Chapter I

CHAPTER I

BIMATRICES, NEUTROSOPHIC BIMATRICES, FUZZY BIMATRICES, INTERVAL BIMATRICES, FUZZY INTERVAL BIMATRICES AND NEUTROSOPHIC INTERVAL BIMATRICES.

SECTION 1.1

BIMATRICES

Definition 1.1.1

A **bimatrix** A_B is defined as the union of two rectangular away of numbers A_1 and A_2 arranged into rows and columns. It is written as follows $A_B = A_1 \cup A_2$ where $A_1 \neq A_2$ with

$$A_1 = \begin{bmatrix} a^1_{11} & a^1_{12} & \cdots & a^1_{1n} \\ a^1_{21} & a^1_{22} & \cdots & a^1_{2n} \\ \cdots & & & \cdots \\ a^1_{m1} & a^1_{m2} & \cdots & a^1_{mn} \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} a^2_{11} & a^2_{12} & \cdots & a^2_{1n} \\ a^2_{21} & a^2_{22} & \cdots & a^2_{2n} \\ \cdots & & & \cdots \\ a^2_{m1} & a^2_{m2} & \cdots & a^2_{mn} \end{bmatrix}$$

'$\cup$' is just the notational convenience (symbol) only.

The above array is called a m by n bimatrix since each of A_i ($i = 1, 2$) has m rows and n columns. It is to be noted that a bimatrix has no numerical value associated with it. It is only a convenient way of representing a pair of arrays of numbers.

Note 1.1.2

If $A_1 = A_2$ then $A_B = A_1 \cup A_2$ is not a bimatrix. A bimatrix A_B is denoted by $(a^1_{ij}) \cup (a^2_{ij})$. If both A_1 and A_2 are m×n matrices then the bimatrix A_B is called the m×n **rectangular bimatrix**. But we make an assumption the **zero bimatrix** is a union of two zero matrices even if A_1 and A_2 are one and the same;
i.e., $A_1 = A_2 = (0)$.

Example 1.1.3

The following are bimatrices

$$A_B = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

is a 2x3 bimatrix.

Definition 1.1.4

Let $A_B = A_1 \cup A_2$ be a bimatrix. If both A_1 and A_2 are square matrices then A_B is called the **square bimatrix**. If one of the matrices in the bimatrix $A_B = A_1 \cup A_2$ is square and other is rectangular or if both A_1 and A_2 are rectangular matrices say $m_1 \times n_1$ and $m_2 \times n_2$ with $m_1 \neq m_2$ or $n_1 \neq n_2$ then we say A_B is a **mixed bimatrix**.

The following are examples of a square bimatrix and a mixed bimatrix.

Example 1.1.5

Given $A_B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$

is a 2x2 square bimatrix.

Example 1.1.6

Let $A_B = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

then A_B is a mixed bimatrix.

BIMATRIX OPERATIONS :

Definition 1.1.7

Consider the bimatrices $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$.

Then we define the following

(1) $A_B = C_B$

Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two bimatrices. We say A_B and C_B are equal written as $A_B = C_B$ if and only if A_1 and C_1 are identical and A_2 and C_2 are identical i.e., $A_1 = C_1$ and $A_2 = C_2$.

(2) $A_B \neq C_B$

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$, we say A_B is not equal to C_B we write $A_B \neq C_B$ if and only if $A_1 \neq C_1$ or $A_2 \neq C_2$.

Example:

$$\text{Let } A_B = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$C_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Clearly $A_B \neq C_B$.

(3) λ AB

Given a bimatrix $A_B = A_1 \cup B_1$ and a scalar λ, the product of λ and A_B written λA_B is defined to be

$$\lambda A_B = \begin{bmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \cdots & & \cdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix} \cup \begin{bmatrix} \lambda b_{11} & \cdots & \lambda b_{1n} \\ \cdots & & \cdots \\ \lambda b_{m1} & \cdots & \lambda b_{mn} \end{bmatrix}$$

Example:

$$\text{Let } A_B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

and $\lambda=3$ then

$$3A_B = \begin{bmatrix} 6 & 0 & 3 \\ 9 & 9 & 3 \end{bmatrix} \cup \begin{bmatrix} 0 & 3 & 3 \\ 6 & 3 & 0 \end{bmatrix}$$

(4) $A_B + C_B$

Let $A_B = A_1 \cup B_1$ and $C_B = A_2 \cup B_2$ be any two m×n bimatrices. The sum D_B of the bimatrices A_B and C_B is defined as

$$D_B = A_B + C_B = [A_1 \cup B_1] + [A_2 \cup A_2] = (A_1 + A_2) \cup (B_2 + B_2);$$ where

$A_1 + A_2$ and $B_1 + B_1$ are the usual addition of matrices i.e., if

$$A_B = (a^1_{ij}) \cup (b^1_{ij})$$

and $$C_B = (a^2_{ij}) \cup (b^2_{ij})$$

then $A_B + C_B = D_B = (a^1_{ij} + a^2_{ij}) \cup (b^1_{ij} + b^2_{ij})$ $(\forall ij)$.

If we write in detail

$$A_B = \begin{bmatrix} a^1_{11} & \cdots & a^1_{1n} \\ \cdots & & \cdots \\ a^1_{m1} & \cdots & a^1_{mn} \end{bmatrix} \cup \begin{bmatrix} b^1_{11} & \cdots & b^1_{1n} \\ \cdots & & \cdots \\ b^1_{m1} & \cdots & b^1_{mn} \end{bmatrix}$$

$$C_B = \begin{bmatrix} a^2_{11} & \cdots & a^2_{1n} \\ \cdots & & \cdots \\ a^2_{m1} & \cdots & a^2_{mn} \end{bmatrix} \cup \begin{bmatrix} b^2_{11} & \cdots & b^2_{1n} \\ \cdots & & \cdots \\ b^2_{m1} & \cdots & b^2_{mn} \end{bmatrix}$$

$$A_B + C_B = \begin{bmatrix} a^1_{11}+a^2_{11} & \cdots & a^1_{1n}+a^2_{1n} \\ \cdots & & \cdots \\ a^1_{m1}+a^2_{m1} & \cdots & a^1_{mn}+a^2_{mn} \end{bmatrix} \cup \begin{bmatrix} b^1_{11}+b^2_{11} & \cdots & b^1_{1n}+b^2_{1n} \\ \cdots & & \cdots \\ b^1_{m1}+b^2_{m1} & \cdots & b^1_{mn}+b^2_{mn} \end{bmatrix}$$

The expression is abbreviated to

$$\begin{aligned} D_B &= A_B + C_B \\ &= (A_1 \cup B_1) + (A_2 \cup B_2) \\ &= (A_1 + A_2) \cup (B_1 + B_2) \end{aligned}$$

Thus two bimatrices are added by adding the corresponding elements only when compatibility of usual matrix addition exists.

