SOME INTERESTING GENERALIZATIONS OF PROXIMITY STRUCTURES

BY

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Introduction

INTRODUCTION

There are no rules of architecture for a castle in the clouds.

- G.K.Chesterton.

The study of proximity spaces has attracted the attention of many famous topologists-Efremovich, Hayashi, Smirnov, Kelley, Naimpally and Warrack. The aim of this thesis is to study some interesting generalizations of proximity structures. Some interesting generalized proximities discussed in this thesis are

1. Paraproximity (2.5)
2. Closure-proximity (2.21)
3. Normal proximity (2.28)
4. Perfect proximity (2.30)
5. Weak normal proximity (2.35)
6. Compact proximity (2.38)
7. Metacompact proximity (2.42)
8. Lodato or LO-proximity (3.1)
9. Leader or LE-proximity (3.1)
10. Pervin or P-proximity (3.3)
11. Separation or S-proximity (3.8)
12. E-discrete proximity (3.10)
13. E-discrete proximity of Alexandroff (3.11)
14. R-proximity (4.8)
15. RC-proximity (4.15)

The first chapter is devoted to fundamental definitions.

In the second chapter, different generalizations of Efremovich proximities are discussed. Efremovich showed that every proximity (2.1) on a set R yields a completely regular space. In 1964, Hayashi [8] gave new axioms for a proximity, slightly different from the axioms of Efremovich proximity named as paraproximity (2.5). This paraproximity yields a completely normal space.

In 1984, Mira Sarkar [14], defined a closure proximity (2.21) on X, the axioms of which differ from those of a paraproximity in some respects. There is a 1-1 correspondence between the closure proximities and T_1-topologies on X. Adding certain axioms to a closure proximity, we get some new proximities. They are

1. Normal proximity (2.28)
2. Perfect proximity (2.30)
3. Weak normal proximity (2.35)
4. Compact proximity (2.38)
5. Metacompact proximity (2.42)

Every normal proximity induces a completely normal topology and every completely normal space has a normal proximity structure. This yields a 1-1 correspondence between the set of normal proximities and the set of completely normal topologies on a set X. A similar result is true in the case of perfect proximity and perfect normality. These generalizations are discussed in detail in chapter two. The weakest proximity compatible with a hereditarily compact (a metacompact) topology on X is constructed in this chapter.

The third chapter is devoted to the study of two important generalized proximities-Lodato or LO-proximity and Leader or LE-proximity (3.1). In this chapter, the definitions of some interesting generalized proximities are given. These include the following.

1. Pervin or P-proximity (3.3)
2. Separation or S-proximity (3.8)
3. E-discrete proximity (3.10)
4. E-discrete proximity of Alexandroff (3.11)

The interrelations between these proximities are also discussed in this chapter. Further more, it is proved here that, in general Lodato proximities (3.20) are not covered.

Fourth chapter deals with R-proximities (4.8) and RC-proximities (4.15) which are also generalizations of Efremovich proximity. D.Harris [6] has introduced R-proximities in order to investigate regular-closed extensions of regular topological spaces. In connection with these proximities, the following two papers are discussed in detail.

1. Regular-closed spaces and proximities [6]
2. A note on R-proximities [20]

The important results discussed in this connection are as follows;

1. A topology is regular iff it is the topology induced by an R – proximity (4.12)
2. Every RC-regular space has its topology induced by an RC-proximity (4.18)

3. If an RC-proximity space is absolutely closed, then its induced topology is a regular-closed topology, and the proximity is given by : A and B are far iff they have disjoint closures. (4.20).
4. An RC-proximity space is absolutely closed iff the induced topology is regular closed. (4.21).
5. A topological space is regular-closed iff it has the topology induced by an absolutely closed RC-proximity. (4.22)
6. There is precisely one RC-proximity that induces the topology of a regular-closed space. (4.23).

Further, a technique for identifying the supremum of any non-empty family of compatible proximities on any R_o-space is established. As an application of this, Alexandrov proximity is identified as the supremum of the compatible R-proximities on any regular space.

Review of Literature

REVIEW OF LITERATURE

The first important paper on proximity structure was published by Efremovich [7] in 1952. In this paper he axiomatically characterised the proximity relation "A is near B" for subsets A and B of any set X. A set X together with a proximity relation is called proximity space. Every proximity structure induces a completely regular topological structure. In every completely regular space, a compatible proximity structure can be defined.

Some authors have worked with weaker axioms, which enabled them to introduce an arbitrary topology on the underlying set.

Some important contributions to the study of proximity spaces are published by Efremovich, Kelley, Leader, Nagata, Naimpally and Warrack.

Since the number of papers published on proximity spaces is numerous, we restrict ourselves to a few papers in this review of literature. We have given a brief review of nine important papers published in this topic.

1. Smirnov. Y.M.

"On Proximity Spaces [21]"

In this paper, the author discovered the connection between Hausdorff compactification of a Tychonoff space and the compatible proximity relation, showing that "a topological space admits a compatible proximity relation iff it is a subspace of a compact Hausdorff space".

2.Pervin. W.J.

"Quasi-proximities for topological spaces [19]"

The author studied the generalizations of Efremovich's original set of axioms for a proximity. He neglected the symmetric condition and obtained a space called quasi-proximity space. The author showed that every topological space can be derived from a quasi-proximity space (X, δ) by defining the binary relation δ as follows:

$$A \ \delta \ B \text{ iff } A \cap \overline{B} \neq \phi.$$

Conversely, every quasi-proximity space (X, δ) becomes a topological space if the closure operator is defined by $cl(A) = \{x : \{x\} \delta A\}$ for every $A \subset X$.

3. Leader. S

"On products of proximity space [11]"

Here also the generalizations of Efremovich's proximity were studied. Omitting the symmetry condition from the original set of Efremovich axioms and also including a weakened form of the "strong axiom", the author defined a space called topological d-space. It was shown that every topological space can be derived from a topological d-space and conversely.

4. Lodato.M.W

"On topologically induced generalized proximity relations [12]".

The author added symmetry to Leader's set of axioms to obtain a symmetric binary relation named as Lodato proximity. He proved that every set with a Lodato proximity defined on it satisfies the R_o axiom (i.e every open set contains the closure of each of its points), and that given any R_o-space we obtain a Lodato proximity compatible with the given topology if we define A δ B iff $\overline{A} \cap \overline{B} \neq \phi$.

5. Chattopathyay.K.C

"Basic quasiproximities, grills and compactifications [2]"

In this paper a basic quasiproximity π on a set X is a binary relation on the power set of X satisfying the following conditions:

1. $(A, B \cup C) \in \pi$ iff $(A,B) \in \pi$ or $(A,C) \in \pi$
2. $(A \cup B, C) \in \pi$ iff $(A,C) \in \pi$ or $(B,C) \in \pi$
3. $A \cap B \neq \phi \Rightarrow (A,B) \in \pi$
4. $(A,B) \in \pi \Rightarrow A \neq \phi \neq B$. The author develops the theory of basic quasiproximity spaces in a manner similar to that of W.J. Thron for proximities (Maths. Ann. 206(1973), 35-62). Thus the notions of stacks and grills as well as filters and ultrafilters are used to advance the theory.

6.Csaszar. K.

"Separation axioms for proximity and closure spaces [4]"

A semiproximity δ is a relation on the powerset exp X of a set X satisfying the following conditions:

(P1) δ is symmetry, (P2) $\phi\ \hat{\delta}\ X$, (P3) $A \cap B \neq \phi \Rightarrow A\ \delta\ B$, (P4) $A\ \delta\ B$, $A \subseteq C$ and $B \subseteq D$ imply $C\ \delta\ D$. If δ also satisfies (P5) $(A \cup B)\ \delta\ C \Rightarrow A\ \delta\ C$ or $B\ \delta\ C$, then δ is a proximity. A semiclosure on X is a function C from exp X into itself satisfying (C1) $C(\phi) = \phi$, (C2) $A \subseteq C(A)$, for all $A \subseteq X$ and (C3) $A \subseteq B$ implies

$C(A) \subseteq C(B)$. When C also satisfies $C(A \cup B) = C(A) \cup C(B)$, then C is a closure on X. Given a (semi-) proximity δ for X, if C_δ is defined by $x \in C_\delta(A)$ iff $\{x\}\ \delta\ A$, then C_δ is a (semi-) closure on X and δ is compatible with C_δ.

A number of separation conditions for semiclosures and for semiproximities are introduced and interrelationships among these are studied. For eg, a semiclosure C satisfies.

(SC′) : $x \notin C(A) \Rightarrow C\{x\} \cap C(A) = \phi$ iff the relation δ_c defined by $A\ \delta_c\ B$ iff $C(A) \cap C(B) \neq \phi$ is a compatible semiproximity for C. Also if C is a closure, then δ_c is a proximity whenever (SC′) holds.

Relationships between separation conditions for semiproximities and Lodato proximities are also considered. A number of examples are included which distinguish among the various separation conditions.

7.Dimitrijevic. R.S

"Base and Subbase of R-proximity [5]"

The author introduces a notion of R-proximity base and subbase. For a non empty set X let b be a binary relation satisfying the following axioms:

(B1) $A\ b\ B \Rightarrow B\ b\ A$ (B2): $A \cap B \neq \phi \Rightarrow A\ b\ B$, (B3) $A\ b\ B \Rightarrow A \neq \phi \neq B$, (B4) $A\ b\ B$ and $A \subset C$, $B \subset D \Rightarrow C\ b\ D$, (B5) $A\ \bar{b}\ B \Rightarrow \exists\ E \subset X$;$A\ \bar{b}\ E$ and $(X-E)\ \bar{b}\ B$.

Let δ be a relation defined as follows: $A\ \delta\ B \Leftrightarrow$ for any two finite covers P,Q of the sets A,B there exists a pair $(A,B) \in P \times Q$ such that $A\ b\ B$. Then δ is an R-proximity relation, (i.e) the following properties hold: (R1) = (B1), (R2) = (B2), (R3) = (B3), (R4) $A \, \delta \, (B \cup C) \Leftrightarrow A \, \delta \, B$ or $A \, \delta \, C$ and (R5)$x \, \bar{\delta} \, B \Rightarrow \exists \, E \subset X$; $x \, \bar{\delta} \, E$ and $(X - E) \, \bar{\delta} \, B$. The relation δ is the coarsest R-proximity on X finer than the relation b. R-proximity subbase axioms are considered. The author writes that "several theorems in R-proximity spaces can be simplified by using the R-proximity base and subbase".

