

CHAPTER 1

CHAPTER 1

GENERALIZED CLOSED SETS IN TOPOLOGY

In this chapter the notion of generalized closed sets due to Levine [21] and g^* -closed sets due to Veerakumar [50] are discussed. Properties, characterizations and applications are analyzed.

SECTION 1.1

g -CLOSED SETS

In this section generalized closed (g -closed) sets due to Levine [21] is studied. Properties, characterizations and applications are analyzed.

Definition 1.1.1

A subset A of a space (X, τ) is called g -closed iff $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Proposition 1.1.2

Every closed set is g -closed.

Theorem 1.1.3

A set A is g -closed iff $\text{cl}(A) - A$ contains no non-empty closed set.

Proof

Let A be g -closed and let F be a closed subset of $\text{cl}(A) - A$. Then $A \subseteq F^c$ and since A is g -closed, $\text{cl}(A) \subseteq F^c$. That is $F \subseteq (\text{cl}(A))^c$. Thus $F \subseteq \text{cl}(A) \cap (\text{cl}(A))^c = \phi$. Therefore F is empty.

To prove the sufficiency, let us assume that $A \subseteq U$ and that U is open. If $\text{cl}(A) \not\subseteq U$, then $\text{cl}(A) \cap (U)^c$ is a non-empty closed subset of $\text{cl}(A) - A$. This is the contradiction to the assumption. Hence $\text{cl}(A) \subset U$. Therefore A is g-closed.

Corollary 1.1.4

A g-closed set A is closed iff $\text{cl}(A) - A$ is closed.

Proof

If A is closed then $\text{cl}(A) - A = \phi$. Therefore $\text{cl}(A) - A$ is closed. Conversely, assume that $\text{cl}(A) - A$ closed. As A is g-closed and $\text{cl}(A) - A$ closed, by Theorem (1.1.3) $\text{cl}(A) - A = \phi$ and hence $\text{cl}(A) = A$.

Theorem 1.1.5

If A and B are g-closed then $A \cup B$ is g-closed.

Proof

Let $A \cup B \subseteq U$ where U is open. Since A and B are g-closed, $\text{cl}(A) \subset U$ and $\text{cl}(B) \subset U$. Therefore $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B) \subseteq U$. Therefore $A \cup B$ is g-closed.

Remark 1.1.6

The intersection of two g-closed sets need not be a g-closed set.

Example 1.1.7

Let $X = \{a, b, c\}$ and let $\tau = \{\phi, \{a\}, X\}$. Let $A = \{a, b\}$ and $B = \{a, c\}$. Then A and B are each g-closed, but $A \cap B$ is not g-closed.

Theorem 1.1.8

Suppose that $B \subseteq A \subseteq X$, B is a g-closed set relative to A and that A is a g-closed subset of X . Then B is g-closed relative to X .

Proof

Let $B \subseteq A$ and U are open in X . Then $B \subseteq A \cap U$ and hence $\text{cl}(B) \subseteq A \cap U$. Then $A \cap \text{cl}(B) \subseteq A \cap U$ and $A \subseteq U \cup (\text{cl}(B))^c$. Since A is g-closed in X , $\text{cl}(A) \subseteq U \cup (\text{cl}(B))^c$. Therefore $\text{cl}(B) \subseteq U \cup (\text{cl}(B))^c$ and hence $\text{cl}(B) \subseteq U$. Therefore B is g-closed relative to X .

Corollary 1.1.9

Let A be a g-closed set and suppose that F is closed. Then $A \cap F$ is a g-closed set.

Proof

$A \cap F$ is closed in A and hence g-closed in A . By Theorem (1.1.8), $A \cap F$ is g-closed set.

Theorem 1.1.10

If A is g-closed and $A \subseteq B \subseteq \text{cl}(A)$, then B is g-closed.

Proof

As A is g-closed, $\text{cl}(A) - A$ has no non-empty closed subset. As $\text{cl}(B) - B \subseteq \text{cl}(A) - A$ and $\text{cl}(B) - B$ has no non-empty closed subset. By Theorem (1.1.3), B is g-closed.

Theorem 1.1.11

Let $A \subseteq Y \subseteq X$ and suppose that A is g-closed in X , then A is g-closed relative to Y .

