Chapter 4

Associated Spaces using λ_g^δ-closed sets

4.1 Intoduction

This Chapter comprises of the application of λ_g^δ-closed sets namely separation spaces which is considered to be interesting because of its property of reversing the dependence relations. Five new spaces are introduced and their properties are analyzed.

4.2 Separation Spaces

Definition 4.2.1. A topological space (X, τ) is said to be a

(i) $_{\lambda_g^\delta}T_\delta$ ***-space*** if every λ_g^δ-closed subset of (X, τ) is δ-closed in (X, τ).

(ii) $_{\delta g^*}T_{\lambda_g^\delta}$ ***-space*** if every δg^*-closed subset of (X, τ) is λ_g^δ-closed in (X, τ).

(iii) $_{g\delta}T_{\lambda_g^\delta}$ ***-space*** if every $g\delta$-closed subset of (X, τ) is λ_g^δ-closed in (X, τ).

(iv) $_{\delta gs}T_{\lambda_g^\delta}$ ***-space*** if every δgs-closed subset of (X, τ) is λ_g^δ-closed in (X, τ).

(v) $_{g\delta s}T_{\lambda_g^\delta}$ ***-space*** if every $g\delta s$-closed subset of (X, τ) is λ_g^δ-closed in (X, τ).

Proposition 4.2.2. If a topological space (X, τ) is $_{\lambda_g^\delta}T_\delta$ with $A \subseteq X$ then the following are equivalent.

(i) A is λ_g^δ-closed;

(ii) A is δ-closed.

Proof. From Proposition 2.2.3(i) and the Definition 4.2.1 (i), the proof follows. □

Proposition 4.2.3. If a topological space (X, τ) is ${}_{\delta g^*}T_{\lambda_g^\delta}$ with $A \subseteq X$ then the following are equivalent.

(i) A is λ_g^δ-closed;

(ii) A is δg^*-closed.

Proof. From Proposition 2.2.6 and Definition 4.2.1 (ii), the proof follows. □

Proposition 4.2.4. If a topological space (X, τ) is ${}_{g\delta}T_{\lambda_g^\delta}$ with $A \subseteq X$ then the following are equivalent.

(i) A is λ_g^δ-closed;

(ii) A is δg^*-closed;

(iii) A is $g\delta$-closed.

Proof. From Proposition 4.2.3, Proposition 2.2.3(ii) and Definition 4.2.1 (iii), the proof follows. □

Proposition 4.2.5. If a topological space (X, τ) is ${}_{\delta gs}T_{\lambda_g^\delta}$ with $A \subseteq X$ then the following are equivalent.

(i) A is λ_g^δ-closed;

(ii) A is δg^*-closed;

(iii) A is δgs-closed.

Proof. From Proposition 4.2.3, Proposition 2.2.3(iii) and Definition 4.2.1 (iv), the proof follows. □

Proposition 4.2.6. If a topological space (X, τ) is ${}_{g\delta s}T_{\lambda_g^\delta}$ with $A \subseteq X$ then the following are equivalent.

(i) A is λ_g^δ-closed;

(ii) A is δg^*-closed(resp. δgs-closed);

(iii) A is $g\delta s$-closed.

Proof. (i) $\Leftrightarrow$ (ii) follows from Proposition 4.2.3(resp. 4.2.3.
(i) $\Rightarrow$ (iii) follows from Proposition 2.2.3(iv).
(iii) $\Rightarrow$ (i) follows from Definition 4.2.1 (v). □

Proposition 4.2.7. For a topological space (X, τ) with subset A, the following conditions hold:

(i) If (X, τ) is a ${}_{\lambda_g^\delta}T_\delta$ -space, $\lambda_g^\delta cl(A) = cl_\delta(A)$.

(ii) If (X, τ) is a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space, $g\delta scl(A) = \lambda_g^\delta cl(A)$.

Proposition 4.2.8. Every ${}_{g\delta s}T_{\lambda_g^\delta}$ -space is a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space.

