{r}} and δgC(Y,ζ)={Y,ϕ,{r},{p,r},{q,r}}.Then f is J-continuous as JC(Y,ζ)= P(Y) −{p}but not δg-continuous. Because for the closed set {r} in (Z,σ), f^{-1}({r}) = {q} is not a δg-closed in (Y,ζ).

Proposition 5.2.9. A g-continuous function f :(Y,ζ) →(Z,σ) is a J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a g-continuous function. Let U be any closed set in (Z,σ).Since f is g-continuous, f^{-1}(U) is g-closed in (Y,ζ).By **Proposition 2.3.10.**, f^{-1}(U) is J-closed in (Y,ζ).Hence f is J-continuous.

Counter Example 5.2.10. Let f : (Y,ζ)→ (Z,σ) be the many-one into function defined by f(p) = q, f(q) = r, f(r) = q.Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p},{p,q}} and σ = {Z, ϕ,{p,q}}.Here σ^c={Z,ϕ,{r}} and gC(Y,ζ)={Y,ϕ,{r},{p,r},{q,r}}.Then f is J-continuous as JC(Y,ζ)= P(Y) −{p}but not g-continuous. Because for the closed set {r} in (Z,σ), f^{-1}({r}) = {q} is not g-closed in (Y,ζ).

Proposition 5.2.11. A ĝ-continuous function f : (Y,ζ) →(Z,σ) is a J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a ĝ-continuous function. Let U be any closed set in (Z,σ).Since f is ĝ-continuous, f^{-1}(U) is ĝ-closed in (Y,ζ).By **Proposition 2.3.16.**, f^{-1}(U) is J-closed in (Y,ζ).Hence f is J-continuous.

Counter Example 5.2.12. Let f : (Y,ζ)→ (Z,σ) be the many-one into function defined by f(p) = q, f(q) = r, f(r) = q. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p},{p,q}} and σ = {Z, ϕ,{p,q}}.Here σ^c={Z,ϕ,{r}} and ĝC(Y,ζ)={Y,ϕ,{r},{q,r}}.Then f is J-continuous as JC(Y,ζ)= P(Y) −{p}but not ĝ-continuous. Because for the closed set {r} in (Z,σ), f^{-1}({r}) = {q} is not ĝ-closed in (Y,ζ).

Proposition 5.2.13. A J-continuous function f :(Y,ζ) →(Z,σ) is a gδ-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a J-continuous function. Let U be any closed set in (Z,σ).Since f is J-continuous, f^{-1}(U) is J-closed in (Y,ζ).By **Proposition 2.3.12.**, f^{-1}(U) is gδ-closed in (Y,ζ).Hence f is gδ-continuous.

Counter Example 5.2.14. Let f : (Y,ζ)→ (Z,σ) be the bijective function defined by f(p) = r, f(q) = q, f(r) = p. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p},{p,q}} and σ = {Z, ϕ,{p,q}}.Here JC(Y,ζ)= P(Y) −{p},σ^c={Z,ϕ,{r}}.Then f is gδ-continuous as gδC(Y,ζ)=P(Y) but not J-continuous. Because for the closed set {r} in (Z,σ), f^{-1}({r}) = {p} is not J-closed in (Y,ζ).

Theorem 5.2.15. A J-continuous function f :(Y,ζ) →(Z,σ) is a

(i) rg-continuous function

(ii) gpr-continuous function

(iii) rwg-continuous function

(iv) gspr-continuous function

(v) πg-continuous function

(vi) πgp-continuous function

(vii) πgs-continuous function

(viii) πgsp-continuous function

(ix) πgα-continuous function.

Proof Obvious.

Remark 5.2.16. The converse of above Theorem 5.2.15. is not true. It can be seen from the following Counter Examples.

Counter Example 5.2.17. In the above **Counter Example 5.2.14.** rgC(Y,ζ)=gprC(Y,ζ)=rwgC(Y,ζ)=gsprC(Y,ζ)=πgC(Y,ζ)=πgpC(Y,ζ)= πgsC(Y,ζ)=πgspC(Y,ζ)=πgαC(Y,ζ)=P(Y).Then f is rg-continuous,gpr-continuous, rwg-continuous,gspr-continuous, πg-continuous and πgp-continuous, πgs-continuous, πgsp-continuous,πgα-continuous respectively but not J-continuous. Because for the closed set {r} in (Z,σ), f^{-1}({r}) = {p} is not J-closed in (Y,ζ).

Theorem 5.2.18. A function f : (Y,ζ)→ (Z,σ) is J-continuous if and only if the inverse image of every open set in (Z,σ) is J-open in (Y,ζ).

Proof Necessity Let f : (Y,ζ)→ (Z,σ) be J-continuous and U be an open set in (Z,σ). Then Z – U is closed in (Z,σ). Since f is J-continuous, f^{-1}(Z – U) = Z−f^{-1}(U) is J-closed in (Y,ζ) and hence f^{-1}(U) is J-open in (Y,ζ).

Sufficiency Assume that f^{-1}(V) is J-open in (Y,ζ) for each open set V in (Z,σ). Let V be a closed set in (Z,σ). Then Z− V is open in (Z,σ).By assumption, f^{-1}(Z− V) = Z−f^{-1}(V) is J-open in (Y,ζ)which implies that f^{-1}(V) is J-closed in (Y,ζ). Hence f is J-continuous.

Remark 5.2.19. The following Counter Examples show that J-continuous function is independent of gs-continuous ,g*s-continuous and δ-continuous.