Proof

Let $A \subseteq Y \cap U$ and U is open in X . Then $A \subseteq U$. Since A is g -closed, $\text{cl}(A) \subseteq U$. It follows that $Y \cap \text{cl}(A) \subseteq Y \cap U$. Hence A is g -closed relative to Y .

Theorem 1.1.12

Let (X, τ) be a compact topological space and suppose that A is a g -closed subset of X . Then A is compact.

Proof

Let F be an open cover of A . Then $A \subseteq \bigcup_{U \in F} U$. Since A is g -closed, $\text{cl}(A) \subseteq \bigcup_{U \in F} U$. As $\text{cl}(A)$ is compact, $A \subseteq \text{cl}(A) \subseteq U_1 \cup \dots \cup U_n$ for some finite number of $U_i \in F$. Therefore A is compact.

Proposition 1.1.13

Let (X_i, τ) be Lindelof and suppose that A is a g -closed subset of X . Then A is Lindelof.

Definition 1.1.14

A subset A of a space (X, τ) is called generalized open iff A^c is g -closed.

Proposition 1.1.15

A set A is g -open iff $F \subseteq \text{int}(A)$ whenever F is closed and $F \subseteq A$.

Theorem 1.1.16

If A and B are separated g -open sets, then $A \cup B$ is g -open.

Proof

Let F be a closed subset of $A \cup B$. Then $F \cap \text{cl}(A) \subseteq A$ and hence by Proposition (1.1.15), $F \cap \text{cl}(A) \subseteq \text{int}(A)$. Similarly, $F \cap \text{cl}(B) \subseteq \text{int}(B)$. Now $F = F \cap (A \cup B) \subseteq F \cap \text{cl}(A) \cap (F \cap \text{cl}(B)) \subseteq \text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$. $F \subseteq \text{int}(A \cup B)$ and by Proposition (1.1.15), $A \cup B$ is g -open.

Remark 1.1.17

The union of two g -open sets is generally not g -open.

Theorem 1.1.18

A set A is g -open in (X, τ) iff $U = X$ whenever U is open and $\text{int}(A) \cup (A)^c \subseteq U$.

Proof

Let A be g -open and U is open and $\text{int}(A) \cup (A)^c \subseteq U$. Then $(U)^c \subseteq \text{cl}(A^c) \cap A = \text{cl}(A^c) - A^c$. Since $(U)^c$ is closed and $(A)^c$ is g -closed, by Theorem (1.1.3), $(U)^c = \emptyset$ or $X = U$.

Conversely, assume that F is a closed set and $F \subseteq A$. By Proposition (1.1.15), it suffices to show that $F \subseteq \text{int}(A)$. Now $\text{int}(A) \cup (A)^c \subseteq \text{int}(A) \cup (F)^c$ and hence $\text{int}(A) \cup (F)^c = X$. Hence $F \subseteq \text{int}(A)$. Therefore A is g -open.

Theorem 1.1.19

If $A \subseteq B \subseteq X$, where A is g -open relative to B and B is g -open relative to X , then A is g -open relative to X .

Proof

Let F be a closed set and suppose that $F \subseteq A$. Then F is closed relative to B and hence $F \subseteq \text{int}_B A$. Therefore there exists an open set U

such that $F \subseteq U \cap B \subseteq A$. But $F \subseteq U^* \subseteq B$ for some open set U^* since B is g -open in X . Thus $F \subseteq U^* \cap U \subseteq B \cap U \subseteq A$. It follows that $F \subseteq \text{int}(A)$. By Proposition (1.1.15), A is g -open in X .

Example 1.1.20

Let $X = \{a, b, c\}$ and suppose that $\tau = \{\emptyset, \{a\}, X\}$. If $A = \{b\}$, $Y = \{a, b\}$ then A is g -open in both X and Y .

Theorem 1.1.21

If $\text{int}(A) \subseteq B \subseteq A$ and if A is g -open then B is g -open.

Proof

$(A)^c \subseteq (B)^c \subseteq \text{cl}(A^c)$ and since $(A)^c$ is g -closed, it follows that $(B)^c$ is g -closed by Theorem (1.1.10), B is g -open.

Theorem 1.1.22

A set A is g -closed iff $\text{cl}(A) - A$ is g -open.

