

On Subtraction Algebras

Jayanthi, M
(12PMA007)

Thesis Submitted to
Avinashilingam Institute for Home Science and Higher Education for
Women,
Coimbatore-641 043

In Partial Fulfilment of the Requirements for the
Degree of Master of Science in Mathematics

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Signature of Head of the Department


Signature of the Supervisor

Acknowledgement

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Introduction

INTRODUCTION

In 1998, Schein, B.M. [32] considered systems of the form (Φ, \circ, \setminus) , where Φ is a set of functions closed under the composition " \circ " of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction " \setminus " (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of Abbott, J.C. [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. In 1995, Zelinka, B. [35] discussed a problem proposed by Schein, B.M. concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras.

The aim of our thesis is to discuss a few interesting articles on subtraction algebras. The following articles are chosen for our discussion.

- (1) "Ideal theory of subtraction algebras" by Jun, Y.B., Kim, H.S. and Roh, E.H. [17]
- (2) "Prime and irreducible ideals in subtraction algebras" by Jun, Y.B. and Kim, K.H. [15]
- (3) "Quotient subtraction algebras" by Çeven, Y., and Küçükkoç, Ş., [6]
- (4) "Homomorphism theorems in subtraction algebra" by Öztürk, M.A. and Hasret Yazarli [27]
- (5) "Subtraction algebras with pseudo-valuations" by Kim, K.H. and Yon, Y.H. [22]
- (6) "On derivations of subtraction algebras" by Yon, Y.H and Kim, K.H. [34]
- (7) "A note on multipliers of subtraction algebras" by Sang Deok Lee and Kim, K.H. [30]
- (8) "States on subtraction algebras" by Sang Moon Lee and Kim, K.H. [31]

This thesis is divided into four chapters.

In the First chapter, Preliminary definitions and properties of subtraction algebras are presented. Also this chapter deals with the study of Ideals, Prime and Irreducible ideals in subtraction algebras due to Jun, Y.B. et al. [14, 15]

In this chapter, the following important results are discussed.

(1) If A be an ideal of a subtraction algebra X , then the set

$$A_w = \{x \in X \mid x - w \in A\}, w \in X$$

is the least ideal of X containing A and w .

(2) Let A be an ideal of subtraction algebra X . Then the following are equivalent.

1. A is a prime ideal.
2. A is a maximal ideal.
3. $x - y \in A$ or $y - x \in A$ for all $x, y \in X$.

(3) Let A be an ideal of a subtraction algebra X and let $w \in X \setminus A$. Then there exists an irreducible ideal M of X such that $A \subset M$ and $w \notin M$.

Chapter-II deals with the study of Quotient subtraction algebras and Pseudo-valuations of subtraction algebras due to Kim, K.H. and Yon, Y.H. [22].

In this chapter, the following interesting results are discussed.

(1) If I and J be any two ideals of subtraction algebra of X and $I \subset J$, then the following conditions are satisfied:

- (a) I is an ideal of subalgebra J
- (b) J/I is an ideal of quotient algebra of X/I .

(2) If J is an ideal of quotient subtraction algebra of X/I , then $f^{-1}(J)$ is an ideal of X and $I \subseteq f^{-1}(J)$.

(3) Let a real-valued function φ be a pseudo-valuation on a subtraction algebra X . Then the set

$$I = \{x \in X \mid \varphi(x) = 0\}$$

is an ideal of X .

- (4) Let X be a subtraction algebra. If $\varphi: X \rightarrow \mathbb{R}$ is a valuation on X , then (X, d_φ) is a metric space.

Chapter-III is devoted to the study of derivation of subtraction algebras due to Yon, Y.H. and Kim, K.H. [34].

In this chapter, the following important results are discussed.