Counter Example 5.2.20. Let f : (Y,ζ)→ (Z,σ) be the bijective function defined by f(p) = r, f(q) = q, f(r) = p.Consider Y = Z = {p,q,r} with ζ = {Y, ϕ,{p,q}} and σ = {Z, ϕ,{p}}.Here gsO(Y,ζ)=g*sO(Y,ζ)={Y, ϕ,{p},{q},{p,q}}.Then f is J-continuous as JO(Y,ζ)= P(Y) but not gs-continuous and g*s-continuous respectively. Because for the open set {p} in (Z,σ), f^{-1}({p}) = {r} is not gs-open and g*s-open in (Y,ζ).

Counter Example 5.2.21. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = r, f(q) = q, f(r) = p.Consider Y = Z = {p,q,r} with ζ = {Y, ϕ,{p},{q},{p,q}} and σ = {Z, ϕ,{p,q}}.Here gsO(Y,ζ)= g*sO(Y,ζ)={Y,ϕ,{p},{q},{p,q},{p,r},{q,r}}.Then f is gs-continuous and g*s-continuous respectively but not J-continuous as JO(Y,ζ)= ζ.Because for the open set {p,q} in (Z,σ), f^{-1}({p,q}) = {q,r} is not J-open in (Y,ζ).

Counter Example 5.2.22. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = q, f(q) = r, f(r) = p. Consider Y = {p,q,r}, Z = {p,q,r,s} with ζ = {Y, ϕ, {p},{q},{p,q}} and σ = {Z, ϕ,{p}}.Here σ^c = {Z, ϕ,{q,r,s}} and δC(Y,ζ)={Y, ϕ,{r},{p,r},{q,r}}.Then f is not J-continuous as JC(Y,ζ)= {Y, ϕ,{r},{p,r},{q,r}}but it is δ-continuous because δC(Z,σ)={Z, ϕ}only.

Counter Example 5.2.23. Let f : (Y,ζ)→ (Z,σ) be an identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z, ϕ,{p},{q},{p,q}}= δC(Z,σ).Here σ^c = {Z, ϕ,{r},{q,r},{p,r}} and δC(Y,ζ)={Y, ϕ}.Then f is J-continuous as JC(Y,ζ)= P(Y) −{p}but not δ-continuous. Because for the δ-closed set {q,r} in (Z,σ),the inverse image is {q,r} not δ-closed in (Y,ζ).

Proposition 5.2.24. A super continuous function f : (Y,ζ)→ (Z,σ) is a J-continuous function but the converse is not true.

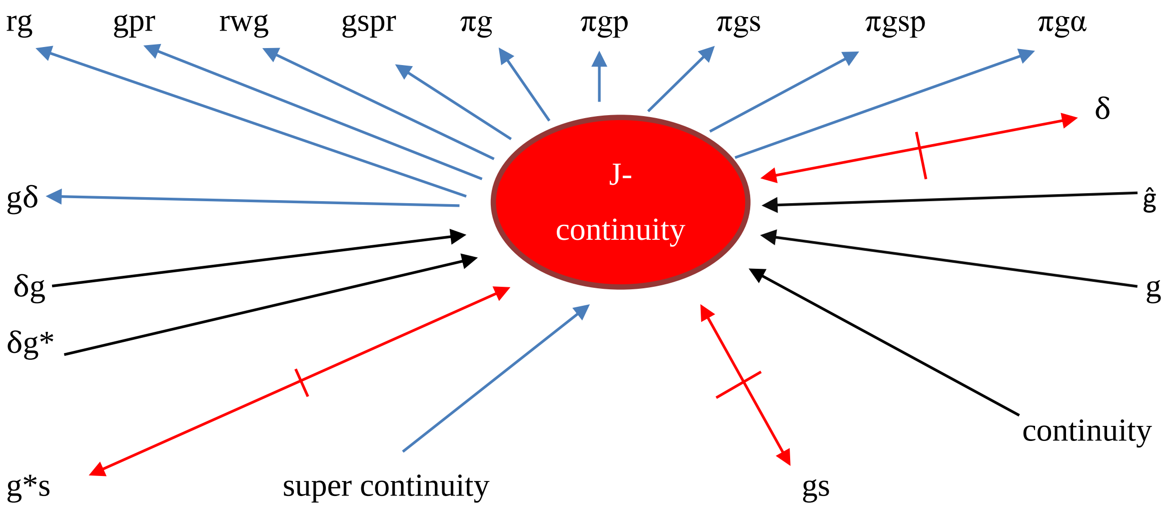
Proof Given f : (Y,ζ) →(Z,σ) is a super continuous function. Let U be any closed set in (Z,σ).Since f is a super continuous function, f^{-1}(U) is δ-closed in (Y,ζ).By **Proposition 2.3.4.**, f^{-1}(U) is J-closed in (Y,ζ). Hence f is J-continuous.

Remark 5.2.25. The converse of the above Proposition is disproved from the following Counter Example.

Counter Example 5.2.26. Let f : (Y,ζ)→ (Z,σ) be the bijective function defined by f(p) = q, f(q) = p, f(r) = r. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p,q}} and σ = {Z,

ϕ,{p}}.Here σ^c = {Z, ϕ,{q,r}} and δC(Y,ζ)={Y, ϕ}.Then f is J-continuous as JC(Y,ζ)= P(Y) but not super continuous. Because for the closed set {q,r} in (Z,σ), f^{-1}({q,r}) = {p,r} is not δ-closed in (Y,ζ).

Remark 5.2.27. From the above discussions, we have the following diagram.



Proposition 5.2.28. A totally continuous function f : (Y,ζ)→ (Z,σ) is a J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a totally continuous function. Let U be any open set in (Z,σ).Since f is a totally continuous function, f^{-1}(U) is clopen in (Y,ζ).By **Proposition 2.3.81.**, f^{-1}(U) is J-open in (Y,ζ). Hence f is J-continuous.

Proposition 5.2.29. A strongly continuous function f : (Y,ζ)→ (Z,σ) is a J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a strongly continuous function. Let U be any closed subset in (Z,σ).Since f is a strongly continuous function, f^{-1}(U) is clopen in (Y,ζ).By **Proposition 2.3.81.**, f^{-1}(U) is J-open in (Y,ζ). Hence f is J-continuous.