Proof. Let (X, τ) be a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space and A be a δgs-closed set in (X, τ). Since every δgs-closed set is $g\delta s$-closed, A is $g\delta s$-closed. Since (X, τ) is ${}_{g\delta s}T_{\lambda_g^\delta}$, A is λ_g^δ-closed in (X, τ). Hence (X, τ) is a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space. □

Proposition 4.2.9. Every ${}_{\delta gs}T_{\lambda_g^\delta}$ -space is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space but not conversely.

Proof. Let (X, τ) be a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space and A be a$g\delta$-closed set in (X, τ). Since every $g\delta$-closed set is δgs-closed, A is δgs-closed. Since (X, τ) is ${}_{\delta gs}T_{\lambda_g^\delta}$, A is λ_g^δ-closed in (X, τ). Hence (X, τ) is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space. □

Example 4.2.10. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space but not a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space.

Proposition 4.2.11. Every ${}_{g\delta}T_{\lambda_g^\delta}$ -space is a ${}_{\delta g^*}T_{\lambda_g^\delta}$ -space but not conversely.

Proof. Let (X,τ) be a ${}_{g\delta}T_{\lambda_g^\delta}$ -space and A be a δg^*-closed set in(X,τ). Then A is $g\delta$-closed as every δg^*-closed set is $g\delta$-closed. Since (X,τ) is ${}_{g\delta}T_{\lambda_g^\delta}$, A is λ_g^δ-closed in (X,τ). Hence (X,τ) is a ${}_{\delta g^*}T_{\lambda_g^\delta}$ -space. □

Example 4.2.12. Let $X = \{a,b,c,d,e\}$ and $\tau = \{X,\phi,\{b\},\{a,b\},\{b,c\},\{a,b,c\}\}$. Then (X,τ) is a ${}_{\delta g^*}T_{\lambda_g^\delta}$ -space but not a ${}_{g\delta}T_{\lambda_g^\delta}$ -space.

Proposition 4.2.13. Every ${}_{g\delta s}T_{\lambda_g^\delta}$ -space is a ${}_{\delta g^*}T_{\lambda_g^\delta}$-space but not conversely.

Proof. Let (X,τ) be a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space and A be a δg^*-closed set in (X,τ). Then A is $g\delta s$-closed as every δg^*-closed set is $g\delta s$-closed. Since (X,τ) is ${}_{g\delta s}T_{\lambda_g^\delta}$, A is λ_g^δ-closed in (X,τ). Hence (X,τ) is a ${}_{\delta g^*}T_{\lambda_g^\delta}$ -space. □

Example 4.2.14. Let X and τ be defined as in Example 4.2.10. Then (X,τ) is a ${}_{\delta g^*}T_{\lambda_g^\delta}$ -space but not a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space.

Proposition 4.2.15. Every ${}_{g\delta s}T_{\lambda_g^\delta}$ -space is a ${}_{g\delta}T_{\lambda_g^\delta}$-space but not conversely.

Proof. Let (X,τ) be a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space and A be a $g\delta$-closed set in (X,τ). Then A is $g\delta s$-closed as every $g\delta$-closed set is $g\delta s$-closed. Since (X,τ) is ${}_{g\delta s}T_{\lambda_g^\delta}$, A is λ_g^δ-closed in (X,τ). Hence (X,τ) is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space. □

Example 4.2.16. Let X and τ be defined as in Example 4.2.10. Then (X,τ) is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space but not a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space.

Proposition 4.2.17. Every ${}_{\delta gs}T_{\lambda_g^\delta}$ -space is a ${}_{\delta g^*}T_{\lambda_g^\delta}$-space but not conversely.

Proof. Let (X,τ) be a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space and A be a δg^*-closed set in (X,τ). Since every δg^*-closed set is δgs-closed, A is δgs-closed. Since (X,τ) is ${}_{\delta gs}T_{\lambda_g^\delta}$, A is λ_g^δ-closed in (X,τ). Hence (X,τ) is a ${}_{\delta g^*}T_{\lambda_g^\delta}$ -space. □

Example 4.2.18. Let X and τ be defined as in Example 4.2.10. Then (X,τ) is a ${}_{\delta g^*}T_{\lambda_g^\delta}$ -space but not a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space.

