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PUBLICATIONS

- Muthamizhselvi S and V.M.Vijayalakshmi, **“Bipolar Fuzzy W-Hausdorff Space”,** Advances and Application in Mathematical Sciences, Vol 21, issue 2, December 2021, pages 819-830.
- Muthamizhselvi S and A. Kalaichelvi, **“Bipolar Fuzzy Gradation of Openness”,** Advances and Application in Mathematical Sciences, Vol 21, issue 4, February 2022, pages 1703-1716.
- Muthamizhselvi S and V.M.Vijayalakshmi, **“Bipolar Fuzzy S-Hausdorff Space”,** Indian Journal of Natural Sciences, Vol 12, issue 70, February 2022, pages 38416-38422.
- Muthamizhselvi. S and V. M. Vijayalakshmi, **“A Study on First and Second Order Bipolar Fuzzy Topological Spaces and Crisp Topological Spaces and Analyzing the Connections Between Them”,** International Journal of Neutrosophic Science, Vol.24, No.4, pg.133-150. (Scopus Indexed)
- Muthamizhselvi. S, V. M. Vijayalakshmi, M.Nila, Vidhyapriya. P**, “Second Order Bipolar Fuzzy Matrix and its Application in Medical Diagnosis”,** Communications on Applied Nonlinear Analysis, 31(**7**), 560 – 573. (Scopus Indexed)

Avinashilingam Institute for HomeScience and Higher Education for Women
(Deemed to be University Estd. u/s 3 of UGC Act 1956, Category 'A' by MHRD
Re-accredited with A++ Grade by NAAC. CGPA 3.65/4, Category I by UGC
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Appendix L2

(Item No 5 of Check List)

Details of Research Publications

S.No	Article	Journal	Other Details Vol/No/Page No/ Year	Published in UGC- CARE / Scopus Indexed/ Web of Science
1	A Study on First and Second Order Bipolar Fuzzy Topological Spaces and Crisp Topological Spaces and Analyzing the Connections Between Them	International Journal of Neutrosophic Science	Vol.24/No.4/ Page No.133-150/ Year - 2024	Scopus Indexed
2	Second Order Bipolar Fuzzy Matrix and its Application in Medical Diagnosis	Communications on Applied Nonlinear Analysis	Accepted	Scopus Indexed

*Proof of list of Journals from Internet to be attached along with copies of reprints.

Scholar :

Supervisor : V. M.

The scholar Miss. Muthamizhselvi, S (19PHMAF004) has published article/paper accepted in the following journals:
. International Journal of Neutrosophic Science - indexed in Scopus,
2. Communications on Applied Nonlinear Analysis - indexed in Scopus.
This may be considered. 21.08.24

Checked By: 23/08/24 (Head i/c

HoD/Dean of Respective School

A Study on First and Second Order Bipolar Fuzzy Topological Spaces and Crisp Topological Spaces and Analyzing the Connections Between Them

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Abstract

In our previous paper we discussed about the concept of SOBPFS, SOBPFT and its mathematical modelling in medical diagnosis. In this paper, the detailed study about SOBPFT accordance with FOBPFT and crisp topological spaces are analysed and also some natural examples of SOBPFT are provided. In third section, the connections between FOBPFT and SOBPFT under five different cases are discussed. And last section tells that, from a crisp topology τ on X there exists three different SOBPFT denoted by $\widehat{\omega(\tau)}$, $\widehat{\omega_*(\tau)}$ and $\widehat{\omega_\varepsilon(\tau)}$ and from a SOBPFT on X there exists three crisp topologies denoted by $i(\hat{\tau}_{\mathfrak{B}})$, $i^*(\hat{\tau}_{\mathfrak{B}})$ and $i_\varepsilon(\hat{\tau}_{\mathfrak{B}})$.

Keywords: Fuzzy set (FS); fuzzy topology (FT); First order bipolar-fuzzy set (FOBPFS); first order bipolar-fuzzy topological spaces (FOBPFT); second order bipolar fuzzy set (SOBPFS); second order bipolar fuzzy topological spaces (SOBPFT).

1. Introduction

Fuzzy set theory was introduced by Zadeh [16] in 1965 which revolutionalized the field of mathematics and artificial intelligence by providing a framework to handle uncertainty and vagueness in decision-making processes. Whereas crisp topology deals with crisp distinctions between elements. (i.e..,) being either fully in or fully out of a set . The important contributions to the study of fuzzy set theory were made by Zimmerman [19] (1996), Yager [15] (1980), Chang [2] (1968), Lowen [9] (1976), Gougen [3] (1967), Gottwald [4] (1993), Hohle [5] (1978), Kaufmann [7] (1975) and many others. In 1968, based on fuzzy sets Chang introduced the concept of fuzzy topological spaces and defined some basic concepts such as open fuzzy sets, closed fuzzy sets, neighbourhood of a fuzzy set, interior of a fuzzy set, fuzzy continuity and fuzzy compactness. In order to study deeper into the structure of fuzzy topological spaces, in 1976 Lowen modified the concept of fuzzy topological spaces of Chang. Also, the author introduced two functors $\tilde{\omega}$ and $\tilde{\iota}$ to establish the connection between fuzzy topological spaces and topological spaces. In 1994, Zhang [18] introduced the notion of bipolar fuzziness. In 2019, Kim [8] et al., introduced the concept of bipolar fuzzy topology and defined some basic concepts such as bipolar fuzzy point, bipolar fuzzy base, bipolar fuzzy subbase, bipolar fuzzy subspace, bipolar fuzzy quotient space, bipolar fuzzy neighbourhood, bipolar fuzzy initial topology, bipolar fuzzy continuity and bipolar fuzzy compactness and obtained some basic properties of each concepts. In 1975, Zadeh [17] introduced the concept of fuzzy set of type 2 (second order fuzzy set) as an extension of a fuzzy set. A detailed study of second order fuzzy sets was done by Mizumuto and Tanaka [10,11] (1976, 1981). Norwich and Turksen [13,14] (1981,1984)

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have used the concept of second order fuzzy sets in their stochastic fuzzy model. In 2007, Kalaichelvi [6] introduced the concept of second order fuzzy topological spaces using fuzzy sets of type 2 defined by Zadeh (1975) and studied some second order fuzzy structures. In 2024, Muthamizhselvi and Vijayalakshmi [12] introduced the new concept of second order bipolar fuzzy topology and its application in medical diagnosis.

2. Preliminary definitions

Definition : 2.1

Let **X** be an arbitrary nonempty set. Let $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$. A **FS** in **X** is a map from **X** into I**.**

Definition : 2.2

A subset $\tau \subset I^X$ is said to **FT** X iff

(i) The **0** & **1** belong to τ
(ii) $f_\lambda \in \tau \ \forall \ \lambda \in \Lambda$ implies $\bigvee_{\lambda \in \Lambda} f_\lambda \in \tau$
(iii) $f, g \in \tau$ implies $f \wedge g \in \tau$

Then (X, τ) is called **FT space**.

Definition : 2.3

A pair $A_{bp} = (A_{bp}{}^{+}, A_{bp}{}^{-})$ is called **FOBPFS** in X, where $A_{bp}{}^{+}: X \to [0,1]$ and $A_{bp}{}^{-}: X \to [-1,0]$.

Definition : 2.4

A pair $\hat{A}_{bp} = \left(\hat{A}_{bp}{}^{+}, \hat{A}_{bp}{}^{-}\right)$ is called a **SOBPFS** in X where $\hat{A}_{bp}{}^{+}: X \to [0,1]^{[0,1]}$ such that $\hat{A}_{bp}{}^{+}(x)(\alpha) \in [0,1]$ and $\hat{A}_{bp}{}^{-}: X \to [-1,0]^{[0,1]}$ such that $\hat{A}_{bp}{}^{-}(x)(\alpha) \in [-1,0]$, where $\alpha \in I$ & $x \in X$.

Definition : 2.5

A collection $\hat{\tau}_{\mathfrak{B}}$ SOBPFS on X defines a **SOBPFT** on X iff

(i) $\hat{0}_{bp}, \hat{1}_{bp} \in \hat{\tau}_{\mathfrak{B}}$
(ii) $\left(\hat{A}_{bp}\right)_\lambda \in \hat{\tau}_{\mathfrak{B}}$, for each $\lambda \in \Lambda$ implies $\bigcup_{\lambda \in \Lambda} \left(\hat{A}_{bp}\right)_\lambda \in \hat{\tau}_{\mathfrak{B}}$.
(iii) $\left(\hat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$, for each $i = 1$ to m implies that $\bigcap_{i=1}^{n} \left(\hat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$.

The pair $(X, \hat{\tau}_{\mathfrak{B}})$ is called a SOBPFT.

3. Connections between first order bipolar fuzzy and second order bipolar fuzzy topological spaces

Theorem : 3.1

Every FOBPFT $\tau_{\mathfrak{B}} = \{\left(A_{bp}\right)_\lambda / \lambda \in \Lambda\}$ on X defines a SOBPFT (Lowen) $\hat{\tau}_{\mathfrak{B}} = \{\left(\hat{A}_{bp}\right)_\lambda / \left(A_{bp}\right)_\lambda \in \tau_{\mathfrak{B}}\}$ on X, where $\left(\hat{A}_{bp}{}^{+}\right)_\lambda (x)(\alpha) = \left(A_{bp}{}^{+}\right)_\lambda (x)$ and $\left(\hat{A}_{bp}{}^{-}\right)_\lambda (x)(\alpha) = \left(A_{bp}{}^{-}\right)_\lambda (x)$ for every $x \in X$ and for every $\alpha \in I$. The correspondence $\tau_{\mathfrak{B}} \to \hat{\tau}_{\mathfrak{B}}$ is denoted as $\mathbb{C}_1$.

Proof:

To prove $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT (Lowen) on X. By the definition of $\left(\hat{A}_{bp}\right)_\lambda$, the correspondence $\left(A_{bp}\right)_\lambda \to \left(\hat{A}_{bp}\right)_\lambda$ is one-one

(i) Since $0_{bp}, 1_{bp}, \alpha_{bp} \in \tau_{\mathfrak{B}}$, $\hat{0}_{bp}, \hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

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(ii) To prove :-$\hat{\tau}_{\mathfrak{B}}$ is closed with respect to arbitrary union
Given $(\hat{\mathrm{A}}_{\mathrm{bp}})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda_0 \subseteq \Lambda$.
To prove: $\left(\bigcup_{\lambda\in\Lambda_0}(\hat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right) \in \hat{\tau}_{\mathfrak{B}}$
$(\hat{\mathrm{A}}_{\mathrm{bp}})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$ for $\lambda \in \Lambda_0 \subseteq \Lambda$ implies $(\mathrm{A}_{\mathrm{bp}})_{\lambda} \in \tau_{\mathfrak{B}}$, for $\lambda \in \Lambda_0 \subseteq \Lambda$

implies$\bigcup_{\lambda\in\Lambda_0}(\mathrm{A}_{\mathrm{bp}})_{\lambda} \in \tau_{\mathfrak{B}}$

implies $\bigcup_{\lambda\in\Lambda_0}\widehat{(\mathrm{A}_{\mathrm{bp}})}_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$

implies$\left(\bigvee_{\lambda\in\Lambda_0}\widehat{(\mathrm{A}_{\mathrm{bp}}{}^{+})}_{\lambda}, \bigwedge_{\lambda\in\Lambda_0}\widehat{(\mathrm{A}_{\mathrm{bp}}{}^{-})}_{\lambda}\right) \in \hat{\tau}_{\mathfrak{B}}$

Consider

For every $\mathrm{x} \in \mathrm{X}$ and for every $\alpha \in \mathrm{I}$

$$\begin{aligned}\left(\bigvee_{\lambda\in\Lambda_0}\widehat{(\mathrm{A}_{\mathrm{bp}}{}^{+})}_{\lambda}\right)(\mathrm{x})(\alpha) &= \left(\bigvee_{\lambda\in\Lambda_0}(\mathrm{A}_{\mathrm{bp}}{}^{+})_{\lambda}\right)(\mathrm{x})\\ &= \bigvee_{\lambda\in\Lambda_0}\left((\mathrm{A}_{\mathrm{bp}}{}^{+})_{\lambda}(\mathrm{x})\right)\\ &= \bigvee_{\lambda\in\Lambda_0}\left((\hat{\mathrm{A}}_{\mathrm{bp}}{}^{+})_{\lambda}(\mathrm{x})(\alpha)\right)\\ &= \left(\bigvee_{\lambda\in\Lambda_0}(\hat{\mathrm{A}}_{\mathrm{bp}}{}^{+})_{\lambda}\right)(\mathrm{x})(\alpha)\end{aligned}$$

Therefore$\left(\bigvee_{\lambda\in\Lambda_0}\widehat{(\mathrm{A}_{\mathrm{bp}}{}^{+})}_{\lambda}\right) = \bigvee_{\lambda\in\Lambda_0}(\hat{\mathrm{A}}_{\mathrm{bp}}{}^{+})_{\lambda}$ and

$$\begin{aligned}\left(\bigwedge_{\lambda\in\Lambda_0}\widehat{(\mathrm{A}_{\mathrm{bp}}{}^{-})}_{\lambda}\right)(\mathrm{x})(\alpha) &= \left(\bigwedge_{\lambda\in\Lambda_0}(\mathrm{A}_{\mathrm{bp}}{}^{-})_{\lambda}\right)(\mathrm{x})\\ &= \bigwedge_{\lambda\in\Lambda_0}\left((\mathrm{A}_{\mathrm{bp}}{}^{-})_{\lambda}(\mathrm{x})\right)\\ &= \bigwedge_{\lambda\in\Lambda_0}\left((\hat{\mathrm{A}}_{\mathrm{bp}}{}^{-})_{\lambda}(\mathrm{x})(\alpha)\right)\\ &= \left(\bigwedge_{\lambda\in\Lambda_0}(\hat{\mathrm{A}}_{\mathrm{bp}}{}^{-})_{\lambda}\right)(\mathrm{x})(\alpha)\end{aligned}$$

Therefore $\left(\bigwedge_{\lambda\in\Lambda_0}\widehat{(\mathrm{A}_{\mathrm{bp}}{}^{-})}_{\lambda}\right) = \left(\bigwedge_{\lambda\in\Lambda_0}(\hat{\mathrm{A}}_{\mathrm{bp}}{}^{-})_{\lambda}\right)$

implies $\bigcup_{\lambda\in\Lambda_0}\widehat{(\mathrm{A}_{\mathrm{bp}})}_{\lambda} = \left(\bigcup_{\lambda\in\Lambda_0}(\hat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right)$

