Chapter V

CHAPTER V

IDEMPOTENT MATRICES OVER SEMIRINGS

In this chapter, we consider a semiring which is either a general Boolean algebra $\mathbb{B}_k$ or a chain semiring $\mathbb{K}$.

Definition: 5.1

Let A be a matrix in $M_n(\mathbb{B}_1)$. For i = 1,..........,n, we define an **i^{th} row matrix $r_i[A]$** of A as a matrix whose i^{th} row is the same as the i^{th} row of A and the other rows are zero. Similarly, we can define a **j^{th} column matrix $c_j[A]$** of A for j = 1,.....,n. A **line matrix** is an i^{th} row matix or a j^{th} column matrix of a matrix; Cells E_1, ... E_k are called **collinear** if $\sum_{i=1}^{k} E_i$ is dominated by a line matrix of J_n.

Definition: 5.2

Let $A \in M_n(\mathbb{B}_1)$. For indices i, j $\in$ {1,........,n}, $r_i[A]$ and $c_j[A]$ are said to be **(i, j)-disjoint** if $XY = O_n$ for all off-diagonal cells $X \subseteq r_i[A]$ and $Y \subseteq c_j[A]$.

Lemma: 5.3

Let A be idempotent in $M_n(\mathbb{B}_1)$. If $r_i[A]$ and $c_j[A]$ are not (i, j)-disjoint, then $E_{i,j} \subseteq A$.

Definition: 5.4

Let E_1, E_2, E_3 and E_4 be four distinct cells in $M_n(\mathbb{B}_1)$. Then their sum is called **a frame** if the four 1's constitute a rectangle with atleast one entry on diagonal. That is, there exist indices i, j, k ∈ (1, ...n} with i ≠ j, k such that

$$\sum_{i=1}^{4} E_l = E_{i,i} + E_{i,j} + E_{k,i} + E_{k,j}.$$

Lemma: 5.5

Let E_1, E_2,.......,E_m and F be cells in an idempotent matrix X∈ $M_n(\mathbb{B}_1)$, where m ≥ 2. Then

(i) If E_1, E_2,.......E_m is not zero, then it is a cell in X;

(ii) If F is off-diagonal, then there exist distinct cells G and H in A such that F = GH. Moreover if both cells G and H are off-diagonal, then three cells F, G and H are mutually distinct.

Corollary: 5.6

If all cells in A∈ $M_n(\mathbb{B}_1)$ are off-diagonal, then A is not idempotent.

Theorem: 5.7

Let A be idempotent in $M_n(\mathbb{B}_1)$. If F is an off-diagonal cell in A such that F is not collinear with any diagonal cell in A, then F is in a frame with one diagonal cell and two additional off-diagonal cells in A.

Proof:

Let ϕ_1 = {E_1,. ....,E_m} be the set of all distinct diagonal cells in A and ϕ_2 be the set of all off-diagonal cells in A. By Corollary 5.7, we have $m \geq 1$. Let us denote $E_i = E_{a_i,a_i}$ for all i = 1,......,m and F = $E_{b,c}$. Since F and E_i are not collinear for all i, it follows that a_1, ...,$a_{m,b}$, and c are mutually distinct indices. Assume that F is not in a frame with one diagonal cell in ϕ_1 and two off-diagonal cells in ϕ_2. Then we can construct an infinite subset of cells in ϕ_2 applying the induction process as in the Proof of 5.7.

The base of induction: Since A is idempotent, by Lemma 5.6 (ii), there exist two distinct cells $E_{b,x_i}, E_{x_1,c} \subseteq A$ such that $E_{b,c} = E_{b,x_1}E_{x_1,c}$ for some index x_1. If $E_{b,x_1} \in \phi_1$ or $E_{x_1,c} \in \phi_1$, then F = $E_{b,c}$ is collinear with a diagonal cell. This is a contradiction. Thus $E_{b,x_1}, E_{x_1,c} \in \phi_2$. If $x_1 = a_i$ for some i, then we obtain that $E_{x_1,x_1} + E_{b,x_1} + E_{x_1,c} + E_{b,c}$ is a frame, a contradiction to the assumption. Hence, $x_1 \neq a_i$ for all i.