Remark 4.2.19. ${}_{\lambda_g^\delta}T_\delta$ -space and ${}_{g\delta s}T_{\lambda_g^\delta}$ -space are independent of each other.

Example 4.2.20. Let X and τ be defined as in Example 4.2.10. Then (X,τ) is a

${}_{\lambda_g^\delta}T_\delta$ -space but not a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space.

Example 4.2.21. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$ then (X, τ) is a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space but not a ${}_{\lambda_g^\delta}T_\delta$-space.

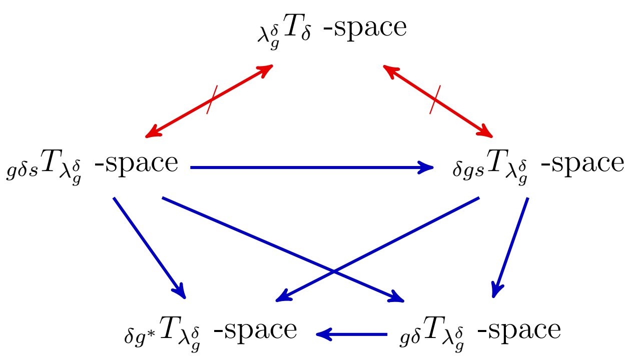
Remark 4.2.22. ${}_{\lambda_g^\delta}T_\delta$ -space and ${}_{\delta gs}T_{\lambda_g^\delta}$ -space are independent of each other.

Example 4.2.23. Let X and τ be defined as in Example 4.2.10. Then (X, τ) is a ${}_{\lambda_g^\delta}T_\delta$ -space but not a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space.

Example 4.2.24. Let X and τ be defined as in Example 4.2.21. Then (X, τ) is a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space but not a ${}_{\lambda_g^\delta}T_\delta$ -space.

Remark 4.2.25. The following figure represents the interrelationship between the five newly introduced separation spaces.

Figure 4.1:



4.3 Relationship between the newly introduced spaces and some of the existing spaces

Proposition 4.3.1. Every T_δ -space is a ${}_{\lambda_g^\delta}T_\delta$ -space

Proof. Let A be a λ_g^δ-closed set and let (X, τ) be a T_δ -space. Since every λ_g^δ-closed set is a $g\delta$-closed set, A is $g\delta$-closed. As (X, τ) is a T_δ -space, A is δ-closed. Hence (X, τ) is a ${}_{\lambda_g^\delta}T_\delta$ -space. □

Proposition 4.3.2. Every T_δ -space is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space but not conversely.

Proof. Let (X, τ) be a T_δ -space and A be a $g\delta$-closed set in (X, τ). Since (X, τ) is a T_δ -space, A is δ-closed in (X, τ). Since every δ-closed set is λ_g^δ-closed, A is λ_g^δ-closed in (X, τ). Hence (X, τ) is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space. □

Example 4.3.3. Let X andτ be defined as in Example 4.2.21. Then (X, τ) is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space but not a T_δ -space.

Proposition 4.3.4. Every T_δ -space is a ${}_{\delta g^*}T_{\lambda_g^\delta}$-space but not conversely.

Proof. Follows from Proposition 4.3.2 and Proposition 4.2.11. □

Example 4.3.5. Let X and τ be defined as in Example 4.2.21. Then (X, τ) is a ${}_{\delta g^*}T_{\lambda_g^\delta}$-space but not a T_δ -space.

Remark 4.3.6. T_δ -space is independent of both ${}_{g\delta s}T_{\lambda_g^\delta}$ -space as well as ${}_{\delta gs}T_{\lambda_g^\delta}$ -space.

Example 4.3.7. Let X and τ be defined as in Example 4.2.10. Then (X, τ) is a T_δ -space but neither a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space nor a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space.

Example 4.3.8. Let X and τ be defined as in Example 4.2.21. Then (X, τ) is a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space as well as a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space but not a T_δ -space.