8.Naimpally. S.A; Di Maio.G

"Proximity Relation [17]"

The authors present an example which shows that S.Leader's construction for the Smirnov compactification of an Efremovich proximity fails for a Lodato proximity space. However, using nearness and contiguity structures, they provide a method of getting a proximity extension of a Lodato space.

9.Csaszar. A

"RE-proximities [3]"

D. Harris [Pacific J.Math. 34 (1970), 675 - 685] introduced R-proximities to investigate regular-closed extensions of regular topological spaces. An R-proximity on a set X is a basic proximity δ on the power set of X such that $\{x\} \, \bar{\delta} \, (X - V)$ implies the existence of W such that $\{x\} \bar{\delta} \, (X - W)$ and $W \, \bar{\delta} \, (X - V)$ ($\bar{\delta}$ means not δ). If Y is a regular extension of X then we can define an R-proximity on X as follows:

$A \, \delta \, B$ iff $\mathrm{cl}_Y A \cap \mathrm{cl}_y B \neq \phi$. Such an R-proximity is called an RE-proximity. If Y is a regular-closed extension of X, then δ is called an RC-proximity. In this paper the author investigates the proximities defined above using concepts and results from the theory of syntopogenous spaces.

Chapter I

CHAPTER - I

PRELIMINARY DEFINITIONS

DEFINITION 1.1

A Topological space X is called T_2 – space or a Hausdorff space if for each pair x_1,x_2 of distinct points of X, there exist neighbourhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint.

DEFINITION 1.2

Suppose that one-point sets are closed in X. Then X is said to be normal if for each pair A,B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

DEFINITION 1.3

By a T_1- space X, we mean a topological space such that for each pair x_1 and x_2 of distinct points of X, there exists neighbourhoods U of x_1 and V of x_2 such that $x_1 \notin V$ and $x_2 \notin U$.

DEFINITION 1.4

A normal T_1 – space is called a T_4 – space.

DEFINITION 1.5

A space X is said to be compact, if every open covering of X contains a finite subcollection that also covers X.

DEFINITION 1.6

A space X is said to be locally compact at x if there is some compact subset C of X that contains a neighbourhood of x. If X is locally compact at each of its points, X is said to be locally compact.

DEFINITION 1.7

A topological space X is R_o iff for each open set G, $x \in G \Rightarrow cl\{x\} \subset G$.

DEFINITION 1.8

Let X be a topological space. A subset of X is called G_δ - set if it is the intersection of countably many open-sets of X.

DEFINITION 1.9

A perfectly normal space is a normal space in which every closed set is G_δ.

DEFINITION 1.10

A space X is completely normal if and only if for every pair A,B of separated sets in X (that is,sets such that $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi$) , there exist disjoint open sets containing them.

DEFINITION 1.11

Suppose that one-point sets are closed in X. Then X is said to be regular if for every pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

DEFINITION 1.12

A topological space X is said to be completely regular if for every closed set $A \subset X$ and any point $p \notin A$, there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(p) = 0$ and $f(x) = 1$ for all $x \in A$.

DEFINITION 1.13

A subset A of a space X is said to be dense in X if $\overline{A} = X$. ($\overline{A}$ – closure of A in X) .

DEFINITION 1.14

A compactification of a space X is a compact Hausdorff space Y containing X such that X is dense in Y.

DEFINITION 1.15

Let X be a locally compact Hausdorff space. Take some object outside X, denoted by the symbol ∞ for convenience, and adjoin it to X, forming the set $Y=X \cup \{\infty\}$. Topologize Y by defining the collection of open sets in Y to be all sets of the following types

i) U, where U is an open subset of X.

ii) $Y - C$, where C is a compact subset of X.

The space Y is called the one-point compactification of X.

DEFINITION 1.16

If $\{U_\alpha\}_{\alpha \in J}$ is a collection of subsets of X, a collection $\{V_\beta\}_{\beta \in K}$ is said to refine $\{U_\alpha\}$ if for each set V_β, there is atleast one set U_α containing it.

DEFINITION 1.17

A collection $\{U_\alpha\}_{\alpha \in J}$ of subsets of X is point finite if each point of X lies in only finitely many elements of $\{U_\alpha\}_{\alpha \in J}$.

DEFINITION 1.18

A space is called metacompact if every open cover has an open-point finite refinement.

DEFINITION 1.19

Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ (power set of x) be nonempty . $\mathcal{F}$ is called a filter on X if

i) $\phi \notin \mathcal{F}$

ii) $F_1, F_2 \in \mathcal{F} \Rightarrow F1 \cap F2 \in \mathcal{F}$

iii) $F \in \mathcal{F}$ and $F \subseteq G \Rightarrow G \in \mathcal{F}$

DEFINITION 1.20

A regular filter is a filter with a base of open sets and a base of closed sets.

DEFINITION 1.21

Let $X \subseteq Z$ and $\mathcal{F}$ a filter on Z. Trace of $\mathcal{F}$ in X is = $\{ X \cap F : F \in \mathcal{F}$ and $X \cap F \neq \phi\}$.

DEFINITION 1.22

Let (X,τ) be a topological space and $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is said to τ - converge to x if every neighbourhood of x belongs to $\mathcal{F}$.

Chapter II

CHAPTER – II

DIFFERENT GENERALIZATIONS OF EFREMOVICH PROXIMITIES

DEFINITION:2.1 Efremovich [7]

Efremovich [7] defined a relation δ on a set, called a proximity. For a pair of subsets A and B of a point set R we write $A\ \delta\ B$ or $(A, B) \in \delta$ if A and B are proximate, other wise $A \overset{A}{\delta} B$ or $(A,B) \notin \delta$.

A relation δ on a set R is called a **proximity** if it satisfies the following four axioms:

(EP1) : (Symmetry) $(A,B) \in \delta$ iff $(B,A) \in \delta$

(EP2) : Both $(A,C) \notin \delta$ and $(B,C) \notin \delta$ iff $(A \cup B, C) \notin \delta$.

(EP3) : For arbitrary two points $a, b \in R$, $(\{a\},\{b\}) \in \delta$ iff $a = b$.

(EP4) : (Separation) If $(A, B) \notin \delta$ then there are disjoint subsets U and V of R such that $(A, R - U) \notin \delta$ and $(B, R - V) \notin \delta$.

Then (R,δ) is a proximity space.

DEFINITION 2.2

A subset U of R is a neighbourhood of $A \subset R$ iff $(A, R\text{-}U) \notin \delta$.

In terms of open sets the above definition is equivalent to

DEFINITION 2.3 Naimpally & Warrack [18]

A subset G of R is defined to be open iff $(\{x\}, R\text{-}G) \notin \delta$ for every $x \in G$.

NOTE : 2.4

If the topology on a set R is defined as in definition 2.2, then R is a completely regular space. (Efremovich [7]).

DEFINITION 2.5 Hayashi [8]

A **paraproximity** on a set R is a relation δ for pair of subsets of R satisfying the following axioms:

(PP1) : $(A, \phi) \notin \delta$ for every $A \subset R$.

(PP2) : $(A, B \cup C) \in \delta$ iff $(A, B) \in \delta$ or $(A, C) \in \delta$.

(PP3) : For any arbitrary index set Λ, $(\bigcup_{\lambda \in \Lambda} A_\lambda, B) \in \delta$ iff there is an index $\mu \in \Lambda$ satisfying the relation $(A_\mu, B) \in \delta$.

(PP4) : For arbitrary two points $a, b \in R$, $(\{a\}, \{b\}) \in \delta$ iff $a = b$.

(PP5) : If $(A, B) \notin \delta$ and $(B, A) \notin \delta$, then there are two disjoint subsets U and V satisfying.

$(A, R - U) \notin \delta$, $(U, R - U) \notin \delta$

$(B, R - V) \notin \delta$, $(V, R - V) \notin \delta$.

Then (R, δ) is called a paraproximity space.

Given a paraproximity on a set R we can associate a topology. For this purpose we need the following lemmas.

LEMMA: 2.6

If $(A, B) \notin \delta$, then $(A, C) \notin \delta$, for any $C \subset B$.

PROOF:

Since $B = B \cup C$ and $(A, B \cup C) \notin \delta$, it follows that $(A, C) \notin \delta$ by (PP2).

LEMMA : 2.7

If $(A, B) \notin \delta$, then $(C, B) \notin \delta$ for any $C \subset A$

PROOF

Since $A = A \cup C$ and $(A \cup C, B) \notin \delta$, it follows that $(C, B) \notin \delta$.

LEMMA 2.8

If $(A, B) \notin \delta$, then $A \cap B = \phi$.

PROOF

Suppose that there exists a point $x \in A \cap B$. Then by Lemmas 2.6 and 2.7 $(\{x\}, \{x\}) \notin \delta$ for any point $x \in R$, contrary to (PP4).

LEMMA: 2.9

$((R - x), \{x\}) \notin \delta$ for any point $x \in R$.

PROOF:

Let $R - x = \bigcup_{\lambda \in \Lambda} y_\lambda$. Then $y_\lambda \neq x$ for all λ. Suppose that $(R - x, \{x\}) \in \delta$ equivalently $(\bigcup_{\lambda \in \Lambda} y_\lambda, \{x\}) \in \delta$, then there is an index μ satisfying $(\{y_\mu\}, \{x\}) \in \delta$ by (PP3). From (PP4) $y_\mu = x$, which is a contradiction.

REMARK : 2.10

In axiom (PP5), we may choose U and V such that $(U, V) \notin \delta$ and $(V, U) \notin \delta$ (U and V are disjoint implies $V \subset R - U$. Therefore, $(U, R\text{-}U) \notin \delta$, and $V \subset R - U$ implies $(U, V) \notin \delta$ (since from lemma:2.6). Similarly we can prove $(V, U) \notin \delta$).

REMARK 2.11

From axiom (PP5), we may deduce that $A \subset U$ and $B \subset V$. (Suppose U does not contain A, then there is a point $x \in A - U$. That is, $x \in A$ and $x \in R - U$.

Also $(A, R - U) \notin \delta$ implies $(\{x\},\{x\}) \notin \delta$ (since from lemmas 2.6 and 2.7), which is a contradiction. Therefore $A \subset U$. Similarly we can prove $B \subset V$).

Now the topology on a set R is defined as follows

DEFINITION 2.12 Hayashi [8]

A set U is open iff $(U,\ R - U) \notin \delta$

This defines a topology and it is denoted by T_δ.