- (1) Every simple derivation is a derivation and every derivation can be partially a simple derivation on intervals.
- (2) For any derivation d of a subtraction algebra X , $\ker(d)$ and $\text{Im}(d)$ are ideals of X , and $X/\ker(d) \cong \text{Im}(d)$ and $X/\text{Im}(d) \cong \ker(d)$.
- (3) Every subtraction algebra X is embedded in $\text{Im}(d) \times \ker(d)$ for any derivation d of X .

Chapter-IV deals with the study of multipliers of subtraction algebras due to Sang Deok Lee and Kim, K.H. [30] and States on subtraction algebras due to Sang Moon Lee and Kim, K.H. [31].

In this chapter, the following important results are discussed.

- (1) The image and inverse image of a normal ideal of a subtraction algebra are normal ideals.
- (2) The normal ideals of direct products of subtraction algebras are characterized.
- (3) Interconnections between multipliers and weak congruence's are obtained.
- (4) Kernel of a Bosbach state on a subtraction algebra X is an ideal of X .
- (5) Let $f: X \rightarrow Y$ be a bijection homomorphism and $s: X \rightarrow [0,1]$ be a Bosbach state on X . Then there exists an unique Bosbach state $t: Y \rightarrow [0,1]$ such that $s = t \circ f$.

Review of Literature

REVIEW OF LITERATURE

In 1998, Schein, B.M. [32] considered systems of the form (Φ, \circ, \setminus) , where Φ is a set of functions closed under the composition " \circ " of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction " \setminus " (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of Abbott, J.C. [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. In 1995, Zelinka, B. [35] discussed a problem proposed by Schein, B.M. concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In 2004, Jun, Y.B., Kim, H.S. and Roh, E.H. [17] introduced the notion of ideal in subtraction algebras and discussed characterization of ideals. In 2006, Jun, Y.B. and Kim, H.S. [14] established the ideal generated by a set, and discussed related results. In 2009, Ceven, Y. and Ozturk, M.A. [7] introduced some additional concepts on subtraction algebras, so called subalgebra, bounded subtraction algebra and union of subtraction algebras and investigated some related properties. In 2011, Çeven, Y. and Küçükkoç, Ş. [6] introduced quotient subtraction algebras, and investigated some properties.

Several other authors have also contributed to the study of the concepts mentioned above. We give here a brief survey of some of the articles published on subtraction algebras.

(1) Subtraction algebras and BCK-Algebras

Kim, Y.H. and Kim, H.S. (2003) [20]

In this article, the authors showed that a subtraction algebra is equivalent to an implicative BCK-algebra, and a subtraction semigroup is a special case of a BCI-semigroup.

(2) A Relation on subtraction algebras

Ahn, S.S., Kim, Y.H. and Lee, K.J. (e-2006) [3]

As a generalization of a subtraction homomorphism, the notion of a relation on subtraction algebras, called an SA-relation, is introduced.

(3) Vague ideals of subtraction algebra

Jun, Y.B. and Park, C.H. (2007) [18]

The notion of vague ideals in subtraction algebras is introduced by the authors and several properties are investigated.

(4) Subtraction algebras with additional conditions

Jun, Y.B., Kim, Y.H. and Oh, K.A. (2007) [16]

Subtraction algebras with additional conditions, so called complicated subtraction algebras, are introduced, and several properties are investigated. In a complicated subtraction algebra, characterizations of ideals are provided, and showed that the set of all ideals in a complicated subtraction algebra is a complete lattice.

(5) Fuzzy ideals of subtraction algebras

Kim, Y.H., Oh, K.A. and Roh, E.H. (2007) [21]

In this paper, the authors defined the concept of a fuzzy ideal of a subtraction algebra and studied characterizations of a fuzzy ideal. The authors gave some conditions to show that a fuzzy set in a subtraction algebra is a fuzzy ideal of a subtraction algebra. The authors investigated the generalization of some properties of a fuzzy ideal of a subtraction algebra.