Remark 5.2.30. The converse of the above Propositions can be seen to be untrue from the following Counter Examples.

Counter Example 5.2.31. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p,q}} and σ = {Z, ϕ, {p}, {q},{p,q}}.Here σ^c = {Z, ϕ, {r}, {q,r}, {p,r}} and ζ^c= {Y, ϕ, {r}}.Then f is J-continuous as JC(Y,ζ)= P(Y) but not strongly continuous and is not totally continuous. Because for the subset {p,r} in (Z,σ),the inverse image is not clopen in (Y,ζ) and for an open set {p} in (Z,σ),no set is clopen in (Y,ζ) respectively.

Proposition 5.2.32. If f : (Y,ζ)→ (Z,σ) is a J-continuous function and (Y,ζ) is a JTC-space. Then f is continuous.

Proof Given f: (Y,ζ) →(Z,σ) is a J-continuous function. Let U be any closed set in (Z,σ).Since f is J-continuous, f^{-1}(U) is J-closed in (Y,ζ) and (Y,ζ) is a JTC-space. Therefore f^{-1}(U) is closed in (Y,ζ).Hence f is continuous.

Result 5.2.33.(a) If f : (Y,ζ)→ (Z,σ) is a J-continuous function and (Y,ζ) is a JTδ-space. Then f is super continuous.

(b) If f : (Y,ζ)→ (Z,σ) is a J-continuous function and (Y,ζ) is a JTδg*-space. Then f is δg*-continuous.

(c) If f : (Y,ζ)→ (Z,σ) is a J-continuous function and (Y,ζ) is a JTδg-space. Then f is δg-continuous.

(d) If f : (Y,ζ)→ (Z,σ) is a J-continuous function and (Y,ζ) is a JTg-space. Then f is g-continuous.

Proof The proof follow similar to **Proposition 5.2.32.**

Proposition 5.2.34. If f : (Y,ζ)→ (Z,σ) is a gδ-continuous function and (Y,ζ) is a gδTJ-space. Then f is J-continuous.

Proof Given f : (Y,ζ) →(Z,σ) is a gδ-continuous function. Let U be any closed set in (Z,σ).Since f is gδ-continuous, f^{-1}(U) is gδ-closed in (Y,ζ) and (Y,ζ) is a gδTJ-space. Therefore f^{-1}(U) is J-closed in (Y,ζ).Hence f is J-continuous.

Theorem 5.2.35. For every subset D of (Y,ζ), f(JCl(D)) $\subseteq$ Cl(f(D)) if f : (Y,ζ) → (Z,σ) is a J-continuous function.

Proof Given f : (Y,ζ) →(Z,σ) is a J-continuous function and D is any subset of (Y,ζ).Then Cl(f(D)) is a closed set in (Z,σ).Since f is a J-continuous function, we get f^{-1}(Cl(f(D))) is a J-closed set in (Y,ζ) -----(1).We know f(D) $\subseteq$ Cl(f(D)),

$D \subseteq f^{-1}(Cl(f(D)))$.From (1), $f^{-1}(Cl(f(D)))$ is a J-closed set containing D. By **Definition 3.2.1.**,we have $JCl(D) \subseteq f^{-1}(Cl(f(D)))$ which implies that $f(JCl(D)) \subseteq Cl(f(D))$.

Corollary 5.2.36. (a) For every subset D of (Y,ζ), $f(JCl(D)) \subseteq Cl(f(D))$ if $f : (Y,\zeta)\to (Z,\sigma)$ is a super continuous function.

(b) For every subset D of (Y,ζ), $f(JCl(D)) \subseteq Cl(f(D))$ if $f : (Y,\zeta)\to (Z,\sigma)$ is a totally continuous function.

Proof (a) and (b) We know that a super continuous function $f : (Y,\zeta)\to (Z,\sigma)$ is a J-continuous function (by **Proposition 5.2.24.**) and a totally continuous function $f : (Y,\zeta)\to (Z,\sigma)$ is a J-continuous function (by **Proposition 5.2.28.**) and also the proof follows from the previous **Theorem 5.2.35.**

Proposition 5.2.37. Let $f: (Y,\zeta) \to (Z,\sigma)$ be a function. If for each point $x\in Y$ and each open set V in (Z,σ) containing f(x), there exists a J-open set U in (Y,ζ) containing x such that $f(U)\subseteq V$, then for each subset D of (Y,ζ), $f(JCl(D)) \subseteq Cl(f(D))$.

Proof Let D be any subset of $(Y,\zeta)$and $y\in f(JCl(D))$. Therefore $y = f(x)$ for some $x\in JCl(D)\subseteq Y$. Let V be any open set in (Z,σ) such that $f(x)\in V$. Then by hypothesis, there exists a J-open set U in (Y,ζ)containing x with $f(U)\subseteq V$. By **Theorem 3.2.8.**, $U\cap D \neq \phi$, then $f(U\cap D) \neq \phi$ which implies that $V\cap f(D) \neq \phi$. Hence $y\in Cl(f(D))$. Thus $f(JCl(D)) \subseteq Cl(f(D))$.

Composition of Functions

Remark 5.2.38. The composition of two J-continuous functions need not be J-continuous as observed from the following Counter Example.