Therefore$\left(\bigcup_{\lambda\in\Lambda_0}(\hat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right) \in \hat{\tau}_{\mathfrak{B}}$

(i) To prove :- $\hat{\tau}_{\mathfrak{B}}$ is closed with respect to finite intersection

Given $(\hat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{i}} \in \hat{\tau}_{\mathfrak{B}}$, for $\mathrm{i} = 1,2,\ldots \mathrm{m}$

To prove: $\left(\bigcap_{i=1}^{m}(\hat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{i}}\right) \in \hat{\tau}_{\mathfrak{B}}$

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$\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$, for $i = 1,2, \dots m$ implies $\left(A_{bp}\right)_i \in \tau_{\mathfrak{B}}$,for $i = 1,2, \dots m$
implies $\cap_{i=1}^{m}\left(A_{bp}\right)_i \in \tau_{\mathfrak{B}}$

implies $\widehat{\cap_{i=1}^{m}\left(A_{bp}\right)_i} \in \hat{\tau}_{\mathfrak{B}}$

implies $\left(\widehat{\wedge_{i=1}^{m}\left({A_{bp}}^{+}\right)_i}, \widehat{\vee_{i=1}^{m}\left({A_{bp}}^{-}\right)_i}\right) \in \hat{\tau}_{\mathfrak{B}}$
Consider,
For every $x \in X$ and for every $\alpha \in I$

$$\left(\widehat{\wedge_{i=1}^{m}\left({A_{bp}}^{+}\right)_i}\right)(x)(\alpha) = \left(\wedge_{i=1}^{m}\left({A_{bp}}^{+}\right)_i\right)(x)$$

$$= \wedge_{i=1}^{m}\left(\left({A_{bp}}^{+}\right)_i(x)\right)$$

$$= \wedge_{i=1}^{m}\left(\left({\widehat{A}_{bp}}^{+}\right)_i(x)(\alpha)\right)$$

$$= \left(\wedge_{i=1}^{m}\left({\widehat{A}_{bp}}^{+}\right)_i\right)(x)(\alpha)$$

Therefore $\left(\widehat{\wedge_{i=1}^{m}\left({A_{bp}}^{+}\right)_i}\right) = \wedge_{i=1}^{m}\left({\widehat{A}_{bp}}^{+}\right)_i$ and

$$\left(\widehat{\vee_{i=1}^{m}\left({A_{bp}}^{-}\right)_i}\right)(x)(\alpha) = \left(\vee_{i=1}^{m}\left({A_{bp}}^{-}\right)_i\right)(x)$$

$$= \vee_{i=1}^{m}\left(\left({A_{bp}}^{-}\right)_i(x)\right)$$

$$= \vee_{i=1}^{m}\left(\left({\widehat{A}_{bp}}^{-}\right)_i(x)(\alpha)\right)$$

$$= \left(\vee_{i=1}^{m}\left({\widehat{A}_{bp}}^{-}\right)_i\right)(x)(\alpha)$$

Therefore $\left(\widehat{\vee_{i=1}^{m}\left({A_{bp}}^{-}\right)_i}\right) = \left(\vee_{i=1}^{m}\left({\widehat{A}_{bp}}^{-}\right)_i\right)$
implies $\widehat{\cap_{i=1}^{m}\left(A_{bp}\right)_i} = \cap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$
Therefore $\cap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$
Therefore $\hat{\tau}_{\mathfrak{B}}$ defines a SOBPFT (lowen) on X.

Theorem : 3.2

Let $\hat{\tau}_{\mathfrak{B}} = \{\left(\widehat{A}_{bp}\right)_\lambda / \lambda \in \Lambda\}$ be a SOBPFT on X. Fix $x \in X$. Then the collection $(\hat{\tau}_{\mathfrak{B}})_x$= distinct elements of the collection $\{\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_{(x)} / \left(\widehat{A}_{bp}\right)_\lambda \in \hat{\tau}_{\mathfrak{B}}\}$ defines a FOBPFT on I(I is the closed unit interval [0,1]), where $\left(\left({\widehat{A}_{bp}}^{+}\right)_\lambda\right)_{(x)} = \left({\widehat{A}_{bp}}^{+}\right)_\lambda(x)$, $\left(\left({\widehat{A}_{bp}}^{-}\right)_\lambda\right)_{(x)} = \left({\widehat{A}_{bp}}^{-}\right)_\lambda(x)$. The correspondence $\hat{\tau}_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_x$ is denoted as $\mathbb{C}_2$.

Proof:

To prove:$(\hat{\tau}_{\mathfrak{B}})_x$ is a FOBPFT on I. By the definition $(\hat{\tau}_{\mathfrak{B}})_x$, there exists $\Lambda_0 \subseteq \Lambda$ such that for $\lambda, \mu \in \Lambda_0$, $\lambda \neq \mu$.

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$\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{(x)} \neq \left(\left(\widehat{A}_{bp}\right)_{\mu}\right)_{(x)}$ and $(\hat{\tau}_{\mathfrak{B}})_x$ can be written as $(\hat{\tau}_{\mathfrak{B}})_x = \{\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{(x)} / \lambda \in \Lambda_0\}$.

(i) To prove :- $\left(\hat{0}_{bp}\right)_{(x)}, \left(\hat{1}_{bp}\right)_{(x)}, \left(\hat{\alpha}_{bp}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

Given $\hat{0}_{bp} = (\hat{0}_{bp}{}^{+}, \hat{0}_{bp}{}^{-}) \in \hat{\tau}_{\mathfrak{B}}$,

Let $\left(\hat{0}_{bp}\right)_{(x)} = \left(\left(\hat{0}_{bp}{}^{+}\right)_{(x)}, \left(\hat{0}_{bp}{}^{-}\right)_{(x)}\right)$

$\hat{0}_{bp}{}^{+}(x) = \left(\hat{0}_{bp}{}^{+}\right)_{(x)} = \mathbf{0}, \hat{0}_{bp}{}^{-}(x) = \left(\hat{0}_{bp}{}^{-}\right)_{(x)} = \mathbf{0}$

Therefore $\left(\hat{0}_{bp}\right)_{(x)} = 0_{bp} \in (\hat{\tau}_{\mathfrak{B}})_x$

Similarly $\hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\hat{1}_{bp}\right)_{(x)}, \left(\hat{\alpha}_{bp}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

(ii) To prove : $(\hat{\tau}_{\mathfrak{B}})_x$ is closed with respect to arbitrary union

Given $\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$, for $\lambda \in \Lambda_1 \subseteq \Lambda_0$

To prove: $\bigcup_{\lambda\in\Lambda_1} \left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

$\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{(x)} \in \hat{\tau}_{\mathfrak{B}_x}$, for $\lambda \in \Lambda_1 \subseteq \Lambda_0$ implies $\left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda_1 \subseteq \Lambda_0$

implies $\bigcup_{\lambda\in\Lambda_1}\left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, where $\bigcup_{\lambda\in\Lambda_1}\left(\widehat{A}_{bp}\right)_{\lambda} = \left(\bigvee_{\lambda\in\Lambda_1}\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}, \bigwedge_{\lambda\in\Lambda_1}\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)$

Let $\widehat{B}_{bp} = \bigcup_{\lambda\in\Lambda_1}\left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$,

where $\widehat{B}_{bp} = \left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right) = \left(\bigvee_{\lambda\in\Lambda_1}\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}, \bigwedge_{\lambda\in\Lambda_1}\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)$

$\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\widehat{B}_{bp}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

Consider ,

$$\begin{aligned}
\left(\widehat{B}_{bp}{}^{+}\right)_{(x)} &= \widehat{B}_{bp}{}^{+}(x) \\
&= \left(\bigvee_{\lambda\in\Lambda_1}\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x) \\
&= \bigvee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)\right) \\
&= \bigvee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{(x)}
\end{aligned}$$

implies $\left(\widehat{B}_{bp}{}^{+}\right)_{(x)} = \bigvee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{(x)}$

$\left(\widehat{B}_{bp}{}^{-}\right)_{(x)} = \widehat{B}_{bp}{}^{-}(x)$

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$$= \left(\wedge_{\lambda\in\Lambda_1}\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)$$
$$= \wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)\right)$$
$$= \wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{(x)}$$

implies $\left(\widehat{B}_{bp}{}^{-}\right)_{(x)} = \wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{(x)}$

Therefore $\left(\widehat{B}_{bp}\right)_{(x)} = \left(\vee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{(x)}, \wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{(x)}\right) \in (\hat{\tau}_{\mathfrak{B}})_x$

implies $\cup_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

(iii) To prove :$(\hat{\tau}_{\mathfrak{B}})_x$ is closed with respect to finite intersection.

Given $\left(\left(\widehat{A}_{bp}\right)_i\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$, for $i = 1,2, \dots m$

To prove: $\cap_{i=1}^{m}\left(\left(\widehat{A}_{bp}\right)_i\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

$\left(\left(\widehat{A}_{bp}\right)_i\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$, for $i = 1$ to m implies$\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$, for $i = 1,2, \dots m$

implies $\cap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$

Let $\widehat{B}_{bp} = \cap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$, where

$\widehat{B}_{bp} = \left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right) = \left(\wedge_{i=1}^{m}\left(\widehat{A}_{bp}{}^{+}\right)_i, \vee_{i=1}^{m}\left(\widehat{A}_{bp}{}^{-}\right)_i\right)$

$\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\widehat{B}_{bp}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

For $\alpha \in I$

Consider

$\left(\widehat{B}_{bp}{}^{+}\right)_{(x)} = \widehat{B}_{bp}{}^{+}(x)$

$$= \left(\wedge_{i=1}^{m}\left(\widehat{A}_{bp}{}^{+}\right)_i\right)(x)$$
$$= \wedge_{i=1}^{m}\left(\left(\widehat{A}_{bp}{}^{+}\right)_i(x)\right)$$
$$= \wedge_{i=1}^{m}\left(\left(\widehat{A}_{bp}{}^{+}\right)_i\right)_{(x)}$$

implies $\left(\widehat{B}_{bp}{}^{+}\right)_{(x)} = \wedge_{i=1}^{m}\left(\left(\widehat{A}_{bp}{}^{+}\right)_i\right)_{(x)}$ and

$\left(\widehat{B}_{bp}{}^{-}\right)_{(x)} = \widehat{B}_{bp}{}^{-}(x)$

$$= \left(\vee_{i=1}^{m}\left(\widehat{A}_{bp}{}^{-}\right)_i\right)(x)$$

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$$= \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}^{-}\right)_i(x)\right)$$
$$= \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}^{-}\right)_i\right)_{(x)}$$

implies $\left(\hat{B}_{bp}^{-}\right)_{(x)} = \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}^{-}\right)_i\right)_{(x)}$

$$\left(\hat{B}_{bp}\right)_{(x)} = \left(\left(\hat{B}_{bp}^{+}\right)_{(x)}, \left(\hat{B}_{bp}^{-}\right)_{(x)}\right) = \left(\wedge_{i=1}^{m}\left(\left(\hat{A}_{bp}^{+}\right)_i\right)_{(x)}, \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}^{-}\right)_i\right)_{(x)}\right) \in (\hat{\tau}_{\mathfrak{B}})_x$$

implies $\cap_{i=1}^{m}\left(\left(\hat{A}_{bp}\right)_i\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

Hence $(\hat{\tau}_{\mathfrak{B}})_x$ is a FOBPFT on I.

Theorem : 3.3

Let $\hat{\tau}_{\mathfrak{B}} = \{(\hat{A}_{bp})_\lambda / \lambda \in \Lambda\}$ be a SOBPFT (Lowen) on X. Fix $\alpha \in I$, then the collection $(\hat{\tau}_{\mathfrak{B}})_\alpha$ = distinct elements of the collection $\{\left((A_{bp})_\lambda\right)_\alpha / (\hat{A}_{bp})_\lambda \in \hat{\tau}_{\mathfrak{B}}\}$ defines a FOBPFT on X where $\left(\left(\hat{A}_{bp}^{+}\right)_\lambda\right)_\alpha : X \to [0,1]$ such that $\left(\left(\hat{A}_{bp}^{+}\right)_\lambda\right)_\alpha(x) = \left(\hat{A}_{bp}^{+}\right)_\lambda(x)(\alpha)$ for every $x \in X$ and $\left(\left(\hat{A}_{bp}^{-}\right)_\lambda\right)_\alpha : X \to [-1,0]$ such that $\left(\left(\hat{A}_{bp}^{-}\right)_\lambda\right)_\alpha(x) = \left(\hat{A}_{bp}^{-}\right)_\lambda(x)(\alpha)$ for every $x \in X$. The correspondence $\hat{\tau}_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_\alpha$ is denoted as $\mathbb{C}_3$.

Proof:-

To prove $(\hat{\tau}_{\mathfrak{B}})_\alpha$ is a FOBPFT on X.

By the definition of $(\hat{\tau}_{\mathfrak{B}})_\alpha$, there exists $\Lambda_0 \subseteq \Lambda$ such that for $\lambda, \mu \in \Lambda_0$, $\lambda \neq \mu$.

$\left((\hat{A}_{bp})_\lambda\right)_\alpha \neq \left((\hat{A}_{bp})_\mu\right)_\alpha$ and $(\hat{\tau}_{\mathfrak{B}})_\alpha$ can be written as $(\hat{\tau}_{\mathfrak{B}})_\alpha = \left\{\left((\hat{A}_{bp})_\lambda\right)_\alpha / \lambda \in \Lambda_0\right\}$.

(i)To prove : $(\hat{0}_{bp})_\alpha, (\hat{1}_{bp})_\alpha, (\hat{\alpha}_{bp})_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$
Let $(\hat{0}_{bp})_\alpha = \left(\left(\hat{0}_{bp}^{+}\right)_\alpha, \left(\hat{0}_{bp}^{-}\right)_\alpha\right)$

Given $\hat{0}_{bp} = \left(\hat{0}_{bp}^{+}, \hat{0}_{bp}^{-}\right) \in \hat{\tau}_{\mathfrak{B}}$. Then

$$\left(\hat{0}_{bp}^{+}\right)_\alpha(x) = \left(\hat{0}_{bp}^{+}\right)(x)(\alpha) = 0 = 0_{bp}^{+}(x) \Rightarrow \left(\hat{0}_{bp}^{+}\right)_\alpha = 0_{bp}^{+}$$

$$\left(\hat{0}_{bp}^{-}\right)_\alpha(x) = \left(\hat{0}_{bp}^{-}\right)(x)(\alpha) = 0 = 0_{bp}^{-}(x) \Rightarrow \left(\hat{0}_{bp}^{-}\right)_\alpha = 0_{bp}^{-}$$

implies $\left(\hat{0}_{bp}\right)_\alpha = 0_{bp} \in (\hat{\tau}_{\mathfrak{B}})_\alpha$

Similarly $\hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\hat{1}_{bp}\right)_\alpha, \left(\hat{\alpha}_{bp}\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$

(ii)To prove:$(\hat{\tau}_{\mathfrak{B}})_\alpha$ is closed with respect to arbitrary union

Given $\left((\hat{A}_{bp})_\lambda\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$ for $\lambda \in \Lambda_0 \subseteq \Lambda$ implies $\left(\hat{A}_{bp}\right)_\lambda \in \hat{\tau}_{\mathfrak{B}}$ for $\lambda \in \Lambda_0 \subseteq \Lambda$

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implies $\cup_{\lambda \in \Lambda_0}\left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$.