NOTE 2.13

Definition:2.12 is equivalent to Definition : 2.3

PROOF:

Let U be open as in definition 2.12. Then from $(U,R\text{-}U) \notin \delta$ and Lemma 2.7 it follows that $(\{x\}, R - U) \notin \delta$ for every $x \in U$. Hence U is open as in definition 2.3.

Conversely, let U be open as in definition : 2.3 Put $U = \bigcup_\lambda x_\lambda$. Then, $(\{x_\lambda\}R - U) \notin \delta$ for every λ .From (PP3) it follows $(U, R - U) = (\bigcup_\lambda x_\lambda, R - U) \notin \delta$. Therefore U is open as in definition 2.12.

THEOREM 2.14

Let R be a set with paraproximity δ satisfying (PP1 – PP5). Then the set R is a completely normal space if R is topologized as in definition :2.12

PROOF:

Consider $(R, R\text{-}R) = (R, \phi) \notin \delta$ (by (PP1))

$\Rightarrow$ whole space R is open.

Let U and V be open.

Then $(U, R-U) \notin \delta$ and $(V, R-V) \notin \delta$.

By Lemma: 2.7, it follows that

$$(U \cap V, R-U) \notin \delta \text{ and } (U \cap V, R-V) \notin \delta.$$

Hence $(U \cap V, R-(U \cap V)) = (U \cap V, (R-U) \cup (R-V)) \notin \delta$ by (PP2).

Therefore $U \cap V$ is open. That is, finite intersection of open sets is open.

Suppose that U_λ is open, that is $(U_\lambda, R - U_\lambda) \notin \delta$ for every $\lambda \in \Lambda$.

By (PP3), $(\bigcup\limits_{\lambda} U_\lambda,\ R - U_\lambda) \notin \delta$ for every λ.

Therefore$(\bigcup\limits_{\lambda} U_\lambda, R - \bigcup\limits_{\lambda} U_\lambda) \notin \delta$ by lemma 2.6

$\Rightarrow \bigcup\limits_{\lambda} U_\lambda$ is open.

That is, arbitrary union of open sets is open.

Therefore, R is a topological space.

From Lemma:2.9 it is easy to show that R satisfies the T_1 separation axiom. That is, for every point x of R, R - {x} is open.

It remains to show that any subset of R satisfies the T_4 separation axiom. It is sufficient to prove that if A and B are separated in the T_1 space R, then there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Since $R - \bar{A}$ is open, $(R-\bar{A}, \bar{A}) \notin \delta$. Also $R - \bar{A} \supset B$ (since A and B are separated).

Therefore $(B, \bar{A}) \notin \delta$ by lemma 2.7

Hence by lemma 2.6, it follows that $(B,A) \notin \delta$.

Similarly we can prove that $(A,B) \notin \delta$.

As a consequence of (PP5) and remark :2.11, we can find the required open sets U and V. Hence the theorem.

COROLLARY: 2.15

Let R be a set with a relation δ satisfying (PP1 – PP3). If the topology of R is defined as in definition: 2.12, then R is a topological space. Moreover, if a relation δ satisfies (PP1 – PP4), then R is a T_1 – space.

PROOF:

The proof of this corollary follows from the proof of the above theorem(2.14).

NOTE 2.16

In a proximity space R, $\overline{A} \cap \overline{B} \neq \phi$ implies $(A,B) \in \delta$. The converse implication holds if a space R is a compact proximity space (Efremovich [7]).

We have a corresponding theorem in the case of paraproximity spaces.

THEOREM : 2.17

Let (R, δ) be a paraproximity space. Then $(A,B) \in \delta$ implies $A \cap \overline{B} \neq \phi$.

PROOF:

Assume that $A \cap \overline{B} = \phi$, and so $A \subset R - \overline{B}$. If we choose all open sets O_λ which contain the closed set $\overline{B}$, then $\bigcap_\lambda \overline{O}_\lambda = \overline{B}$ by the regularity of R. Therefore $A \subset R - \overline{B} = R - \bigcap_\lambda \overline{O}_\lambda = \bigcup_\lambda (R - \overline{O}_\lambda)$. Since all sets $R - \overline{O}_\lambda$ are open, $(R - \overline{O}_\lambda, \overline{O}_\lambda) \notin \delta$ for all λ.

Then by (PP3), $(\bigcup_\lambda (R - \overline{O}_\lambda), \overline{O}_\lambda) \notin \delta$ for all λ.

Consequently it follows from lemmas 2.6 and 2.7 that $(A,B) \notin \delta$, a contradiction to our assumption that $(A,B) \in \delta$. Therefore $A \cap \overline{B} \neq \phi$.

COROLLARY: 2.18

Let (R, δ) be a paraproximity space Let x be a point of R and A be a subset of R. Then $(A, \{x\}) \in \delta$ iff $x \in A$. If $(\{x\},A) \in \delta$, then $x \in \overline{A}$.

The following theorem (2.19) shows that a paraproximity δ can be introduced in a completely normal space R.

THEOREM : 2.19

If R is a completely normal space and the relation δ is defined by setting "$(A, B) \in \delta$ if and only if $A \cap \overline{B} \neq \phi$", then δ is a paraproximity for R. (Of course, $\overline{B}$ is closure of B for the original topology of R).

PROOF:

(PP1) : For any $A \subset R$, $A \cap \overline{\phi} = \phi$ and so $(A,\phi) \notin \delta$.

(PP2) : $(A, B \cup C) \in \delta$ iff $(A,B) \in \delta$ or $(A, C) \in \delta$.

(PP3) : For an arbitrary index set Λ, $(\bigcup_{\lambda \in \Lambda} A_\lambda, B) \in \delta$ iff there is an index $\mu \in \Lambda$ satisfying the relation $(A_\mu, B) \in \delta$.

(PP4) : For any point $a \in R$ it follows that $a \cap \overline{a} = a \neq \phi$, which means $(\{a\},\{a\}) \in \delta$.

Conversely, if $a \cap \overline{b} \neq \phi$ then $a = b$ since $b = \overline{b}$

(PP5) : Suppose that $(A,B) \notin \delta$ and $(B,A) \notin \delta$. Because $A \cap \overline{B} = \phi$, $\overline{A} \cap B = \phi$ and R is completely normal, there are two disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Since $\overline{R - U} = R - U$, $A \cap \overline{(R - U)} = A \cap (R - U) = \phi$ and so $(A, R - U) \notin \delta$. Similarly $(B, R - V) \notin \delta$. Also $(U, R - U) \notin \delta$ and $(V, R - V) \notin \delta$ (since U and V are open in R).

COROLLARY: 2.20

If R is a topological space and δ is defined as above, then δ satisfies (PP1–PP3). Moreover, if R is a T_1 space, then δ satisfies (PP1 – PP4).

Next we show that the T_1 topologies on X are found to be in 1-1 correspondence with the closure proximities (Mira sarkar [14]) on X.

DEFINITION: 2.21 Mira Sarakar[14]

Let X be any set and let δ be a binary relation on the power set $\mathcal{P}(X)$ satisfying the following axioms;

(CP1) : $(A,\phi) \notin \delta$ and $(\phi, A) \notin \delta$, for each $A \subset X$.

(CP2) : $(A, B \cup C) \in \delta \Leftrightarrow (A,B) \in \delta$ or $(A,C) \in \delta$

(CP3) : $(\bigcup\limits_{\lambda\in\Lambda} U_\lambda, B) \in \delta \Leftrightarrow (U_\lambda, B) \in \delta$ for some $\lambda \in \Lambda$ where

Λ is an arbitrary index set.

(CP4) : $(\{x\},\{y\}) \in \delta \Leftrightarrow x = y$, for each pair of points $x, y \in X$

(CP5) : $(\{x\}, A) \in \delta$ and $(\{a\}, B) \in \delta$ for each $a \in A$ together imply $(\{x\}, B) \in \delta$ whenever $x \in X$.

Then δ is a **closure – proximity** and (X, δ) a closure proximity space.

RESULT: 2.22

If $A^\delta = \{x \in X : (\{x\}, A) \in \delta\}$, then δ is a Kuratowski closure operator on X.

(Let X be a set with power set $\mathcal{P}(X)$. A mapping f of $\mathcal{P}(X)$ into $\mathcal{P}(X)$ is called Kuratowski closure operator for X if and only if f satisfies conditions.

i) $f(\phi) = \phi$

ii) For each A, $A \subset f(A)$

iii) For each A, $f(f(A)) = f(A)$ and

iv) for each A and B, $f(A \cup B) = f(A) \cup f(B)$.

THEOREM: 2.23

The induced topology (defined by 2.12) of a closure proximity is T_1, where $(A,B) \in \delta$ iff $A \cap \overline{B} \neq \phi$.

PROOF:

From (CP1 – CP3) it can be shown that $(A,B) \in \delta \Rightarrow A \cap \overline{B} \neq \phi$. Hence the proof follows.

DEFINITION: 2.24

If X is a topological space and δ is a proximity on X, then δ is compatible to the topology if for every $A \subseteq X$, $cl\,(A) = \overline{A} \equiv \{x: \{x\}\ \delta\ A\}$.

LEMMA 2.25

The relation γ defined on $\mathcal{P}(X)$ by $(A, B) \in \gamma \Leftrightarrow A \cap \overline{B} \neq \phi$ where (X, τ) is a T_1 – space, is a closure-proximity compatible with τ.

PROOF:

(CP1 – CP4) follow from (PP1 – PP4)

(CP5): $(\{x\},A) \in \gamma \Rightarrow x \in \overline{A}$ and $(\{a\}, B) \in \delta \Rightarrow a \in \overline{B}\ \forall\, a \in A$.

Therefore $A \subset \overline{B}$ and so $x \in \overline{B} \Rightarrow (\{x\},B) \in \gamma$

COROLLARY: 2.26

The T_1 topologies on X have a 1-1 correspondence with the closure proximities on it.

PROOF:

The proof of this corollary follows from Theorem :2.23 and Lemma: 2.25

COROLLARY : 2.27

For a closure – proximity space, the following are equivalent.

i) (X, τ_δ) is discrete

ii) δ is symmetric

iii) $(A,B) \in \delta \Leftrightarrow (\bar{A}, \bar{B}) \in \delta$, for each $A,B \subset X$.

iv) (X, τ_δ) has a locally finite base.