Counter Example 5.2.39. Let $f : (Y,\zeta) \to (Z,\sigma)$ be the bijective function defined by f(p) = q,f(q) = r,f(r) = p. Consider Y = Z = P ={p,q,r} with $\zeta = \{Y, \phi, \{p\}\}$ and $\sigma = \{Z, \phi, \{r\}\}$, $\mu = \{P, \phi,\{p,q\}\}$.Then f is J-continuous as $JC(Y,\zeta)= P(Y) -\{p\}$. Let $g : (Z,\sigma) \to(P,\mu)$ be the one-one onto function defined by g(p) = p, g(q) = r, g(r) = q. Then g is J-continuous as $JC(Z,\sigma) = P(Z)-\{r\}$. Consider the composition function $g \circ f :(Y,\zeta)\to(P,\mu)$ such that $(g\circ f)(p) = g(f(p)) = r$,$(g\circ f)(q) = g(f(q)) = q$ and $(g\circ f)(r) =$

g(f(r)) = p. But their composition g∘f is not J-continuous. Because for the closed set {r} in (P,μ), $f^{-1}(g^{-1}(\{r\}))$ = {p} is not J-closed in (Y,ζ).

Proposition 5.2.40. If f : (Y,ζ)→ (Z,σ) is a J-continuous function and g : (Z,σ) →(P,μ) is a J-continuous function, then g ∘ f : (Y,ζ)→ (P, μ) is a J-continuous function when (Z,σ) is a JTC-space.

Proof Given g : (Z,σ) →(P,μ) is a J-continuous function. Let U be any closed set in (P,μ). Hence $g^{-1}(U)$ is J-closed in (Z,σ).Consider (Z,σ) is a JTC-space. Therefore $g^{-1}(U)$ is closed in (Z,σ). Given f: (Y,ζ)→ (Z,σ) is a J-continuous function.So $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is J-closed in (Y,ζ). Hence g ∘ f is a J-continuous function.

Remark 5.2.41. In the above Proposition if (Z,σ) is a T_δ-space, we get the composition of J-continuous function is also a J-continuous function from the following Proposition.

Proposition 5.2.42. If f : (Y,ζ)→ (Z,σ) is a J-continuous function and g : (Z,σ) →(P,μ) is a J-continuous function, then g ∘ f : (Y,ζ)→ (P, μ) is a J-continuous function when (Z,σ) is a T_δ-space.

Proof Given g : (Z,σ) →(P,μ) is a J-continuous function. Let U be any closed set in (P,μ).Hence $g^{-1}(U)$ is J-closed in (Z,σ).By **Proposition 2.3.12.**,$g^{-1}(U)$ is gδ-closed. Consider (Z,σ) is a T_δ-space. Therefore $g^{-1}(U)$ is δ-closed in (Z,σ). Hence $g^{-1}(U)$ is closed in (Z,σ). Given f : (Y,ζ)→ (Z,σ) is a J-continuous function. So $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is J-closed in (Y,ζ). Hence g ∘ f is a J-continuous function.

Proposition 5.2.43. If f : (Y,ζ)→ (Z,σ) is a J-continuous function and g : (Z,σ) →(P,μ) is a super continuous function, then g ∘ f : (Y,ζ)→ (P,μ) is a J-continuous function.

Proof Given g : (Z,σ) →(P,μ) is a super continuous function. Let U be any closed set in (P,μ).Hence $g^{-1}(U)$ is δ-closed in (Z,σ) which implies $g^{-1}(U)$ is closed in (Z,σ). Given f : (Y,ζ)→ (Z,σ) is a J-continuous function. So $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is J-closed in (Y,ζ). Hence g ∘ f is a J-continuous function.

Proposition 5.2.44. If f : (Y,ζ)→ (Z,σ) is a super continuous function and g : (Z,σ) →(P,μ) is a super continuous function, then g ∘ f : (Y,ζ)→ (P, μ) is a J-continuous function.

Proof Given g : (Z,σ) →(P,μ) is a super continuous function. Let U be any closed set in (P,μ).Hence g^{-1}(U) is δ-closed in (Z,σ) which implies g^{-1}(U) is closed in (Z,σ). Given f : (Y,ζ)→ (Z,σ) is a super continuous function. So $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is δ-closed in (Y,ζ). By **Proposition 2.3.4.**,$f^{-1}(g^{-1}(U))$ is J-closed in (Y,ζ). Hence g ∘ f is a J-continuous function.

Proposition 5.2.45. If f : (Y,ζ)→ (Z,σ) is a J-continuous function and g : (Z,σ) →(P,μ) is a continuous function, then g ∘ f : (Y,ζ)→ (P, μ) is a J-continuous function.

Proof Given g : (Z,σ) →(P,μ) is a continuous function. Let U be any closed set in (P,μ).Hence g^{-1}(U) is closed in (Z,σ).Given f : (Y,ζ)→ (Z,σ) is a J-continuous function. So $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is J-closed in (Y,ζ). Hence g ∘ f is a J-continuous function.

§ 5.3. Quasi J-Continuous Functions

Definition 5.3.1. A function f : (Y,ζ) → (Z,σ) is said to be **quasi J-continuous** if the inverse image of every J-closed set in (Z,σ) is closed in (Y,ζ).

Example 5.3.2. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = p, f(q) = p, f(r) = p and f(s) = p. Consider Y = {p,q,r,s},Z ={p,q,r} with ζ = {Y, ϕ,{p}} and σ = {Z, ϕ, {p}, {q}, {p,q}}.Here JC(Z,σ)={Z,ϕ,{r},{q,r},{p,r}} and ζ^c= {Y,ϕ,{q,r,s}}.Then f is quasi J-continuous.

Proposition 5.3.3. A strongly continuous function f : (Y,ζ)→ (Z,σ) is a quasi J-continuous function but the converse is not true.

Proof Let U be any J-closed set in (Z,σ). Since f : (Y,ζ) →(Z,σ) is a strongly continuous function. For any subset U in (Z,σ), f^{-1}(U) is clopen in (Y,ζ) which implies f^{-1}(U) is closed in (Y,ζ).Hence f is quasi J-continuous.

Remark 5.3.4. The converse of the above Proposition can be seen from the following Counter Example.

