Chapter 7

CHAPTER VII

NON-MARKOVIAN GENERAL BULK SERVICE QUEUE WITH REPEATED WORKING VACATIONS

Introduction

As a generalization of classical queueing models with server vacations, working vacation queues were introduced in 2002 by Servi and Finn where the server serves the customers at lower rate than completely stopping service during the vacation period. In recent years the research on working vacation queues has considerable developments.

In chapter VI some, Markovian bulk arrival and bulk service queueing models under server working vacations are analysed. In this chapter, a bulk service queueing model, in which the inter arrival times form an independent identically distributed sequence of random variables having general distribution function, is studied under servers multiple working vacation. Baba (2005) has considered a GI / M / 1queue with multiple working vacation and derived the steady state distributions for the number of customers in the system both at arrival and arbitrary epochs. The model studied in this chapter is an extension of the model of Baba (2005) and a generalization of the $M / M_{(a,\ b)}/1/$ MWV model considered in section (6.3). The queueing system studied here is formulated as an embedded Markovian chain by choosing arrival epochs as embedded points. The steady state distribution for the number of customers in queue at arrival epoch is derived by analyzing the embedded Markov chain. The steady state results of Baba (2005) and the queue size distribution of $M/M_{(a,\ b)}/1/$MWV model of section (6.3) are obtained as particular cases.

7.1. Mathematical Analysis of the system

I Model Description

The model studied here differs from the model of section (6.3) only in the inter- arrival time distribution. Consider a $GI/M_{(a,\ b)}/1$ queue with multiple working vacation, in which the server serves the customers in batches according to the general bulk service rule introduced by Neuts (1967). The server takes multiple working vacation whenever the system contains less than '**a**' customers. During vacation time if the queue length exceeds '**a**' then the server starts serving the customers with a lower service rate μ_V. This type of service continues until the vacation time of the server terminates. When the working vacation period terminates, and the system consists of more than '**a**' customers, the server switches to regular service period. The distribution of the service times during regular service period and the service time during a working vacation are both exponential, but their rates are different.

The server repeats another working vacation if he finds less than '**a**' customers at the vacation termination point. The batch rules followed during regular service period and working vacation period are the same. The distribution of service times in the respective period is exponential with the rate μ and μ_V. According to this rule the server starts a service with a minimum of '**a**' customers. At each service completion point if the server finds $a \leq x \leq b$ customers, he takes them all in a batch and if he finds more than '**b**' customers, he takes only '**b**' customers in a batch for service.

II Embedded Markovian Chain of Queue Length

Let t_n be the arrival epoch of the n^{th} customer with $t_0 = 0$. The inter-arrival times $\{t_n,\ n \geq 1\}$ are independent and identically distributed with a general distribution function, denoted by A(t) with mean $1/\lambda$ and the LST of A(t) is denoted by $A^*(\theta)$. The working vacation times, the service times during a regular busy period and the service times during working vacation are all exponentially distributed with rates η, μ and μ_v respectively. The service time distribution in each case is assumed to be independent of the size of the batch. The system is studied at time $t_n - 0$ and this model is denoted by **GI / M$_{(a,\,b)}$ / 1 / MWV.**

For the GI/M$_{(a,\,b)}$/1/MWV queue, let $N_q(t)$ denote the number of customers in the queue at time t and

$$J_n = \begin{cases} 0, & \text{if the } n^{th} \text{ arrival occurs during idle vacation} \\ 1, & \text{if the } n^{th} \text{ arrival occurs during working vacation} \\ 2, & \text{if the } n^{th} \text{ arrival occurs during regular busy period} \end{cases}$$

Since the working vacation times, the service times during regular and working vacation busy period are exponentially distributed, the process $\{(N(t_n - 0), J_n) \; ; \; n \geq 1\}$ is an embedded Markov chain with the state space $\{(n \geq 0) \; ; \; j = 1, 2\} \cup \{0 \leq n \leq a-1 \; ; \; j = 0\}$. The steady state queue size probabilities are defined by

$$P_n = \underset{k\to\infty}{\text{lt}} \Pr\{N_q(t_k - 0) = n, J_k = 1\} \qquad n \geq 0$$

$$Q_n = \underset{k\to\infty}{\text{lt}} \Pr\{N_q(t_k - 0) = n, J_k = 2\} \qquad n \geq 0$$

$$R_n = \underset{k\to\infty}{\text{lt}} \Pr\{N_q(t_k - 0) = n, J_k = 0\} \qquad 0 \leq n \leq a-1$$