Proposition 4.3.9. Every ${}_\delta T_{3/4}$ -space is a ${}_{\lambda_g^\delta}T_\delta$ -space but not conversely.

Proof. Let (X, τ) be a ${}_\delta T_{3/4}$ -space and A be a λ_g^δ-closed set in (X, τ). As every λ_g^δ-closed set is a $g\delta s$-closed set and since (X, τ) is ${}_\delta T_{3/4}$, A is δ-closed in (X, τ). Hence (X, τ) is a ${}_{\lambda_g^\delta}T_\delta$ -space. □

Example 4.3.10. Let X and τ be defined as in Example 4.2.10 then (X, τ) is a ${}_{\lambda_g^\delta}T_\delta$ -space but not a ${}_\delta T_{3/4}$ -space.

Proposition 4.3.11. Every ${}_\delta T_{3/4}$ -space is a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space but not conversely.

Proof. Let (X, τ) be a ${}_\delta T_{3/4}$ -space and A be a $g\delta s$-closed set in (X, τ). Since (X, τ)

is ${}_{\delta}T_{3/4}$, A is δ-closed in (X,τ). Since every δ-closed set is λ_g^δ-closed, A is λ_g^δ-closed in (X,τ). Hence (X,τ) is a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space. □

Example 4.3.12. Let X and τ be defined as in Example 4.2.21 then (X,τ) is a ${}_{g\delta s}T_{\lambda_g^\delta}$ -space but not a ${}_{\delta}T_{3/4}$ -space.

Proposition 4.3.13. Every ${}_{\delta}T_{3/4}$ -space is a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space but not conversely.

Proof. Follows from Proposition 4.3.11 and Proposition 4.2.8. □

Example 4.3.14. Let X and τ be defined as in Example 4.2.21 then (X,τ) is a ${}_{\delta gs}T_{\lambda_g^\delta}$ -space but not a ${}_{\delta}T_{3/4}$ -space.

Proposition 4.3.15. Every ${}_{\delta}T_{3/4}$ -space is a ${}_{\delta g^*}T_{\lambda_g^\delta}$-space but not conversely.

Proof. Follows from Proposition 4.3.13 and Proposition 4.2.17. □

Example 4.3.16. Let X and τ be defined as in Example 4.2.10 then (X,τ) is a ${}_{\delta g^*}T_{\lambda_g^\delta}$ -space but not a ${}_{\delta}T_{3/4}$ -space.

Proposition 4.3.17. Every ${}_{\delta}T_{3/4}$ -space is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space.

Proof. Follows from Proposition 4.3.11 and Proposition 4.2.15. □

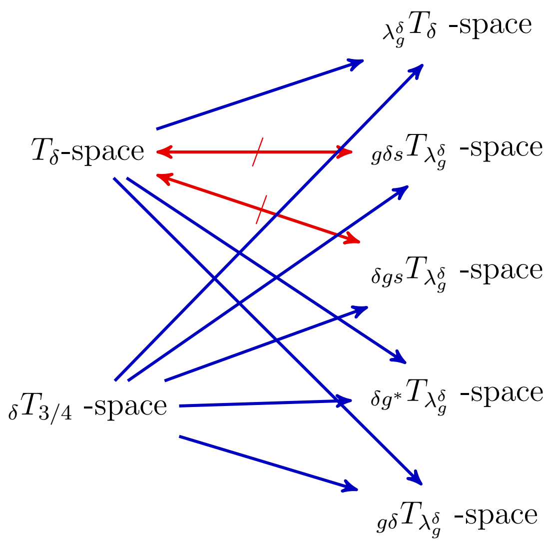
Example 4.3.18. Let X and τ be defined as in Example 4.2.10 then (X,τ) is a ${}_{g\delta}T_{\lambda_g^\delta}$ -space but not a ${}_{\delta}T_{3/4}$ -space.

Remark 4.3.19. The following diagram portrays the relationship between the five newly introduced spaces and two of the already existing spaces namely T_δ -space and ${}_{\delta}T_{3/4}$ -space.