Counter Example 5.3.5. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ,{p},{q},{p,q},{p,r}} and σ= {Z, ϕ,{p},{q},{p,q}}.Here ζ^c= {Y, ϕ,{q},{r},{p,r},{q,r}}.Then f is quasi J-continuous as JC(Z,σ) ={Z, ϕ,{r},{q,r},{p,r}}but not strongly continuous. Because for the set {p} in (Z,σ), f $^{-1}$({p}) = {p} is not clopen in (Y,ζ).

Proposition 5.3.6. A totally continuous function f : (Y,ζ)→ (Z,σ) is a quasi J-continuous function if (Z,σ) is a JTC-space.

Proof Let U be any J-closed set in (Z,σ). Since (Z,σ) is a JTC-space, U is closed in (Z,σ) and f : (Y,ζ) →(Z,σ) is a totally continuous function. For any closed U in (Z,σ),f $^{-1}$(U) is clopen in (Y,ζ) which implies f $^{-1}$(U) is closed in (Y,ζ).Hence f is quasi J-continuous.

Remark 5.3.7. If (Z,σ) is a JTC-space is not used in the above Proposition, then it does not hold.

Counter Example 5.3.8. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ={Y,ϕ,{p},{q,r}} and σ ={Z,ϕ,{p}}.Here JC(Z,σ) = P(Y) –{p}and ζ^c={Y, ϕ,{p},{q,r}}.Then f is totally continuous but not quasi J-continuous if (Z,σ) is not a JTC-space. Because for the J-closed set {r} in (Z,σ), the inverse image of J-closed set is not closed in (Y,ζ).

Proposition 5.3.9. A quasi J-continuous function f : (Y,ζ)→ (Z,σ) is a J-continuous function but the converse is not true.

Proof Let U be closed set in (Z,σ). By **Proposition 2.3.2.**,U is J-closed in (Z,σ).Since f is a quasi J-continuous function, f $^{-1}$(U) is closed in (Y,ζ) which implies f $^{-1}$(U) is J-closed in (Y,ζ) (by **Proposition 2.3.2.)**.Hence f is a J-continuous function.

Counter Example 5.3.10. Let f : (Y,ζ)→ (Z,σ) be an identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ,{p}} and σ = {Z, ϕ,{p},{p,q}}.Here ζ^c= {Y, ϕ,{q,r}} and σ^c= {Y, ϕ,{q,r},{r}}.Then f is J-continuous as JC(Y,ζ) = JC(Z,σ) = P(Y) –{p}but not quasi J-continuous. Because for the J-closed set {p,q} in (Z,σ), f $^{-1}$({p,q}) = {p,q} is not closed in (Y,ζ).

§ 5.4. Totally J-Continuous Functions and Strongly J-Continuous Functions

Definition 5.4.1. A function f : (Y,ζ) → (Z,σ) is said to be **totally J-continuous** if the inverse image of every closed set in (Z,σ) is J-clopen in (Y,ζ).

Example 5.4.2. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = q, f(q) = r, f(r) = p. Consider Y = Z ={p,q,r}with ζ = {Y, ϕ, {p},{p,q},{p,r}} and σ = {Z, ϕ, {p,q}}.Here σ^c= {Y, ϕ,{r}}and JC(Y,ζ)=JO(Y,ζ)=P(Y).Then f is totally J-continuous.

Proposition 5.4.3. A totally J-continuous function f : (Y,ζ)→ (Z,σ) is a J-continuous function but the converse is not true.

Proof Let U be closed set in (Z,σ). Since f is a totally J-continuous function, f^{-1}(U) is J-clopen in (Y,ζ) which implies f^{-1}(U) is J-closed in (Y,ζ).Hence f is a J-continuous function.

Counter Example 5.4.4. Counter Example 5.3.10. is J-continuous as JC(Y,ζ) = P(Y) –{p}, JO(Y,ζ) = P(Y) –{q,r}but not totally J-continuous. Because for the closed set {q,r} in (Z,σ), f^{-1}({q,r}) = {q,r} is not J-clopen in (Y,ζ).

Definition 5.4.5. A function f : (Y,ζ) → (Z,σ) is said to be **strongly J-continuous** if the inverse image of every subset in (Z,σ) is J-clopen in (Y,ζ).

Example 5.4.6. From the above **Example 5.4.2.**, for every subset in (Z,σ) is J-clopen in (Y,ζ).Then f is strongly J-continuous.

Result 5.4.7. A strongly J-continuous function f : (Y,ζ)→ (Z,σ) is a totally J-continuous function but the converse is not true.

Proof It is clear from the definitions of strongly J-continuous function and totally J-continuous function respectively.

Remark 5.4.8. The converse is not true which can be seen from the following Counter Example.

Counter Example 5.4.9. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = p, f(q) = r, f(r) = q. Consider Y = Z ={p,q,r}with ζ = {Y, ϕ, {p}} and σ = {Z, ϕ, {p,q}}.Here σ^c= {Y, ϕ,{r}}and JC(Y,ζ)=P(Y)−{p}and JO(Y,ζ)= P(Y)−{q,r}.Then f is totally J-continuous but not strongly J-continuous. Since for the subset {p} in (Z,σ),the inverse image is {p} in (Y,ζ) is not J-clopen in (Y,ζ).

Theorem 5.4.10. A function f : (Y,ζ)→ (Z,σ) is totally J-continuous if and only if the inverse image of every open set in (Z,σ) is J-clopen in (Y,ζ).

Proof It is clear.

Proposition 5.4.11. A totally continuous function f : (Y,ζ)→ (Z,σ) is a totally J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a totally continuous function. Let U be any open set in (Z,σ). Since f is a totally continuous function, $f^{-1}(U)$ is clopen in (Y,ζ).By **Proposition 2.3.75.** and **Proposition 2.3.2.**, $f^{-1}(U)$ is J-open and J-closed respectively in (Y,ζ). Hence f is totally J-continuous.