Since A is idempotent and $E_{b,x_1} \in \phi_2$ by Lemma 5.6 (i), we can find two cells E_{b,x_2} and E_{x_2,x_1} in $\phi_1 \cup \phi_2$ in such that $E_{b,x_1} = E_{b,x_2}E_{x_2,x_1}$ for some index x_2. Then we have $E_{x_2,c} = E_{x_2,x_1}E_{x_1,c} \subseteq A$ by Lemma 5.6 (i). If $x_2 = x_1$, then the four cells $E_{x_2,x_1}, E_{b,x_2}E_{x_1,c}$ and $F = E_{b,c}$ are in a frame, which contradicts the assumption. If $x_2 = a_i$ for some i, then $E_{x_2,x_2}, E_{b,x_2}E_{x_2,c}$ and $F = E_{b,c}$ are in a frame, which also contradicts the assumption. Thus $x_2 \neq a_i$ for all i and $x_2 \neq x_1$.

The induction step: Assume that for certain $k \geq 2$, the set of cells $\{E_{b,x_1}, ...E_{b,x_k}, E_{x_2,x_1}, ...E_{x_k,x_{k-1}}\} \subseteq \phi_2$ was already constructed. Then we may add new elements to this set as follows. By Lemma 5.6 (ii), since A is idempotent, there exist two cells $E_{b,x_{k+1}}$ and E_{x_{k+1},x_k} in $\phi_1 \cup \phi_2$ such that $E_{b,x_k} = E_{b,x_{k+1}}, E_{x_{k+1},x_k}$ for some index x_{k+1}. Thus by Lemma 5.6 (i),

$$E_{x_{k+1},c} = E_{x_{k+1},x_k} \ldots\ldots\ldots E_{x_2,x_1} E_{x_1,c} \subseteq A .$$

Now, we show that x_{k+1} is neither b nor x_i for all i = 1,………,k. Note that $x_{k+1} \neq b$ since $F = E_{b,c}$ is not collinear with any diagonal cell. Assume that $x_{k+1} = x_i$ for some $i \in \{1,\ldots\ldots\ldots,k\}$. Then $E_{x_i,c} = E_{x_{k+1},c} \subseteq A$ and by Lemma 5.6 (i) $E_{x_i,x_i} = E_{x_{k+1},x_i} = E_{x_{k+1},xk} \ldots\ldots\ldots E_{x_{i+1},x_i} \subseteq A$. Therefore, the four cells $E_{x_i,x_i}, E_{b,x_i}, E_{x_i,c}$ and $F = E_{b,c}$ are in a frame, which contradicts the assumption. Thus $x_{k+1} \neq x_i$ for all i = 1,………..k and we have constructed the sets.

$$\{E_{b,x_1}, \ldots\ldots\ldots, E_{b,x_{k+1}}, E_{x_2,x_1}, \ldots\ldots\ldots\ldots, E_{x_{k+1},k}\} \subseteq \phi_2 .$$

Therefore we obtain an infinite set of off-diagonal cells $\{E_{b,x_i} / i \in N\}$ on the b^{th} row, which is impossible.

Lemma: 5.8

Let A be an idempotent matrix in $M_n(\mathbb{B}_1)$ with $E_{i,i} \subseteq A$. If $w(r_i[A]) = s+1$ and $w(c_i[A]) = t+1$, then there exist exactly s• t frames in A dominating $E_{i,i}$.

Definition: 5.9

For a matrix $A \in \mathbb{M}_n(\mathbb{B}_1)$, let $\{E_{i,i}, E_{j_1,i}, \ldots\ldots\ldots, E_{j_s,i}\}$ and $\{E_{i,i}, E_{i,i_1}, \ldots\ldots\ldots, E_{i,i_t}\}$ be the sets of cells in $c_i[A]$ and $r_i[A]$, respectively, where $s, t \geq 1$. If $E_{j_k,i_l} \subseteq A$ for all k = 1,…….,s and l = 1,…………t, then

$\sum_{k=1}^{s} \sum_{l=1}^{t} \left(E_{i,i} + E_{j_k,i} + E_{i,i_l} + E_{j_k,i_l}\right)$ is called an i^{th} **rectangle part of A** and denoted by $RP_i[A]$.

Remark: 5.10

Let A be idempotent in $M_n(\mathbb{B}_1)$ with $E_{i,i} \subseteq A$. If $w(r_i[A]) > 1$ and $w(c_i[A]) > 1$, then Lemma 5.9 shows that the i^{th} rectangle part $RP_i[A]$ of A exists.