We now add axioms to a closure-proximity one after another and examine the induced topologies.

DEFINITION: 2.28

Let a closure – proximity δ satisfy the following axiom.

(CP6) : $(A,B) \notin \delta$ and $(B,A) \notin \delta$ together imply that there exist disjoint neighbourhoods C,D of X such that $(A, X - C) \notin \delta$ and $(B, X - D) \notin \delta$

Then δ is called a **normal proximity** and (X, δ) a normal proximity space.

THEOREM : 2.29

Let δ be a normal proximity. Then (X, τ_δ) is completely normal.

PROOF:

If $A \cap \bar{B} = \phi$ and $B \cap \bar{A} = \phi$, then both $(A,B) \notin \delta$ and $(B,A) \notin \delta$.

Also $(A, X\text{-}C) \notin \delta. \Rightarrow A \cap \overline{(X - C)} = \phi$, as δ is a closure proximity.

Therefore $A \subset \text{Int } C$. Similarly $B \subset \text{Int } D$.

Since (X, τ_δ) is T_1 also (Theorem:2.23), the result follows.

COROLLARY: 2.29(a)

A paraproximity on X is a normal proximity and conversely.

DEFINITION 2.30

Let a normal proximity δ satisfy the additional axiom.

(CP7) : $(X-A, A) \notin \delta \Rightarrow \exists$ a sequence $\{G_n, n \in N\}$ of subsets of X, where $(G_n, X - G_n) \notin \delta$ and $(A, X - G_n) \notin \delta$ for all n. Moreover, if $(\{x\}, A) \notin \delta$, then $(G_n, \{x\}) \notin \delta$ for some $n \in N$.

Then δ is called a **perfect proximity** and (X, δ) a perfect proximity space.

THEOREM: 2.30(a)

Let δ be a perfect proximity. Then (X, τ_δ) is a perfectly normal space.

PROOF:

Let $A \subset X$ be closed. Then $(X-A, A) \notin \delta$. So (CP7) implies $A \subset \bigcap_{n \in N} G_n$, where each G_n is an open subset of X.

Since $x \notin A \Rightarrow (\{x\},A) \notin \delta$. We have $(G_n, \{x\}) \notin \delta$ and so $x \notin G_n$ for some $n \in N$, that is $A = \bigcap_{n \in N} G_n$. Therefore (X, τ_δ) is a perfectly normal space.

THEOREM : 2.31

Let (X, τ) be a perfectly normal space. Then the relation γ on $\mathcal{P}(X)$ defined by $(A,B) \in \gamma$ iff $A \cap \overline{B} \neq \phi$, for $A,B \subset X$, is a perfect proximity compatible with τ.

PROOF:

We need to verify only (CP6) and (CP7).

$(A,B) \notin \gamma$ and $(B,A) \notin \gamma \Rightarrow A \cap \overline{B} = \phi$ and $B \cap \overline{A} = \phi$. So there exist $C, D \in \tau$ where $C \cap D = \phi$ such that $A \subset C$ and $B \subset D$, as (X, τ) is completely normal also.

Thus $A \cap \overline{(X - C)} = \phi$ and so $(A, X - C) \notin \gamma$.

Similarly, $(B, X - D) \notin \gamma$. Hence (CP6).

$(X - A, A) \notin \gamma \Rightarrow (X - A) \cap \overline{A} = \phi \Rightarrow A$ is closed.

$\Rightarrow \exists$ a sequence $(G_n, n \in N\}$, with $G_n \in \tau$, such that

$A = \bigcap_{n \in N} G_n$, as (X, τ) is perfectly normal.

That is, $A \cap \overline{(X - G_n)} = \phi \Rightarrow (A, X - G_n) \notin \gamma$, for all $n \in N$.

Also, $(G_n, X - G_n) \notin \gamma$, $\forall\, n \in N$. Also $(\{x\}, A) \notin \gamma \Rightarrow x \notin A \Rightarrow x \notin G_n$, for some $n \in N$, which implies $G_n \cap \overline{\{x\}} = \phi$, as (X, τ) is $T_1 \Rightarrow (G_n, \{x\}) \notin \gamma$.

Hence (CP7).

COROLLARY : 2.32.

If (X, τ) be completely normal, then γ on $\mathcal{P}(X)$ defined above is a normal proximity compatible with τ.

The following corollaries 2.33 and 2.34 follow from corollaries 2.26, 2.32 and theorem : 2.23

COROLLARY: 2.33

The completely normal topologies on X have a 1-1 correspondence with the normal proximities (paraproximities) on it.

COROLLARY : 2.34

The perfectly normal topologies on X have a 1-1 correspondence with the perfect proximities on it.

REMARK: 2.35

If (CP5) and (CP6) of a normal proximity are replaced by the following axiom.

(CP8) : $(A,B) \notin \delta$, $(X-A, A) \notin \delta$ and $(X-B, B) \notin \delta$ together imply that there exist $C, D \subset X$, $C \cap D = \phi$, such that $(A, X-C) \notin \delta$, $(C, X-C) \notin \delta$, $(B, X-D) \notin \delta$ and $(D, X-D) \notin \delta$. Then (X, τ_δ) is a normal space. But in this case, the space need not be completely normal. This δ is called a **weak normal proximity**.

NOTE 2.36

A normal proximity is a weak normal proximity.

THEOREM : 2.37

Let (X, τ) be a normal space. Then the relation α on $\mathcal{P}(X)$ defined by $(A,B) \in \alpha \Leftrightarrow A \cap \overline{B} \neq \phi$, is a weak normal proximity compatible with τ. Moreover, it is the weakest such relation compatible with τ.

PROOF:

Since $(X-A, A) \notin \alpha \Rightarrow A$ is closed.

$(A,B) \notin \alpha$ and $(X-B, B) \notin \alpha$ together imply that A and B are disjoint closed subsets of X. Then (CP8) is obvious.

If β is any other weak normal proximity on X, then $(A,B) \in \beta \Rightarrow A \cap \overline{B} \neq \phi$. Then $(A,B) \in \alpha$, which proves that α is the weakest such relation compatible with τ.

DEFINITION: 2.38

Let ξ satisfy axioms (CP1 – CP3) together with the following:

(CP9) : For every family $\{F_\alpha\}$, $\alpha \in \Lambda$ of subsets of X, where $(X - F_\alpha, F_\alpha) \notin \xi$

$\forall \alpha \in \Lambda$ and for each $A \subset X$, $(A, \bigcap_{\alpha\in\Lambda} F_\alpha) \notin \xi \Rightarrow (A, \bigcap_{\alpha\in F} F_\alpha) \notin \xi$, for some finite subset F of Λ.

Then ξ is called a **compact proximity** and (X, ξ) a compact proximity space.

THEOREM : 2.39

The induced topology of a compact proximity is hereditarily compact.

PROOF:

Let $\{G_\alpha, \alpha \in \Lambda\}$ be an open cover of an arbitrary $A \subset X$. Then each G_α may be supposed to be open in X.

So, $A \cap (\bigcap_{\alpha\in\Lambda} F_\alpha) = \phi$, where $F_\alpha = X - G_\alpha$.

Therefore, $(A, \bigcap_{\alpha\in\Lambda} F_\alpha) \notin \xi$, as $\cap F_\alpha$ is closed.

So, $(A, \bigcap_{\alpha\in F} F_\alpha) \notin \xi$, where $F \subset \Lambda$ is finite, as $(X - F_\alpha, F_\alpha) \notin \xi$, $\forall \alpha \in \Lambda$.

Hence $A \subset \bigcup_{\alpha\in F} (X - F_\alpha) \Rightarrow A$ is compact in X.

REMARK : 2.40

If axiom (CP3) is replaced by axiom

(CP-10): $(A \cup B, C) \in \xi \Leftrightarrow (A,C) \in \xi$ or $(B,C) \in \xi$, then this new compact proximity yields a compact topology, which may not be hereditarily compact.

To find a compact proximity compatible with the topology of a space, we have the following theorem.

THEOREM: 2.41

If (X, τ) be a hereditarily compact space, then γ' defined on $\mathcal{P}(X)$ by $(A,B) \in \gamma' \Leftrightarrow A \cap \overline{B} \neq \phi$, is the weakest compact proximity compatible with τ.

DEFINITION: 2.42

Let η satisfy axioms (CP1 – CP3) together with axiom.

(CP-11): For every family $\{F_\alpha, \alpha\in \Lambda\}$ of subsets of X, with $(X - F_\alpha, F_\alpha) \notin \eta$, $\forall\ \alpha\in\Lambda$ and for each $A\subset X$, the result $(A, \bigcap_{\alpha\in\Lambda} F_\alpha) \notin \eta$ implies there exists $\{K_\lambda, \lambda\in\mu\}$ with $K_\lambda \subset X$ such that $(A, \bigcap_{\lambda\in\mu} K_\lambda) \notin \eta$ and for a fixed $\lambda \in \mu$, $(X - K_\lambda, F_{\alpha_\lambda}) \notin \eta$, for some $\alpha_\lambda\in\Lambda$. Moreover, $x \in A \Rightarrow$ there exists $\lambda_x \subset \mu$, where $\mu - \lambda_x$ is finite, such that $(X- K_\lambda,x) \notin \eta$, for every $\lambda\in\lambda_x$.

Then η is called a **metacompact proximity** and (X,η) a metacompact proximity space.

THEOREM: 2.43

The induced topology of a metacompact proximity is hereditarily metacompact.

THEOREM: 2.44

If (X, τ) be T_1 and hereditarily metacompact, then γ'' defined on $\mathcal{P}(X)$ by $(A,B) \in \gamma'' \Leftrightarrow A \cap \overline{B} \neq \phi$, is the weakest compatible metacompact proximity.

NOTE: 2.45

If μ is a finite set, then metacompact proximity reduces to a compact proximity.

Chapter III

CHAPTER III

LO – PROXIMITY AND LE-PROXIMITY

DEFINITION 3.1

A binary relation α defined on the power set of X is called a **Leader or LE-proximity** on X iff it satisfies the following conditions.

(LP1) : $A\,\alpha\,(B \cup C)$ iff $A\,\alpha\,B$ or $A\,\alpha\,C$ and $(A \cup B)\,\alpha\,C$ iff $A\,\alpha\,C$ or $B\,\alpha\,C$.

(LP2) : $A\,\alpha\,B \Rightarrow A \neq \phi,\ B \neq \phi$.