Example:

Let $A_B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

and $C_B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

then $D_B = A_B + C_B$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

(5) A_B-C_B

Subtraction is defined in terms of operations already considered for if

$A_B = A_1 \cup A_2$

and $C_B = C_1 \cup C_2$

then

$$A_B - C_B = A_B + (-C_B)$$
$$= (A_1 \cup A_2) + (-C_1 \cup -C_2)$$
$$= (A_1 - C_1) \cup (A_2 - C_2)$$
$$= [A_1 + (-C_1)] \cup [A_2 + (-C_2)].$$

(6) Bimatrix multiplication is not defined when the bimatrices are not square bimatrix. Secondly in case of mixed bimatrix, multiplication is defined when both A_1 and A_2 are square matrices or when $A_B = A_1^{m \times n} \cup A_2^{p \times s}$ and $C_B = B_1^{n \times u} \cup B_2^{s \times t}$.

Note 1.1.8

Addition of bimatrices are defined if and only if both the bimatrices are m×n bimatrices.

Let $A_B = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

$$C_B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

The addition of A_B with C_B is not defined for A_B is a 2×3 bimatrix where as C_B is a 3×2 bimatrix.

Clearly $A_B + C_B = C_B + A_B$ when both A_B and C_B are m× n matrices. Also if A_B, C_B, D_B be any three m × n bimatrices then

$$A_B + (C_B + D_B) = (A_B + C_B) + D_B.$$

Note 1.1.9

Now we have defined addition and subtraction of bimatrices. Unlike in matrices we cannot say if we add two bimatrices the sum will be a bimatrix. That is in general sum of two bimatrices is not a bimatrix.

Example 1.1.10

Let $A_B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

and

$$C_B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \end{bmatrix}$$

$$A_B + C_B = \left\{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right\} \cup \left\{\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

Clearly $A_B + C_B$ is not a bimatrix as $A_1 \cup C_1 = A_2 \cup C_2$

where $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$.

Theorem 1.1.11

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be two m × n bimatrices.

$A_B + B_B = (A_1 + B_1) \cup (A_2 + B_2)$is a bimatrix if and only if $A_1 + B_1 \neq A_2 + B_2$.

Proof :

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be two m×n bimatrices.

If $A_B + B_B = (A_1 + B_1) \cup (A_2 + B_2)$ is a bimatrix then $A_1 + B_1 \neq A_2 + B_2$.

On the other hand in $A_B + B_B$ if $A_1 + B_1 \neq A_2 + B_2$ then clearly $A_B + B_B$ is a bimatrix. Hence the theorem.

Corollary 1.1.12

If $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be two m×n bimatrices then $A_B - B_B = A_B + (-B_B)$ is a bimatrix if and only if $A_1 + (-B_1) \neq A_2 + (-B_2)$.

Theorem 1.1.13

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be two m × m square bimatrices $A_B \cdot B_B = A_1B_1 \cup A_2B_2$ is a bimatrix if and only if $A_1B_1 \neq A_2B_2$.

Proof:

Given $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ are two m ×m square bimatrices and $A_B B_B$ is a bimatrix i.e.,$A_B B_B = (A_1 \cup A_2)(B_1 \cup B_2) = A_1 B_1 \cup A_2 B_2$ is a bimatrix. Hence $A_1B_1 \neq A_2B_2$.

Clearly if $A_1 B_1 \neq A_2 B_2$ then $A_B B_B = A_1B_1 \cup A_2B_2$ is a m×m bimatrix.

Theorem 1.1.14

Let $A_B = A_1^{m \cdot n} \cup A_2^{p \cdot q}$ be a mixed bimatrix and $B_B = B_1^{n \cdot m} \cup B_2^{q \cdot p}$ be another mixed bimatrix. Then the product is defined and is a mixed square bimatrix as

$$A_B B_B = C_1^{m \cdot m} \cup C_2^{p \cdot p}$$

With $\quad C_1 = C_1^{m \cdot m} \cup C_2^{p \cdot p}$

Where $\quad C_1^{m \cdot m} = A_1^{m \cdot n} \times B_1^{n \cdot m}$

and $\quad C_2^{p \cdot p} = A_2^{p \cdot q} \times B_2^{q \cdot p}$.

Note 1.1.15

If A_B and B_B be two m×m square bimatrices in general $A_B B_B \neq B_B A_B$.

Example 1.1.16

Let $\quad A_B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$

and $\quad B_B = \begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$$A_B B_B = \begin{bmatrix} 15 & 6 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$$

consider $$B_B A_B = \begin{bmatrix} 15 & 2 \\ 3 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$$

Clearly $A_B B_B \neq B_B A_B$.

Note 1.1.17

In some cases for the bimatrices A_B and B_B only one type of product $A_B B_B$ may be defined and $B_B A_B$ may not be even defined.

This is shown by the following example:

Example 1.1.18

Let $$A_B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

and

$$B_B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \\ 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 4 \end{bmatrix}$$

Clearly $$A_B B_B = \begin{bmatrix} 4 & 0 \\ 3 & 0 \\ -2 & -1 \end{bmatrix} \cup \begin{bmatrix} 8 & -1 & 6 \\ 9 & 0 & 3 \\ 7 & -2 & 9 \end{bmatrix}$$

is a mixed bimatrix. But $B_B A_B$ is not even defined.

Note 1.1.19

Bimatrix multiplication is also additive and also distributive.

Definition 1.1.20

We define the transpose of a bimatrix

$A_B = A_1 \cup A_2 = (a^1_{ij}) \cup (a^2_{ij})$ as $A'_B = (a^1_{ji}) \cup (a^2_{ji})$

Note 1.1.21

If A_B and B_B are bimatrices

then (i) $(A_B + B_B)' = A'_B + B'_B$

(ii) $(A_B B_B)' = B'_B A'_B$

Definition 1.1.22

A **symmetric bimatrix** is a matrix A_B for which $A_B = A'_B$ i.e., the component matrices of AB are also symmetric matrices.

Example 1.1.23

$$\text{Let } A_B = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & -5 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 2 \\ 1 & -5 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

is a symmetric bimatrix.

Definition 1.1.24

Let $A_B^{m \times m} = A_1 \cup A_2$ be a m×m square bimatrix i.e., A_1 and A_2 are m×m square matrices. A **skewsymmetric bimatrix** is a bimatrix A_B for which $A_B = - A'_B$. where $- A'_B = \mathbf{-A'_1} \cup \mathbf{-A'_2}$ i.e., the component matrices A_1 and A_2 of A_B are also skew symmetric.

Example 1.1.25

$$\text{Let } A_B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 3 & 13 \\ -3 & 0 & -2 \\ -13 & 2 & 0 \end{bmatrix}$$

is a skewsymmetric bimatrix .

Theorem 1.1.26

Let $A_B^{m \cdot m} = A_1 \cup A_2$ be a square bimatrix. Then $A_B^{m \cdot m}$ can be written as the sum of a symmetric and a skew symmetric bimatrix.