(6) Some questions on fuzzifications of ideals in subtraction algebras

Lee, K.J. and Park, C.H. (2007) [24]

In this paper, the authors introduced the notion of a fuzzy ideal in subtraction algebras, and gave some conditions for a fuzzy set to be a fuzzy ideal in subtraction algebras. The author also pose three questions on fuzzy ideals of subtraction algebras.

(7) A Note on complicated subtraction algebras

Roh, E.H. (2007) [29]

The author provided characterizations of complicated in a subtraction algebra, and the author showed that the complicated subtraction algebra is a ring.

(8) Some topological properties in subtraction algebras

Ahn, S.S., Kim, Y.H. and Oh, K.A. (2008) [4]

In this paper, the authors showed that how certain topologies associated with ideals of subtraction algebras on subtraction algebras. The authors showed subtraction algebras to be topological subtraction algebras with respect to these topologies. Furthermore, the authors showed that how certain standard properties may arise. In addition, the authors demonstrate that it is natural for these topologies to have many clopen sets and thus to be highly disconnected via the ideal theory of subtraction algebras.

(9) Order systems, ideals and right fixed maps of subtraction algebras

Jun, Y.B., Park, C.H. and Roh, E.H. (2008) [19]

Conditions for an ideal to be irreducible are provided. The notion of an order system in a subtraction algebra is introduced, and related properties are investigated. Relations between ideals and order systems are given. The concept of a fixed map in a subtraction algebra is discussed, and related properties are investigated.

(10) Fuzzy ideals in near-subtraction semigroups

Prince Williams, D.R. (2008) [28]

In this article, the author introduced a notion of fuzzy ideals in near-subtraction semigroups and the author studied their related properties.

(11) Subdirect sum of subtraction algebras

Seon Yu Kim, Kang Eun Park and Roh, E.H. (2009) [33]

Subdirect sum of subtraction algebras are introduced by the authors and related properties are investigated.

(12) Some results on subtraction algebras

Çeven, Y. and Öztürk, M.A. (2009) [7]

In this article, some additional concepts relating to subtraction algebras, the so called subalgebra, bounded subtraction algebra and unions of subtraction algebras, are introduced, and some properties are investigated by the authors.

(13) On prime and fuzzy prime ideals of a subtraction algebra

Dheena, P. and Mohanraaj, G. (2009) [8]

In this paper, the authors introduced the notion of m-system, fuzzy ideal, fuzzy prime ideal and fuzzy m-system in a subtraction algebra. The authors have shown that an ideal in a subtraction algebra X is prime iff its complement is a m-system. The authors have also shown that in a subtraction algebra X , for any ideal A which does not intersect a m-system M , then there exists a prime ideal P containing A that does not intersect M . The authors have also shown that a fuzzy ideal is fuzzy prime iff its image contains two elements and its level sets are prime ideals.

(14) Interval valued intuitionistic (S^*, T^*) -fuzzy ideals of subtraction algebras

Jong Geol Lee, Kim, K.H. (2009) [11]

The authors introduced the notion of interval-valued fuzzy bi-ideals with respect to t-norm T^* and s-norm S^* and they investigated some of the properties. Also, the homomorphism image and inverse image are investigated.

(15) Derivations on subtraction algebras

Öztürk, M.A. and Çeven, Y., (2009) [26]

In this paper, the authors introduced the notions of a derivation and a generalized derivation determined by a derivation for a complicated subtraction algebra. The authors gave some related properties and equivalent conditions which derivations hold.

(16) \mathcal{N} -ideals of subtraction algebras

Jun, Y.B., Jacob Kavikumar, and Keum Sook So (2010) [13]

Using \mathcal{N} -structures, the notion of an \mathcal{N} -ideal in a subtraction algebra is introduced. Characterizations of an \mathcal{N} -ideals are discussed by the authors and conditions for an \mathcal{N} -structure to be an \mathcal{N} -ideal are provided. The description of a created \mathcal{N} -ideal is established.