Let $\widehat{B}_{bp} = \cup_{\lambda \in \Lambda_0}\left(\widehat{A}_{bp}\right)_{\lambda}$,

where $\widehat{B}_{bp} = \left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right) = \left(\vee_{\lambda \in \Lambda_0}\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}, \wedge_{\lambda \in \Lambda_0}\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right) \in \hat{\tau}_{\mathfrak{B}}$

$\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\widehat{B}_{bp}\right)_{\alpha} \in (\hat{\tau}_{\mathfrak{B}})_{\alpha}$

For $x \in X$,

Consider,

$$\left(\widehat{B}_{bp}{}^{+}\right)_{\alpha}(x) = \left(\widehat{B}_{bp}{}^{+}\right)(x)(\alpha)$$

$$= \left(\vee_{\lambda \in \Lambda_0}\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)(\alpha)$$
$$= \vee_{\lambda \in \Lambda_0}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)\right)$$
$$= \vee_{\lambda \in \Lambda_0}\left(\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{\alpha}(x)\right)$$

$$= \left(\vee_{\lambda \in \Lambda_0}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{\alpha}\right)(x)$$

Therefore $\left(\widehat{B}_{bp}{}^{+}\right)_{\alpha} = \vee_{\lambda \in \Lambda_0}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{\alpha}$

$$\left(\widehat{B}_{bp}{}^{-}\right)_{\alpha}(x) = \left(\widehat{B}_{bp}{}^{-}\right)(x)(\alpha)$$

$$= \left(\wedge_{\lambda \in \Lambda_0}\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)(\alpha)$$

$$= \wedge_{\lambda \in \Lambda_0}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)\right)$$

$$= \wedge_{\lambda \in \Lambda_0}\left(\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{\alpha}(x)\right)$$

$$= \left(\wedge_{\lambda \in \Lambda_0}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{\alpha}\right)(x)$$

implies $\left(\widehat{B}_{bp}{}^{-}\right)_{\alpha} = \wedge_{\lambda \in \Lambda_0}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{\alpha}$

Therefore $\left(\widehat{B}_{bp}\right)_{\alpha} = \left(\left(\widehat{B}_{bp}{}^{+}\right)_{\alpha}, \left(\widehat{B}_{bp}{}^{-}\right)_{\alpha}\right)$

$$= \left(\vee_{\lambda \in \Lambda_0}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{\alpha}, \wedge_{\lambda \in \Lambda_0}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{\alpha}\right)$$

$$= \cup_{\lambda \in \Lambda_0}\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{\alpha} \in (\hat{\tau}_{\mathfrak{B}})_{\alpha}$$

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Therefore $\cup_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$

The proof for the finite intersection can also be proved in similar manner.
Therefore $(\hat{\tau}_{\mathfrak{B}})_\alpha$ is a FOBPFT (Lowen) on X.

Theorem : 3.4

Let $\tau_{\mathfrak{B}} = \left\{\left(A_{bp}\right)_\lambda / \lambda \in \Lambda\right\}$ be a FOBPFT on I. Then the collection $(\hat{\tau}_{\mathfrak{B}})_I = \{\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I / \left(A_{bp}\right)_\lambda \in \tau_{\mathfrak{B}}\}$ defines a SOBPFT (Lowen) on a non-empty set X where $\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I = \left(\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I, \left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I\right)$ such that $\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I(x) = \left(A_{bp}{}^{+}\right)_\lambda$, $\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I(x) = \left(A_{bp}{}^{-}\right)_\lambda$ for every $x \in X$. The correspondence $\tau_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_I$ is denoted as $\mathbb{C}_4$.

Proof:

To prove $(\hat{\tau}_{\mathfrak{B}})_I$ is a SOBPFT (Lowen) over X.

By the definition of $(\hat{\tau}_{\mathfrak{B}})_I$, there exists $\Lambda_0 \subseteq \Lambda$ such that for $\lambda, \mu \in \Lambda_0, \lambda \neq \mu$,

$\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I \neq \left(\left(\widehat{A}_{bp}\right)_\mu\right)_I$ and $(\hat{\tau}_{\mathfrak{B}})_I$ can be written as $(\hat{\tau}_{\mathfrak{B}})_I = \{(\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I / \lambda \in \Lambda_0\}$.

(i) To prove : $(\hat{0}_{bp})_I, (\hat{1}_{bp})_I, (\hat{\alpha}_{bp})_I \in (\hat{\tau}_{\mathfrak{B}})_I$

Given $0_{bp} = (0_{bp}{}^{+}, 0_{bp}{}^{-}) \in \tau_{\mathfrak{B}}$

Let $(\hat{0}_{bp})_I = \left((\hat{0}_{bp}{}^{+})_I, (\hat{0}_{bp}{}^{-})_I\right)$, then

$\left(\hat{0}_{bp}{}^{+}\right)_I(x) = 0_{bp}{}^{+} = \mathbf{0} = \hat{0}_{bp}{}^{+}(x) \Rightarrow \left(\hat{0}_{bp}{}^{+}\right)_I = \hat{0}_{bp}{}^{+}$

$\left(\hat{0}_{bp}{}^{-}\right)_I(x) = 0_{bp}{}^{-} = \mathbf{0} = \hat{0}_{bp}{}^{-}(x) \Rightarrow \left(\hat{0}_{bp}{}^{-}\right)_I = \hat{0}_{bp}{}^{-}$

implies $(\hat{0}_{bp})_I = \hat{0}_{bp} \in (\hat{\tau}_{\mathfrak{B}})_I$

Similarly $\hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\hat{1}_{bp}\right)_I, \left(\hat{\alpha}_{bp}\right)_I \in (\hat{\tau}_{\mathfrak{B}})_I$

(ii) To prove: $(\hat{\tau}_{\mathfrak{B}})_I$ is closed with respect to arbitrary union.

Consider $\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I \in (\hat{\tau}_{\mathfrak{B}})_I$ for $\lambda \in \Lambda_1 \subseteq \Lambda_0$

implies $\left(A_{bp}\right)_\lambda \in \tau_{\mathfrak{B}}$, for $\lambda \in \Lambda_1 \subseteq \Lambda_0$ implies $\cup_{\lambda\in\Lambda_1}\left(A_{bp}\right)_\lambda \in \tau_{\mathfrak{B}}$

Let $B_{bp} = \cup_{\lambda\in\Lambda_1}\left(A_{bp}\right)_\lambda \in \tau_{\mathfrak{B}}$

where $B_{bp} = \left(B_{bp}{}^{+}, B_{bp}{}^{-}\right) = \left(\vee_{\lambda\in\Lambda_1}\left(A_{bp}{}^{+}\right)_\lambda, \wedge_{\lambda\in\Lambda_1}\left(A_{bp}{}^{-}\right)_\lambda\right)$.

$B_{bp} \in \tau_{\mathfrak{B}}$ implies $\left(B_{bp}\right)_I \in (\hat{\tau}_{\mathfrak{B}})_I$

For $x \in X$,

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Consider,

$$\left(\widehat{B}_{bp}{}^{+}\right)_{I}(x) = B_{bp}{}^{+}$$

$$= \vee_{\lambda\in\Lambda_1}\left(A_{bp}{}^{+}\right)_{\lambda}$$

$$= \vee_{\lambda\in\Lambda_1}\left(\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{I}(x)\right)$$

$$= \left(\vee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{I}\right)(x)$$

implies $\left(\widehat{B}_{bp}{}^{+}\right)_{I} = \vee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{I}$

$$\left(\widehat{B}_{bp}{}^{-}\right)_{I}(x) = B_{bp}{}^{-}$$

$$= \wedge_{\lambda\in\Lambda_1}\left(A_{bp}{}^{-}\right)_{\lambda}$$

$$= \wedge_{\lambda\in\Lambda_1}\left(\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{I}(x)\right)$$

$$= \left(\wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{I}\right)(x)$$

implies$\left(\widehat{B}_{bp}{}^{-}\right)_{I} = \wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{I}$

Therefore $(\widehat{B}_{bp})_I = \left(\left(\widehat{B}_{bp}{}^{+}\right)_{I}, \left(\widehat{B}_{bp}{}^{-}\right)_{I}\right) = \left(\vee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{I}, \wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{I}\right)$

Therefore$(\widehat{B}_{bp})_I = \cup_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{I} \in (\hat{\tau}_{\mathfrak{B}})_I$.
The proof for the finite intersection can also be proved in similar manner.

Theorem : 3.5

Let $\hat{\tau}_{\mathfrak{B}} = \{(\widehat{A}_{bp})_{\lambda}/\lambda \in \Lambda\}$ be a SOBPFT (lowen) on X. Then the collection $(\hat{\tau}_{\mathfrak{B}})_c = \left\{\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{c}/\left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\}$ is also a SOBPFT (Lowen) on X where

$\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{c}(x)(\alpha) = \left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(1-\alpha)$ and

$\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{c}(x)(\alpha) = \left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(1-\alpha)$. The correspondence $\hat{\tau}_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_c$ is denoted as $\mathbb{C}_5$.

Proof:

The proof of this theorem is obvious.

4. Connections between crisp topological spaces and second order bipolar fuzzy topological spaces

Definition : 4.1

Let $\widehat{A}_{bp} = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right) \in$ SOBPFS

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For $\varepsilon \in (0,1)$, define

$$\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} = \left\{x \in X : \left(\widehat{A}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\widehat{A_{bp}}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I\right\}.$$

Definition : 4.2

Let $\widehat{A}_{bp} = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right) \in$ SOBPFS. Define

$$\left(L_{\widehat{A}_{bp}}\right) = \left\{x \in X : \left(\widehat{A}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\widehat{A}_{bp}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I, \text{for some } \varepsilon \in (0,1)\right\}$$

Remark : 4.3

1) $\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} \subseteq \left(L_{\widehat{A}_{bp}}\right)$, for every $\varepsilon \in (0,1)$
2) $\left(L_{\widehat{A}_{bp}}\right) = \cup_{\varepsilon\in(0,1)} \left(L_{\widehat{A}_{bp}}\right)_{\varepsilon}$

Proposition : 4.4

Let $(X, \hat{\tau}_{\mathfrak{B}})$ be a SOBPFTS. Then the collection $\left\{\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} / \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$ is closed with respect to finite intersection.

Proof:

Consider $\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} \cap \left(L_{\widehat{B}_{bp}}\right)_{\varepsilon}$

$$= \{x \in X : \left(\widehat{A}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\widehat{A}_{bp}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I \text{ and}$$

$$\left(\widehat{B}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\widehat{B}_{bp}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I, \text{ for every } \varepsilon \in (0,1)\}$$

$$= \{x \in X : \widehat{A}_{bp}{}^{+}(x)(\alpha) > \varepsilon, \widehat{A}_{bp}{}^{-}(x)(\alpha) < -\varepsilon \text{ and } \widehat{B}_{bp}{}^{+}(x)(\alpha) > \varepsilon, \widehat{B}_{bp}{}^{-}(x)(\alpha) < -\varepsilon$$

$$\text{for every } \varepsilon \in (0,1) \text{ and for every } \alpha \in I\}$$

$$= \{x \in X : \left(\widehat{A}_{bp}{}^{+}(x)(\alpha) \wedge \widehat{B}_{bp}{}^{+}(x)(\alpha)\right) > \varepsilon, \left(\widehat{A}_{bp}{}^{-}(x)(\alpha) \vee \widehat{B}_{bp}{}^{-}(x)(\alpha)\right) < -\varepsilon,$$

$$\text{for every } \varepsilon \in (0,1) \text{ and for every } \alpha \in I\}$$

$$= \{x \in X : \left(\widehat{A}_{bp}{}^{+} \wedge \widehat{B}_{bp}{}^{+}\right)(x)(\alpha) > \varepsilon, \left(\widehat{A}_{bp}{}^{-} \vee \widehat{B}_{bp}{}^{-}\right)(x)(\alpha) < -\varepsilon, \text{ for every } \varepsilon \in (0,1)$$

$$\text{and for every } \alpha \in I\}$$

$$= \{x \in X : \left(\left(\widehat{A}_{bp}{}^{+} \wedge \widehat{B}_{bp}{}^{+}\right)(x)\right)^{-1}(\varepsilon, 1] = I, \left(\left(\widehat{A}_{bp}{}^{-} \vee \widehat{B}_{bp}{}^{-}\right)(x)\right)^{-1}[-1, -\varepsilon) = I\}$$

$= \left(L_{(\widehat{A}_{bp}\cap\widehat{B}_{bp})}\right)_{\varepsilon}$ (Therefore the given collection is closed with respect to finite intersection)

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Proposition : 4.5

Let $(X, \hat{\tau}_{\mathfrak{B}})$ be a SOBPFTS Then the collection $\left\{\left(L_{\hat{A}_{bp}}\right) \,/\, \hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$ is closed with respect to finite intersection.

Proof:

The proof of this proposition is similar as above.

Definition : 4.6

Let (X, τ) be a topological space

(1) For $\varepsilon \in (0,1)$, define $\left(\hat{K}_{bp}\right)_{\varepsilon} = \{\hat{A}_{bp} \in SBPF(X) / \left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau\}$
(2) Define $\hat{K}_{bp} = \{\hat{A}_{bp} \in SBPF(X) / \left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau$, for every $\varepsilon \in (0,1)\}$.
(3) Define $\left(\hat{K}_{bp}\right)_{*} = \{\hat{A}_{bp} \in SBPF(X) / \left(L_{\hat{A}_{bp}}\right) \in \tau\}$

Proposition : 4.7

Each of the above three sets $\left(\hat{K}_{bp}\right)_{\varepsilon}, \hat{K}_{bp}, \left(\hat{K}_{bp}\right)_{*}$ is closed with respect to finite intersection.

The proof is immediate from the preposition 2.3.4 and preposition 2.3.5.