Then R_n, Q_n and P_n respectively denote the probability that the queue contains n customers and the server is in idle vacation state, busy vacation state and in regular busy state respectively. During idle vacation period, the number of customers in the system and queue are the same, whereas in working vacation and regular busy period n denotes the number of customers in the queue and the system contains k $(a \leq k \leq b)$ customers in service and n in queue.

To derive the queue size equations satisfied by the steady state probabilities the following probabilities are introduced.

Let b_k denote the probability that k batches are served at regular service rate μ in an inter arrival time. Then

$$b_k = \int_0^\infty e^{-\mu t} \frac{(\mu t)^k}{k!} dA(t) \qquad (k \geq 0)$$

Let c_k denote the probability that, the working vacation time is greater than an inter arrival time and the service completion of k-batches occur at rate μ_v in an inter arrival time.

$$\text{Then } c_k = \int_0^\infty e^{-\eta t} \frac{e^{-\mu_v t} (\mu_v t)^k}{k!} dA(t) \qquad (k \geq 0)$$

Let d_k denote the probability that, the server returns from vacation in an inter arrival time and k service completions occur in an inter arrival time.

$$\text{Then } d_k = \int_0^\infty \sum_{i=0}^{k} \left\{ \int_0^t \eta\, e^{-\eta x} \frac{(\mu_v x)^i}{i!} e^{-\mu_v x} \frac{(\mu (t-x))^{k-i}}{(k-i)!} e^{-\mu(t-x)} dx \right\} dA(t) \quad (k \geq 0)$$

(i.e.) k services occur in an inter arrival time such that i $(0 \leq i \leq k)$ service completions occur at rate μ_v (till the server returns from vacation) and the remaining $(k - i)$ service completions occur at rate μ.

III Steady State Equations

The steady state queue size equations at pre-arrival epochs, are obtained by noting the transitions between the states of Markov chain and are given by

During Working Vacation Period

$$Q_n = \sum_{k=0}^{\infty} Q_{kb+n-1}\, c_k \qquad n \geq 1 \tag{7.1}$$

$$Q_0 = \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j}\, c_k + R_{a-1}\, c_0 \tag{7.2}$$

During Regular Busy Period

$$P_n \quad = \quad \sum_{k=0}^{\infty} P_{kb+n-1}\, b_k + \sum_{k=0}^{\infty} Q_{kb+n-1}\, d_k \qquad n \geq 1 \qquad (7.3)$$

$$P_0 \quad = \quad \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} P_{(k-1)b+j}\, b_k + \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j}\, d_k + R_{a-1}\, d_0 \qquad (7.4)$$

During Idle Vacation Period

$$R_n \quad = \quad R_{n-1} + \sum_{k=0}^{\infty} Q_{kb+n-1}\left(1 - \sum_{i=0}^{k} (c_i + d_i)\right)$$

$$+ \sum_{k=0}^{\infty} P_{kb+n-1}\left(1 - \sum_{i=0}^{k} b_i\right) \qquad 1 \leq n \leq a-1 \qquad (7.5)$$

$$R_0 \quad = \quad \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j}\left(1 - \sum_{i=0}^{k} (c_i + d_i)\right)$$

$$+ \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} P_{(k-1)b+j}\left(1 - \sum_{i=0}^{k} b_i\right) + R_{a-1}\,(1 - (d_0 + c_0)) \qquad (7.6)$$

IV Steady State Solution

Let E denote the forward displacement operator then $E(p_n) = p_{n+1}$ and $E(Q_n) = Q_{n+1}$. The equations (7.1) and (7.3) can be respectively written as

$$(E - \sum_{k=0}^{\infty} c_k E^{kb})\, Q_n = 0 \qquad n \geq 0 \qquad (7.7)$$

and $$(E - \sum_{k=0}^{\infty} b_k E^{kb})\, p_n = \sum_{k=0}^{\infty} Q_{kb+n}\, d_k \qquad n \geq 0 \qquad (7.8)$$