Proof:

Given $A_B^{m \cdot m} = A_1 \cup A_2$ is a square bimatrix. Clearly A_1 and A_2 are just square matrices.If A is any square matrix then

$$A = A + \frac{A'}{2} - \frac{A'}{2}$$

$$A = \frac{A + A'}{2} + \frac{A - A'}{2}$$

so each A_1 and A_2 can be represented by the above equation

i.e.,

$$A_1 = \frac{A_1 + A'_1}{2} + \frac{A_1 - A'_1}{2}$$

and

$$A_2 = \frac{A_2 + A'_2}{2} + \frac{A_2 - A'_2}{2}$$

Now $A_B^{m \times m} A_1 \cup A_2$

$$= \frac{A_1 + A'_1}{2} + \frac{A_1 - A'_1}{2} \cup \frac{A_2 + A'_2}{2} + \frac{A_2 - A'_2}{2}$$

$$= \frac{A_1 + A'_1}{2} + \frac{A_1 - A'_1}{2} \cup \frac{A_2 + A'_2}{2} + \frac{A_2 - A'_2}{2}$$

$$= \left[\left(\frac{A_1 + A'_1}{2}\right) \cup \left(\frac{A_2 + A'_2}{2}\right)\right] + \left[\left(\frac{A_1 - A'_1}{2}\right) \cup \left(\frac{A_2 - A'_2}{2}\right)\right]$$

is the sum of symmetric bimatrix and skew symmetric bimatrix.

Theorem 1.1.27

Let $A_B = A_1^{n \times n} \cup A_2^{m \times m}$ be a mixed square bimatrix. A_B can be written as a sum of mixed symmetric bimatrix and mixed skew symmetric bimatrix.

Proof:

Let $A_B = A_1^{n \times n} \cup A_2^{m \times m}$ be a mixed square bimatrix.

Now $A_1^{n \times n}$ is a square matrix so let

$$A_1^{n \times n} = A = \frac{A + A'}{2} + \frac{A - A'}{2}$$

where $\frac{A + A'}{2}$ and $\frac{A - A'}{2}$ are symmetric and skew symmetric $n \times n$ square matrices.

Let $A_2^{n \times n} = B$ clearly B is a square matrix; So B can be written as a sum of the symmetric and skew symmetric square matrices each of order m i.e.,

$$B=\frac{B+B'}{2}+\frac{B-B'}{2}$$

So now

$$A_B=\left(\frac{A+A'}{2}+\frac{A-A'}{2}\right)\cup\left(\frac{B+B'}{2}+\frac{B-B'}{2}\right)$$

$$=\left(\frac{A+A'}{2}\cup\frac{B+B'}{2}\right)+\left(\frac{A-A'}{2}\cup\frac{B-B'}{2}\right)$$

Thus A_B is a is a sum of a mixed square symmetric bimatrix and a mixed square skew symmetric bimatrix.

Definition 1.1.28

Let A_B be any bimatrix i.e., $A_B = A_1^{m \cdot n} \cup A_2^{p \cdot q}$. If we cross out all but k_1 rows and s_1 columns of the $m\times n$ matrix A_1 and cross out all but k_2 rows and s_2 columns of the $p\times q$ matrix A_2 the resulting $k_1\times s_1$ and $k_2\times s_2$ bimatrix is called a **subbimatrix** of A_B.

Example 1.1.29

$$\text{Let} \quad A_B = \begin{bmatrix} 3 & 4 & 7 & 1 \\ 7 & 0 & 1 & 2 \\ 1 & 2 & 3 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 & 3 & 4 & -1 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 1 & 7 & 0 & -1 & 0 \end{bmatrix}$$

$A_B = A_1 \cup A_2$

then a subbimatrix of A_B is given by

$$A_B = \begin{bmatrix} 3 & 4 & 7 \\ 1 & 2 & 3 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}$$

Definition 1.1.30

Let $A_B = A_1 \cup A_2$ be a square bimatrix. The **bideterminant of a square bimatrix** is an ordered pair (d_1, d_2) where $d_1 = |A_1|$ and $d_2 = |A_2|$. $|A_B| = (d_1, d_2)$ where d_1 and d_2 are reals may be positive or negative or even zero. ($|A|$ denotes determinant of A).

Example 1.1.31

$$\text{Let} \quad A_B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 4 & 5 \\ -2 & 0 \end{bmatrix}$$

The bideterminant of this bimatrix is the pair (0, 10) and denoted by $|A_B|$.

Definition 1.1.33

Given a square bimatrix $A_B = A_1 \cup A_2$, if there exists a square bimatrix $A_B^{-1} = A_1^{-1} \cup A_2^{-1}$ which satisfies the identity

$$A_B A_B^{-1} = A_B^{-1} A_B = A_1 A_1^{-1} \cup A_2 A_2^{-1} = I \cup I$$

then A_B^{-1} is called the **biinverse** or **bireciprocal** of A_B.

Example 1.1.34

$$\text{Let} \quad A_B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\text{then} \quad A_B^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \cup \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

Result 1.1.35

$$(A_B B_B)^{-1} = B_B^{-1} A_B^{-1}$$

$$(A_B^{-1})^{-1} = A_B$$

SECTION 1.2

NEUTROSOPHIC BIMATRICES

Definition 1.2.1

In the neutrosophic logic every logical variable x is described by an ordered triple x = (T, I, F) where T is the degree of truth, F is the degree of false and I the level of indeterminacy.

(A). To maintain consistency with the classical and fuzzy logics and with probability there is the special case where

$$T + I + F = 1.$$

(B). But to refer to intuitionistic logic, which means incomplete information on a variable proposition or event one has

$$T + I + F < 1.$$

(C). Analogically referring to Paraconsistent logic, which means contradictory sources of information about a same logical variable, proposition or event one has

$$T + I + F > 1.$$

Example 1.2.2

From a pool of refugees, waiting in a political refugee camp in Turkey to get the American visa, a% have the chance to be accepted – where a varies in the set A, r% to be rejected – where r varies in the set R, and p% to be in pending (not yet decided) – where p varies in P.

Say, for example, that the chance of someone Popescu in the pool to emigrate to USA is (between) 40-60% (considering different criteria of emigration one gets different percentages, we have to take care of all of them), the chance of being rejected is 20-25% or 30-35%, and the chance of being in 15 pending is 10% or 20% or 30%. Then the neutrosophic probability that Popescu emigrates to the Unites States is

NP (Popescu) = ((40-60) (20-25) $\cup$ (30-35), {10,20,30}), closer to the life.

Notion 1.2.3

Throughout this section by 'I' we denote the indeterminacy of any notion/ concept/ relation. That is when we are not in a position to associate a relation between any two concepts then we denote it as indeterminacy.

Further we assume all fields to be real fields of characteristic 0 all vector spaces are taken as real spaces over reals

Definition 1.2.4

Let K be the field of reals. We call the **field** generated by $K \cup I$ to be the **neutrosophic field** for it involves the indeterminacy factor in it. We define $I^2 = I$, $I + I = 2I$ i.e., $I + \ldots + I = nI$, and if $k \in K$ then $k.I = kI$, $0I = 0$. We denote the neutrosophic field by K(I) which is generated by $K \cup I$ that is $K(I) = \langle K \cup I \rangle$

Example 1.2.5

Let R be the field of reals. The neutrosophic field is generated by $\langle R \cup I \rangle$ i.e. R(I) clearly $R \cup \langle R \cup I \rangle$.