Proof

Suppose that A is g -closed and that $F \subseteq \text{cl}(A) - A$, F being closed. By Theorem (1.1.3), $F = \emptyset$ and hence $F \subseteq \text{int}(\text{cl}(A) - A)$. By Proposition (1.1.14), $\text{cl}(A) - A$ is g -open.

Suppose $A \subseteq U$, where U is an open set. Now $\text{cl}(A) \cap (U^c) \subseteq \text{cl}(A) \cap (A^c) = \text{cl}(A) - A$ and since $\text{cl}(A) \cap (U^c)$ is closed and $\text{cl}(A) - A$ is g -open, it follows that $\text{cl}(A) \cap (U^c) \subseteq \text{int}(\text{cl}(A) - A) = \emptyset$. Therefore $\text{cl}(A) \cap (U^c) = \emptyset$ or $\text{cl}(A) \subseteq U$. Thus A is g -closed.

Definition 1.1.23

A topological space X is said to be $T_{1/2}$ space iff every g -closed set is closed.

Proposition 1.1.24

Every $T_{1/2}$ space is T_0 .

The converse of the Proposition (1.1.24) need not be true as seen from the following example.

Example 1.1.25

Let $X = \{a, b, c\}$ and suppose that $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then (X, τ) is a T_0 space, but not a $T_{1/2}$ space. Since $\{a, c\}$ is g-closed but not closed.

Proposition 1.1.26

Every T_1 space is $T_{1/2}$.

The converse of the Proposition (1.1.26) need not be true as seen from the following example.

Example 1.1.27

Let $X = \{a, b\}$ and suppose that $\tau = \{\emptyset, \{a\}, X\}$. Then (X, τ) , $T_{1/2}$ space which is not T_1 .

Corollary 1.1.28

The property $T_{1/2}$ is strictly between T_0 and T_1 .

SECTION 1.2

g^* -CLOSED SETS

In this section a new class of sets called g^* -closed sets due to Veerakumar [51] is discussed. Properties, characterizations and and A is a g^* -closed set but not a closed set of (X, τ) .

Proposition 1.2.1

Every g^* -closed set is a g -closed set.

Remark 1.2.2

The class of g^* -closed sets is properly contained in the class of g -closed sets.

Remark 1.2.3

The converse of Proposition (1.2.1) need not be true.

Definition 1.2.4

A space (X, τ) is called a $T_{1/2}^*$ space if every g^* -closed set is closed.

Proposition 1.2.5

Every $T_{1/2}$ space is $T_{1/2}^*$ space.

The converse of the Proposition (1.2.5) need not be true as seen from the following example.

Example 1.2.6

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$, $G^*C(X, \tau) = \{\{b, c\}, \emptyset, X\} = C(X, \tau)$. So (X, τ) is not a $T_{1/2}$ space. Since $\{b\}$ is a g-closed set but not a closed set of (X, τ) .

Definition 1.2.7

A topological space (X, τ) is said to be a T_b -space if every gs-closed set in X is closed.

Proposition 1.2.8

Every T_b space is a $T_{1/2}^*$ space.

The converse of the Proposition (1.2.8) need not be true as seen from the following example.

Example 1.2.9

Let (X, τ) be as in the Example (1.2.8). Already we have seen that (X, τ) is a $T_{1/2}^*$ space (X, τ) is not a T_b space ,since it is not even $T_{1/2}$ space.

Theorem 1.2.10

For a space (X, τ) , the following conditions are equivalent.

- 1) (X, τ) is a $T_{1/2}^*$ space.
- 2) Every singleton of X is either g-closed or open.

Definition 1.2.11

A space (X, τ) is called a $^*T_{1/2}$ space if every g-closed set of (X, τ) is a g^* -closed set.

Theorem 1.2.12

Every $T_{1/2}$ space is a ${}^*T_{1/2}$ space.

The converse of the Proposition (1.1.12) need not be true as seen from the following example.

Example 1.2.13

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$. Let $A = \{a, b\}$ then A is ${}^*T_{1/2}$ space but not $T_{1/2}$ space.

Theorem 1.2.14

A space (X, τ) is a $T_{1/2}$ space iff it is ${}^*T_{1/2}$ and $T_{1/2}^*$.

Theorem 1.2.15

If (X, τ) is a ${}^*T_{1/2}$ space, then for each $x \in X$. $\{x\}$ is either closed or g^* -open.

Proposition 1.2.16

Every T_b space is a T_d space.