Definition : 4.8

Let (X, τ) be a topological space. Then define

(i) $\widehat{\omega_{\varepsilon}(\tau)}$ to be a SOBPFT generated by $\left(\hat{K}_{bp}\right)_{\varepsilon}$
(ii) $\widehat{\omega(\tau)}$ to be a SOBPFT generated by $\hat{K}_{bp}$
(iii) $\widehat{\omega_{*}(\tau)}$ to be a SOBPFT generated by $\left(\hat{K}_{bp}\right)_{*}$

Theorem : 4.9

Let (X, τ) be a topological space. Then

(i) $\widehat{\omega(\tau)} \subseteq \widehat{\omega_{\varepsilon}(\tau)}$, for every $\varepsilon \in (0,1)$.
(ii) $\widehat{\omega(\tau)} \subseteq \widehat{\omega_{*}(\tau)}$.

Proof:

(i) Consider $\hat{A}_{bp} \in \hat{K}_{bp}$
Then by the definition $\left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau$, for every $\varepsilon \in (0,1)$
implies $\hat{A}_{bp} \in \left(\hat{K}_{bp}\right)_{\varepsilon}$ for every $\varepsilon \in (0,1)$

Therefore $\widehat{\omega(\tau)} \subseteq \widehat{\omega_{\varepsilon}(\tau)}$, for every $\varepsilon \in (0,1)$.

(ii) Consider $\hat{A}_{bp} \in \hat{K}_{bp}$

implies $\left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau$, for every $\varepsilon \in (0,1)$

implies $\bigcup_{\varepsilon \in (0,1)} \left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau$

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implies $\left(L_{\hat{A}_{bp}}\right) \in \tau$ (from remark (2.3.3))

implies $\hat{A}_{bp} \in \left(\hat{K}_{bp}\right)_*$

Therefore $\widehat{\omega(\tau)} \subseteq \widehat{\omega_*(\tau)}$.

Proposition : 4.10

Let τ_1, τ_2 be two topologies on X such that $\tau_1 \subseteq \tau_2$. Then

(i) $\widehat{\omega(\tau_1)} \subseteq \widehat{\omega(\tau_2)}$
(ii) $\widehat{\omega_\varepsilon(\tau_1)} \subseteq \widehat{\omega_\varepsilon(\tau_2)}$
(iii) $\widehat{\omega_*(\tau_1)} \subseteq \widehat{\omega_*(\tau_2)}$

Proof:

(i) Let $\hat{K}_{bp} \in \widehat{\omega(\tau_1)}$.

implies $\left(L_{\hat{A}_{bp}}\right)_\varepsilon \in \tau_1$, for any $\varepsilon \in (0,1)$

implies $\left(L_{\hat{A}_{bp}}\right)_\varepsilon \in \tau_2$, for any $\varepsilon \in (0,1)$ (since $\tau_1 \subseteq \tau_2$)

implies $\hat{K}_{bp} \in \widehat{\omega(\tau_2)}$

Therefore $\widehat{\omega(\tau_1)} \subseteq \widehat{\omega(\tau_2)}$

The proofs of the other are similar.

Definition: 4.11

Let $(X, \hat{\tau}_{\mathfrak{B}})$ SOBPFT. Define

(i) $i_\varepsilon(\hat{\tau}_{\mathfrak{B}})$ to be the topology generated by the collection $\left\{\left(L_{\hat{A}_{bp}}\right)_\varepsilon \,/\, \hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$
(ii) $i^*(\hat{\tau}_{\mathfrak{B}})$ to be the topology generated by the collection $\left\{\left(L_{\hat{A}_{bp}}\right) \,/\, \hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$
(iii) $i(\hat{\tau}_{\mathfrak{B}})$ to be the topology generated by the collection $\left\{\left(L_{\hat{A}_{bp}}\right)_\varepsilon \,/\, \hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}, \varepsilon \in (0,1)\right\}$ as a subbasis.

Theorem : 4.12

Let (X, τ) be a topological space. Then

(i) $\tau \subseteq i^*\left(\widehat{\omega_*(\tau)}\right)$
(ii) For $\varepsilon \in (0,1)$, $\tau \subseteq i_\varepsilon\left(\widehat{\omega_\varepsilon(\tau)}\right)$
(iii) $\tau \subseteq i\left(\widehat{\omega(\tau)}\right)$

Proof:

(i) Consider $M \in \tau$

Consider the SOBPF characteristic function

$\left(\hat{\chi}_{bp}\right)_M = \left(\left(\hat{\chi}_{bp}{}^{+}\right)_M, \left(\hat{\chi}_{bp}{}^{-}\right)_M\right)$ on X as follows:

$$\left(\hat{\chi}_{bp}{}^{+}\right)_M(x) = \mathbf{1}\text{ , if } x \in M$$

$$= \mathbf{0}\text{ , if } x \notin M$$

$$\left(\hat{\chi}_{bp}{}^{-}\right)_M(x) = -\mathbf{1}\text{ , if } x \in M$$

$$= \mathbf{0}\text{ , if } x \notin M$$

Consider,

$$\left(L_{\left(\hat{\chi}_{bp}\right)_M}\right)$$

$$= \left\{x \in X \,/\, \left(\left(\hat{\chi}_{bp}{}^{+}\right)_M(x)\right)^{-1}(\varepsilon, 1] = I\,, \left(\left(\hat{\chi}_{bp}{}^{-}\right)_M(x)\right)^{-1}[-1, -\varepsilon) = I \text{ for some } \varepsilon \in (0,1)\right\}$$

$$= \left\{x \in X \,/\, \left(\hat{\chi}_{bp}{}^{+}\right)_M(x)(\alpha) > \varepsilon\,, \left(\hat{\chi}_{bp}{}^{-}\right)_M(x)(\alpha) < -\varepsilon\,, \text{for some } \varepsilon \in (0,1) \text{ and for every } \alpha \in I\right\}$$

$$= \left\{x \in X \,/\, \left(\hat{\chi}_{bp}{}^{+}\right)_M(x)(\alpha) \neq 0\,, \left(\hat{\chi}_{bp}{}^{-}\right)_M(x)(\alpha) \neq 0, \text{for every } \alpha \in I\right\}$$

$$= \left\{x \in X \,/\, \left(\hat{\chi}_{bp}{}^{+}\right)_M(x) \neq \mathbf{0}\,, \left(\hat{\chi}_{bp}{}^{-}\right)_M(x) \neq \mathbf{0}\right\}$$

$$= \left\{x \in X \,/\, \left(\hat{\chi}_{bp}{}^{+}\right)_M(x) = \mathbf{1}\,, \left(\hat{\chi}_{bp}{}^{-}\right)_M(x) = -\mathbf{1}\right\}$$

$$= M$$

$M \in \tau$ implies $\left(L_{\left(\hat{\chi}_{bp}\right)_M}\right) \in \tau$

implies $\left(\hat{\chi}_{bp}\right)_M \in \left(\hat{K}_{bp}\right)_*$

implies $\left(\hat{\chi}_{bp}\right)_M$ is a basis element of $\widehat{\omega_*(\tau)}$

implies $\left(L_{\left(\hat{\chi}_{bp}\right)_M}\right)$ is a basis element of $i^*\left(\widehat{\omega_*(\tau)}\right)$

implies M is a basis element of $i^*\left(\widehat{\omega_*(\tau)}\right)$

implies $\tau \subseteq i^*\left(\widehat{\omega_*(\tau)}\right)$

Proofs of (ii) and (iii) are similar.

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Example : 4.13

Let $\hat{\tau}_{\mathfrak{B}} = \{\hat{0}_{bp}\} \cup \{\hat{A}_{bp} \in \mathrm{SBPF}(X)\ /\mathrm{supp}\left(\mathbf{1} - \hat{A}_{bp}{}^{+}(x)\right), \mathrm{supp}\left(-\mathbf{1} - \hat{A}_{bp}{}^{-}(x)\right)$

is finite for every $x \in X\}$.

Then $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Proof:

(i) Since $\mathrm{supp}\left(\mathbf{1} - \hat{1}_{bp}{}^{+}(x)\right) = \emptyset$, $\mathrm{supp}\left(-\mathbf{1} - \hat{1}_{bp}{}^{-}(x)\right) = \emptyset$, $\hat{1}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

(ii) $\left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda$

implies $\mathrm{supp}\left(\mathbf{1} - \left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)\right)$, $\mathrm{supp}\left(-\mathbf{1} - \left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)\right)$ are finite

Consider $\mathrm{supp}\left(\mathbf{1} - \left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)\right) = \mathrm{supp}\left(\mathbf{1} - \vee_{\lambda\in\Lambda}\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)\right)\right)$

$\subseteq \mathrm{supp}\left(\mathbf{1} - \left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)\right)$

implies $\mathrm{supp}\left(\mathbf{1} - \left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)\right)$ is finite

Similarly $\mathrm{supp}\left(-\mathbf{1} - \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)\right) = \mathrm{supp}\left(-\mathbf{1} - \wedge_{\lambda\in\Lambda}\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)\right)\right)$

$\supset \mathrm{supp}\left(-\mathbf{1} - \left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)\right)$

implies $\mathrm{supp}\left(-\mathbf{1} - \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)\right)$ is finite

Therefore $\cup_{\lambda\in\Lambda}\left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$

(iii) $\left(\hat{A}_{bp}\right)_{i} \in \hat{\tau}_{\mathfrak{B}}$, for $i = 1,2, \dots m$

implies $\mathrm{supp}\left(\mathbf{1} - \left(\hat{A}_{bp}{}^{+}\right)_{i}(x)\right)$, $\mathrm{supp}\left(-\mathbf{1} - \left(\hat{A}_{bp}{}^{-}\right)_{i}(x)\right)$ are finite

Consider $\mathrm{supp}\left(\mathbf{1} - \left(\wedge_{i=1}^{m}\left(\hat{A}_{bp}{}^{+}\right)_{i}\right)(x)\right) = \mathrm{supp}\left(\mathbf{1} - \wedge_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{+}\right)_{i}(x)\right)\right)$

There exists k such that

$\mathrm{supp}\left(\mathbf{1} - \wedge_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{+}\right)_{k}(x)\right)\right) \subseteq \mathrm{supp}\left(\mathbf{1} - \left(\hat{A}_{bp}{}^{+}\right)_{K}(x)\right)$ which is finite

Similarly $\mathrm{supp}\left(-\mathbf{1} - \left(\vee_{i=1}^{m}\left(\hat{A}_{bp}{}^{-}\right)_{i}\right)(x)\right) = \mathrm{supp}\left(-\mathbf{1} - \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{-}\right)_{i}(x)\right)\right)$

There exists k such that

$\mathrm{supp}\left(-\mathbf{1} - \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{-}\right)_{k}(x)\right)\right) \supset \mathrm{supp}\left(-\mathbf{1} - \left(\hat{A}_{bp}{}^{-}\right)_{K}(x)\right)$ which is also finite

Therefore $\cap_{i=1}^{m}\left(\hat{A}_{bp}\right)_{i} \in \hat{\tau}_{\mathfrak{B}}$

Therefore $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Example : 4.14

Let $\hat{\tau}_{\mathfrak{B}} = \{\hat{1}_{bp}\} \cup \{\hat{A}_{bp} \in \mathrm{SBPF}(X)/\ \hat{A}_{bp}{}^{+}(x)(\alpha) = 0, \hat{A}_{bp}{}^{-}(x)(\alpha) = 0$, for every $x \in X$ and

for $\alpha \neq \frac{r}{n}$, $r = 0,1,2,\dots n\}$

Then $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Proof:

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(i) Since $\hat{0}_{bp}{}^{+}(x)(\alpha) = 0, \hat{0}_{bp}{}^{-}(x)(\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$.
implies $\hat{0}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

(ii) $(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda$
$(\hat{A}_{bp}{}^{+})_{\lambda}(x)(\alpha) = 0, (\hat{A}_{bp}{}^{-})_{\lambda}(x)(\alpha) = 0$, for every $x \in X$, for $\alpha \neq \frac{r}{n}$,
$r = 0,1,2,\ldots n$
implies $\vee_{\lambda\in\Lambda}\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)\right) = 0,\ \wedge_{\lambda\in\Lambda}\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)\right) = 0$
for every $x \in X$ and for $\alpha \neq \frac{r}{n}$, $r = 0,1,2,\ldots n$
implies $\left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)(\alpha) = 0, \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)(\alpha) = 0$,
for every $x \in X$ and for $\alpha \neq \frac{r}{n}$, $r = 0,1,2,\ldots n$
Therefore $\cup_{\lambda\in\Lambda}(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$

(iii) Similarly we can prove for finite intersection.
Therefore $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Example : 4.15

Le $\hat{\tau}_{\mathfrak{B}} = \{\hat{A}_{bp} \in SBPF(X) /$ for every $x \in X$, either $\hat{A}_{bp}{}^{+}(x) = \mathbf{0}, \hat{A}_{bp}{}^{-}(x) = \mathbf{0}$ (or)

$\hat{A}_{bp}{}^{+}(x)(\alpha) > 0,\ \hat{A}_{bp}{}^{-}(x)(\alpha) < 0$, for every $\alpha \in I\}$

Then $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Proof:

(i) Obviously $\hat{0}_{bp}, \hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

(ii) Consider $(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda$
Suppose $\left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x) = \mathbf{0}, \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x) = \mathbf{0}$, for every $x \in X$
Then $\cup_{\lambda\in\Lambda}(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$
Suppose $\left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x) \neq \mathbf{0}, \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x) \neq \mathbf{0}$,
Then $\left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)(\alpha) > 0, \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)(\alpha) < 0$, for some $\alpha \in I$
there exists a $\lambda \in \Lambda$ such that
$(\hat{A}_{bp}{}^{+})_{\lambda}(x)(\alpha) > 0, (\hat{A}_{bp}{}^{-})_{\lambda}(x)(\alpha) < 0$ for some $\alpha \in I$
for that $\lambda \in \Lambda$, $(\hat{A}_{bp}{}^{+})_{\lambda}(x)(\alpha) > 0, (\hat{A}_{bp}{}^{-})_{\lambda}(x)(\alpha) < 0$,for every $\alpha \in I$
implies $\left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)(\alpha) > 0, \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)(\alpha) < 0$, for every $\alpha \in I$
Therefore $\cup_{\lambda\in\Lambda}(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$

(iii) Consider $(\hat{A}_{bp})_{i} \in \hat{\tau}_{\mathfrak{B}}$, for i = 1 to m
Suppose $\left(\wedge_{i=1}^{m}\left(\hat{A}_{bp}{}^{+}\right)_{i}\right)(x) = \mathbf{0}, \left(\vee_{i=1}^{m}\left(\hat{A}_{bp}{}^{-}\right)_{i}\right)(x) = \mathbf{0}$
Then $\cap_{i=1}^{m}(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$
Suppose $\left(\wedge_{i=1}^{m}\left(\hat{A}_{bp}{}^{+}\right)_{i}\right)(x) \neq \mathbf{0}, \left(\vee_{i=1}^{m}\left(\hat{A}_{bp}{}^{-}\right)_{i}\right)(x) \neq \mathbf{0}$
Then $\left(\wedge_{i=1}^{m}\left(\hat{A}_{bp}{}^{+}\right)_{i}\right)(x)(\alpha) > 0, \left(\vee_{i=1}^{m}\left(\hat{A}_{bp}{}^{-}\right)_{i}\right)(x)(\alpha) < 0$, for some $\alpha \in I$
Therefore $(\hat{A}_{bp}{}^{+})_{i}(x)(\alpha) > 0, (\hat{A}_{bp}{}^{-})_{i}(x)(\alpha) < 0$, for some $\alpha \in I$ & for $i = 1,2, \ldots m$
Therefore $(\hat{A}_{bp}{}^{+})_{i}(x)(\alpha) > 0, (\hat{A}_{bp}{}^{-})_{i}(x)(\alpha) < 0$, for every $\alpha \in I$ & for $i = 1,2, \ldots m$
Therefore $\left(\wedge_{i=1}^{m}\left(\hat{A}_{bp}{}^{+}\right)_{i}\right)(x)(\alpha) > 0, \left(\vee_{i=1}^{m}\left(\hat{A}_{bp}{}^{-}\right)_{i}\right)(x)(\alpha) < 0$, for every $\alpha \in I$

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Therefore $\bigcap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$

Therefore $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Example : 4.16

Consider the closed unit interval I. For $\alpha \in I$, define $\left(\widehat{A}_{bp}\right)_{\alpha}: I \rightarrow I^{I}$ such that

$\left(\widehat{A}_{bp}{}^{+}\right)_{\alpha}(\beta)(\gamma) = \alpha\beta\gamma$, $\left(\widehat{A}_{bp}{}^{-}\right)_{\alpha}(\beta)(\gamma) = -\alpha\beta\gamma$, for every $\beta, \gamma \in I$.