(LP3) : $A\,\alpha\,B$ and $b\,\alpha\,C$ for each $b \in B$ implies $A\,\alpha\,C$.

(LP4) : $A \cap B \neq \phi$ implies $A\,\alpha\,B$.

If in addition α satisfies.

(LP5) : $A\,\alpha\,B$ iff $B\,\alpha\,A$

then α is called a **Lodato or LO-proximity**. The pair (X, α), where α is a LO-proximity is referred to as a Lodato space.

NOTE : 3.2

The closure-proximity (Definition:2.21 of Chapter II) is an LE-proximity.

DEFINITION :3.3

A binary relation β defined on the power set of X is called a **Pervin or P-Proximity** on X iff β satisfies LP1, LP2, LP4 and

(LP3′) : $(A,B) \notin \beta \Rightarrow$ there exists an $E \subset X \ni (A,E) \notin \beta$.

And $(X - E, B) \notin \beta$.

THEOREM : 3.4

Every P-proximity β on X is a LE-proximity on X.

PROOF:

It is sufficient to show β satisfies (LP3).

Suppose $A\, \beta\, B$ and $b\, \beta\, C$ for every $b \in B$.

If $(A,C) \notin \beta$ then by (LP3′) there exists $E \subset X$ such that $(A, E) \notin \beta$ and

$(X\text{-}E, C) \notin \beta$.

$(A, E) \notin \beta$ and $A\, \beta\, B$ together imply $B \not\subset E$.

That is, $B \cap (X - E) \neq \phi$.

If $b \in B \cap (X - E)$, then $b\, \beta\, C \Rightarrow (X - E)\, \beta\, C$, a contradiction.

Therefore $A\, \beta\, C$. Hence (LP3).

THEOREM: 3.5

Every topological space (X,τ) has a compatible LE-or P-Proximity ξ_o given by $A\, \xi_o\, B$ iff $A \cap \overline{B} \neq \phi$. Moreover, ξ_o is the largest compatible LE- or P-proximity.

PROOF:

In view of theorem 3.4, we need to verify only (LP3′).

If $(A,B) \notin \xi_o$, then $A \cap \overline{B} = \phi$.

Put $E = \overline{B}$. Then $A \cap \overline{E} = \phi$ and $(X - E) \cap \overline{B} = \phi$.

That is, $(A,E) \notin \xi_o$ and $(X - E, B) \notin \xi o$.

Hence (LP3′).

$\tau = \tau(\xi_o)$ follows from the fact that $x\, \xi_o\, A$ iff $x \in \overline{A}$.

Hence τ and ξ_o are compatible.

If ξ is any LE- or P-proximity, then from $A \,\xi\, B$ iff $A \,\xi\, B^{\xi}$, we have that $(A,B) \notin \xi$ implies $A \cap \overline{B} = \phi$. That is $(A,B) \notin \xi_o$.

THEOREM : 3.6

If α is any LO-proximity (3.1), then $\tau(\alpha)$ is necessarily R_o. Conversely, a compatible LO-Proximity (3.1) α_1 can be defined on every R_o-space by

$$A \,\alpha_1\, B \text{ iff } \overline{A} \cap \overline{B} \neq \phi .$$

Further more, α_1 is the largest compatible LO-proximity (3.1).

PROOF:

$\tau(\alpha)$ is R_o follows from the fact that $x \in \overline{Y}$ iff $x \,\alpha\, y$ iff $y \,\alpha\, x$ iff $y \in \overline{X}$.

To prove α_1 is a LO-proximity, it is sufficient to verify (LP3).

Suppose $A \,\alpha_1\, B$ and $b \,\alpha_1\, C$ for each $b \in B$.

Then $\overline{A} \cap \overline{B} \neq \phi$ and $\overline{b} \cap \overline{C} \neq \phi$ for each $b \in B$.

That is, there exists a $c \in C$ such that $c \in \overline{b}$. Since X is R_o, $b \in \overline{c} \subset \overline{C}$ and hence $\overline{A} \cap \overline{C} \neq \phi$, which means $A\alpha_1 C$. Hence (LP3).

Since $x \in A^{\alpha_1}$ iff $\overline{x} \cap \overline{A} \neq \phi$ iff $x \in \overline{A}$, it follows that $\tau = \tau(\alpha_1)$

For every LO-proximity α, $A\alpha\, B$ if $\overline{A} \,\alpha\, \overline{B}$, and thus α_1 is the largest compatible LO-proximity.

COROLLARY :3.7

There exists a LO-proximity (3.1) which is not a P-proximity.

PROOF:

There exists R_o-spaces which are not regular and $A\alpha_1 B$ iff $\overline{A} \cap \overline{B} \neq \phi$ shows that if α_1 were a P-proximity, then it would be an Efremovich proximity, which is impossible.

DEFINITION 3.8

A binary relation γ defined on the power set of X is called a **separation or S-proximity** iff the following conditions are satisfied.

(SP1) : $A \gamma B \Rightarrow B \gamma A$.

(SP2) : $(A \cup B) \gamma C$ iff $A \gamma C$ or $B \gamma C$

(SP3) : $A \gamma B \Rightarrow A \neq \phi, B \neq \phi$.

(SP4) : $x \gamma B$ and $b \gamma C$ for every $b \in B \Rightarrow x \gamma C$.

(SP5) : $A \cap B \neq \phi \Rightarrow A \gamma B$.

(SP6) : $x \gamma y$ implies $x = y$.

THEOREM: 3.9

Every LO-proximity (3.1) on X is also an S-proximity on X.

DEFINITION: 3.10

A binary relation α defined on the power set of X is called an **E-discrete proximity** on X if it is a LO-proximity (3.1) and it satisfies in addition axiom.

(DP1) : For $x \in X$, $B \subset X$, $(x,b) \notin \alpha$, $\forall\, b \in B \Rightarrow (x, B) \notin \alpha$.

DEFINITION :3.11

A binary relation α is defined on the power set of X is called an **E-discrete proximity of Alexandroff** on X if it is a LE-proximity (3.1) and it satisfies in addition axiom (DP1).

PROPOSITION : 3.12

If α is any E-discrete proximity (E-discrete proximity of Alexandroff) then the induced topology by α is E-discrete (E-discrete of Alexandroff). Conversely, every E-discrete space (E-discrete of Alexandroff) has a compatible E-discrete proximity (E-discrete proximity of Alexandroff) α_o given by

$$A \, \alpha_o \, B \text{ iff } \overline{A} \cap \overline{B} = \phi$$

$$(A \, \alpha_o \, B \text{ iff } A \cap \overline{B} = \phi)$$

Next we show that, in general, Lodato proximities are not covered.

To show this we need the following definitions and results.

DEFINITION: 3.13

A closed filter $\mathcal{F}$ on a topological space (X, τ) is a proper filter (that is, a filter which does not contain the empty set) which has a base consisting of only closed sets.

DEFINITION 3.14

Maximal (with respect to set indusion) closed filters are all called ultraclosed filters.

DEFINITION :3.15

Ultrafilters are maximal proper filters on a set.

DEFINITION :3.16

Grills are exactly the unions of ultrafilters.

DEFINITION: 3.17

A **basic proximity** π on a set X is a symmetric binary relation on the power set $\mathcal{P}(X)$ of X satisfying the conditions:

$(A, B \cup C) \in \pi \Leftrightarrow (A,B) \in \pi$ or $(A,C) \in \pi$

$A \cap B \neq \phi \Rightarrow (A,B) \in \pi$

$(A, \phi) \notin \pi, \forall A \subset X.$

The pair (X, π) is called a basic proximity space provided π is a basic proximity on X.

DEFINITION: 3.18

For a basic proximity π on X, we define $C_\pi(A) = \{x \in X: (\{x\},A) \in \pi\}$ for all $A \subset X$.

NOTE: 3.19

C_π is a symmetric (Čech) closure operator). For a basic proximity π, C_π need not be a Kuratowski closure operator.

DEFINITION: 3.20 K. C. CHATTOPADHYAY [1]

A basic proximity π on X is called a **Lodato proximity** if the following condition is satisfied:

$(C_\pi(A), C_\pi(B)) \in \pi \Rightarrow (A,B) \in \pi$

If π is a Lodato proximity on X then C_π is a Kuratowski closure operator on X and hence (X,C_π) is a topological space.

DEFINITION : 3.21

Let (X,π) be a basic proximity space and $\mathcal{G}$ be a grill on X. Then $\mathcal{G}$ is called a π-clan if $(A,B) \in \pi$ for all A,B in $\mathcal{G}$.

DEFINITION: 3.22

Let π be a Lodato proximity (3.20) on X. A Wallman π-clan is a π-clan which contains some ultraclosed filter.

DEFINITION:3.23

The Lodato proximity (3.20) π is said to be covered if for each $(A,B) \in \pi$ there exists a Wallman π-clan $\mathcal{G}$ such that $\{A,B\} \subset \mathcal{G}$.

PROPOSITION: 3.24

Let $\mathcal{U}$ be an ultraclosed filter on (X,τ) and $\mathcal{A}$ a base of $\mathcal{U}$ consisting of closed sets. If F is a closed set and $F \cap A \neq \phi$ for all A in $\mathcal{A}$, then $F \in \mathcal{U}$.

PROOF:

Let $\mathcal{B}$ be the collection of all finite intersections of members of the family $\mathcal{A} \cup \{F\}$. Then $\mathcal{B}$ is a filter base consisting of closed sets. Let $\mathcal{U}_o$ be the filter generated by $\mathcal{B}$ as a base. Then $\mathcal{U}_o$ is a closed filter and $\mathcal{U}_o \supset \mathcal{U} \cup \{F\}$. By the maximality of $\mathcal{U}$, it follows that $F \in \mathcal{U}$.

COROLLARY: 3.25

Let $\mathcal{U}$ be an ultraclosed filter on (X,τ) and V an open set such that $V \cap F \neq \phi$ for all F in $\mathcal{U}$. Then $V \in \mathcal{U}$.

PROOF:

Suppose $V \notin \mathcal{U}$. Let $\mathcal{A}$ be a base of $\mathcal{U}$ consisting of closed sets. Then $V \not\supset A$ for all $A \in \mathcal{A}$. Thus $(X - V) \cap A \neq \phi$ for all $A \in \mathcal{A}$. Since $X\text{-}V$ is closed, by the above proposition: 3.24, it follows that $X\text{-}V \in \mathcal{U}$ and hence $V \cap (X\text{-}V) \neq \phi$, a contradiction. Therefore $V \in \mathcal{U}$.