Let $G(z^b)$ and $H(z^b)$ respectively denote the PGF of c_k and b_k then,

$$C(z^b) = \sum_{k=0}^{\infty} c_k z^{kb} = A^*(\eta + \mu_v (1 - z^b)) \qquad (7.9)$$

and $$B(z^b) = \sum_{k=0}^{\infty} b_k z^{kb} = A^*(\mu(1 - z^b)) \qquad (7.10)$$

Then the equation $z = B(z^b)$ has a unique root r inside (0,1)

if $B'(1) = \left(\frac{b\mu}{\lambda}\right) > 1$ and $z = C(z^b)$ has only one root r_1 inside (0,1).

Lemma1

If $\rho = \left(\frac{\lambda}{b\mu}\right) < 1$, then the equation $z = B(z^b)$ has unique root r inside (0,1).

The result follows from the corresponding result of GI/M/1 model of Gross and Harris (1985) (i.e.,) $B(r^b) = r$ with $0 < r < 1$.

Lemma 2

If $\eta > 0$, then the equation $z = A^*(\eta + \mu_v (1 - z^b))$ has a unique root r_1 inside (0, 1).

Let $\psi(z) = A^*(\eta + \mu(1 - z^b))$

then $0 < \psi(0) = A^*(\eta + \mu) < \psi(1) = A^*(\eta) < 1$

and for $0 < z < 1$

$$\psi'(z) = b\,\mu \int_0^\infty t\; e^{-(\eta + \mu(1 - z^b))t}\, d(A(t)) > 0$$

$$\psi''(z) = (b\,\mu)^2 \int_0^\infty t^2\; e^{-(\eta + \mu(1 - z^b))t}\, d(A(t)) > 0$$

$\therefore$ $z = A^*(\eta + \mu(1 - z^b))$ has a unique root say r_1 in the interval (0,1) (i.e.,) $C(r_1^b) = r_1$ with $0 < r_1 < 1$.

Hence the homogeneous difference equation (7.7) has solution

$$Q_n = r_1^n Q_0 \qquad n \geq 0 \qquad (7.11)$$

and the non-homogeneous difference equation (7.8) has solution

$$P_n = \left(A\, r^n + \left(\sum_{k=0}^{\infty} r_1^{kb}\, d_k \right) \frac{r_1^n}{(r_1 - B(r_1^b))} \right) Q_0 \quad \text{where } r_1 \neq r \qquad (7.12)$$

$$\text{Let} \quad D(r_1^b) = \sum_{k=0}^{\infty} r_1^{kb}\, d_k \quad \text{then}$$

$$P_n = (A\, r^n + B_d\, r_1^n)\, Q_0 \qquad n \geq 0 \qquad (7.13)$$

$$\text{where} \quad B_d = \left(\frac{D(r_1^b)}{r_1 - B(r_1^b)} \right) \qquad (7.14)$$

To find the remaining probabilities, the equations (7.5) and (7.6) are used.

Substituting for Q_n's and P_n's from (7.12) and (7.13) equation (7.5) implies,

$$R_n = R_{n-1} + \sum_{k=0}^{\infty} r_1^{kb+n-1} \left(1 - \sum_{i=0}^{k} (c_i + d_i)\right) + \sum_{k=0}^{\infty} \left(A r^{kb+n-1} + k r_1^{kb+n-1}\right)\left(1 - \sum_{i=0}^{k} b_i\right)$$

(i.e)

$$R_n = R_{n-1} + r_1^{n-1} \left(\frac{1 - C(r_1^b) - D(r_1^b)}{1 - r_1^b} \right) + k r_1^{n-1} \left(\frac{1 - B(r_1^b)}{1 - r_1^b} \right) + A r^{n-1} \left(\frac{1 - B(r^b)}{1 - r^b} \right)$$

$$\text{since} \quad \sum_{k=0}^{\infty} x^{kb} \sum_{i=0}^{k} c_i = \frac{\sum_{k=0}^{\infty} c_k\, x^{kb}}{(1 - x^b)}.$$

Lemmas 1 and 2 imply, $C(r_1^b) = r_1$ and $B(r^b) = r$

and the definition of B_d implies $D(r_1^b) = B_d(r_1 - B(r_1^b))$.