Counter Example 5.4.12. In the above **Example 5.4.2.**,it is totally J-continuous but not totally continuous. Because for the open set {p,q}in (Z,σ),their inverse is {p,r} is not clopen in (Y,ζ).

Proposition 5.4.13. A strongly continuous function f : (Y,ζ)→ (Z,σ) is a strongly J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a strongly continuous function. Let U be any subset in (Z,σ).Since f is a strongly continuous function, $f^{-1}(U)$ is clopen in (Y,ζ).By **Proposition 2.3.75. and Proposition 2.3.2.**, $f^{-1}(U)$ is J-open and J-closed respectively in (Y,ζ). Hence f is strongly J-continuous.

Counter Example 5.4.14. In the above **Example 5.4.2.**,it is strongly J-continuous but not strongly continuous. Because for the open set {p,q}in (Z,σ),their inverse is {p,r} is not clopen in (Y,ζ).

Proposition 5.4.15. A strongly J-continuous function f : (Y,ζ)→ (Z,σ) is a J-continuous function but the converse is not true.

Proof Let U be closed set in (Z,σ). Since f is a strongly J-continuous function, $f^{-1}(U)$ is J-clopen in (Y,ζ) which implies $f^{-1}(U)$ is J-closed in (Y,ζ).Hence f is a J-continuous function.

Counter Example 5.4.16. Counter Example 5.4.4. is J-continuous but not strongly J-continuous.

Result 5.4.17. If (Y,ζ) is a discrete topological space,the following results are equivalent. (i) f is J-continuous (ii) f is totally J-continuous (iii) f is totally continuous (iv) f is continuous.

Note 5.4.18. When (Z,σ) is a discrete topological space,then the converse of above Result 5.4.7. is true. It can be seen from the following Result.

Result 5.4.19. If (Z,σ) is a discrete topological space,the following results are equivalent. (i) f is totally J-continuous (ii) f is strongly J-continuous.

§ 5.5. Quasi Totally J-Continuous Functions

Definition 5.5.1. A function f : (Y,ζ) → (Z,σ) is said to be **Quasi totally J-continuous** if the inverse image of every J-open set in (Z,σ) is a clopen set in (Y,ζ).

Example 5.5.2. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = q, f(q) = p, f(r) = p. Consider Y = Z ={p,q,r}with ζ = {Y,ϕ, {p},{q,r}} and σ = {Z, ϕ,{p},{q},{p,q}}.Here JO(Z,σ) = σ and ζ^c= {Y, ϕ,{p},{q,r}}.Then f is Quasi totally J-continuous .

Note 5.5.3. A Quasi totally J-continuous function is stronger than a totally continuous function. It can be seen from the following Proposition.

Proposition 5.5.4. A Quasi totally J-continuous function f : (Y,ζ)→ (Z,σ) is a totally continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a Quasi totally J-continuous function. Let U be any open set in (Z,σ).Then U is a J-open set in (Z,σ) by **Theorem 2.3.75.**Since f is a Quasi totally J-continuous function, f^{-1}(U) is clopen in (Y,ζ). Hence f is totally continuous.

Counter Example 5.5.5. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = r, f(q) = p, f(r) = q. Consider Y = Z ={p,q,r}with ζ = {Y, ϕ, {p},{q},{p,q}{p,r}} and σ = {Z, ϕ,{p}}.Here JO(Z,σ) = P(Z)−{q,r} and ζ^c= {Y, ϕ,{q},{r},{q,r},{p,r}}.Then f is totally continuous but not Quasi totally J-continuous . Because for the J-open set {q} in (Z,σ), the corresponding inverse image {r} is not clopen in (Y,ζ).

Theorem 5.5.6. A function f : (Y,ζ)→ (Z,σ) is Quasi totally J-continuous if and only if the inverse image of every J-closed set in (Z,σ) is clopen in (Y,ζ).

Proof Obvious.

Proposition 5.5.7. A Quasi totally J-continuous function f : (Y,ζ)→ (Z,σ) is a quasi J-continuous function but the converse is not true.

Proof Let U be any J-closed set in (Z,σ).Since f : (Y,ζ) →(Z,σ) is a Quasi totally J-continuous function.Then f^{-1}(U) is clopen in (Y,ζ) which implies f^{-1}(U) is closed in (Y,ζ).Hence f is quasi J-continuous.

Remark 5.5.8. The converse of the above Proposition 5.5.7. does not hold good.It can be seen from the following Counter Example.

Counter Example 5.5.9. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ,{p},{q},{p,q},{p,r}} and σ= {Z, ϕ,{p},{q},{p,q}}.Here ζ^c= {Y, ϕ,{q},{r},{p,r},{q,r}}.Then f is quasi J-continuous as JC(Z,σ) ={Z, ϕ,{r},{q,r},{p,r}}but not Quasi totally J-continuous . Because for the J-open set {p} in (Z,σ),f^{-1}({p}) = {p} is not clopen in (Y,ζ).

Proposition 5.5.10. A strongly continuous function f : (Y,ζ)→ (Z,σ) is a Quasi totally J-continuous function but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a strongly continuous function. Let U be J-open set in (Z,σ).Since f is a strongly continuous function, f^{-1}(U) is clopen in (Y,ζ). Hence f is Quasi totally J-continuous.

Proposition 5.5.11. A Quasi totally J-continuous function f : (Y,ζ)→ (Z,σ) is a J-continuous function but the converse is not true.

Proof Let U be an open set in (Z,σ).By **Theorem 2.3.75.**,U is J-open in (Z,σ).Since f is a Quasi totally J-continuous function, f^{-1}(U) is clopen in (Y,ζ) which implies f^{-1}(U) is open in (Y,ζ).Hence f^{-1}(U) is J-open in (Y,ζ) (by **Theorem 2.3.75.)**.Hence f is a J-continuous function.