Example 1.2.6

Let Q be the field of rationals. The neutrosophic field is generated by Q and I i.e. $Q \cup I$ denoted by Q(I).

Definition 1.2.7

Let K(I) be a neutrosophic field we say K(I) is a **prime neutrosophic field** if K(I) has no proper subfield which is a neutrosophic field.

Example 1.2.8

Q(I) is a prime neutrosophic field where as R(I) is not a prime eutrosophic field for $Q(I) \cup R(I)$.

It is very important to note that all neutrosophic fields are of characteristic zero. Likewise we can define neutrosophic subfield.

Definition 1.2.9

Let K(I) be a neutrosophic field, $P \cup K(I)$ is a neutrosophic subfield of P if P itself is a neutrosophic field. K(I) will also be called as the **extension neutrosophic field of the neutrosophic field P.**

Definition 1.2.10

Let $M_{n \times m} = \{(a_{ij}) / a_{ij} \in K(I)\}$, where K (I),is a neutrosophic field.We call $M_{n \times m}$ to be the **neutrosophic matrix.**

Example 1.2.11

Let $Q(I) = \langle Q \cup I \rangle$ be the neutrosophic field.

$$M_{3 \times 2} = \begin{pmatrix} 0 & 1 \\ -2 & 4I \\ 1 & -I \end{pmatrix}$$

is the neutrosophic matrix, with entries from rationals and the indeterminacy I.

Note 1.2.12

We define product of two neutrosophic matrices whenever the product is defined as follows:

Let $A = \begin{pmatrix} -1 & 2 & -1 \\ 3 & 1 & 0 \end{pmatrix}_{2 \times 3}$

and $B = \begin{pmatrix} -I & 1 & 2 & 4 \\ 1 & I & 0 & 2 \\ 5 & -2 & 3I & -I \end{pmatrix}_{3 \times 4}$

$$AB = \begin{bmatrix} -6I+2 & -1+4I & -2-3I & I \\ -2I & 3+I & 6 & 12+2I \end{bmatrix}_{2 \times 4}$$

(we use the fact $I^2 = I$).

Definition 1.2.13

Let $A = A_1 \cup A_2$ where A_1 and A_2 are two distinct neutrosophic matrices with entries from a neutrosophic field. Then $A = A_1 \cup A_2$ is called the **neutrosophic bimatrix.**

Example 1.2.14

$$\text{Let } A_N = \begin{bmatrix} 0 & I \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & I \\ I & 0 \end{bmatrix}$$

A_N is the 2×2 neutrosophic bimatrix

Definition 1.2.15

If both A_1 and A_2 are neutrosophic matrices we call A a neutrosophic bimatrix. If only one of A_1 or A_2 is a neutrosophic matrix and other is not a neutrosophic matrix then we all $A = A_1 \cup A_2$ as the **semi neutrosophic bimatrix**. (It is clear all neutrosophic bimatrices are trivially semi neutrosophic bimatrices).

Definition 1.2.16

It both A_1 and A_2 are $m \times n$ neutrosophic matrices then we call $A = A_1 \cup A_2$ a $m \times n$ **neutrosophic bimatrix** or a **rectangular neutrosophic bimatrix**.

Example 1.2.17

$$\text{Let } A_N = \begin{bmatrix} 3 & I \\ 1 & 2 \\ I & 0 \end{bmatrix} \cup \begin{bmatrix} 3 & 3 \\ 1 & I \\ 4 & I \end{bmatrix}$$

A_N is the 3×2 rectangular neutrosophic bimatrix.

Definition 1.2.18

If $A = A_1 \cup A_2$ be such that A_1 and A_2 are both $n \times n$ neutrosophic matrices then we call $A = A_1 \cup A_2$ a **square** or a **$n \times n$ neutrosophic bimatrix**.

Example 1.2.19

$$\text{Let } A_N = \begin{bmatrix} 2 & 0 \\ 4 & I \end{bmatrix} \cup \begin{bmatrix} 3 & I \\ 0 & 0 \end{bmatrix}$$

A_N is a square neutrosophic bimatrix.

Definition 1.2.20

If in the neutrosophic bimatrix $A = A_1 \cup A_2$ both A_1 and A_2 are square matrices but of different order say A_1 is a $n \times n$ matrix and A_2 a $s \times s$ matrix then we call $A = A_1 \cup A_2$ a **mixed neutrosophic square bimatrix**. (Similarly one can define **mixed square semi neutrosophic bimatrix**).

Example 1.2.21

$$\text{Let } A_N = \begin{bmatrix} 2 & 0 \\ 4 & I \end{bmatrix} \cup \begin{bmatrix} 3 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix}$$

A_N is a mixed neutrosophic square bimatrix.

Example 1.2.22

$$\text{Let } A_N = \begin{bmatrix} 3 & 1 \\ 0 & I \end{bmatrix} \cup \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

A_N is a mixed square semi neutrosophic bimatrix.

Definition 1.2.23

Likewise in $A = A_1 \cup A_2$ if both A_1 and A_2 are rectangular matrices say A_1 is a m×n matrix and A_2 is a p×q matrix then we call $A = A_1 \cup A_2$ a **mixed neutrosophic rectangular bimatrix**. (If $A = A_1 \cup A_2$ is a semi neutrosophic bimatrix then we call A the **mixed rectangular semi neutrosophic bimatrix**).

Example 1.2.24

$$\text{Let } A_N = \begin{bmatrix} 3 & 1 & 1 \\ I & 0 & 2 \end{bmatrix} \cup \begin{bmatrix} I & 2 & 0 \\ 3 & 1 & 2 \\ 4 & I & 0 \\ 3 & 3 & 1 \end{bmatrix}$$

A_N is a mixed neutrosophic rectangular bimatrix.

Example 1.2.25

$$\text{Let } A_N = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \cup \begin{bmatrix} I & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

A_N is a mixed rectangular semi *neutrosophic bimatrix.*

Definition 1.2.26

A bimatrix (weak neutrosophic bimatrix) $M = M_1 \cup M_2$ is said to be a **square neutrosophic bimatrix (weak neutrosophic bimatrix)** if both M_1 and M_2 are square neutrosophic bimatrices of same order.

They are said to be **mixed square bimatrices** of M_1 and M_2 are square matrices of different order .A neutrosophic bimatrix $M = M_1 \cup M_2$ is said to be a $m \times n$ **rectangular neutrosophic bimatrix** if both M_1 and M_2 are $m \times n$ neutrosophic rectangular matrices.

A neutrosophic bimatrix $M = M_1 \cup M_2$ is said to be a **mixed rectangular neutrosophic bimatrix** if both M_1 and M_2 are rectangular neutrosophic matrices of different orders.

A neutrosophic birow vector

$M = (M_1^1, M_2^1, \ldots M_t^1) \cup (M_1^2, M_2^2, \ldots M_n^2)$, m_i^k, $k = 1,2$

are reals with atleast one m_p^1 and m_q^2 to be I.