Then the collection $\hat{\tau}_{\mathfrak{B}} = \left\{\left(\widehat{A}_{bp}\right)_{\alpha} / \alpha \in I\right\} \cup \left\{\hat{1}_{bp}\right\}$ is a SOBPFT on I.

Proof:

(i) Since $\hat{0}_{bp}{}^{+} = \left(\widehat{A}_{bp}{}^{+}\right)_{0} = \hat{0}$, $\hat{0}_{bp}{}^{-} = \left(\widehat{A}_{bp}{}^{-}\right)_{0} = \hat{0}$
implies $\hat{0}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

(ii) Let $\left(\widehat{A}_{bp}\right)_{\alpha} \in \hat{\tau}_{\mathfrak{B}}$ for α belongs to an arbitrary set $S \subseteq I$
For $\beta, \gamma \in I$, Consider
Now $\bigcup_{\alpha\in S}\left(\widehat{A}_{bp}\right)_{\alpha} = \left(\left(\bigvee_{\alpha\in S}\left(\widehat{A}_{bp}{}^{+}\right)_{\alpha}\right), \left(\bigwedge_{\alpha\in S}\left(\widehat{A}_{bp}{}^{-}\right)_{\alpha}\right)\right)$

$$\left(\bigvee_{\alpha\in S}\left(\widehat{A}_{bp}{}^{+}\right)_{\alpha}\right)(\beta)(\gamma) = \bigvee_{\alpha\in S}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\alpha}(\beta)(\gamma)\right)$$
$$= \bigvee_{\alpha\in S}(\alpha\beta\gamma)$$
$$= \left(\bigvee_{\alpha\in S}\alpha\right)\beta\gamma$$
$$= \alpha'\beta\gamma$$
$$= \left(\widehat{A}_{bp}{}^{+}\right)_{\alpha'}$$

$$\left(\bigwedge_{\alpha\in S}\left(\widehat{A}_{bp}{}^{-}\right)_{\alpha}\right)(\beta)(\gamma) = \bigwedge_{\alpha\in S}\left(\widehat{A}_{bp}{}^{-}\right)_{\alpha}(\beta)(\gamma)$$
$$= \bigwedge_{\alpha\in S}(-\alpha\beta\gamma)$$
$$= \left(\bigwedge_{\alpha\in S}(-\alpha)\right)\beta\gamma$$
$$= (-\alpha')\beta\gamma$$
$$= \left(\widehat{A}_{bp}{}^{-}\right)_{\alpha'}$$

Therefore $\bigcup_{\alpha\in S}\left(\widehat{A}_{bp}\right)_{\alpha} = \left(\widehat{A}_{bp}\right)_{\alpha'} \in \hat{\tau}_{\mathfrak{B}}$

(iii) The proof for finite intersection is similar
Therefore $\hat{\tau}_{\mathfrak{B}}$ is a s SOBPFT on I.

5. Conclusion:

In this paper, the detailed study about SOBPFT relating to FOBPFT and crisp topology are established and some examples for SOBPFT are provided. The connections between FOBPFT and SOBPFT under five different cases are discussed. And from a crisp topology τ on X there exists three different SOBPFT denoted by $\widehat{\omega(\tau)}$, $\widehat{\omega_{*}(\tau)}$ and $\widehat{\omega_{\varepsilon}(\tau)}$ and from a SOBPFT on X there exists three crisp topologies denoted by $i(\hat{\tau}_{\mathfrak{B}})$, $i^{*}(\hat{\tau}_{\mathfrak{B}})$ and $i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})$.

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Second Order Bipolar Fuzzy Matrix and its Application in Medical Diagnosis

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Abstract

In this research article, firstly a new concept of second order bipolar fuzzy matrix (SOBPFM) is introduced and their basic properties and results are proved with examples. Secondly, the concept of symmetrical difference operator (SDO) over SOBPFM is introduced and various properties of SDO are discussed and verified over SOBPFM. Finally, an application on SOBPFM to develop a methodology in handling decision-making problem in medical field which enables clinicians to effectively assess and classify medical conditions, particularly in scenarios involving complex symptoms and overlapping disease manifestation.

Keywords : second order bipolar fuzzy set (SOBPFS), second order bipolar fuzzy matrix (SOBPFM), symmetrical difference operator (SDO)

1. Introduction

The notion of fuzzy set theory was introduced by Zadeh [14] in 1965, which is an universality of classical set theory. The utility of fuzzy set theory in decision making was first demonstrated by Bellman and Zadeh [2] in 1970. Since then many researchers have been working the process of dealing with decision making problems by applying fuzzy set theory.

In 1975, Zadeh [15] proposed the concept of second order fuzzy set which is defined as a map from a set X to I^I. Using second order fuzzy sets Kalaichelvi [3] (2007) extended first order fuzzy topology of Chang and Lowen to second order fuzzy topology.

The matrix is a fundamental concept in mathematics and computer science. It is widely used in various fields including physics, engineering, economics and computer graphics. The concept of fuzzy matrix was first defined by Thomsan [13] in 1977. As an extension of Boolean matrices, the theory of fuzzy matrices were developed by Kim and Roush [4]. In 2022, Saranya et al [12] studied the basic operations and properties of fuzzy matrices. A fuzzy matrix extends the concept of a traditional matrix by allowing entries to have degrees of membership in a set rather than precise value.

In 1994, Zhang [16] proposed the concept of bipolar fuzzy set. Bipolar fuzzy set is an extension of the traditional fuzzy set theory allowing for the representation of uncertainty in both position and negative directions whose membership degree range in$[-1,1]$ where the satisfactory degree of a certain property associated to the fuzzy set locates in the interval $[0,1]$ and the satisfactory degree of counter-property to the concerned fuzzy set locates in the interval$[-1,0]$. In 2024, Muthamizhselvi and Vijayalakshmi [6] proposed the concept of Second order bipolar fuzzy set and second order bipolar fuzzy topological space.

In 2001, Kuratowski [5] defined new concept of symmetrical difference over ordinary sets through union, intersection and negation. In 2004, Anton Antonov [1] introduced SDO over IFM and shown that associativity was not true for IFM. In 2021, Muthuraji [10] analysed commutative monoid on symmetrical difference operator over IFM.

In 2019, M. Pal and Sanjib Mondal [11] introduced bipolar fuzzy matrix. In 2023, Muthuraji and Anitha [7] proposed the recent technology using the bipolar fuzzy matrices with the aid of score function in agriculture. Muthuraji and Punitha Elizabeth [9] proposed application of bipolar fuzzy matrices in flood damage. Application of Bipolar Fuzzy Matrix in the Research of Crops on Agriculture was proposed by Muthuraji and Anitha [8] in 2023. In this paper, second order bipolar fuzzy set, second order bipolar fuzzy matrix and the concept of symmetrical difference operator (SDO) over SOBPFM are introduced. Also an application of second order bipolar fuzzy matrix is presented into a decision making problem and a general algorithm has been constructed to solve the problems in medical diagnosis.

2. Preliminaries

2.1 Definition

Let **X** be an arbitrary nonempty set. Let $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$. A **fuzzy set** in **X** is a map from **X** into I.

2.2 Definition

A **second order fuzzy set** on X is a map $\hat{f} : X \to I^I$ where I is the closed unit interval [0,1].

2.3 Definition

Let X be a nonempty set. Then a pair $A_{bp} = (A_{bp}{}^{+}, A_{bp}{}^{-})$ is called **bipolar-valued fuzzy set or bipolar fuzzy set** in X, where $A_{bp}{}^{+}: X \to [0,1]$ and $A_{bp}{}^{-}: X \to [-1,0]$. The set of all bipolar fuzzy set in X is denoted as BPF(X).

2.4 Definition

A pair $\widehat{A}_{bp} = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)$ is called a **second order bipolar fuzzy set** in X where $\widehat{A}_{bp}{}^{+}: X \to [0,1]^{[0,1]}$ such that $\widehat{A}_{bp}{}^{+}(x)(\alpha) \in [0,1]$ and $\widehat{A}_{bp}{}^{-}: X \to [-1,0]^{[0,1]}$ such that $\widehat{A}_{bp}{}^{-}(x)(\alpha) \in [-1,0]$, where $\alpha \in I$ & $x \in X$.

2.5 Definition

Let F be a matrix, $F = [F_{ij}]_{u\times v}$, where $[F_{ij}]_{u\times v} \in [0,1]$, $1 \le i \le u$ and $1 \le j \le v$, then F is called **fuzzy matrix.**

2.6 Definition

A **bipolar fuzzy matrix** $\left((A_{bp})_{ij}\right)_{p\times q}$ is defined as $\left((A_{bp})_{ij}\right)_{p\times q} = \left((A_{bp}{}^{+}, A_{bp}{}^{-})_{ij}\right)_{p\times q}$ where $(A_{bp}{}^{+})_{ij}(x) \in [0,1]$ and $(A_{bp}{}^{-})_{ij}(x) \in [-1,0]$ $\forall i, j$, where $x \in X$ and p, q denotes the rows and columns of the matrix.

2.7 Definition (Operations in bipolar fuzzy matrix)

Let $\left((A_{bp})_{ij}\right)_{p\times q} = \left(({A_{bp}}^{+}, {A_{bp}}^{-})_{ij}\right)_{p\times q}$ &

$\left((B_{bp})_{ij}\right)_{p\times q} = \left(({B_{bp}}^{+}, {B_{bp}}^{-})_{ij}\right)_{p\times q}$ be two bipolar fuzzy matrices. Define

(i) $\left((A_{bp})_{ij} + (B_{bp})_{ij}\right)_{p\times q} = \left((C_{bp})_{ij}\right)_{p\times q}$ where $C_{bp} = ({C_{bp}}^{+}, {C_{bp}}^{-})$ and is defined as
${C_{bp}}^{+}(x) = \max\{{A_{bp}}^{+}(x), {B_{bp}}^{+}(x)\}$ &
${C_{bp}}^{-}(x) = \min\{{A_{bp}}^{-}(x), {B_{bp}}^{-}(x)\}$, $\forall i, j$, for every $x \in X$.

(ii) $\left((A_{bp})_{ij} * (B_{bp})_{ij}\right)_{p\times q} = \left((D_{bp})_{ij}\right)_{p\times q}$ where $D_{bp} = ({D_{bp}}^{+}, {D_{bp}}^{-})$ and is defined as
${D_{bp}}^{+}(x) = \min\{{A_{bp}}^{+}(x), {B_{bp}}^{+}(x)\}$ &
${D_{bp}}^{-}(x) = \max\{{A_{bp}}^{-}(x), {B_{bp}}^{-}(x)\}$, $\forall i, j$, for every $x \in X$.

(iii) $\left(\left((A_{bp})_{ij}\right)^{c}\right)_{p\times q} = \left(\left(({A_{bp}}^{+})^{c}, ({A_{bp}}^{-})^{c}\right)_{ij}\right)_{p\times q}$ is defined as

$({A_{bp}}^{+})^{c}(x) = 1 - {A_{bp}}^{+}(x)$, $\forall i, j$, for every $x \in X$.

$({A_{bp}}^{-})^{c}(x) = -1 - {A_{bp}}^{-}(x)$, $\forall i, j$, for every $x \in X$.

3. Second order bipolar fuzzy matrix

3.1 Definition

Let X be a nonempty set. A **second order bipolar fuzzy matrix** (SBPFM) $\left((\widehat{A}_{bp})_{ij}\right)_{p\times q}$ is defined as $\left((\widehat{A}_{bp})_{ij}\right)_{p\times q} = \left(\left({\widehat{A}_{bp}}^{+}, {\widehat{A}_{bp}}^{-}\right)_{ij}\right)_{p\times q}$ where $\left({\widehat{A}_{bp}}^{+}\right)_{ij}(x)(\alpha) \in [0,1]$ and $\left({\widehat{A}_{bp}}^{-}\right)_{ij}(x)(\alpha) \in [-1,0]$, $\forall i, j$, where $\alpha \in I$ & $x \in X$.