Hence, R_n can be simplified as,

$$R_n = R_{n-1} + \left[\frac{A(1-r)}{(1-r^b)}\, r^{n-1} + \left(\frac{1 - r_1}{1 - r_1^b} \right)(B_d + 1)\, r_1^{n-1} \right] Q_0 \qquad 1 \leq n \leq a-1 \qquad (7.15)$$

By adding equations (7.2), (7.4) and (7.6) we get

$$Q_0 + P_0 + R_0 = \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j} \left(1 - \sum_{j=0}^{k-1} (c_i + d_i) \right)$$

$$+ \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} P_{(k-1)b+j} \left(1 - \sum_{j=0}^{k-1} b_i \right) + R_{a-1}$$

Substituting for Q_n's and P_n's and after simplification it is found that

$$R_0 = R_{a-1} + \left[\frac{A\,(r^{a-1} - 1)}{(1 - r^b)} + (B_d + 1)\left(\frac{r_1^{a-1} - 1}{(1 - r_1^b)} \right) \right] Q_0 \qquad (7.16)$$

Adding equations (7.15) and (7.16) over 0 to n

$$R_n = R_{a-1} + \left[\frac{A\,(r^{a-1} - r^n)}{(1 - r^b)} + (B_d + 1)\left(\frac{r_1^{a-1} - r_1^n}{(1 - r_1^b)} \right) \right] Q_0$$

$$0 \le n \le a-1 \qquad (7.17)$$

Equation (7.2) implies,

$$R_{a-1}\, c_0 = Q_0 - \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j}\, c_k$$

Substituting for Q_n's

$$R_{a-1} = \left[\frac{r_1^b - r_1^a}{c_0\, r_1^b\,(1 - r_1)} + \frac{r_1^{a-1} - r_1^b}{r_1^b\,(1 - r_1)} \right] Q_0 \qquad (7.18)$$

Therefore, R_n for $0 \le n \le a-1$ are given by

$$R_n = \left[A\,\frac{(r^{a-1} - r^n)}{(1 - r^b)} + (B_d + 1)\left(\frac{r_1^{a-1} - r_1^n}{1 - r_1^b} \right) + \frac{1}{r_1^b\,(1 - r_1)}\left(\frac{r_1^b - r_1^a}{c_0} + r_1^{a-1} - r_1^b \right) \right] Q_0$$

$$0 \le n \le a-1 \qquad (7.19)$$

Thus the steady state queue size probabilities are expressed in terms of Q_0 and the constant A.

By using equation (7.4), A can be calculated as

$$A \left[\frac{r^a - r^b}{r^b\,(1 - r)} + b_0\, \frac{r^b - r^{a-1}}{(1 - r)\, r^b} \right] + B_d \left[\frac{r_1^a - r_1^b}{r_1^b\,(1 - r_1)} + b_0\, \frac{r_1^b - r_1^{a-1}}{r_1^b\,(1 - r_1)} \right] + d_0\, \frac{(r_1^b - r_1^a)}{c_0\, r_1^b\,(1 - r_1)} = 0$$

Let $f(x) = \dfrac{(x^a - x^b)}{x^b\,(1 - x)} + b_0\, \dfrac{(x^b - x^{a-1})}{(1 - x)\, x^b}$ then the above equation becomes,

$$A\, f(r) + B_d\, f(r_1) = \frac{d_0}{c_0} \left(\frac{r_1^a - r_1^b}{r_1^b\,(1 - r_1)} \right) \qquad (7.20)$$

Now the value of Q_0 can be calculated by using the normalizing condition,

$$\sum_{n=0}^{\infty} P_n + \sum_{n=0}^{\infty} Q_n + \sum_{n=0}^{a-1} R_n = 1 \text{ , as,}$$

$$Q_0^{-1} = A\,g(r) + (B_d + 1)\,g\,(r_1) + \frac{a}{r_1^b\,(1-r_1)}\left(\frac{r_1^b - r_1^a}{c_0} + (r_1^{a-1} - r_1^b)\right) \tag{7.21}$$

$$\text{where } g(x) = \frac{1}{(1-x^b)}\left(\frac{x^a - x^b}{1-x} + a\,x^{a-1}\right) \tag{7.22}$$