A neutrosophic column bivector

$$C = \begin{bmatrix} a_1^1 \\ \vdots \\ a_r^1 \end{bmatrix} \cup \begin{bmatrix} a_1^2 \\ \vdots \\ a_s^2 \end{bmatrix}$$

With atleast one a_i^1 and a_j^2 to be I.

Example 1.2.27

Let B = (1 0 6 I -1 1) ∪ (-1 I 4 0 I 1 -1 4 -2) is a neutrosophic row bivector.

$$C = \begin{bmatrix} 0 \\ -1 \\ I \\ 2 \\ 3 \\ -4 \end{bmatrix} \cup \begin{bmatrix} I \\ 1 \\ -4 \\ 0 \\ 2 \\ -5 \end{bmatrix}$$

is a neutrosophic column bivector.

B' = (1 1 0 6 -1 I -9) ∪ (1 2 3 4 5 -3) is a weak neutrosophic row bivector.

$$C' = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \\ 0 \\ 5 \end{bmatrix} \cup \begin{bmatrix} 1 \\ 1 \\ 0 \\ 5 \\ 3 \end{bmatrix}$$

is a weak neutrosophic column bivector.

SECTION 1.3

FUZZY BIMATRICES

Definition 1.3.1

A **fuzzy matrix** is a matrix with elements having values in the closed unit interval [0,1].

Definition 1.3.2

Let $R = [r_{ij}]$ and $S = [s_{ij}]$ be $n \times n$ fuzzy matrices. Then

(i) $R \vee S = [r_{ij} \vee s_{ij}]$, where $X \vee Y = \max\{X,Y\}$

(ii) $R \wedge S = [r_{ij} \wedge s_{ij}]$, where $X \wedge Y = \min\{X,Y\}$

(iii) $R \Theta S = [r_{ij} \Theta s_{ij}]$, where $X \Theta Y = \begin{cases} X & \text{for } X>Y \\ O & \text{for } X<Y \end{cases}$

(iv) $R\times S = [r_{i1} \wedge s_{1j}] \vee [(r_{i2} \wedge s_{2j}) \vee \ldots \vee (r_{in} \wedge s_{nj})]$ $R\times S$ can be written as RS. If $R\times R = R$, then R is called idempotent.

(v) Denote $R^k = [r_{ij}^k]$, $k = 2,3,\ldots$. Then

$$r_{ij}^k = \bigvee_{j_1=1}^{n} \bigvee_{j_2=1}^{n} \cdots \bigvee_{j_{k-1}=1}^{n} (r_{ij_1} \wedge r_{j_1j_2} \wedge \ldots \wedge r_{j_{k-1}j})$$

(vi) $R^{k+1} = R^k \times R$, $k = 1,2,3,\ldots$.

(vii) $R^o = I$, where I is the unit $n\times n$ matrix (all diagonal entries 1 and all other 0)

(viii) R^T (or) $R' = [r_{ij}]$ (the transpose of R)

(ix) $\Delta R = R' \Theta R'$, $\nabla R = R \wedge R'$

(x) $R \le S$ iff $(r_{ij} \le s_{ij})$ for all $i,j \in \{1,2,\ldots,n\}$.

Definition 1.3.3

Let $A = A_1 \cup A_2$ where A_1 and A_2 are two distinct fuzzy matrices with entries from the interval [0,1]. Then $A = A_1 \cup A_2$ is called the **fuzzy bimatrix**.

Example 1.3.4

$$\text{Let } A_F = \begin{bmatrix} 0 & .1 & 0 \\ .1 & .2 & .1 \\ .3 & .2 & .1 \end{bmatrix} \cup \begin{bmatrix} .2 & .1 & .1 \\ .1 & 0 & .1 \\ .2 & .1 & .2 \end{bmatrix}$$

A_F is the 3×3 fuzzy bimatrix.

Definition 1.3.5

If both A_1 and A_2 are fuzzy matrices we call A a **fuzzy bimatrix**. If only one of A_1 or A_2 is a fuzzy matrix and other is not a fuzzy matrix then we call $A = A_1 \cup A_2$ as the **semi fuzzy bimatrix**. (It is clear all fuzzy matrices are trivially semi fuzzy matrices).

Definition 1.3.6

It both A_1 and A_2 are $m \times n$ fuzzy matrices then we call $A = A_1 \cup A_2$ a **m×n fuzzy bimatrix** or a rectangular **fuzzy bimatrix**.

Example 1.3.7

$$\text{Let } A_F = \begin{bmatrix} .3 & 1 \\ 1 & .2 \\ .5 & 0 \end{bmatrix} \cup \begin{bmatrix} .3 & .7 \\ 1 & 1 \\ .4 & 1 \end{bmatrix}$$

A_F is the 3×2 rectangular fuzzy bimatrix.

Definition 1.3.8

If $A = A_1 \cup A_2$ is such that A_1 and A_2 are both $n \times n$ fuzzy matrices then we call $A = A_1 \cup A_2$ a **square** or a **n×n fuzzy bimatrix**.

Example 1.3.9

$$\text{Let } A_F = \begin{bmatrix} .2 & 0 \\ .4 & .2 \end{bmatrix} \cup \begin{bmatrix} .3 & 1 \\ 1 & .2 \end{bmatrix}$$

A_F is square fuzzy bimatrix.

Definition 1.3.10

If in the fuzzy bimatrix $A = A_1 \cup A_2$ both A_1 and A_2 are square matrices but of different order say A_1 is a $n \times n$ matrix and A_2 a $s \times s$ matrix then we call $A = A_1 \cup A_2$ a **mixed fuzzy square bimatrix**. (Similarly one can define **mixed square semi fuzzy bimatrix**).

Example 1.3.11

$$\text{Let } A_F = \begin{bmatrix} .2 & 0 \\ .4 & .2 \end{bmatrix} \cup \begin{bmatrix} .3 & 1 & .2 \\ 1 & .2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A_F is mixed square fuzzy bimatrix.

Example 1.3.12

$$\text{Let } A_F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cup \begin{bmatrix} .3 & 1 & .2 \\ 1 & .2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A_F is mixed square semi fuzzy bimatrix.

Definition 1.3.13

Likewise in $A = A_1 \cup A_2$ if both A_1 and A_2 are rectangular matrices say A_1 is a $m \times n$ matrix and A_2 is a $p \times q$ matrix then we call $A = A_1 \cup A_2$ a **mixed fuzzy rectangular bimatrix**. (If $A = A_1 \cup A_2$ is a semi fuzzy bimatrix then we call A the **mixed rectangular semi fuzzy bimatrix**).

Example 1.3.14

$$\text{Let } A_F = \begin{bmatrix} .3 & 1 & .5 & 1 \\ .6 & 0 & .2 & .3 \end{bmatrix} \cup \begin{bmatrix} 1 & .2 & 0 \\ .3 & 1 & .2 \\ .4 & 1 & 0 \\ .3 & .3 & .2 \end{bmatrix}$$

A_F is a mixed fuzzy rectangular bimatrix.