Example: 5.11

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in M_4(\mathbb{B}_1).$$

$$\text{Then } RP_1[A] = RP_3[A] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

But there is neither $RP_2[A]$ nor $RP_4[A]$.

Definition: 5.12

Let A be a matrix in $M_n(\mathbb{B}_1)$ with $E_{i,i} \subseteq A$. If $w(r_i[A]) = 1$ or $w(c_i[A]) = 1$, then $r_i[A] + c_i[A]$ is called an **i^{th} line part of A**, and denoted by $LP_i[A]$. That is, $A \in M_n(\mathbb{B}_1)$ has the i^{th} line part $LP_i[A] = r_i[A] + c_i[A]$ if and only if $r_i[A] = E_{i,i}$ or $c_i[A] = E_{i,i}$.

Example: 5.13

Let $B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be a matrix in $M_n(\mathbb{B}_1)$. Then B has 1st and 4th line parts, which are $LP_1[B] = E_{1,1} + E_{1,3} + E_{1,4}$ and $LP_4[B] = E_{4,4} + E_{1,4}$, respectively. But B has neither $LP_2[B]$ nor $LP_3[B]$.

Corollary: 5.14

If A is idempotent in $M_n(\mathbb{B}_1)$, then every cell dominated by A is either in a rectangle part or in a line part of A.

Proof:

It follows directly from Theorem 5.7 and Lemma 5.9.

Definition: 5.15

Suppose that $A \in M_n(\mathbb{B}_1)$ has i^{th} and j^{th} rectangle parts $RP_i[A]$ and $RP_j[A]$ for some i and j with $i \neq j$. We say that $RP_i[A]$ and $RP_j[A]$ are **disjoint** if $r_i[A]$ and $c_j[A]$ are (i, j)–disjoint, or $r_j[A]$ and $c_i[A]$ are (j, i)–disjoint or both.

Theorem: 5.16

Let A be idempotent in $M_n(\mathbb{B}_1)$. Then any two rectangle parts of A are either disjoint or identical.

Proof:

It is straight forward that disjoint parts are not identical. Suppose that the i^{th} and j^{th} rectangle parts of A are not disjoint. By definition, we have $r_i[A]$ and $c_j[A]$ are not (i, j)–disjoint, and $r_j[A]$ and $c_i[A]$ are not (j, i)–disjoint.

Therefore, $E_{i,,j} \subseteq A$ and $E_{j,,i} \subseteq A$ by Lemma 5.3. Then in this case we claim that for any cell E, we have $E \subseteq RP_i[A]$ if and only if $E \subseteq RP_j[A]$. It is straight forward that the four cells $E_{i,I}$, $E_{j,j}$, $E_{i,j}$, $E_{j,i} \subseteq RP_t[A]$ for t = i, j. Suppose $E \subseteq RP_i[A]$. We first consider the case $E \subseteq r_i[A]$ say $E = E_{i,a}$. Then we have $E_{j,a} = E_{i,j}\, E_{i,a} \subseteq A$ by Lemma 5.6 (i), and the four cells $E_{i,a}$, $E_{i,j}$, $E_{j,a}$ and $E_{j,j}$ from a frame. Therefore, $E = E_{i,a} \subseteq RP_j[A]$. Similarly for the case $E \subseteq c_i[A]$ it follows that $E \subseteq RP_j[A]$.

Next, consider the case $E \not\subseteq r_i[A]$ and $E \not\subseteq c_i[A]$, say $E = E_{c,d}$. Since $E \subseteq RP_i[A]$, there exist two off-diagonal cells $E_{i,x} \subseteq r_i[A]$ and $E_{y,i} \subseteq c_i[A]$ such that $E_{c,d} = E_{y,i} = E_{i,x}$. Therefore, we have that c = y and d = x by the result which states that, for any cells $E_{i,j}$ and $E_{u,v}$; we have $E_{i,j}\, E_{u,v} = E_{i,v}$ or O_n according as j = u or $j \neq u$, since A is idempotent, we obtain by Lemma 5.6 (i) that

$$E_{c,j} = E_{yj,} = E_{y,i}\, E_{i,j} \subseteq A \text{ and } E_{j,d} = E_{j,x} = E_{j,i}\, E_{i,x} \subseteq A.$$

Hence the four cells $E_{c,d}$, $E_{c,j}$, $E_{j,d}$ and $E_{j,j}$ form a frame. Therefore, we have $E = E_{c,d} \subseteq RP_j[A]$, then we have that $E \subseteq RP_i[A]$. Therefore, the two rectangle parts $RP_i[A]$ and $RP_j[A]$ are identical.