3.2 Definition (Operations on second order bipolar fuzzy matrix)

Let $\left((\widehat{A}_{bp})_{ij}\right)_{p\times q} = \left(\left({\widehat{A}_{bp}}^{+}, {\widehat{A}_{bp}}^{-}\right)_{ij}\right)_{p\times q}$ &

$\left((\widehat{B}_{bp})_{ij}\right)_{p\times q} = \left(\left({\widehat{B}_{bp}}^{+}, {\widehat{B}_{bp}}^{-}\right)_{ij}\right)_{p\times q}$ be two second order bipolar fuzzy matrices. Define

(iv) $\left((\widehat{A}_{bp})_{ij} + (\widehat{B}_{bp})_{ij}\right)_{p\times q} = \left((\widehat{C}_{bp})_{ij}\right)_{p\times q}$ where $\widehat{C}_{bp} = \left({\widehat{C}_{bp}}^{+}, {\widehat{C}_{bp}}^{-}\right)$ and is defined as
${\widehat{C}_{bp}}^{+}(x)(\alpha) = \max\left\{{\widehat{A}_{bp}}^{+}(x)(\alpha), {\widehat{B}_{bp}}^{+}(x)(\alpha)\right\}$ &
${\widehat{C}_{bp}}^{-}(x)(\alpha) = \min\{{\widehat{A}_{bp}}^{-}(x)(\alpha), {\widehat{B}_{bp}}^{-}(x)(\alpha)\}$, $\forall i, j$, for every $x \in X$ for every $\alpha \in I$.

(v) $\left((\widehat{A}_{bp})_{ij} * (\widehat{B}_{bp})_{ij}\right)_{p\times q} = \left((\widehat{D}_{bp})_{ij}\right)_{p\times q}$ where $\widehat{D}_{bp} = \left({\widehat{D}_{bp}}^{+}, {\widehat{D}_{bp}}^{-}\right)$ and is defined as
${\widehat{D}_{bp}}^{+}(x)(\alpha) = \min\left\{{\widehat{A}_{bp}}^{+}(x)(\alpha), {\widehat{B}_{bp}}^{+}(x)(\alpha)\right\}$ &
${\widehat{D}_{bp}}^{-}(x)(\alpha) = \max\{{\widehat{A}_{bp}}^{-}(x)(\alpha), {\widehat{B}_{bp}}^{-}(x)(\alpha)\}$, $\forall i, j$, for every $x \in X$ for every $\alpha \in I$.

(vi) $\left(\left(\left(\widehat{A}_{bp}\right)_{ij}\right)^{c}\right)_{p\times q} = \left(\left(\left(\widehat{A}_{bp}{}^{+}\right)^{c}, \left(\widehat{A}_{bp}{}^{-}\right)^{c}\right)_{ij}\right)_{p\times q}$ is defined as

$\left(\widehat{A}_{bp}{}^{+}\right)^{c}(x)(\alpha) = 1 - \widehat{A}_{bp}{}^{+}(x)(\alpha)$, $\forall i, j$, for every $x \in X$ for every $\alpha \in I$.

$\left(\widehat{A}_{bp}{}^{-}\right)^{c}(x)(\alpha) = -1 - \widehat{A}_{bp}{}^{-}(x)(\alpha)$, $\forall i, j$, for every $x \in X$ for every $\alpha \in I$.

3.3 Example

Let $\left(\widehat{A}_{bp}\right)_{ij}, \left(\widehat{B}_{bp}\right)_{ij}$ be two 3×3 second order bipolar fuzzy matrices.

$$\left(\widehat{A}_{bp}\right)_{ij} = \begin{bmatrix} (0.7,-0.3) & (0.5,-0.1) & (0.6,-0.4) \\ (0.2,-0.8) & (0.1,-0.5) & (0.9,-0.3) \\ (0.4,-0.7) & (0.3,-0.2) & (0.8,-0.6) \end{bmatrix}_{3\times 3}$$

$$\left(\widehat{B}_{bp}\right)_{ij} = \begin{bmatrix} (0.9,-0.2) & (0.4,-0.5) & (0.7,-0.3) \\ (0.5,-0.1) & (0.6,-0.4) & (0.8,-0.6) \\ (0.1,-0.8) & (0.2,-0.7) & (0.5,-0.9) \end{bmatrix}_{3\times 3}.$$

Then compute (i) $\left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{B}_{bp}\right)_{ij}$

(ii) $\left(\widehat{A}_{bp}\right)_{ij} + \left(\widehat{B}_{bp}\right)_{ij}$

(iii) $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)^{c}$.

Solution:

(i) $$\left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{B}_{bp}\right)_{ij} = \begin{bmatrix} (0.7,-0.2) & (0.4,-0.1) & (0.6,-0.3) \\ (0.2,-0.1) & (0.1,-0.4) & (0.8,-0.3) \\ (0.1,-0.7) & (0.2,-0.2) & (0.5,-0.6) \end{bmatrix}_{3\times 3}$$

(ii) $$\left(\widehat{A}_{bp}\right)_{ij} + \left(\widehat{B}_{bp}\right)_{ij} = \begin{bmatrix} (0.9,-0.3) & (0.5,-0.5) & (0.7,-0.4) \\ (0.5,-0.8) & (0.6,-0.5) & (0.9,-0.6) \\ (0.4,-0.8) & (0.3,-0.7) & (0.8,-0.9) \end{bmatrix}_{3\times 3}$$

(iii) $$\left(\left(\widehat{A}_{bp}\right)_{ij}\right)^{c} = \begin{bmatrix} (0.3,-0.7) & (0.5,-0.9) & (0.4,-0.6) \\ (0.8,-0.2) & (0.9,-0.5) & (0.1,-0.7) \\ (0.6,-0.3) & (0.7,-0.8) & (0.2,-0.4) \end{bmatrix}_{3\times 3}.$$

3.4 Definition

The **transpose of second order bipolar fuzzy m × n matrix** $\left(\left(\widehat{A}_{bp}\right)_{ij}\right) = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)_{ij}$ is defined as the $n \times m$ second order bipolar fuzzy matrix $\left(\left(\widehat{B}_{bp}\right)_{ij}\right) = \left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right)_{ij}$ with $\left(\left(\widehat{B}_{bp}\right)_{ij}\right) = \left(\left(\widehat{A}_{bp}\right)_{ji}\right)$ for all $1 \le i \le m$ and $1 \le j \le n$. The **transpose** of $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)$ is denoted as $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)^{T}$.

3.5 Example

$$\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right) = \begin{bmatrix}(0.8,-0.7)\\(0.1,-0.6)\\(0.5,-0.5)\end{bmatrix}$$

Then the transpose is given by

$$\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)^{\mathrm{T}} = [(0.8,-0.7)\quad(0.1,-0.6)\quad(0.5,-0.5)]$$

3.6 Definition

A square matrix $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right) = \left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}, \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\mathrm{ij}}$ with $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right) = \begin{cases}(1,-1) & \text{if } \mathrm{i}=\mathrm{j}\\(0,0) & \text{if } \mathrm{i}\neq\mathrm{j}\end{cases}$ is called **second order bipolar fuzzy identity matrix**, denoted as $(\hat{\mathrm{I}}_{\mathrm{bp}})_{\mathrm{n}}$.

3.7 Example

$$(\hat{\mathrm{I}}_{\mathrm{bp}})_{\mathrm{n}} = \begin{bmatrix}(1,-1) & (0,0) & (0,0)\\(0,0) & (1,-1) & (0,0)\\(0,0) & (0,0) & (1,-1)\end{bmatrix}$$

3.8 Remark

Let $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right) = \left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}, \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\mathrm{ij}}$ be square matrix of order n and $\hat{\mathrm{I}}_{\mathrm{bp}}$ be **second order bipolar fuzzy identity matrix**. Then

$$(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} * \hat{\mathrm{I}}_{\mathrm{bp}} = \hat{\mathrm{I}}_{\mathrm{bp}} * (\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} = (\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}$$

3.9 Definition

Let $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right) = \left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}, \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\mathrm{ij}}$ be square matrix (order n). Then the **trace of second order bipolar fuzzy matrix** $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)$ is denoted by $\mathrm{tr}(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}$ and is defined by

$$\mathrm{tr}(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} = \left(\max\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{\mathrm{ij}}, \min\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\mathrm{ij}}\right), \text{ where } \mathrm{i}=\mathrm{j}$$

3.10 Example

$(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} = \begin{bmatrix}(0.1,-0.2) & (0.7,-0.9)\\(0.3,-0.5) & (0.5,-0.1)\end{bmatrix}$ Then the trace of SOBPFM $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)$ is given by

$$\begin{aligned}\mathrm{tr}(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} &= \left(\max\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{\mathrm{ij}}, \min\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\mathrm{ij}}\right)\\&= (\max\{0.1,0.5\}, \min\{-0.2,-0.1\})\\&= (0.5,-0.2)\end{aligned}$$

3.11 Properties of second order bipolar fuzzy matrix

Let $(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}, (\widehat{\mathrm{B}}_{\mathrm{bp}})_{\mathrm{ij}}$ and $(\widehat{\mathrm{C}}_{\mathrm{bp}})_{\mathrm{ij}}$ be three SOBPFM of order $\mathrm{m}\times\mathrm{n}, \mathrm{n}\times\mathrm{p}$ and $\mathrm{p}\times\mathrm{q}$ respectively, then

(i) $(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} * \left((\widehat{\mathrm{B}}_{\mathrm{bp}})_{\mathrm{ij}} * (\widehat{\mathrm{C}}_{\mathrm{bp}})_{\mathrm{ij}}\right) = \left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} * (\widehat{\mathrm{B}}_{\mathrm{bp}})_{\mathrm{ij}}\right) * (\widehat{\mathrm{C}}_{\mathrm{bp}})_{\mathrm{ij}}$ (Associativity)

(ii) $\left(\widehat{A}_{bp}\right)_{ij} * \left(\left(\widehat{B}_{bp}\right)_{ij} + \left(\widehat{C}_{bp}\right)_{ij}\right) = \left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{B}_{bp}\right)_{ij} + \left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{C}_{bp}\right)_{ij}$ (Distributive law)

The same results hold for second order bipolar fuzzy complement matrices

(i) $\left(\widehat{A}_{bp}{}^{c}\right)_{ij} * \left(\left(\widehat{B}_{bp}{}^{c}\right)_{ij} * \left(\widehat{C}_{bp}{}^{c}\right)_{ij}\right) = \left(\left(\widehat{A}_{bp}{}^{c}\right)_{ij} * \left(\widehat{B}_{bp}{}^{c}\right)_{ij}\right) * \left(\widehat{C}_{bp}{}^{c}\right)_{ij}$ (Associativity)

(ii) $\left(\widehat{A}_{bp}{}^{c}\right)_{ij} * \left(\left(\widehat{B}_{bp}{}^{c}\right)_{ij} + \left(\widehat{C}_{bp}{}^{c}\right)_{ij}\right) = \left(\widehat{A}_{bp}\right)^{c}{}_{ij} * \left(\widehat{B}_{bp}{}^{c}\right)_{ij} + \left(\widehat{A}_{bp}{}^{c}\right)_{ij} * \left(\widehat{C}_{bp}{}^{c}\right)_{ij}$ (Distributive law)

3.12 Example

(i) $\left(\widehat{A}_{bp}\right)_{ij} = \begin{bmatrix} (0.7,-0.1) & (0.8,-0.5) \\ (0.1,-0.8) & (0.3,-0.2) \end{bmatrix}$ $\left(\widehat{B}_{bp}\right)_{ij} = \begin{bmatrix} (0.8,-0.5) & (0.2,-0.7) \\ (0.6,-0.4) & (0.3,-0.9) \end{bmatrix}$

$\left(\widehat{C}_{bp}\right)_{ij} = \begin{bmatrix} (0.1,-0.3) & (0.3,-0.8) \\ (0.4,-0.5) & (0.9,-0.7) \end{bmatrix}$

$\left(\widehat{B}_{bp}\right)_{ij} * \left(\widehat{C}_{bp}\right)_{ij} = \begin{bmatrix} (0.1,-0.3) & (0.2,-0.7) \\ (0.4,-0.4) & (0.3,-0.7) \end{bmatrix}$

$\left(\widehat{A}_{bp}\right)_{ij} * \left(\left(\widehat{B}_{bp}\right)_{ij} * \left(\widehat{C}_{bp}\right)_{ij}\right) = \begin{bmatrix} (0.1,-0.1) & (0.2,-0.5) \\ (0.1,-0.4) & (0.3,-0.2) \end{bmatrix}$

$\left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{B}_{bp}\right)_{ij} = \begin{bmatrix} (0.7,-0.1) & (0.2,-0.5) \\ (0.1,-0.4) & (0.3,-0.2) \end{bmatrix}$

$\left(\widehat{A}_{bp}\right)_{ij} * \left(\left(\widehat{B}_{bp}\right)_{ij} * \left(\widehat{C}_{bp}\right)_{ij}\right) = \begin{bmatrix} (0.1,-0.1) & (0.2,-0.5) \\ (0.1,-0.4) & (0.3,-0.2) \end{bmatrix}$

Property (i) holds

(ii) $\left(\widehat{A}_{bp}\right)_{ij} = \begin{bmatrix} (0.7,-0.1) & (0.4,-0.2) \\ (0.3,-0.5) & (0.6,-0.9) \end{bmatrix}$ $\left(\widehat{B}_{bp}\right)_{ij} = \begin{bmatrix} (0.1,-0.3) & (0.5,-0.4) \\ (0.7,-0.1) & (0.8,-0.5) \end{bmatrix}$

$\left(\widehat{C}_{bp}\right)_{ij} = \begin{bmatrix} (0.8,-0.7) & (0.1,-0.5) \\ (0.2,-0.9) & (0.7,-0.6) \end{bmatrix}$

$\left(\widehat{B}_{bp}\right)_{ij} + \left(\widehat{C}_{bp}\right)_{ij} = \begin{bmatrix} (0.8,-0.7) & (0.5,-0.5) \\ (0.7,-0.9) & (0.8,-0.6) \end{bmatrix}$

$\left(\widehat{A}_{bp}\right)_{ij} * \left(\left(\widehat{B}_{bp}\right)_{ij} + \left(\widehat{C}_{bp}\right)_{ij}\right) = \begin{bmatrix} (0.7,-0.1) & (0.4,-0.2) \\ (0.3,-0.5) & (0.6,-0.6) \end{bmatrix}$

$\left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{B}_{bp}\right)_{ij} = \begin{bmatrix} (0.1,-0.1) & (0.4,-0.2) \\ (0.3,-0.1) & (0.6,-0.5) \end{bmatrix}$

$\left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{C}_{bp}\right)_{ij} = \begin{bmatrix} (0.7,-0.1) & (0.1,-0.2) \\ (0.2,-0.5) & (0.6,-0.6) \end{bmatrix}$

$\left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{B}_{bp}\right)_{ij} + \left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{C}_{bp}\right)_{ij} = \begin{bmatrix} (0.7,-0.1) & (0.4,-0.2) \\ (0.3,-0.5) & (0.6,-0.6) \end{bmatrix}$

Property (ii) holds

Similarly we can prove for complement of SOBPFM.