Example 1.3.15

$$\text{Let } A_F = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 \\ .3 \\ .4 \\ .3 \end{bmatrix}$$

A_F is a mixed rectangular semi fuzzy bimatrix.

SECTION 1.4

INTERVAL BIMATRICES

Definition 1.4.1

Given matrices $B = (b_{ij})$ and $C = (c_{ij})$ of order n such that $b_{ij} \le c_{ij}$, $i, j = 1, 2, \ldots, n$. Then the **interval matrix** $A_I = [B, C]$ is defined by

$A_I = [B, C] = \{A = (a_{ij}) \mid b_{ij} \le a_{ij} \le c_{ij} \; ; \; i, j = 1, 2, \ldots, n\}$. (interval vectors and matrices are vectors and matrices whose elements are interval numbers. The superscript I being used to indicate such a vector or matrix)

Example 1.4.2

Let $A_I = [B,C]$ where

$$B = \begin{bmatrix} 2 & 7 \\ 0 & -5 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 12 & 20 \\ 15 & 0 \end{bmatrix}$$

All matrices $D = (d_{ij})$with

$2 \le d_{11} \le 12$,

$7 \le d_{12} \le 20$,

$0 \le d_{21} \le 15$ and

$-5 \le d_{22} \le 0$

are in the interval matrix $A_I = [B,C]$

Definition 1.4.3

Let $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ where $[A_1, B_1]$ and $[A_2, B_2]$ interval matrices defined on the same interval$[a, b]$ or on different intervals $[a_1, b_1]$ and $[a_2, b_2]$. We call $[A,B] = [A_1, B_1] \cup [A_2, B_2]$ to be the interval bimatrix defined on the same interval $[a, b]$ or on different intervals $[a_1, b_1] \cup [a_2, b_2]$; with no mathematical meaning attached to the symbol '$\cup$';we say [A, B] is the **interval bimatrix** defined on the bi-interval $[a_1, b_1] \cup [a_2, b_2]$ or on the bi-interval $[a, b] \cup [a, b]$.

Definition 1.4.4

If both the interval matrices are m×n matrices then we say $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ is a m×n **interval bimatrix**; if $m \neq n$ we say the interval bimatrix is a **rectangular interval bimatrix**. If m = n, we call $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ to be a **square interval bimatrix** defined on the bi-interval $[a, b] \cup [a, b]$ or $[a_1, b_1] \cup [a_2, b_2]$.

If $[A_1, B_1]$ is a m×n rectangular interval matrix and $[A_2, B_2]$ is a p×q rectangular interval matrix then we call $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ to be a **mixed rectangular interval bimatrix** defined on the bi-interval $[a_1, b_1] \cup [a_2, b_2]$.

If the interval bimatrix $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ is such that $[A_1, B_1]$ is a m×m square interval matrix and $[A_2, B_2]$ is a p×p $(p \neq m)$ square interval matrix then we call [A, B] a **mixed interval square bimatrix** defined on the bi-interval $[a_1, b1] \cup [a_2, b_2]$. If one of the interval matrix in a interval bimatrix is a square interval matrix and the other interval matrix $[A_2, B_2]$ is a rectangular interval matrix then we just call [A, B] the **mixed interval bimatrix** defined on the bi-interval $[a_1, b_1] \cup [a_2, b_2]$.

Example 1.4.5

$$\text{Let } [A,B] = \begin{bmatrix} 3 & 2 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

is a interval rectangular 3×4 bimatrix defined on the bi-interval $[a_1, b_1] \cup [a_2, b_2]$ where $[a_1, b_1] = [a_2, b_2] = [-R, R]$, R the reals.

SECTION 1.5

FUZZY INTERVAL BIMATRICES

Definition 1.5.1

Let [A, B] be an interval matrix with entries from [0, 1] or [a, b] with $0 \leq a < b \leq 1$. [A, B] is called the **fuzzy interval matrix** defined on the interval [a, b] or [0, 1]. If [A, B] contains only n×n fuzzy matrices then we call [A, B] the **fuzzy interval of fuzzy rectangular matrices**.

Example 1.5.2

Let [A, B] be a interval of 2×2 fuzzy square matrices where

$$A = \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0.8 \\ 0.9 & 1 \end{bmatrix}$$

[A, B] is the fuzzy interval 2×2 square matrix.

Example 1.5.3

Let [A, B] be an interval of 3×1 fuzzy rectangular matrix defined on the fuzzy interval [0, 0.6] where

$$A = \begin{bmatrix} 0 \\ 0.1 \\ 0.2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0.5 \\ 0.6 \\ 0.4 \end{bmatrix}$$

Definition 1.5.4

Let $[A_1, B_1]$ and $[A_2, B_2]$ be any two interval fuzzy matrices defined on the fuzzy intervals say $[a_1, b_1]$ and $[a_2,b_2]$ respectively. Let $[A, B] = [A_1, B_1]\cup[A_2,B_2]$ where '$\cup$' is just a symbol used for notational convenience, then [A, B] contains all fuzzy bimatrices of the form $M_1 \cup M_2$ where M_1 is the fuzzy matrix from the fuzzy interval matrix $[A_1, B_1]$ defined on the interval $[a_1, b_1]$ and M_2 is the fuzzy matrix from the fuzzy interval matrix $[A_2, B_2]$ defined on the interval $[a_2, b_2]$, $M = M_1 \cup M_2$ is a fuzzy bimatrix defined on the fuzzy bi-interval $[a_1, b_1] \cup [a_2, b_2]$ where M is called the **fuzzy bimatrix of the fuzzy interval bimatrix [A, B]**. ($[a_1, b_1]\cup[0, 1]$ and $[a_2, b_2]\cup[0, 1]$)

They are 5 types of fuzzy interval bimatrices, namely,

1. Fuzzy interval square bimatrices where both the fuzzy interval matrices will contain only m×m fuzzy square matrices.

Example 1.5.5

Let $[A, B] = [A_1, B_1] \cup[A_2, B_2]$ be a interval fuzzy bimatrices defined on the fuzzy bi-interval $[0.2, 1] \cup[0,0.7]$ where $[A_1, B_1]$ contains all 3×3 interval fuzzy matrices defined on the fuzzy intervals $[0.2, 1]\cup[0, 1]$ and $[A_2, B_2]$ contains all 3×3 interval fuzzy matrices defined on the fuzzy interval
$[0, 0.7] \cup[0, 1]$.

We call [A, B] the interval fuzzy square 3×3 bimatrix defined on the bi-interval $[0.2, 1]\cup[0, 0.7]$. Any element M in [A, B] will be of the form
$M = M_1 \cup M_2$ where

$$M_1 = \begin{bmatrix} 0.2 & 0.7 & 1 \\ 0.5 & 0.2 & 0.7 \\ 1 & 0.3 & 0.4 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0.7 & 0.2 \\ 0.6 & 0 & 0 \\ 0.5 & 0.6 & 0 \end{bmatrix} = M_2.$$

2. Fuzzy interval rectangular bimatrices where both the fuzzy interval matrices will contain p×q fuzzy rectangular matrices.