Theorem: 3.17

Let A be a matrix in $M_n(\mathbb{B}_1)$. Then A is idempotent if and only if the following two conditions are satisfied:

(i) there exist integers r, $l \geq 0$ such that A is a sum of r disjoint rectangle parts and l line parts,

(ii) if for some $i \neq j$ $r_i[A]$ and $c_j[A]$ are not (i, j)-disjoint, then $E_{i,j}$ is a cell in A.

Proof:

Let A be a matrix in $M_n(\mathbb{B}_1)$. It is routine to check that a matrix satisfying the two conditions is idempotent. To show the opposite implication, without loss of generality, we can assume that A has r rectangle parts and l line parts, where $r, l \geq 0$. Let F be an off-diagonal cell in A. By Corollary 5.14, F is in some rectangle part or some line part of A. Therefore, A is the sum of r rectangle parts and l line parts of A. It follows from Theorem 5.16 that r-rectangle part of A are disjoint. By Lemma 5.3, if $r_i[A]$ and $c_j[A]$ are not (i, j)-disjoint, then $E_{i,j}$ is a cell in A.

Definition: 5.18

Let $\mathbb{K}$ be a chain semiring. Let α be a fixed members of $\mathbb{K}$, other than 1. For each $x \in \mathbb{K}$, define $x^\alpha = 0$ if $x \leq \alpha$, and $x^\alpha = 1$ otherwise. Then the mapping $x \to x^\alpha$ is a homomorphism of $\mathbb{K}$ onto $\mathbb{B}_1$. Its entrywise extension to a mapping $A \to A^\alpha$ of $M_n(\mathbb{K})$ onto $M_n(\mathbb{B}_1)$ preserves matrix sums and products and multiplication by scalars. We call A^α the **α–patten** of A.

Theorem: 5.19

Let $A = [a_{i,\,j}]$ be a matrix in $M_n(\mathbb{K})$. Then A is idempotent if and only if all $a_{i,j}$-patterns of A are idempotent in $M_n(\mathbb{B}_1)$.

Proof:

Let A be a idempotent in $M_n(\mathbb{K})$. Then all a_{ij}-patterns of A are idempotent in $M_n(\mathbb{B}_1)$ because each a_{ij}-pattern of A homomorphism of $M_n(\mathbb{K})$ onto $M_n(\mathbb{B}_1)$.

Conversely, assume that each a_{ij}-pattern $A^{a_{i,j}}$ of A is idempotent in $M_n(\mathbb{B}_1)$. If $A^2 \neq A$, then for some (i,j)th entries of A and A^2, we have

$$a_{i,j} \neq \sum_{k=1}^{n} a_{i,k} a_{k,j} . \qquad (*)$$

If $a_{i,j} < \sum_{k=1}^{n} a_{i,k} a_{k,j}$, then the (i,j)th entry of $A^{a_{ij}}$ is 0, but that of $\left(A^{a_{ij}}\right)^2$ is 1, a contradiction to the fact that a_{ij} –pattern of A is idempotent in $M_n(\mathbb{B}_1)$. Hence we have $a_{i,j} > \sum_{k=1}^{n} a_{i,k} a_{k,j}$. We notice that the right side of (∗) is just $a_{i,k} a_{k,j}$ for some k = {1,……….n}. Furthermore we have $a_{i,k} a_{k,j} = a_{i,k}$ or $a_{k,j}$. If $a_{i,k} a_{k,j} = a_{i,k}$, then $a_{i,j} > \sum_{k=1}^{n} a_{i,k} a_{k,j}$, and hence the (i,j)th entry of $A^{a_{ik}}$ is 1, but that of $\left(A^{a_{i,k}}\right)^2$ is 0, a contradiction. Similarly if $a_{i,k} a_{k,j} = a_{i,k}$, then we have $\left(A^{a_{k,j}}\right)^2 \neq A^{a_{kj}}$, a contradiction. Therefore A is idempotent in $M_n(\mathbb{K})$.