(17) On f-derivations of complicated subtraction algebras

Jong Geol Lee, Kim, H.J. and Kim, K.H. (2011) [12]

In this article, the authors introduced the notion of f-derivation of subtraction algebra, which is a generalization of derivation in complicated subtraction algebras, and to study some properties of f-derivation of complicated subtraction algebras.

(18) On fuzzy ideals in near-subtraction ordered semigroups

Lekkoksung, S. (2012) [25]

In this article, some properties of fuzzy ideal in near-subtraction ordered semigroup have been investigated.

Chapter - I

CHAPTER I

IDEAL THEORY OF SUBTRACTION ALGEBRAS

SECTION: 1.1

PRELIMINARIES ON SUBTRACTION ALGEBRAS

Definition: 1.1.1 [9]

A BCI-algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms for all $x, y, z \in X$:

$$(BCI-1) ((x * y) * (x * z)) * (z * y) = 0$$

$$(BCI-2) (x * (x * y)) * y = 0$$

$$(BCI-3) x * x = 0$$

$$(BCI-4) x * y = 0 \text{ and } y * x = 0 \Rightarrow x = y$$

Definition: 1.1.2 [10]

A BCI-algebra $(X, *, 0)$ is called a BCK-algebra if (BCK-1) $0 * x = 0$ for all $x \in X$.

Note: In BCI/BCK-algebra, define a partial ordering " \leq " on X by $x \leq y$ iff $x * y = 0$.

Lemma: 1.1.3

In any BCI-algebra X , we have

- (1) $x * 0 = x$,
- (2) $(x * y) * z = (x * z) * y$,
- (3) $x \leq y$ imply $x * z \leq y * z$ and $z * y \leq z * x$,
- (4) $(x * z) * (y * z) \leq x * y$ for any $x, y, z \in X$.

Proof: Obvious

Definition: 1.1.4 [17]

A *subtraction algebra* is defined as an algebra $(X;-)$ (briefly X) with a single binary operation "-" that satisfies the following identities: for any $x, y, z \in X$,

$$(S1) x - (y - x) = x;$$

$$(S2) x - (x - y) = y - (y - x);$$

$$(S3) (x - y) - z = (x - z) - y.$$

Definition: 1.1.5

A non-empty subset S of a subtraction algebra X is called a *subalgebra* if $x - y \in S \quad \forall x, y \in S$.

Note: 1

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$.

Note: 2

The subtraction determines an order relation on $X : a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$.

Note: 3**(1) Semilattice [1,2]**

A semilattice is a nonempty set with a single binary operation which satisfies an idempotent, commutative and associative laws.

(2) Boolean algebra [2]

A Boolean algebra $B = [A, \wedge, \vee, ', 0, I]$ is a set A with two binary operations (\wedge, \vee) , two universal bounds $(0, I)$, and one unary operation $'$.

The ordered set $(X; \leq)$ is a semi-Boolean algebra. That is, it is a meet semilattice (\wedge -semilattice) with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$. The complement of an element $b \in [0, a]$ is $a - b$, and if $b, c \in [0, a]$, then

$$\begin{aligned}
b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\
&= a - ((a - b) - ((a - b) - (a - c))).
\end{aligned}$$

Proposition: 1.1.6

In subtraction algebra $(X; -)$, the following are true:

- (P1) $(x - y) - y = x - y$.
- (P2) $x - 0 = x$ and $0 - x = 0$.
- (P3) $(x - y) - x = 0$.
- (P4) $x - (x - y) \leq y$.
- (P5) $(x - y) - (y - x) = x - y$.
- (P6) $x - (x - (x - y)) = x - y$.
- (P7) $(x - y) - (z - y) \leq x - z$.

Proof: Obvious

Proposition: 1.1.7 [17]

In a subtraction algebra $(X; -)$, we have

- (P8) $x \leq y$ iff $x = y - w$ for some $w \in X$.
- (P9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
- (P10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

Proof: Let $(X; -)$ be a subtraction algebra.