PROPOSITION: 3.26

On a compact topological space (X,τ) every ultraclosed filter converges.

PROOF:

Let $\mathcal{U}$ be an ultraclosed filter on (X,τ). Since the space is compact it follows that there exists an x in X such that $x \in C(F)$ for all $F \in \mathcal{U}$. Let V be an open neighbourhood of x. Then $V \cap F \neq \phi$ for all $F \in \mathcal{U}$. Thus by the above corollary: 3.25, $V \in \mathcal{U}$. Hence $\mathcal{U}$ converges to x.

PROPOSITION: 3.27

On a T_1-space (X,τ), every convergent ultraclosed filter has the form $\mathcal{U}(x)$, for some $x \in X$, where $\mathcal{U}(x) = \{A \subset X : x \in A\}$.

PROOF:

Let $\mathcal{U}$ be an ultraclosed filter on (X,τ) such that it converges to a point $x \in X$. Obviously, $x \in C(F)$ for all $F \in \mathcal{U}$. Hence, in particular, x belongs to each member of a base of $\mathcal{U}$ consisting of closed sets. Since $\{x\}$ is a closed set it follows by proposition:3.24, that $\{x\} \in \mathcal{U}$. Thus $\mathcal{U} = \mathcal{U}(x)$.

THEOREM: 3.28

Let (X,τ) be a compact T_1- space such that it has two infinite components. Then $\pi = \{(E,F): C(E) \cap C(F) \neq \phi$ or E and F are both infinite$\}$ is a Lodato proximity (3.20) on X such that $C_\pi = C$ and π is not covered.

PROOF:

It is easy to verify that π is indeed a Lodato proximity on X such that $C_\pi = C$.

Let A,B be two infinite components of (X,τ). Obviously $(A,B) \in \pi$. However, no Wallman π-clan can contain both A and B. For suppose $\mathcal{G}$ is such a Wallman π-clan. Let $\mathcal{U}$ be an ultraclosed filter such that $\mathcal{U} \subset \mathcal{G}$. Then since (X,τ) is a compact T_1- space it follows, by propositions 3.26 and 3.27 that $\mathcal{U} = \mathcal{U}(x)$ for some $x \in X$. Thus $\{x\}$, A and B are all in $\mathcal{G}$. From this it follows that $x \in C_\pi(A) \cap C_\pi(B) = C(A) \cap C(B) = A \cap B = \phi$, a contradiction. Hence the theorem.

The following are the examples of compact T_1-spaces with two infinite components.

EXAMPLE 3.29

Let X be the union of closed intervals [1,2] and [3,4]. Then X with the topology induced by the usual topology of real line is an example of a compact T_1-space with two infinite components.

EXAMPLE: 3.30

Let $X = A \cup B$ such that A,B are both infinite sets and $A \cap B = \phi$. Define $C:\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by

$$\begin{aligned}\overline{D} = C(D) &= D \text{ if } D \text{ is a finite subset of } X\\ &= A \cup D \text{ if } A \cap D \text{ is infinite and } B \cap D \text{ is finite}\\ &= B \cup D \text{ if } B \cap D \text{ is infinite and } A \cap D \text{ is finite..}\\ &= X \text{ otherwise}\end{aligned}$$

C satisfies requirements for a closure operator. X with this topology is compact T_1 and contains two infinite components A and B.

Chapter IV

CHAPTER IV

R – PROXIMITIES AND RC-PROXIMITIES

DEFINITION 4.1

A space is said to be regular – closed if it is regular, and cannot be nontrivally densely embedded in a regular space.

DEFINITION 4.2

The class of spaces which can be densely embedded in a regular closed space, are called the class of **RC-regular spaces**. These spaces lie properly between the class of regular spaces and the class of completely regular spaces.

DEFINITION 4.3 DOUGLAS HARRIS [6]

A **proximity** on X is a symmetric relation δ between subsets of X satisfying the following four conditions.

(DHP1) : $\phi \overset{A}{\delta} A$ for every $A \subset X$.

(DHP2) : $A \delta A$ for every $A \neq \phi$.

(DHP3) : $A \delta (B \cup C)$ iff $A \delta B$ or $A \delta C$.

(DHP4) : If x and y are distinct points of X, then $\{x\} \overset{A}{\delta} \{y\}$.

NOTE : These axioms are those of the usual theory of proximities with the exception of the axiom of complete regularity.

DEFINITION: 4.4

An operator u on power set $\mathcal{P}(X)$ is defined as $uA = \{x : \{x\} \delta A\}$.

NOTE: 4.5

This operator u has the following properties:

(O1) : $u \phi = \phi$

(O2) : $A \subset uA$ for each $A \subset X$.

(O3) : $u(A \cup B) = uA \cup uB$.

The operator u induces a topology whose closed sets are the sets A which satisfy $uA = A$.

It need not be true that the operator u is a topological closure operator, that is we need not have $uuA = uA$ for each $A \subset X$.

DEFINITION: 4.6

For $A \subset X$ and $B \subset X$, we say that B is a proximal neighbourhood of A or A is a proximal subset of B if $A \overset{A}{\delta} (X - B)$ and we write $A \ll B$.

The relation $\ll$ and the operator u satisfy the following results.

RESULTS:4.7

(R1) : If $A \subset B \ll C \ll D \subset E$, then $A \ll C$, $C \ll E$, $A \ll E$, and $X - E \subset X - D \ll X - C \ll X - B \subset X - A$.

(R2) : If $A \ll B$ and $C \ll D$, then $A \cap C \ll B \cap D$.

(R3) : If $A \ll B$, then $A \subset uA \subset B$ and $A \subset X - u(X\text{-}B) \subset B$.

NOTATION:

We write $x \,\delta\, A$ and $x \ll A$ respectively in place of $\{x\} \,\delta\, A$ and $\{x\} \ll A$.

DEFINITION: 4.8

An **R-proximity** is a proximity satisfying (DHP1 – DHP4) along with

(DHP5): (Axiom of regularity) If $x \ll A$, then there is $B \subset X$ with $x \ll B \ll A$.

DEFINITION: 4.9

A filter is said to be round if for each member V of the filter there is a member W of the filter such that $W \ll V$.

NOTE:

Using the definition of round filters, axiom (DHP5) can be restated as follows:

The filter of proximal neighbourhoods of a point is round.

LEMMA: 4.10

An R-proximity induces a closure operator u such that $uuA = uA$ for each $A \subset X$. The topology induced by the R-proximity is regular, and u is the closure operator induced by the topology.

PROOF:

To prove that $uuA = uA$, we need only to show that $uuA \subset uA$.

If $x \notin uA$, then $x \ \hat{\delta}\ A$. That is, $x << X - A$, so by (DHP5) there is $B \subset X$ with $x << B << X - A$. Applying R3, we get $B \subset X - uA \subset X - A$, and hence from R1, we have $x << X - uA$. That is $x \notin uuA$.

Therefore $uuA \subset uA$.

We have shown that u is the closure operator of the topology that it induces: the closed sets are precisely the sets of the form uA for some $A \subset X$. This fact along with (R3) shows that the proximal neighbourhood filter of each point of X is a regular filter. In particular, the proximal neighbourhood filter of each point is contained in the neighbourhood filter of the point. Since by definition of the topology the converse inclusion also holds we have the equality of the two filters. Therefore, the neighbourhood filter of each point of the space is a regular filter, that is, the topology is regular.

The next lemma is a generalization of the converse of the above lemma:4.10

LEMMA:4.11

Suppose Z is a regular topological space, and that X is a dense subspace of Z. Define a relation between subsets of X by setting $A \, \delta \, B$ if $cl_z A \cap cl_z B \neq \phi$.

a) The relation δ is an R-proximity on X.

b) A filter on X is round iff it is the trace of a filter that is regular on Z.

PROOF:

a) Symmetry, DHP1, DHP2 and DHP4 are immediate, and DHP3 follows from the distributivity of closure with finite union.

Next to check (DHP5) : If V is a neighbourhood (in X) of $x \in X$, there is a closed neighbourhood (in Z) B of $x \in Z$ and an open neighbourhood (in Z) W of $x \in Z$ such that $W \cap X = V$ and $B \subset W$. Setting $A = B \cap X$ we find $x \ll A \ll V$.

b) Suppose γ is a round filter on X, and let ζ be the filter on Z generated by $\{cl_z F : F \in \gamma\}$. Then ζ certainly has a base of closed sets. Now if $F \in \gamma$ and $G \in \gamma$ with $G \ll F$, we have $cl_z G \cap cl_z (X\text{-}F) = \phi$. Since X is dense in Z, we also have $cl_z F \cup cl_z (X\text{-}F) = Z$. It follows that $cl_z\, G \subset Z - cl_z(X - F) \subset cl_z\, F$, and we have thus shown that ζ also has a base of open sets. Thus ζ is a regular filter, and it clearly induces γ on X.

Conversely, suppose ζ is a regular filter on Z. Since X is dense in Z and ζ has a base of open sets, every member of ζ intersects X and so the trace γ on X of ζ exists. If $V \in \gamma$, so that $V = W \cap X$ for some open set $W \in \zeta$, we let P be any member of ζ such that $cl_zP \subset W$. Then if $Q = P \cap X$ we have $cl_zQ \cap cl_z(X - V) = \phi$, so $Q << V$ and $Q \in \gamma$. Thus γ is round.

THEOREM:4.12

A topology is regular if and only if it is the topology induced by an R-proximity.

PROOF:

Follows from Lemma 4.10 and Lemma:4.11.

We establish some properties of round filters with respect to an R-proximity in the following.

LEMMA:4.13

(F1) : Every round filter is a regular filter.

(F2) : Every neighbourhood filter is maximal round

(F3) : Every round filter is contained in a maximal round filter.

(F4) : Distinct maximal round filters contain disjoint open members.

PROOF:

F1 follows from Lemma:4.10 and from R3.

We have the fact that neighbourhood filters are round and maximal regular in a regular space. Along with this fact, F2 follows from F1.

F3 follows from Zorn's lemma which states that "If, in a partially ordered set X, every chain has an upper bound, then X has a maximal element".

From F1 we have all round filters are open filters. Also if the supremum of two round filters is a filter, then by R2, this supremum is a round filter. Thus, if two round filters do not contain disjoint open sets then their supremum is a round filter containing each.