Counter Example 5.5.12. Let f : (Y,ζ)→ (Z,σ) be an identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ,{p}} and σ = {Z, ϕ,{p},{p,q}}.Here ζ^c= {Y, ϕ,{q,r}} and

JO(Z,σ) = P(Y) –{q,r}.Then f is J-continuous as JO(Y,ζ) = P(Y) –{q,r} but not Quasi totally J-continuous . Because for an J-open set {p,q} in (Z,σ), f^{-1}({p,q}) = {p,q} is not clopen in (Y,ζ).

Theorem 5.5.13. If f : (Y,ζ)→ (Z,σ) is a function. Then the following options (a) and (b) are equivalent under some restrictions as (Y,ζ) is a discrete topological space. (a) f is Quasi totally J-continuous `(b) f is quasi J-continuous.

Proof (a)⇒(b) It follows from **Proposition 5.5.7. (b)⇒(a)** Let U be any J-open set in (Z,σ). By hypothesis, f^{-1}(U) is open in (Y,ζ). Since (Y,ζ) is a discrete topological space, f^{-1}(U) is also closed in (Y,ζ).That is f^{-1}(U) is clopen in (Y,ζ). Hence f is Quasi totally J-continuous .

Proposition 5.5.14. If f : (Y,ζ)→ (Z,σ) is a J-continuous function and g : (Z,σ) →(P,μ) is a strongly continuous function, then g ∘ f : (Y,ζ)→ (P, μ) is a J-continuous function.

Proof Given g : (Z,σ) →(P,μ) is a strongly continuous function. Let U be any closed set in (P,μ). Hence g^{-1}(U) is clopen in (Z,σ).Therefore g^{-1}(U) is closed in (Z,σ). Given f : (Y,ζ)→ (Z,σ) is a J-continuous function. So $(g \circ f)^{-1}$(U) = f^{-1}(g^{-1}(U)) is J-closed in (Y,ζ). Hence g ∘ f is a J-continuous function.

Proposition 5.5.15. If f : (Y,ζ)→ (Z,σ) is a continuous function and g : (Z,σ) →(P,μ) is a Quasi totally J-continuous function, then g ∘ f : (Y,ζ)→ (P, μ) is a Quasi totally J-continuous function.

Proof Given g : (Z,σ) →(P,μ) is a Quasi totally J-continuous function. Let U be J-closed set in (P,μ).Hence g^{-1}(U) is clopen in (Z,σ).Given f : (Y,ζ)→ (Z,σ) is a continuous function. So $(g \circ f)^{-1}$(U) = f^{-1}(g^{-1}(U)) is clopen in (Y,ζ). Hence g ∘ f is a Quasi totally J-continuous function.

§ 5.6. Contra J-Continuous Functions

Definition 5.6.1. A function f : (Y,ζ) → (Z,σ) is said to be **contra J-continuous** if the inverse image of every closed set in (Z,σ) is a J-open set in (Y,ζ).

Example 5.6.2. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = q, f(q) = q, f(r) = r. Consider Y = Z ={p,q,r}with ζ = {Y, ϕ, {p}}} and σ = {Z, ϕ,{p},{q},{p,q}}.Here σ^c= {Y, ϕ,{r},{q,r},{p,r}}and JO(Y,ζ)= P(Y) −{q,r}.Then f is contra J-continuous.

Proposition 5.6.3. A strongly J-continuous function f : (Y,ζ)→ (Z,σ) is a contra J-continuous function but the converse is not true.

Proof Let U be any closed set in (Z,σ). Since f : (Y,ζ) →(Z,σ) is a strongly J-continuous function. Then f^{-1}(U) is J-clopen in (Y,ζ) which implies f^{-1}(U) is J-open in (Y,ζ). Therefore Hence f is contra J-continuous.

Remark 5.6.4. The converse of the above Proposition can be seen from the following Counter Example.

Counter Example 5.6.5. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z ={p,q,r}with ζ = {Y, ϕ, {p},{p,q}} and σ = {Z, ϕ,{p,q}}.Here σ^c= {Y, ϕ,{r}}and JO(Y,ζ)= P(Y) –{q,r} and JC(Y,ζ)= P(Y) –{p}.Then f is contra J-continuous but not strongly J-continuous as for arbitrary subset {p} in (Z,σ),the inverse image {p} is J-clopen in (Y,ζ).

Proposition 5.6.6. A totally J-continuous function f : (Y,ζ)→ (Z,σ) is a contra J-continuous function but the converse is not true.

Proof Let U be any closed set in (Z,σ). Since f : (Y,ζ) →(Z,σ) is a totally J-continuous function. Then f^{-1}(U) is J-clopen in (Y,ζ) which implies f^{-1}(U) is J-open in (Y,ζ).Therefore Hence f is contra J-continuous.

Remark 5.6.7. The converse of the above Proposition 5.6.6. can be seen from the following Counter Example.