To prove: (P8) If $x \leq y$, then by taking $w = y - x$ we have

$$x = x - 0 = x - (x - y) = y - (y - x) = y - w.$$

Conversely, if $x = y - w$ for some $w \in X$, then

$$x - y = (y - w) - y = (y - y) - w = 0 - w = 0. \text{ This implies, } x \leq y.$$

To prove: (P9) If $x \leq y$, then $x = y - w$ for some $w \in X$.

Hence $x - z = (y - w) - z = (y - z) - w$, and so, $x - z \leq y - z$ by (P8).

Next if $x \leq y$, then $x - y = 0$. Thus

$$\begin{aligned} (z - y) - (z - x) &= (z - (z - x)) - y = (x - (x - z)) - y \\ &= (x - y) - (x - z) = 0 - (x - z) = 0. \end{aligned}$$

That is, $z - y \leq z - x$.

To prove: (P10) If $x \leq z$, then $x - y \leq z - y$ by (P9).

But $x - y \leq x$, and thus $x - y \leq x \wedge (z - y)$. Let $w = x \wedge (z - y)$.

Then $w \leq x$, and so $w = x \wedge w = x - (x - w)$.

Also, $y \wedge (z - y) = (z - y) - ((z - y) - y) = (z - y) - (z - y) = 0$.

Hence, $w - (w - y) = y \wedge w = x \wedge y \wedge (z - y) = 0$.

Therefore, $w - (x - y) = (w - 0) - (x - y) = (w - (w - (w - y))) - (x - y)$

$$\begin{aligned} &= (w - y) - (x - y) = ((x - (x - w)) - y) - (x - y) \\ &= ((x - y) - (x - w)) - (x - y) = ((x - y) - (x - y)) - (x - w) \\ &= 0 - (x - w) = 0 \end{aligned}$$

and thus $x \wedge (z - y) = w \leq (x - y)$. Consequently, $x - y = x \wedge (z - y)$.

Proposition: 1.1.8 [15, 23]

Let $(X; -)$ be a subtraction algebra. Then for all $x, y, z \in X$,

$$(P11) \quad (x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$$

$$(P12) \quad (x - y) - z = (x - z) - (y - z)$$

Proof: Let $(X; -)$ be a subtraction algebra.

To prove: (P11) For any $x, y, z \in X$, we have

$$(x \wedge y) - (x \wedge z) = (x - (x - y)) - (x - (x - z)) \leq (x - z) - (x - y) \leq y - z \quad (1)$$

$$\text{On the other hand, } (x \wedge y) - (x \wedge z) \leq x \wedge y \leq x \quad (2)$$

Combining (1) and (2), we have $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$

To prove: (P12) For any $x, y, z \in X$, we have

$$\begin{aligned} ((x - z) - (y - z)) - ((x - y) - z) &= (((x - z) - z) - (y - z)) - ((x - y) - z) \text{ by (P1)} \\ &\leq ((x - z) - y) - ((x - y) - z) \text{ by (P7)} \\ &= ((x - y) - z) - ((x - y) - z) \text{ by (S3)} \\ &= 0, \end{aligned}$$

and so $((x - z) - (y - z)) - ((x - y) - z) = 0$.

$$\text{That is, } (x - z) - (y - z) \leq (x - y) - z. \quad (3)$$

Using (S3), (P3) and (P7), we get

$$\begin{aligned} ((x - y) - z) - ((x - z) - (y - z)) &= ((x - z) - y) - ((x - z) - (y - z)) \\ &\leq (y - z) - y = 0, \end{aligned}$$

and therefore, $((x - y) - z) - ((x - z) - (y - z)) = 0$.

$$\text{That is, } (x - y) - z \leq (x - z) - (y - z). \quad (4)$$

By (3) and (4), we get $(x - y) - z = (x - z) - (y - z)$.

Proposition: 1.1.9

Let X be a subtraction algebra and let $x, y \in X$. If $w \in X$ is an upper bound for x and y , then the element

$$x \vee y = w - ((w - y) - x)$$

is a least upper bound for x and y .