DEFINITION : 4.14

A subset B of X surrounds the subset A if every maximal round filter that intersects A (that is, every member of the filter intersects A) contains B.

DEFINITION : 4.15

An **RC-proximity** is a proximity that satisfies (DHP1 – DHP5), along with

(DHP6): (Axiom of RC-regularity). The subset B surrounds the subset A iff $B >> A$.

NOTE: 4.16

Since an RC-proximity is an R-proximity, the induced topology is regular.

The next lemma shows that every RC-regular space has its topology induced by an RC-proximity.

LEMMA: 4.17

Let Z be a regular-closed topological space, and let X be a dense subspace of Z. Let δ be the R-proximity induced on X by Z by setting $A \delta B$ if $cl_z A \cap cl_z B \neq \phi$. Then

a. The relation δ is an RC-proximity on X.

b. The maximal round filters on X are precisely the traces on X of the neighbourhood filters of points of Z.

PROOF:

b) Since Z is regular, by lemma: 4.11(b) the trace γ on X of the neighbourhood filter ζ of a point $z \in Z$ is a round filter. If η is round and $\gamma \subset \eta$, by lemma 4.11(b) there is a regular filter $\wp$ on Z whose trace on X is η. Since ζ is maximal regular, we have $\wp \subset \zeta$ and thus $\eta \subset \gamma$. Conversely, if γ is a maximal round filter it is the trace on X of a regular filter on Z, and since Z is regular-closed this regular filter has a cluster point. The trace on X of the neighbourhood filter of this cluster point must be the given maximal round filter.

a). Suppose that A and B are subsets of X and $A >> B$.

That is $\mathrm{cl}_Z(X - A) \cap \mathrm{cl}_Z B = \phi$ (by definition of proximity).

If γ is a maximal round filter on X then by (b) we know that γ is the trace on X of the neighbourhood filter of some point $z \in Z$. If γ intersects B then $z \in \mathrm{cl}_Z B$, and so there is a neighbourhood V of z disjoint from $X - A$. Therefore $A \in \gamma$. Thus if $A >> B$, then A surrounds B.

Conversely, suppose A and B are subsets of X and A surrounds B. Let $z \in \mathrm{cl}_Z B$ and let γ be the trace of X of the neighbourhood filter of z. Then by (b) γ is a maximal round filter. Since γ intersects B we must have $A \in \gamma$, from which it follows $z \notin \mathrm{cl}_Z(X - A)$. Thus $A >> B$.

This lemma yields the following theorem.

THEOREM: 4.18

Every RC-regular space has its topology induced by an RC-proximity.

DEFINITION: 4.19

An RC-proximity space is absolutely closed if every maximal round filter is the proximal neighbourhood filter of some point of the space (that is, converges in the topology induced by the proximity).

THEOREM: 4.20

If an RC-proximity space is absolutely closed, then its induced topology is a regular-closed topology, and the proximity is given by: A and B are far iff they have disjoint closures.

PROOF:

Suppose A and B are subsets of an RC-proximity space X, and that $A \overset{\wedge}{\delta} B$, that is $A \ll X - B$. By (DHP6), we have that X – B surrounds A, and so every maximal round filter that intersects A contains X –B. By (F1) neighbourhood filters are maximal round, and thus we see that any neighbourhood filter that intersects A fails to intersect B, that is, A and B have disjoint closures.

Conversely, suppose $A \delta B$, that is $A \not\ll X - B$. Then by (DHP6), X – B does not surround A, so there is some maximal round filter that intersects A and intersects B. Since the proximity is absolutely closed, this maximal round filter must be the neighbourhood filter of some point of the space, and this point is in the closure of both A and B.

To prove that the topology is regular-closed.

From F1, every round filter is a regular filter. By the above characterization of the proximity, and also by the fact that every open set containing a closed set is a round neighbourhood of the closed set, we have every regular filter is a round filter. Therefore, every maximal regular filter converges and the topology is regular-closed.

THEOREM: 4.21

An RC-proximity space is absolutely closed iff the induced topology is regular closed.

PROOF:

An absolutely closed RC-proximity induces a regular-closed topology is proved in theorem:4.20

To prove the converse:

A maximal round filter (being a regular filter by F1) must have a cluster point, to which it must converge (since neighbourhood filters are round by F2). Therefore the space is absolutely closed.

From theorems 4.18, 4.20 and 4.21 we have the following two theorems.

THEOREM: 4.22

A topological space is regular-closed iff it has the topology induced by an absolutely closed RC-proximity.

THEOREM:4.23

There is precisely one RC-proximity that induces the topology of a regular-closed space.

Now we establish a technique for identifying the supremum of any non-empty family of compatible proximities on any R_o- space. As an application of this, the Alexandrov proximity is identified as the supremum of the compatible R-proximities on any regular space.

DEFINITION: 4.24 M.C.Rayburn[20]

A **proximity** on a set X is a binary relation δ on the power set $\mathcal{P}(X)$ which satisfies the following properties:

(MRP1): For every $A \subseteq X$, $\phi \overset{\wedge}{\delta} A$

(MRP2): If $A \, \delta \, B$, then $B \, \delta \, A$.

(MRP3): If $A \cap B \neq \phi$, then $A \, \delta \, B$.

(MRP4): $A \, \delta \, (B \cup C)$ iff $A \, \delta \, B$ or $A \, \delta \, C$.

RESULT: 4.25

A topological space has compatible proximities iff it satisfies the R_o-separation axiom.

DEFINITION: 4.26

The compatible proximities are partially ordered as follows: $\delta \leq \rho$ if $A \, \delta \, B \Rightarrow A \, \rho \, B$.

DEFINITION : 4.27

A proximity δ is separated if for every pair of points x and y, $x \neq y$ implies $x \overset{\wedge}{\delta} y$.

RESULT : 4.28

A space is T_1 iff every compatible proximity is separated.

DEFINITION : 4.29 M.C.RAYBURN [20]

A proximity δ on a topological space is a **Lodato proximity** (LO-proximity) if $\overline{A} \, \delta \, B$ implies $A \, \delta \, B$.

DEFINITION: 4.30

A proximity δ_w on a topological space is a **Wallman proximity**, that is a, $A \delta_w B$ if $\overline{A} \cap \overline{B} \neq \phi$.

RESULT: 4.31

The Wallman proximity is Lodato (4.29) indeed δ_w is the smallest Lodato proximity (4.29) in the order $\leq$.

DEFINITION: 4.32

Let (X,τ) be an R_o-topological space and define $A\delta^* B$ if $A \delta_w B$ or if neither cl(A) nor Cl(B) is compact. Then δ^* is a compatible proximity. We shall refer to it as the **Alexandrov proximity.**

REMARK : 4.33

Let X be any non-compact R_o topological space. Pick a point $p \notin X$ and let $X^* = X \cup \{p\}$. Topologize X* by taking any subset to be open if it is an open subset of X or if its complement is closed and compact in X. Since X is open in X, {p} is closed. Since X is R_o, every point of X has compact closure, hence the closure cannot contain p. Thus X* is a compact R_o-space containing X densely. Clearly X* is T_1 iff X is T_1, and X* is T_2 iff X is T_2 and locally compact.

Let δ_w be the Wallman proximity on X*. Let A and B be subsets of X.

Then $A \overset{A}{\delta}_w B$ in X* precisely when $Cl(A) \cap cl(B) = \phi$ and atleast one of cl(A), cl(B) is compact. Thus the subspace proximity induced by the one-point compatification of X is the Alexandrov proximity δ^*, it is a Lodato proximity.

RESULT: 4.34

i) If δ is a proximity on X and $A \ll B$, then $\overline{A} \subseteq B$ and $A \subseteq B^o$.

ii) If $A \ll B_1$ and $A \ll B_2$, then $A \ll B_1 \cap B_2$ and if $A \ll B \subseteq C$, then $A \ll C$.

iii) If $A \ll B$, then $X - B \ll X - A$.

LEMMA: 4.35

Proximity δ is Lodato (4.29) on X iff $A \ll B$ implies that $\overline{A} \ll B^o$.

PROOF:

$A \ll B \Rightarrow \overline{A} \ll B^o$ is equivalent to $\overline{A}\ \delta\ \overline{(X - B)} \Rightarrow A\ \delta\ (X - B)$. Hence the result follows.

DEFINITION: 4.36

For $A \subseteq X$ and δ a proximity on X, let the neighbourhood family of A be $N(A) = \{B: A \subseteq B^o\}$. And let the proximal neighbourhood family of A be $N_\delta(A) = \{B: A \ll B\}$.

RESULT: 4.37

i)If δ is a compatible proximity on X, then for each $A \subseteq X$, $N_\delta(A)$ is a filter of sets. If $A = \{x\}$, then $N_\delta(x) = N(x)$. Moreover, if $B \in N_\delta(A)$, then $(X - A) \in N_\delta(X-B)$

ii)Let X be a set and δ, ρ be compatible proximities on X. Then $\delta \leq \rho$ iff for all $A \subseteq X$, $N_\rho(A) \ll N_\delta(A)$.

THEOREM: 4.38

Let X be an R_o topological space and $\{\mathcal{F}_A : A \subseteq X\}$ be a family of filters on X satisfying the following conditions:

a. For each $A \subseteq X$, if $B \in \mathcal{F}_A$, then $A \subseteq B$.

b. For each $A \subseteq X$, if $B \in \mathcal{F}_A$, then $X - A \in \mathcal{F}_{X-B}$, and

c. For each $x \in X$, $\mathcal{F}_{\{x\}} = N(x)$.

If we define $A \overset{\mathrm{A}}{\delta} B$ to mean $X - B \in \mathcal{F}_A$, then δ is a compatible proximity on X, and for each A, $\mathcal{F}_A = N_\delta(A)$.

COROLLARY: 4.39

Let X be an R_o-topological space and $\mathcal{M}$ be a non-empty family of compatible proximities on X. The supremum of $\mathcal{M}$ is the proximity δ such that for each $A \subseteq X$, $N_\delta(A) = \bigcap_{\mathcal{M}} N(A)$.

THEOREM: 4.40

Let X be a regular topological space. Let A be compact and B be closed and disjoint from A on X. Then for every compatible R-proximity δ on X, $A \overset{\mathrm{A}}{\delta} B$.