Thus the steady state queue size probabilities are given by

$$\left.\begin{aligned}
Q_n &= r_1^n\,Q_0 \qquad\qquad n \geq 0 \\
P_n &= (A\,r^n + B_d\,r_1^n)\,Q_0 \qquad n \geq 0 \\
R_n &= (A\,h_n(r) + (B_d + 1)h_n\,(r_1) + \frac{1}{(r_1^b\,(1-r_1))}\left(\frac{r_1^b - r_1^a}{C_0} + (r_1^{a-1} - r_1^b)\right) Q_0
\end{aligned}\right\} \tag{7.23}$$

$$\text{where } h_n(x) = \frac{x^{a-1} - x^n}{1-x^b}\;,\qquad Af(r) + B_d f(r_1) = \frac{d_0}{c_0}\left(\frac{r_1^a - r_1^b}{r_1^b\,(1-r_1)}\right)$$

$$\text{with} \qquad f(x) = \frac{(x^a - x^b)}{x^b\,(1-x)} + b_0\,\frac{(x^b - x^{a-1})}{(1-x)\,x^b} \quad \text{and} \quad B_d = \left(\frac{D(r_1^b)}{r_1 - B(r_1^b)}\right)$$

and Q_0 is given in equation (7.21)

V Mean Queue Length

The mean queue length L_q of the model, can be calculated by using equation 7.23 .

$$L_q = \sum_{n=0}^{\infty} n\,Q_n + \sum_{n=0}^{\infty} n\,P_n + \sum_{n=0}^{a-1} n\,R_n$$

After simplification it is found that

$$L_q = \left[A\,H(r) + (B_d + 1)\,H(r_1) + \frac{a(a-1)r_1^b(1-r_1)}{2}\left(\frac{(r_1^b - r_1^a)}{c_0} + r_1^{a-1} - r_1^b\right)\right] Q_0$$

$$\text{where } H(x) = \frac{x}{(1-x)^2} + \frac{1}{(1-x^b)}\left[\frac{a(a-1)}{2}\,x^{a-1} + \frac{ax^a\,(1-x) - x\,(1-x^a)}{(1-x)^2}\right]$$

follows from equations (7.21) .Thus the queue size probabilities are given by

$$Q_n \quad = \quad (1-r)\,\beta\, r_1^{n+1} \qquad n \geq 0$$

$$P_n \quad = \quad (1-r)\,\alpha\,\beta\,(r^{n+1} - r_1^{n+1}) \qquad n \geq 0$$

These probabilities coincide with the corresponding probabilities of Baba (2005)

(2) M/M$_{(a,\, b)}$/1 multiple working vacation model (sec 6.3)

If the inter arrival time is exponential with parameter λ,

then $$b_k \quad = \quad \left(\frac{\lambda}{\lambda+\mu}\right)\left(\frac{\mu}{\mu+\lambda}\right)^k$$

$$c_k \quad = \quad \left(\frac{\lambda}{(\lambda+\mu_v+\eta)}\right)\left(\frac{\mu_v^k}{(\lambda+\mu_v+\eta)}\right)^k$$

$$D(r_1^b) \quad = \quad \frac{\eta\, r_1}{\lambda+\mu\,(1-r_1^b)} \quad \text{and}$$

$$d_0 \quad = \quad \frac{\lambda\,\eta}{(\lambda+\mu)(\lambda+\eta+\mu_v)}$$

With these identification it is verified that the constant $B_d = \dfrac{D(r_1^b)}{r_1 - B(r_1^b)}$ exactly coincide with B of M/M$_{(a,\, b)}$/1 multiple working vacation model of previous chapter (sec 6.3) and hence the expressions of the probabilities Q_n, P_n of G1/M$_{(a,\, b)}$/1 and M/M$_{(a,\, b)}$/1 coincide.

By modifying the expression of A given in equation (6.3.13), R_ns of M/M$_{(a,\, b)}$/1 model exactly coincide with the R_ns of GI/M$_{(a,\, b)}$/1 .