3.13 Theorem

Let $\left(\widehat{A}_{bp}\right)_{ij}$ and $\left(\widehat{B}_{bp}\right)_{ij}$ be two SOBPFMs (order n) and λ - scalar such that $0 \leq \lambda \leq 1$. Then

(i) $\mathrm{tr}\left(\widehat{A}_{bp}+\widehat{B}_{bp}\right)_{ij}=\mathrm{tr}\left(\widehat{A}_{bp}\right)_{ij}+\mathrm{tr}\left(\widehat{B}_{bp}\right)_{ij}$

(ii) $\mathrm{tr}\left(\lambda\left(\widehat{A}_{bp}\right)_{ij}\right)=\lambda\mathrm{tr}\left(\widehat{A}_{bp}\right)_{ij}$

(iii) $\mathrm{tr}\left(\widehat{A}_{bp}\right)_{ij}=\mathrm{tr}\left(\left(\widehat{A}_{bp}\right)_{ij}\right)^{T}$

Proof :

(i) Let $\left(\widehat{A}_{bp}\right)_{ij}$ and $\left(\widehat{B}_{bp}\right)_{ij}$ be two SOBPFMs (order n)

$$\mathrm{tr}\left(\widehat{A}_{bp}\right)_{ij}=\left(\max\left(\widehat{A}_{bp}^{\ +}\right)_{ij},\min\left(\widehat{A}_{bp}^{\ -}\right)_{ij}\right)$$

$$\mathrm{tr}\left(\widehat{B}_{bp}\right)_{ij}=\left(\max\left(\widehat{B}_{bp}^{\ +}\right)_{ij},\min\left(\widehat{B}_{bp}^{\ -}\right)_{ij}\right)$$

Then $\left(\widehat{A}_{bp}+\widehat{B}_{bp}\right)_{ij}=\left(\hat{C}_{bp}\right)_{ij}$, where $\hat{C}_{bp}=\left(\hat{C}_{bp}^{\ +},\hat{C}_{bp}^{\ -}\right)$. By the definition of trace of SOBPFM, we have

$$\begin{aligned}\mathrm{tr}\left(\hat{C}_{bp}\right)_{ij}&=\left(\max\left\{\max\left\{\left(\widehat{A}_{bp}^{\ +}\right)_{ij},\left(\widehat{B}_{bp}^{\ +}\right)_{ij}\right\}\right\},\min\left\{\min\left\{\left(\widehat{A}_{bp}^{\ -}\right)_{ij},\left(\widehat{B}_{bp}^{\ -}\right)_{ij}\right\}\right\}\right)\\&=\left(\max\left\{\max\left(\widehat{A}_{bp}^{\ +}\right)_{ij},\max\left(\widehat{B}_{bp}^{\ +}\right)_{ij}\right\},\min\left\{\min\left(\widehat{A}_{bp}^{\ -}\right)_{ij},\min\left(\widehat{B}_{bp}^{\ -}\right)_{ij}\right\}\right)\\&=\mathrm{tr}\left(\widehat{A}_{bp}\right)_{ij}+\mathrm{tr}\left(\widehat{B}_{bp}\right)_{ij}\end{aligned}$$

(ii) $$\begin{aligned}\mathrm{tr}\left(\lambda\left(\widehat{A}_{bp}\right)_{ij}\right)&=\left(\max\left(\lambda\left(\widehat{A}_{bp}^{\ +}\right)_{ij}\right),\min\left(\lambda\left(\widehat{A}_{bp}^{\ -}\right)_{ij}\right)\right)\\&=\lambda\left(\max\left(\widehat{A}_{bp}^{\ +}\right)_{ij},\min\left(\widehat{A}_{bp}^{\ -}\right)_{ij}\right)\\&=\lambda\mathrm{tr}\left(\widehat{A}_{bp}\right)_{ij}\end{aligned}$$

(iii) Proof is obvious.

4. Properties of SDO on SOBPFM

In this section SDO denoted by ⊖ is introduced over SOBPFM. Some properties are discussed.

4.1 Definition

Let $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{p\times q}$ and $\left(\left(\widehat{B}_{bp}\right)_{ij}\right)_{p\times q}$ are two SOBPFM of same order. The SDO⊖ over $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{p\times q}$ and $\left(\left(\widehat{B}_{bp}\right)_{ij}\right)_{p\times q}$ using basic operation $\wedge,\vee$ and complement is defined as

Let $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{p\times q}=\left(\left(\widehat{A}_{bp}^{\ +},\widehat{A}_{bp}^{\ -}\right)_{ij}\right)_{p\times q}$ & $\left(\left(\widehat{B}_{bp}\right)_{ij}\right)_{p\times q}=\left(\left(\widehat{B}_{bp}^{\ +},\widehat{B}_{bp}^{\ -}\right)_{ij}\right)_{p\times q}$ be $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{p\times q}\ominus\left(\left(\widehat{B}_{bp}\right)_{ij}\right)_{p\times q}=\left(\left(\left(\widehat{A}_{bp}\wedge\widehat{B}_{bp}^{\ c}\right)\vee\left(\widehat{A}_{bp}^{\ c}\wedge\widehat{B}_{bp}\right)\right)_{ij}\right)_{p\times q}=\left(\left(\hat{F}_{bp}\right)_{ij}\right)_{p\times q}$

Where $\left(\left(\hat{F}_{bp}\right)_{ij}\right)_{p\times q}=\left(\left(\hat{F}_{bp}^{\ +},\hat{F}_{bp}^{\ -}\right)_{ij}\right)_{p\times q}$

$\widehat{F}_{bp}{}^{+} = \left(\left(\left(\widehat{A}_{bp}{}^{+} \wedge \left(\widehat{B}_{bp}{}^{+}\right)^{c}\right) \vee \left(\left(\widehat{A}_{bp}{}^{+}\right)^{c} \wedge \widehat{B}_{bp}{}^{+}\right)\right)_{ij}\right)_{p\times q}$ and

$\widehat{F}_{bp}{}^{-} = \left(\left(\left(\widehat{A}_{bp}{}^{-} \vee \left(\widehat{B}_{bp}{}^{-}\right)^{c}\right) \wedge \left(\left(\widehat{A}_{bp}{}^{-}\right)^{c} \vee \widehat{B}_{bp}{}^{-}\right)\right)_{ij}\right)_{p\times q}$

4.2 Lemma

For any $\left(\widehat{A}_{bp}\right)_{ij}$ and $\left(\widehat{B}_{bp}\right)_{ij} \in (SOBPFM)_{m\times n}$. Then

(i) $\left(\widehat{A}_{bp}\right)_{ij} \ominus \left(\widehat{B}_{bp}\right)_{ij} = \left(\widehat{B}_{bp}\right)_{ij} \ominus \left(\widehat{A}_{bp}\right)_{ij}$

(ii) $\left(\widehat{A}_{bp}\right)_{ij} \ominus \left(\widehat{1}_{bp}\right)_{ij} = \left(\widehat{A}_{bp}{}^{c}\right)_{ij}$, where $\left(\widehat{1}_{bp}\right)_{ij} = \left(\widehat{1}_{bp}{}^{+}, \widehat{1}_{bp}{}^{-}\right)_{ij}$

(iii) $\left(\widehat{A}_{bp}\right)_{ij} \ominus \left(\widehat{0}_{bp}\right)_{ij} = \left(\widehat{A}_{bp}\right)_{ij}$

(iv) $\left(\widehat{A}_{bp}\right)_{ij} \ominus \left(\widehat{A}_{bp}\right)_{ij} = \left(\widehat{A}_{bp}\right)_{ij} \wedge \left(\widehat{A}_{bp}{}^{c}\right)_{ij}$

Proof:

(i) From the above definition

$$\left(\widehat{A}_{bp}\right)_{ij} \ominus \left(\widehat{B}_{bp}\right)_{ij}$$

$$= \left[\left(\left(\widehat{A}_{bp}{}^{+} \wedge \left(\widehat{B}_{bp}{}^{+}\right)^{c}\right) \vee \left(\left(\widehat{A}_{bp}{}^{+}\right)^{c} \wedge \widehat{B}_{bp}{}^{+}\right), \left(\widehat{A}_{bp}{}^{-} \vee \left(\widehat{B}_{bp}{}^{-}\right)^{c}\right) \wedge \left(\left(\widehat{A}_{bp}{}^{-}\right)^{c} \vee \widehat{B}_{bp}{}^{-}\right)\right)_{ij}\right]$$

$$= \left[\left(\left(\left(\widehat{B}_{bp}{}^{+}\right)^{c} \wedge \widehat{A}_{bp}{}^{+}\right) \vee \left(\widehat{B}_{bp}{}^{+} \wedge \left(\widehat{A}_{bp}{}^{+}\right)^{c}\right), \left(\left(\widehat{B}_{bp}{}^{-}\right)^{c} \vee \widehat{A}_{bp}{}^{-}\right) \wedge \left(\widehat{B}_{bp}{}^{-} \vee \left(\widehat{A}_{bp}{}^{-}\right)^{c}\right)\right)_{ij}\right]$$

$$= \left[\left(\left(\widehat{B}_{bp}{}^{+} \wedge \left(\widehat{A}_{bp}{}^{+}\right)^{c}\right) \vee \left(\left(\widehat{B}_{bp}{}^{+}\right)^{c} \wedge \widehat{A}_{bp}{}^{+}\right), \left(\widehat{B}_{bp}{}^{-} \vee \left(\widehat{A}_{bp}{}^{-}\right)^{c}\right) \wedge \left(\left(\widehat{B}_{bp}{}^{-}\right)^{c} \vee \widehat{A}_{bp}{}^{-}\right)\right)_{ij}\right]$$

$$= \left(\widehat{B}_{bp}\right)_{ij} \ominus \left(\widehat{A}_{bp}\right)_{ij}$$

(ii) $\left(\widehat{A}_{bp}\right)_{ij} \ominus \left(\widehat{1}_{bp}\right)_{ij}$

$$= \left[\left(\left(\widehat{A}_{bp}{}^{+} \wedge \widehat{0}_{bp}{}^{+}\right) \vee \left(\left(\widehat{A}_{bp}{}^{+}\right)^{c} \wedge \widehat{1}_{bp}{}^{+}\right), \left(\left(\widehat{A}_{bp}{}^{-}\right)^{c} \vee \widehat{1}_{bp}{}^{-}\right) \wedge \left(\widehat{A}_{bp}{}^{-} \vee \widehat{0}_{bp}{}^{-}\right)\right)_{ij}\right]$$

$$= \left[\left(\widehat{0}_{bp}{}^{+} \vee \left(\widehat{A}_{bp}{}^{+}\right)^{c}, \left(\widehat{A}_{bp}{}^{-}\right)^{c} \wedge \widehat{0}_{bp}{}^{-}\right)_{ij}\right]$$

$$= \left(\left(\widehat{A}_{bp}{}^{+}\right)^{c}, \left(\widehat{A}_{bp}{}^{-}\right)^{c}\right)_{ij}$$

$$= \left(\left(\widehat{A}_{bp}\right)^{c}\right)_{ij}$$

(iii) Proof is similar as (ii)

(iv) $\left(\hat{A}_{bp}\right)_{ij} \ominus \left(\hat{A}_{bp}\right)_{ij}$

$$= \left[\left(\left(\hat{A}_{bp}{}^{+} \wedge \left(\hat{A}_{bp}{}^{+}\right)^{c}\right) \vee \left(\left(\hat{A}_{bp}{}^{+}\right)^{c} \wedge \hat{A}_{bp}{}^{+}\right), \left(\hat{A}_{bp}{}^{-} \vee \left(\hat{A}_{bp}{}^{-}\right)^{c}\right) \wedge \left(\left(\hat{A}_{bp}{}^{-}\right)^{c} \vee \hat{A}_{bp}{}^{-}\right)\right)_{ij}\right]$$

$$= \left[\left(\hat{A}_{bp}{}^{+} \wedge \left(\hat{A}_{bp}{}^{+}\right)^{c}, \left(\hat{A}_{bp}{}^{-}\right)^{c} \vee \hat{A}_{bp}{}^{-}\right)_{ij}\right]$$

$$= \left(\hat{A}_{bp} \wedge \left(\hat{A}_{bp}\right)^{c}\right)_{ij}$$

4.3 Lemma

For any $\left(\hat{A}_{bp}\right)_{ij}$ and $\left(\hat{B}_{bp}\right)_{ij} \in (\mathrm{SOBPFM})_{m\times n}$. Then

$$\left(\left(\hat{A}_{bp}\right)^{c}\right)_{ij} \ominus \left(\left(\hat{B}_{bp}\right)^{c}\right)_{ij} = \left(\hat{A}_{bp}\right)_{ij} \ominus \left(\hat{B}_{bp}\right)_{ij}$$

Proof :

$$\left(\left(\hat{A}_{bp}\right)^{c}\right)_{ij} \ominus \left(\left(\hat{B}_{bp}\right)^{c}\right)_{ij}$$

$$= \left[\left(\left(\left(\hat{A}_{bp}{}^{+}\right)^{c} \wedge \hat{B}_{bp}{}^{+}\right) \vee \left(\hat{A}_{bp}{}^{+} \wedge \left(\hat{B}_{bp}{}^{+}\right)^{c}\right), \left(\left(\hat{A}_{bp}{}^{-}\right)^{c} \vee \hat{B}_{bp}{}^{-}\right) \wedge \left(\hat{A}_{bp}{}^{-} \vee \left(\hat{B}_{bp}{}^{-}\right)^{c}\right)\right)_{ij}\right]$$

$$= \left[\left(\left(\hat{A}_{bp}{}^{+} \wedge \left(\hat{B}_{bp}{}^{+}\right)^{c}\right) \vee \left(\left(\hat{A}_{bp}{}^{+}\right)^{c} \wedge \hat{B}_{bp}{}^{+}\right), \left(\hat{A}_{bp}{}^{-} \vee \left(\hat{B}_{bp}{}^{-}\right)^{c}\right) \wedge \left(\left(\hat{A}_{bp}{}^{-}\right)^{c} \vee \hat{B}_{bp}{}^{-}\right)\right)_{ij}\right]$$

$$= \left(\hat{A}_{bp}\right)_{ij} \ominus \left(\hat{B}_{bp}\right)_{ij}$$

4.4 Lemma

For any $\left(\hat{A}_{bp}\right)_{ij}$, $\left(\hat{B}_{bp}\right)_{ij}$ and $\left(\hat{C}_{bp}\right)_{ij} \in$ SOBPFM with same order. Then

$$\left(\hat{A}_{bp}\right)_{ij} \ominus \left(\left(\hat{B}_{bp}\right)_{ij} \ominus \left(\hat{C}_{bp}\right)_{ij}\right) = \left(\left(\hat{A}_{bp}\right)_{ij} \ominus \left(\hat{B}_{bp}\right)_{ij}\right) \ominus \left(\hat{C}_{bp}\right)_{ij}$$