Example 1.5.6

Let [A, B] = $[A_1, B_1] \cup [A_2, B_2]$ be a interval fuzzy bimatrix defined on the fuzzy bi-interval $[0, 0.5] \cup [0.3,1]$. $[A_1, B_1]$ is the set of all m×n interval fuzzy matrices defined on the fuzzy interval [0, 0.5] and $[A_2, B_2]$ is the set of all m×n fuzzy interval matrices defined on the fuzzy [0.3, 1]. We call [A, B] the m×n rectangular fuzzy interval bimatrix defined on the bi-interval $[0, 0.5] \cup [0.3, 1]$. If we take m = 2 and n = 4, we would get the set of all 2 × 4 rectangular fuzzy interval bimatrix defined on the bi-interval $[0, 0.5] \cup [0.3, 1]$. Just any element $M = M_1 \cup M_2$ in [A, B] will be of the form

$$M = \begin{bmatrix} 0 & 0.4 & 0.3 & 0 \\ 0.2 & 0 & 0.1 & 0.5 \end{bmatrix} \cup \begin{bmatrix} 0.3 & 1 & 1 & 0.8 \\ 1 & 0.8 & 0.7 & 0.5 \end{bmatrix}$$

3. Fuzzy interval mixed square bimatrices where one of the fuzzy interval matrix will be a m×m fuzzy square matrix where as the other will be a t×t fuzzy square matrix t ≠ m.

Example 1.5.7

Let [A, B] = $[A_1, B_1] \cup [A_2, B_2]$ be any fuzzy interval bimatrix defined on the fuzzy bi-interval $[a_1, b_1] \cup [a_2,b_2]$. To be more specific let $[A_1, B_1]$ contain all fuzzy interval 2×2 square fuzzy matrices with entries from the fuzzy interval [0, 0.7] and $[A_2, B_2]$ contains all fuzzy interval 5×5 square fuzzy matrices with entries from the fuzzy interval [0.5, 1]. Thus any element in the fuzzy interval bimatrix [A, B] will be of the form $M = M_1 \cup M_2$ where

$$M_1 = \begin{bmatrix} 0 & 0.2 \\ 0.7 & 0.5 \end{bmatrix} \text{ and } \begin{bmatrix} 0.5 & 0.6 & 1 & 1 & 0.7 \\ 0.6 & 0.6 & 0.7 & 1 & 1 \\ 1 & 1 & 1 & 0.6 & 0.8 \\ 0.9 & 1 & 1 & 0.8 & 1 \\ 0.8 & 0.6 & 0.7 & 1 & 0.9 \end{bmatrix} = M_2.$$

4. Fuzzy interval mixed rectangular bimatrix will contain fuzzy interval m×n rectangular matrices and p×q fuzzy interval rectangular matrices p ≠ m and or q ≠ n.

Example 1.5.8

Let [A, B] = $[A_1, B_1] \cup [A_2, B_2]$ be a fuzzy interval bimatrix defined on the fuzzy bi-interval $[a_1, b_1] \cup [a_2, b_2]$, where $[A_1, B_1]$ is the fuzzy interval rectangular 3×4 matrix defined on the fuzzy interval [0, 0.6] and $[A_2, B_2]$ is the fuzzy interval rectangular 6×2 matrix defined on the fuzzy interval [0.1, 1].

We call the fuzzy interval bimatrix [A, B] to be the fuzzy interval mixed rectangular bimatrix defined on the bi-interval $[0, 0.6] \cup [0.1, 1]$. Any element M in [A, B] will be of the form

$$M = \begin{bmatrix} 0 & 0.1 & 0.2 & 0 \\ 0.1 & 0 & 0.4 & 0.52 \\ 0.6 & 0.3 & 0 & 0.51 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0.2 \\ 0.3 & 1 \\ 1 & 1 \\ 0.3 & 0.9 \\ 0.6 & 0.2 \\ 0.8 & 1 \end{bmatrix} = M_1 \cup M_2,$$

both the fuzzy interval matrices are rectangular fuzzy matrices.

5. The fuzzy interval mixed bimatrices will contain both fuzzy interval square matrices and fuzzy interval rectangular matrices.

Example 1.5.9

Let $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ be a fuzzy interval mixed bimatrix defined on the fuzzy bi-interval $[a_1, b_1] \cup [a_2, b_2]$. To be more specific where $[A_1, B_1]$ contains the collection of 4×4 fuzzy interval matrix with entries from [0.6,1] and $[A_2, B_2]$ contains all 1×5 fuzzy interval matrix with entries from [0, 0.7]. Any element M in the fuzzy interval bimatrix which we defined as the mixed fuzzy interval bimatrix will be of the form $M = M_1 \cup M_2$ where

$$M_1 \cup M_2 = \begin{bmatrix} 0.6 & 1 & 1 & 1 \\ 1 & 0.5 & 0.6 & 0.7 \\ 0.8 & 0.8 & 1 & 0.6 \\ 0.9 & 1 & 0.9 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 0.1 & 0.7 & 0 & 0.5 \end{bmatrix}.$$

SECTION 1.6

NEUTROSOPHIC INTERVAL BIMATRICES

Definition 1.6.1

Let A = (a_{ij}) and B = (b_{ij}) be two m×n neutrosophic matrices with entries from the neutrosophic rational field $\langle Q \cup I \rangle$. [A, B] will be called the **neutrosophic interval matrices**, where if C = (c_{ij}) is any other m×n neutrosophic matrix with (a_{ij}) ≤ (c_{ij}) ≤ (b_{ij}) then we say C = (c_{ij})∪ [A, B] i.e. the matrix C =(c_{ij}) is a **matrix of the neutrosophic interval matrix [A, B]**.

Example 1.6.2

Let [A, B] be a neutrosophic interval 3×2 matrices, where

$$A = \begin{bmatrix} 2+5I & 0 \\ 5 & 7-I \\ 8 & 8I-9 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 10+4 & 12 \\ 10 & 9I+20 \\ 18 & 10I+7 \end{bmatrix}.$$

Now

$$C = \begin{bmatrix} 9I+5 & 8 \\ 4+2I & 6-I \\ 12 & 3I+1 \end{bmatrix}$$

belongs to the neutrosophic interval of matrices [A, B] for we see (a_{ij}) ≤ (c_{ij}) ≤ (b_{ij}); suppose

$$D = \begin{bmatrix} 21I & 45 \\ 0 & 8I+25 \\ 70 & 24I+48 \end{bmatrix}$$

be any 3×2 neutrosophic matrix. Clearly D does not belong to the interval of neutrosophic matrices, [A, B].

Definition 1.6.3

We say a neutrosophic matrix A to be a **fuzzy neutrosophic matrix** if the entries of A are from [a + bI, c+ dI] where a, b, c, d $\cup$ [0, 1].

Example 1.6.4

$$\text{Let } A = \begin{bmatrix} 0.2+I & 0.7+1I \\ 1+0.I & 0.2+0.3I \\ 1+I & 0 \end{bmatrix}$$

Clearly A is a fuzzy neutrosophic matrix defined on the interval [0, 1+I].