PROOF:

For each $a \in A$, $a \overset{\mathrm{A}}{\delta} B$. So $\{a\} \ll X - B$, and there is a set N_a such that $\{a\} \ll N_a \ll X - B$. Indeed $\{a\} \subseteq N_a^{\circ} \subseteq N_a \ll X - B$. Then $\{N_a^{\circ}: a \in A\}$ is an open cover of compact A, so it has a finite subcover $\{N_k^{\circ}: 1 \leq k \leq n\}$. Let $N = \cup\{N_k: 1 \leq k \leq n\}$. Then $A \subseteq N \ll X - B$. So $A \overset{\mathrm{A}}{\delta} B$.

COROLLARY: 4.41

Let δ be an R-proximity and K be a compact set of X. Then $N_\delta(K) = N(K)$.

DEFINITION: 4.42

A proximity which is both an R-proximity and a Lodato proximity (4.29) will be called an **LR-proximity**.

THEOREM: 4.43

Let X be a topological space. The following are equivalent:

a. X is regular.

b. The Wallman proximity δ_w is a compatible R-proximity.

c. There exists a compatible LR-proximity

d. There exists a compatible R-proximity

THEOREM: 4.44

Let δ be a compatible LR-proximity on X. Then for every closed compact $K \subseteq X$, $N_\delta(K)$ is round.

PROOF:

Let $K << A$. Then, for each $p \in K$, $p << A$; so there is some B_p with $p << B_p << A$. Then $\{B_p^o : p \in k\}$ is an open cover of compact K, so there is a finite subcover $K \subseteq \bigcup_{k=1}^{n} B_k^o << A$. Finally, $K = \overline{K} \subseteq (\bigcup_{k=1}^{n} B_k^o)^o$, so since δ is Lodato, $K << \bigcup_{k=1}^{n} B_k^o$.

THEOREM: 4.45

Let X be a regular space with compatible R-proximity δ. If ρ is any compatible proximity with $\rho \leq \delta$, then ρ is an R-proximity.

THEOREM: 4.46

Let X be a regular space and δ be a compatible R-proximity. Then $\delta \leq \delta^*$, the Alexandrov proximity.

PROOF:

Let $A, B \subseteq X$ with $A \overset{A}{\delta^*} B$. Then $A \overset{A}{\delta_w} B$ and atleast one of cl(A), cl(B) is compact. Then by theorem: 4.40, $A \overset{A}{\delta} B$ for every compatible R-proximity on X.

THEOREM: 4.47

Let X be an R_o topological space. If A and B are disjoint, closed and not compact subsets of X, then there is a compatible Lodato proximity δ (4.29) on X such that $A \delta B$. Moreover, if X is regular, then δ is an LR-proximity.

PROOF:

Let A be closed and not compact in X. There is a decreasing sequence $\Phi = \{F_n\}_1^\infty$ of closed subsets of A for which $\bigcap\limits_{n=1}^{\infty} F_n = \phi$. Since Φ is a filterbase, there is an ultrafilter μ with $\Phi \subseteq \mu$. Let $\sigma_a = \{D : \overline{D} \in \mu\}$. Then $A \in \sigma_a$ and since $\Phi \subseteq \sigma_a$, $\cap \sigma_a = \phi$.

There is a corresponding family σ_b with $B \in \sigma_b$ and $\cap \sigma_b = \phi$. Let $\gamma = \sigma_a \cup \sigma_b$ and define δ on X by $C \delta D$ if either $C \delta_w D$ or C and D are both in γ. It can be easily verified that δ is a proximity.

Since σ_a and σ_b each have empty intersection, neither can contain a singleton, so for any $x \in X$ and $C \subseteq X$, $x \delta C$ iff $x \delta_w C$, and δ is compatible. If $\overline{C} \delta D$, then either $\overline{C} \delta_w D$, so $C \delta_w D$ and $\overline{C} \delta D$, or $\overline{C}$ and D are in γ. If $C \in \sigma_a$, say then by definition of σ_a, $C \in \sigma_a$. Thus $C \delta D$ and δ is Lodato. Since A and B are in γ, $A \delta B$.

Finally, suppose X is regular. Let $x \in X$ and suppose $x \in G^o$. Since $A \cap B = \phi$, x is not in atleast one, say B. There is an n with $x \notin F_n \subseteq A$ (where $F_n \in \phi$). So without loss of generality, $G^o \subseteq X - (B \cup F_n)$. By regularity, there is some D with $x \in D^o \subseteq \overline{D} \subseteq G^o$. Clearly $D \overset{A}{\delta}_w (X - G)$ and since by choice of G^o, $D \overset{A}{\delta}_w B$, and $D \overset{A}{\delta}_w F_n$, we have $D \notin \gamma$. Therefore $D \overset{A}{\delta} (X - G)$, so $x << D << G$ and δ is an R-proximity.

COROLLARY: 4.48

Let X be regular and A,B be disjoint closed sets in X.Then $A \overset{A}{\delta} B$ for every compatible R-proximity iff A or B is compact.

THEOREM: 4.49

The Alexandrov proximity δ^* is the supremum of the R-proximities on a regular space. Hence if δ^* is an R-proximity, then the R-proximities are precisely those proximities δ with $\delta \leq \delta^*$.

PROOF:

If $A \subseteq X$ has $\overline{A}$ compact, then $A \,\delta^*\, B \Leftrightarrow A \,\delta_w\, B$, so $N_{\delta^*}(A) = N_{\delta_w}(A)$.

Therefore $N_{\delta^*}(A) = \cap\{N_\delta(A)$: δ is a compatible R-proximity}.

Suppose $\overline{A}$ is not compact. Then $N_{\delta^*}(A) = \{B : B \in N_{\delta_w}(A)$ and $X - B^o$ is compact}. If $\overline{A}$ and $\overline{(X - B)}$ are disjoint and not compact, there is some compatible LR-proximity δ with $\overline{A} \,\delta\, \overline{(X - B)}$, hence $A \,\delta\, (X - B)$. Thus $B \notin N_\delta(A)$, and $\{B : B \in N_{\delta_w}(A)$ and $X - B^o$ is compact) $= \cap \{N_\delta(A)$: δ is a compatible R-proximity}. Hence by corollary : 4.39, δ^* is the supremum of the compatible R-proximities.

LEMMA: 4.50

Let δ be a proximity on topological space X. The following are equivalent:

a. δ is an Efremovich proximity.

b. Whenever $A \hat{\delta} B$, then there are sets U and V with $A << U$, $B << V$ and $U\hat{\delta} V$.

c. δ is a Lodato proximity (4.29) and $N_\delta(F)$ is round for every closed set F.

PROOF:

(a) $\Leftrightarrow$ (b)

To prove (b) $\Rightarrow$ (a)

Assume (b). To prove (a), it is sufficient to show δ satisfies "strong axiom", which states that $A \hat{\delta} B$ implies there exists a subset $E \ni A \hat{\delta} E$ and $(X\text{-}E) \hat{\delta} B$.

If $U \hat{\delta} V$, then $U \subset (X\text{-}V)$.

Put $E = X - U$, then $A \hat{\delta} E$ and $(X\text{-}E) \hat{\delta} B$. Hence (a).

Assume (a)

Then $A \hat{\delta} B \Rightarrow$ implies there exists a subset V such that $A \hat{\delta} V$ and $(X - V) \hat{\delta} B$.

Moreover, $\exists$ a $U \ni A \hat{\delta} (X - U)$ and $U \hat{\delta} V$. Therefore, $A << U$, $B << V$ and $U \hat{\delta} V$. Hence (b).

(b) $\Rightarrow$ (c):

Suppose $A \hat{\delta} B$.

Let $A << U$ and $B << V$ and $U \hat{\delta} V$.

Since $\overline{A} \subseteq U$ and $\overline{B} \subseteq V$, $\overline{A} \hat{\delta} \overline{B}$, hence δ is Lodato.

(c) ⇒ (a)

Let $A \ll B$. Since δ is Lodato, $\overline{A} \ll B^\circ$. Thus there exist a set C with $\overline{A} \ll C \ll B^\circ$. Hence $A \ll C \ll B$.

THEOREM : 4.51

For any R_o topological space, the following are equivalent.

a. The Alexandrov proximity δ^* is an R-proximity.

b. The Alexandrov proximity is an Efremovich proximity.

c. The space is locally compact and T_2.

PROOF:

Since a compact, completely regular space has only one compatible LR-proximity, which is Efremovich, we need only prove (a) ⇔ (b) in this theorem.

Let F be closed in X and take $B \in N_\delta(F)$. Since δ^* is Lodato and F is closed, $F \ll B^\circ$. Let $D = X - B^\circ$, closed. Then $F \overset{A}{\delta} D$, so $F\ \delta_w\ D$ and atleast one of F and D must be compact.

If F is compact, since δ^* is R, $N_\delta(F)$ is round by theorem :4.44. Suppose D is compact, then $D \ll X - F$, so since δ^* is R, there is a set T with $A \ll T \ll X - F$. Hence $F \ll X - T \ll X - D = B^\circ$ and $N_\delta(F)$ is round. Thus δ^* is Lodato proximity, every closed set F has $N_{\delta^*}(F)$ round and from lemma: 4.50 it follows that δ^* is Efremovich.

COROLLARY: 4.52

If X is locally compact T_2 space, the compatible R-proximities are precisely those proximities $\delta \le \delta^*$.

Summary and Conclusion

SUMMARY AND CONCLUSION

Some interesting generalizations of proximity structures are studied in this thesis. Mainly different generalizations of Efremovich proximities are discussed. Some interesting generalized proximities discussed in this thesis are

1. Paraproximity
2. Closure-proximity
3. Normal proximity
4. Perfect proximity
5. Weak normal proximity
6. Compact proximity
7. Metacompact proximity
8. Lodato or LO-proximity
9. Leader or LE-proximity
10. Pervin or P-proximity
11. Separation or S-proximity
12. E-discrete proximity
13. E-discrete proximity of Alexandroff
14. R-proximity
15. RC-proximity

Chapter I deals with fundamental definitions.

Chapter II is devoted to the study of first seven generalized proximities listed above. Here it is proved that a paraproximity yields a completely normal space. Also there is 1-1 correspondence between closure-proximities (normal proximities, perfect proximities) and T_1-topologies (completely normal topologies, perfectly normal topologies).

The third chapter deals with the generalized proximities (numbered 8-13 in the above list). The interrelations between these proximities are obtained.

In the fourth chapter, R-proximities and RC-proximities are discussed.

For further research, one can study different combination of these generalized proximities. It is a good research problem to study all these concepts for bitopological space and fuzzy topological spaces.

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