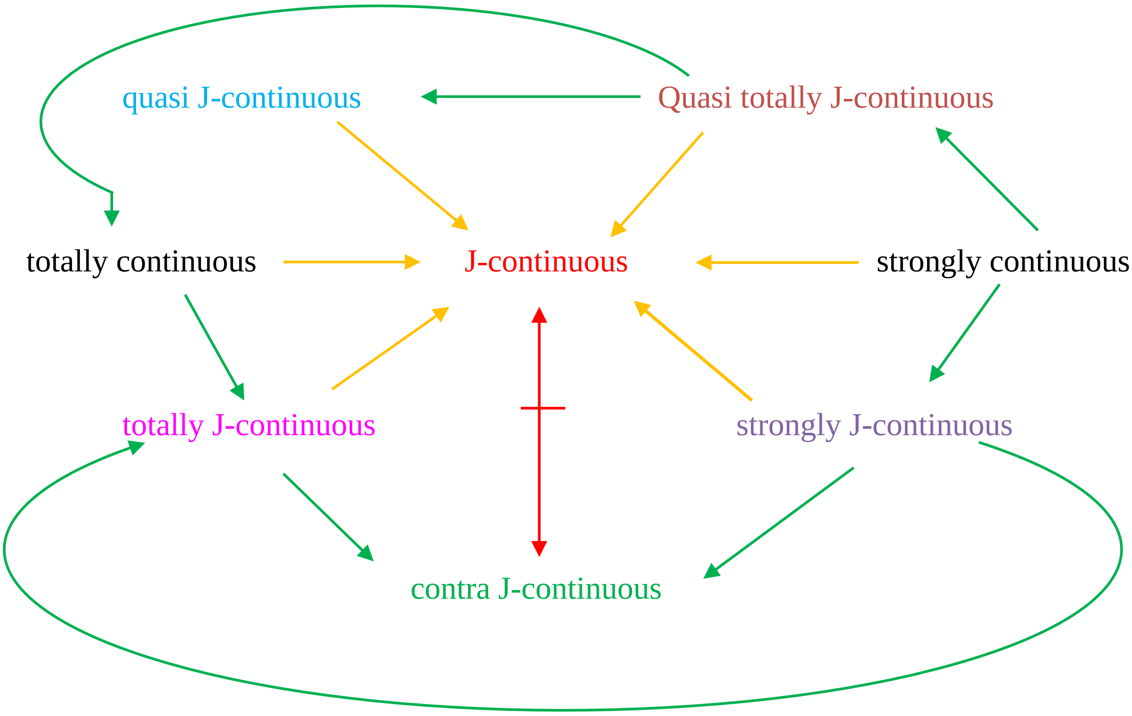
Counter Example 5.6.8. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = r, f(q) = q, f(r) = p. Consider Y = Z ={p,q,r}with ζ = {Y, ϕ, {p},{p,q} and σ = {Z, ϕ,{p,q}}.Here σ^c= {Z, ϕ,{r}} and JO(Y,ζ)= P(Y)−{q,r},JC(Y,ζ)= P(Y)−{p}.Then f is contra J-continuous but not totally J-continuous. Because for the closed set {r} in (Z,σ), the corresponding inverse image {p} is not J-clopen in (Y,ζ).

Remark 5.6.9. contra J-continuous function and J-continuous functions are independent.

Counter Example 5.6.10. Let f : $(Y,\zeta)\rightarrow(Z,\sigma)$ be the function defined by f(p) = s, f(q) = r, f(r) = p,f(s) = q. Consider Y = Z ={p,q,r,s}with ζ = {Y, ϕ,{p,q} and σ = {Z, ϕ,{p,q,r}}.Here σ^c= {Z, ϕ,{s}} and JO(Y,ζ)= P(Y)−{{r,s},{q,r,s},{p,r,s}},JC(Y,ζ)= P(Y)−{{p},{q},{p,q}}.Then f is contra J-continuous but not J-continuous. Because for the closed set {s} in (Z,σ),the corresponding inverse image {p} is not J-closed in (Y,ζ).

Counter Example 5.6.11. Let f : $(Y,\zeta)\rightarrow(Z,\sigma)$ be an identity function.Consider Y = Z ={p,q,r}with ζ = {Y, ϕ,{p} and σ = {Z, ϕ,{p},{p,q}}.Here σ^c= {Z, ϕ,{r},{q,r}} and JO(Y,ζ)= P(Y)−{{q,r}},JC(Y,ζ)= P(Y)−{{p}}.Then f is J-continuous but not contra J-continuous. Because for the closed set {q,r} in (Z,σ),the corresponding inverse image {q,r} is not J-open in (Y,ζ).

Result 5.6.12. From the above discussions we get the following diagram.



The composition of two contra J-continuous functions need not be a contra J-continuous function. This can be seen from the following Counter Example.

Counter Example 5.6.13. Let f : $(Y,\zeta) \rightarrow (Z,\sigma)$ be the function defined by f(p) = p, f(q) = r, f(r) = q. Consider Y = Z = P ={p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z, ϕ, {p,q}}, μ = {P, ϕ,{r}}.Then f is contra J-continuous as JO(Y,ζ)= P(Y)−{q,r}.Let g: $(Z,\sigma) \rightarrow (P,\mu)$ be the function defined by g(p) = r, g(q) = q, g(r) = p. Then g is contra J-continuous as JO(Z,σ) = P(Z).Consider the composition function g ∘ f : $(Y,\zeta) \rightarrow (P,\mu)$ such that (g∘ f)(p) = g(f(p)) = r, (g∘f)(q) = g(f(q)) = p and (g∘ f)(r) = g(f(r)) = q. But their composition g∘f is not contra J-continuous. Because for the closed set {p,q} in (P,μ), $f^{-1}(g^{-1}(\{p,q\}))$ = {q,r} is not J-open in (Y,ζ).

Proposition 5.6.14. If f : $(Y,\zeta) \rightarrow (Z,\sigma)$ is a contra J-continuous function and g : $(Z,\sigma) \rightarrow (P,\mu)$ is a continuous function, then g ∘ f : $(Y,\zeta) \rightarrow (P,\mu)$ is a contra J-continuous function.

Proof Given g : $(Z,\sigma) \rightarrow (P,\mu)$ is a continuous function. Let U be closed set in (P,μ).Hence $g^{-1}(U)$ is closed in (Z,σ).Given f : $(Y,\zeta) \rightarrow (Z,\sigma)$ is a contra J-continuous function. So $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is J-open in (Y,ζ). Hence g ∘ f is a contra J-continuous function.