Proof:

The above property is proved by an illustrative example as follows:

$$\left(\hat{A}_{bp}\right)_{ij} = \begin{bmatrix} (0.7,-0.1) & (0.8,-0.5) \\ (0.1,-0.8) & (0.3,-0.2) \end{bmatrix} \quad \left(\left(\hat{A}_{bp}\right)^{c}\right)_{ij} = \begin{bmatrix} (0.3,-0.9) & (0.2,-0.5) \\ (0.9,-0.2) & (0.7,-0.8) \end{bmatrix}$$

$$\left(\hat{B}_{bp}\right)_{ij} = \begin{bmatrix} (0.8,-0.5) & (0.2,-0.7) \\ (0.6,-0.4) & (0.3,-0.9) \end{bmatrix} \quad \left(\left(\hat{B}_{bp}\right)^{c}\right)_{ij} = \begin{bmatrix} (0.2,-0.5) & (0.8,-0.3) \\ (0.4,-0.6) & (0.7,-0.1) \end{bmatrix}$$

$$\left(\hat{C}_{bp}\right)_{ij} = \begin{bmatrix} (0.1,-0.3) & (0.3,-0.8) \\ (0.4,-0.5) & (0.9,-0.7) \end{bmatrix} \quad \left(\left(\hat{C}_{bp}\right)^{c}\right)_{ij} = \begin{bmatrix} (0.9,-0.7) & (0.7,-0.2) \\ (0.6,-0.5) & (0.1,-0.3) \end{bmatrix}$$

$$\left(\hat{B}_{bp}\right)_{ij} \ominus \left(\hat{C}_{bp}\right)_{ij} = \begin{bmatrix} (0.8,-0.5) & (0.3,-0.3) \\ (0.6,-0.5) & (0.7,-0.3) \end{bmatrix}$$

$$(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} \ominus \left((\widehat{\mathrm{B}}_{\mathrm{bp}})_{\mathrm{ij}} \ominus (\widehat{\mathrm{C}}_{\mathrm{bp}})_{\mathrm{ij}}\right) = \begin{bmatrix} (0.3,-0.5) & (0.7,-0.5) \\ (0.6,-0.5) & (0.7,-0.3) \end{bmatrix}$$

$$(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} \ominus (\widehat{\mathrm{B}}_{\mathrm{bp}})_{\mathrm{ij}} = \begin{bmatrix} (0.3,-0.5) & (0.8,-0.5) \\ (0.6,-0.6) & (0.3,-0.8) \end{bmatrix}$$

$$(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} \ominus \left((\widehat{\mathrm{B}}_{\mathrm{bp}})_{\mathrm{ij}} \ominus (\widehat{\mathrm{C}}_{\mathrm{bp}})_{\mathrm{ij}}\right) = \begin{bmatrix} (0.3,-0.5) & (0.7,-0.5) \\ (0.6,-0.5) & (0.7,-0.3) \end{bmatrix}$$

Hence $(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} \ominus \left((\widehat{\mathrm{B}}_{\mathrm{bp}})_{\mathrm{ij}} \ominus (\widehat{\mathrm{C}}_{\mathrm{bp}})_{\mathrm{ij}}\right) = \left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}} \ominus (\widehat{\mathrm{B}}_{\mathrm{bp}})_{\mathrm{ij}}\right) \ominus (\widehat{\mathrm{C}}_{\mathrm{bp}})_{\mathrm{ij}}$.

5. An application on second order bipolar fuzzy matrix in medical diagnosis

In this paper, we propose an algorithm to find what type of fever affects the different group of people the most.

Let $\mathrm{F} = \{\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3, \mathrm{f}_4\}$ be the set of all fevers where $\mathrm{f}_1 =$ Dengue, $\mathrm{f}_2 =$ Swine flu, $\mathrm{f}_3 =$ Malaria, $\mathrm{f}_4 =$ Typhoid. Let $\mathrm{S} = \{\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3, \mathrm{s}_4\}$ be the set of all symptoms caused by the above types of fever where $\mathrm{s}_1 =$ Decrease in platelet count, $\mathrm{s}_2 =$ fatique, $\mathrm{s}_3 =$ Lose of apetite, $\mathrm{s}_4 =$ Eye pain.

Let $\mathrm{P} = \{\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3, \mathrm{p}_4\}$ be the set of all patients of different age groups where

$\mathrm{p}_1 -$ Kids aging between 1-12

$\mathrm{p}_2 -$ Teenagers aging between 13-19

$\mathrm{p}_3 -$ Adult aging between 20-50

$\mathrm{p}_4 -$ Old people aging between 50 & above

Now three second order bipolar fuzzy matrices $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_0$ which gives information about the types of fever for the common symptoms, $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_S$ which indicates common symptoms that occur in the different age group of people and $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_{¢}$ which shows which common symptom indicates the intensity of the fever are constructed by a decision-maker. Then the event intensity relation $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_1$, conformability intensity relation $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_2$, the non-event intensity relation $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_3$ and non-symptom intensity relation $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_4$ are determined.

5.1 Algorithm

Step 1: Enter the SOBPFM $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_0$ which indicates an event of types of fever for the common symptoms.

Step 2: Enter the SOBPFM $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_S$ which indicates common symptoms that occur in the different age group of people.

Step 3: Enter the second order bipolar fuzzy matrix $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_{¢}$ which indicates the confirmative relation given by $\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{ij}}\right)_{¢} = \mathrm{S} * \mathrm{F}$.

Step 4: Compute the event intensity relation $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{1} = \left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{S} * \left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{0}$.

Step 5: Compute conformability intensity relation $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{2} = \left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{S} * \left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{¢}$.

Step 6: Compute non-event intensity relation $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{3} = \left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{S} * \left(\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{0}\right)^{c}$.

Step 7: Compute non-symptom intensity relation $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{4} = \left(\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{S}\right)^{c} * \left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{0}$.

Step 8: Separate the positive and negative values in $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{4}$

5.2 Problem

Symptoms with fever:

Symptoms	Temperature Reference level	Temperature level in patient	Temperature level in [0,1]
s_1	101^0-105^0 for $\mathbf{f_1}$	102.3^0	0.4
s_2	102^0-106^0 for $\mathbf{f_2}$	101.2^0	0
s_3	99^0-105^0 for $\mathbf{f_3}$	104.3^0	0.9
s_4	103^0-104^0 for $\mathbf{f_4}$	103.8^0	0.8

Solution:

Step 1: Matrix for event relation $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{0} = S \times F$ shows the types of fever for the common symptoms

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad f_1 & \qquad f_2 & \qquad f_3 & \qquad f_4 \end{array} \\ \begin{bmatrix} (0.7,-0.6) & (0.6,-0.8) & (0.9,-0.5) & (0.2,-0.9) \\ (0.5,-0.4) & (0.7,-0.7) & (0.8,-0.9) & (0.1,-0.6) \\ (0.4,-0.2) & (0.1,-0.8) & (0.3,-0.4) & (0.8,-0.4) \\ (0.5,-0.3) & (0.4,-0.2) & (0.6,-0.7) & (0.7,-0.1) \end{bmatrix} \end{array}$$

Step 2: The following SOBPFM $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{S}$ indicates the common symptoms that occurs in different age group of people.

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad s_1 & \qquad s_2 & \qquad s_3 & \qquad s_4 \end{array} \\ \begin{bmatrix} (0.6,-0.7) & (0.8,-0.5) & (0.4,-0.6) & (0.1,-0.7) \\ (0.6,-0.8) & (0.3,-0.1) & (0.1,-0.2) & (0.2,-0.6) \\ (0.7,-0.6) & (0.5,-0.4) & (0.2,-0.4) & (0.6,-0.1) \\ (0.9,-0.2) & (0.6,-0.3) & (0.5,-0.5) & (0.9,-0.8) \end{bmatrix} \end{array}$$

Step 3: Assuming that the matrix for confirmative relation $\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{\text{₵}} = \mathrm{S} \times \mathrm{F}$

$$\begin{array}{cc} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} & \begin{bmatrix} (0.5,-0.7) & (0.7,-0.6) & (0.3,-0.8) & (0.1,-0.2) \\ (0.4,-0.2) & (0.5,-0.1) & (0.1,-0.3) & (0.8,-0.7) \\ (0.1,-0.3) & (0.3,-0.5) & (0.1,-0.6) & (0.2,-0.5) \\ (0.5,-0.2) & (0.9,-0.8) & (0.7,-0.4) & (0.6,-0.3) \end{bmatrix} \end{array}$$

The following four types of relations can be constructed using the relations $\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{0}, \left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{\mathrm{S}}$ and $\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{\text{₵}}$.

Step 4: The event intensity relation$\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{1} = \left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{\mathrm{S}} * \left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{0}$

$$\begin{array}{cc} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} & \begin{bmatrix} (0.6,-0.6) & (0.6,-0.5) & (0.4,-0.5) & (0.1,-0.7) \\ (0.5,-0.4) & (0.3,-0.1) & (0.1,-0.2) & (0.1,-0.6) \\ (0.4,-0.2) & (0.1,-0.4) & (0.2,-0.4) & (0.6,-0.1) \\ (0.5,-0.2) & (0.4,-0.2) & (0.5,-0.5) & (0.7,-0.1) \end{bmatrix} \end{array}$$

Step 5: The conformability intensity relation $\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{2} = \left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{\mathrm{S}} * \left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{\text{₵}}$

$$\begin{array}{cc} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} & \begin{bmatrix} (0.5,-0.7) & (0.7,-0.5) & (0.3,-0.6) & (0.1,-0.2) \\ (0.4,-0.2) & (0.3,-0.1) & (0.1,-0.2) & (0.2,-0.6) \\ (0.1,-0.3) & (0.3,-0.4) & (0.1,-0.4) & (0.2,-0.1) \\ (0.5,-0.2) & (0.6,-0.3) & (0.5,-0.4) & (0.6,-0.3) \end{bmatrix} \end{array}$$

Step 6: The non-event intensity relation $\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{3} = \left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{\mathrm{S}} * \left(\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{ij}}\right)_{0}\right)^{\mathrm{c}}$

$$\left(\left(\left(\hat{A}_{bp}\right)_{ij}\right)_{0}\right)^{C} = \begin{array}{cc} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} & \begin{bmatrix} (0.3,-0.4) & (0.4,-0.2) & (0.1,-0.5) & (0.8,-0.1) \\ (0.5,-0.6) & (0.3,-0.3) & (0.2,-0.1) & (0.9,-0.4) \\ (0.6,-0.8) & (0.9,-0.2) & (0.7,-0.6) & (0.2,-0.6) \\ (0.5,-0.7) & (0.6,-0.8) & (0.4,-0.3) & (0.3,-0.9) \end{bmatrix} \end{array}$$

$$\left(\left(\hat{A}_{bp}\right)_{ij}\right)_{3} = \begin{array}{cc} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} & \begin{bmatrix} (0.3,-0.4) & (0.4,-0.5) & (0.1,-0.5) & (0.1,-0.1) \\ (0.5,-0.6) & (0.3,-0.1) & (0.1,-0.1) & (0.2,-0.4) \\ (0.6,-0.6) & (0.5,-0.2) & (0.2,-0.4) & (0.2,-0.1) \\ (0.5,-0.2) & (0.6,-0.3) & (0.4,-0.3) & (0.3,-0.8) \end{bmatrix} \end{array}$$

Step 7: Eventually, the non-symptom intensity relation

$$\left(\left(\hat{A}_{bp}\right)_{ij}\right)_4 = \left(\left(\left(\hat{A}_{bp}\right)_{ij}\right)_S\right)^c * \left(\left(\hat{A}_{bp}\right)_{ij}\right)_0$$

$$\left(\left(\left(\hat{A}_{bp}\right)_{ij}\right)_S\right)^C = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{bmatrix} (0.4,-0.3) & (0.2,-0.5) & (0.6,-0.4) & (0.9,-0.3) \\ (0.4,-0.2) & (0.7,-0.9) & (0.9,-0.8) & (0.8,-0.4) \\ (0.3,-0.4) & (0.5,-0.6) & (0.8,-0.6) & (0.4,-0.9) \\ (0.1,-0.8) & (0.4,-0.7) & (0.5,-0.5) & (0.1,-0.2) \end{bmatrix} \end{array}$$

$$\left(\left(\hat{A}_{bp}\right)_{ij}\right)_4 = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} (0.4,-0.3) & (0.2,-0.5) & (0.6,-0.4) & (0.2,-0.3) \\ (0.4,-0.2) & (0.7,-0.7) & (0.8,-0.8) & (0.1,-0.4) \\ (0.3,-0.2) & (0.1,-0.6) & (0.3,-0.4) & (0.4,-0.4) \\ (0.1,-0.3) & (0.4,-0.2) & (0.5,-0.5) & (0.1,-0.1) \end{bmatrix} \end{array}$$

Step 8: Separate the positive and negative values in $\left(\left(\hat{A}_{bp}\right)_{ij}\right)_4$

$$\left(\left(\hat{A}_{bp}{}^{+}\right)_{ij}\right)_4 = \begin{bmatrix} 0.4 & 0.2 & 0.6 & 0.2 \\ 0.4 & 0.7 & 0.8 & 0.1 \\ 0.3 & 0.1 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.5 & 0.1 \end{bmatrix}$$

$$\left(\left(\hat{A}_{bp}{}^{-}\right)_{ij}\right)_4 = \begin{bmatrix} -0.3 & -0.5 & -0.4 & -0.3 \\ -0.2 & -0.7 & -0.8 & -0.4 \\ -0.2 & -0.6 & -0.4 & -0.4 \\ -0.3 & -0.2 & -0.5 & -0.1 \end{bmatrix}$$

6. Conclusion

In this article, the definition of second order bipolar fuzzy matrix is presented and their properties are discussed. Also symmetrical difference operation has been extended over SOBPFM and its properties are studied. In addition to this, a decision making problem in the medical field has been presented where the resulting SOBPFM $\left(\left(\hat{A}_{bp}{}^{+}\right)_{ij}\right)_4$ concludes that the patient p_2 has the high intensity of swine flu and malaria and patient p_4 has low intensity of Dengue and typhoid. It is to be noted that intensity of fever in the different age group of people may vary depending upon the temperature levels recorded.

7. References

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