Remark 4.3.20. In general, δg-closed set is independent of λ_g^δ-closed set. However, this independence relation is altered by the separation spaces as observed from the following proposition.

Proposition 4.3.21. Every δg-closed set is a λ_g^δ-closed set in the following spaces:

Figure 4.2:



(i) $T_{3/4}$ -space,

(ii) $_{\delta gs}T_{\lambda_g^\delta}$ -space,

(iii) $_{\delta g^*}T_{\lambda_g^\delta}$ -space,

(iv) $_{g\delta s}T_{\lambda_g^\delta}$ -space.

Proof. (i) In a $T_{3/4}$ -space, every δg-closed set is δ-closed and by Proposition 2.2.3(i), the result follows.

(ii) Let (X, τ) be a topological space and A be a δg-closed set in (X, τ). Every δg-closed set is δgs-closed and hence A is δgs-closed in (X, τ). Since (X, τ) is a $_{\delta gs}T_{\lambda_g^\delta}$-space, A is λ_g^δ-closed.

(iii) Similar to (ii).

(iv) Similar to (ii).

□

Corollary 4.3.22. In a digital lone or Khalimsky line, which is a $T_{3/4}$-space, every δg-closed set is λ_g^δ-closed.

Proposition 4.3.23. If a topological space (X,τ) is ${}_{g\delta}T_{\lambda_g^\delta}$ as well as ${}_{\lambda_g^\delta}T_\delta$ then it is a T_δ -space.

Proof. Let (X,τ) be a ${}_{g\delta}T_{\lambda_g^\delta}$ as well as a ${}_{\lambda_g^\delta}T_\delta$ -space. Let A be a $g\delta$-closed set in (X,τ). Since (X,τ) is ${}_{g\delta}T_{\lambda_g^\delta}$, A is λ_g^δ-closed in (X,τ). Further, since (X,τ) is ${}_{\lambda_g^\delta}T_\delta$, A is δ-closed in (X,τ). Therefore (X,τ) is a T_δ -space. □

Proposition 4.3.24. If a topological space (X,τ) is ${}_{\delta gs}T_{\lambda_g^\delta}$ as well as ${}_{\lambda_g^\delta}T_\delta$ then it is a $T_{3/4}$ -space.

Proof. Let (X,τ) be a ${}_{\delta gs}T_{\lambda_g^\delta}$ as well as ${}_{\lambda_g^\delta}T_\delta$ -space. Let A be a δg-closed set in (X,τ). Since every δg-closed set is δgs-closed and (X,τ) is ${}_{\delta gs}T_{\lambda_g^\delta}$, A is λ_g^δ-closed in (X,τ). Further, since (X,τ) is ${}_{\lambda_g^\delta}T_\delta$, A is δ-closed in (X,τ). Therefore (X,τ) is a $T_{3/4}$ -space. □

Theorem 4.3.25. *For a topological space (X,τ) which is ${}_{\lambda_g^\delta}T_\delta$, the following conditions are equivalent:*

(i) X is almost weakly Hausdorff;

(ii) Every singleton of (X,τ) is δ-closed or δ-open;

(iii) Every singleton of (X,τ) is λ_g^δ-closed or λ_g^δ-open.

Proof. (i) $\Leftrightarrow$ (ii) is proved in Theorem 5.1 of [].
(ii) $\Rightarrow$ (iii) *Case(a):* Let $\{x\}$ be δ-closed. Since every δ-closed set is λ_g^δ-closed, $\{x\}$ is λ_g^δ-closed.
Case(b): Let $\{x\}$ be δ-open. Since every δ-open set is λ_g^δ-open, $\{x\}$ is λ_g^δ-open.
(iii) $\Rightarrow$ (ii) *Case(a):* Let $\{x\}$ be λ_g^δ-closed. Since (X,τ) is ${}_{\lambda_g^\delta}T_\delta$, $\{x\}$ is δ-closed.
Case(b): Let $\{x\}$ be λ_g^δ-open. Since (X,τ) is ${}_{\lambda_g^\delta}T_\delta$, $\{x\}$ is δ-open. □