Definition 1.6.5

[A, B] will be called as the **fuzzy neutrosophic interval matrices** if each of A and B are m×n fuzzy neutrosophic matrix and $(a_{ij}) \le (b_{ij})$ where $A = (a_{ij})$ and $B= (b_{ij})$, a_{ij}, $b_{ij} \cup$ [a + bI, c + dI] with a, b, c, d $\cup$ [0, 1], $1 \le i \le m$ and $1 \le j \le n$.

If m = n then we see the fuzzy neutrosophic interval matrix is a fuzzy neutrosophic interval square matrix. If m $\neq$ n, then we call [A, B] the **fuzzy neutrosophic interval rectangular matrix**.

We say a fuzzy neutrosophic matrix $C = (c_{ij}) \cup [A, B]$ where $A = (a_{ij})$ and $B = (b_{ij})$ if and only if $a_{ij} \le c_{ij} \le b_{ij}$, $1 \le i, j \le n$. If [A, B] is a fuzzy neutrosophic interval of square matrix if $A =(a_{ij})$ and $B = (b_{ij})$ are p×p square fuzzy neutrosophic matrices.

Example 1.6.6

Let [A, B] be a fuzzy neutrosophic 3×2 interval matrix where

$$A = \begin{bmatrix} 0.5+I & 0.2I \\ 0.3I & 0.4+0.3I \\ I & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0.9+I & 0.8I \\ I & 0.7+I \\ 1+I & 0 \end{bmatrix}.$$

The minimal element is 0 and maximal element is 1 + I.

Definition 1.6.7

Let $[A_1, B_1]$ and $[A_2, B_2]$ be two neutrosophic interval matrices. The neutrosophic interval bimatrix $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ is defined to be the collection of all neutrosophic bimatrices $M = M_1 \cup M_2$ where M_1 is the neutrosophic matrix from the neutrosophic interval matrix $[A_1, B_1]$ and M_2 is the neutrosophic matrix from the neutrosophic interval matrix $[A_2, B_2]$. If both the matrices M_1 and M_2 are $n \times n$ square matrices then we call $[A, B]$ to be the **neutrosophic interval square bimatrix**.

If both the neutrosophic interval matrices $[A_1,B_1]$ and $[A_2,B_2]$ are rectangular $n \times m$ neutrosophic interval matrices then we call $[A,B]$ the neutrosophic interval rectangular bimatrix. If one of $[A_1, B_1]$ is a neutrosophic interval square matrix and $[A_2,B_2]$ is a neutrosophic interval rectangular matrix, then $[A, B] = [A_1,B_1] \cup [A_2,B_2]$ will be defined as the **neutrosophic interval mixed bimatrix**.

If one of $[A_1, B_1]$ is a neutrosophic interval of $n \times n$ square matrix and $[A_2, B_2]$ is a neutrosophic interval of $p \times p$ square matrix $p \neq n$ then we call $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ to be a **neutrosophic interval of mixed square bimatrices**.

If one of $[A_1, B_1]$ is a neutrosophic interval of rectangular matrix say $m \times n$ $(m \neq n)$ and $[A_2, B_2]$ is a neutrosophic interval $p \times q$ $(p \neq q, p \neq m)$ rectangular matrix. Then we call $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ to be the **neutrosophic interval of mixed rectangular bimatrices**.

Example 1.6.8

Let $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ where $[A_1, B_1]$ is a 2×2 neutrosophic interval of square matrices and $[A_2, B_2]$ be the neutrosophic interval of 4×1 rectangular matrices. Thus $[A, B]$ is a neutrosophic interval of mixed bimatrices. Any element M in $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ will be of the form

$M = M_1 \cup M_2$ where

$$M_1 = \begin{bmatrix} 2I & 0 \\ 2+4I & 5-I \end{bmatrix}$$

and

$$M_2 = [20, 25 + 14I, 0, 5 + I].$$

Example 1.6.9

Let $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ where $[A_1, B_1]$ is the neutrosophic interval of 3×5 rectangular matrices and $[A_2, B_2]$ is the neutrosophic interval of 4×2 rectangular matrices. Then [A, B] is the neutrosophic interval of mixed rectangular bimatrices. Any element M in [A, B] will be a mixed rectangular neutrosophic bimatrix where $M = M_1 \cup M_2$, with

$$M_1 = \begin{bmatrix} 0 & 1+I & 7+I & 2I & 5 \\ 3I & 1+I & 8+5I & 4 & 3-5 \\ 2 & 0 & 7I & 3 & 4-I \end{bmatrix} \cup [A_1, B_1]$$

and

$$M_2 = \begin{bmatrix} 2I & 0 \\ 3+I & 4 \\ 5I+2 & 7I \\ 20 & 3I+5 \end{bmatrix} \cup [A_2, B_2].$$

We give yet another example of a neutrosophic interval of rectangular bimatrix.

Definition 1.6.10

Let $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ where $[A_1,B_1]$ and $[A_2, B_2]$ are fuzzy neutrosophic interval matrices, where $A_1 = (a^1_{ij})$, $B_1 = (b^1_{ij})$, $A_2 = (a^2_{ij})$ and $B_2 = (b^1_{ij})$, the minimal elements in $A_i = (a^i_{ij})$ will be the least element of the entries in the fuzzy neutrosophic interval matrices [Ai, Bi] and the maximal elements of $B_i = (b^i_{ij})$ will be the greatest element of fuzzy neutrosophic interval matrices $[A_i, B_i]$, $i = 1, 2$.

Thus [A, B] will contain elements $M = M_1 \cup M_2$ which are fuzzy neutrosophic bimatrices with $M_1 \cup [A_1, B_1]$ and $M_2 \cup [A_2,B_2]$. [A, B] is called the **fuzzy neutrosophic interval bimatrix**.

Example 1.6.11

Let [A, B] = [A_1, B_1] $\cup$ [A_2, B_2] be a fuzzy neutrosophic interval bimatrix where [A_1, B_1] is the fuzzy neutrosophic interval of 2×3 matrices and [A_2, B_2] is a fuzzy neutrosophic interval of 2×2 matrices

$$A_1 = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0 & 0.3I & 0.1 \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} I & 1 & 0.9 \\ 0.8 & 0.9I & 1 \end{bmatrix}$$

where 0 is the real minimum and 0.I is the neutrosophic minimum and 1 is the real maximum and I is the neutrosophic maximum.

Here

$$A_2 = \begin{bmatrix} 0 & 0.2I \\ 0.3I & 0.1 \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} 0.9I & 0.8I \\ 0.3I & 0.7 \end{bmatrix}.$$

Here 0 is the real minimum, 0.7 is the real maximum and 0.2I is the neutrosophic minimum and 0.9I is the neutrosophic maximum. [A, B] = [A_1, B_1] $\cup$ [A_2, B_2] is the fuzzy neutrosophic interval bimatrix we call this fuzzy neutrosophic interval bimatrix to be a **mixed fuzzy neutrosophic interval bimatrix** or **fuzzy neutrosophic interval mixed bimatrix.**