Proof: Let $w \in X$ be an upper bound for x and y .

Since $w - y \leq w$, it follows from Proposition 1.1.7 (P10) that

$$(w - y) - x = (w - y) \wedge (w - x) \leq w - x \text{ which implies from Proposition 1.1.7 (P9) that}$$
$$x = x \wedge w = w - (w - x) \leq w - ((w - y) - x).$$

Similarly, $y \leq w - ((w - x) - y) = w - ((w - y) - x)$.

Hence $w - ((w - y) - x)$ is an upper bound for x and y .

Let a be any other upper bound for x and y .

Using Proposition 1.1.7 (P9), we have

$$w - a \leq w - y \text{ and } w - a \leq w - x.$$

It follows that $w - a \leq (w - y) \wedge (w - x)$.

Applying Proposition 1.1.7 (P9) again, we get

$$w - ((w - y) - x) = w - ((w - y) \wedge (w - x)) \leq w - (w - a) \leq a.$$

Therefore, $x \vee y = w - ((w - y) - x)$ is a least upper bound for x and y .

SECTION: 1.2

IDEALS ON SUBTRACTION ALGEBRAS

Definition: 1.2.1 (Jun et al. [17])

A non-empty subset A of a subtraction algebra X is called an ideal of X if it satisfies

(SI1) $0 \in A$

(SI2) $y \in A$ and $x - y \in A$ imply $x \in A$ for all $x, y \in X$.

Note: Any ideal of a subtraction algebra X is a subalgebra of X .

Theorem: 1.2.2

Let A be a non-empty subset of a subtraction algebra X . Then A is an ideal of X iff it satisfies:

“for any $a, b \in A$, $x - a \leq b$ implies $x \in A$ ”.

Proof: Assume that A is an ideal of X and let $a, b \in A$.

If $x - a \leq b$, then $(x - a) - b = 0 \in A$, and so $x \in A$ by (SI2).

Conversely suppose that A satisfies the condition that for any $a, b \in A$,

$x - a \leq b \Rightarrow x \in A$. Since A is non-empty, we can take $a \in A$, and then $0 - a = 0 \leq a$.

This implies $0 \in A$ (1)

Let $x, y \in X$ be such that $y \in A$ and $x - y \in A$. Since $x - (x - y) \leq y$,

we obtain $x \in A$ (2)

By (1) and (2), we get A is an ideal of X .

Lemma: 1.2.3 [15]

Let A be an ideal of a subtraction algebra X . If $x \leq y$ and $y \in A$. Then $x \in A$.

Proof: Obvious

Lemma: 1.2.4 [31]

Let X be a subtraction algebra and A a non-empty subset of X . Then A is an ideal of X iff it satisfies the following properties:

- (1) $x \in A$ and $y \leq x \Rightarrow y \in A$
- (2) If $x, y \in A$ and $x \vee y$ exists, then $x \vee y \in A$.

Theorem: 1.2.5

If A be an ideal of a subtraction algebra X , then the set

$$A_w = \{x \in X \mid x - w \in A\}, w \in X$$

is the least ideal of X containing A and w .

Proof: Let A be an ideal of a subtraction algebra X . Let $w \in X$. Since $0 - w = 0 \in A$, we have $0 \in A_w$. Let $x, y \in X$ be such that $y \in A_w$ and $x - y \in A_w$. Then $y - w \in A$ and $(x - y) - w \in A$. It follows from (P12) that

$$(x - w) - (y - w) = (x - y) - w \in A$$

So from (SI2) that $x - w \in A$, that is, $x \in A_w$.

Hence A_w is an ideal of X . Obviously A_w contains A and w .

Let B be an ideal of X containing A and w .

If $x \in A_w$, then $x - w \in A \subseteq B$, and hence $x \in B$ by (SI2). Thus $A_w \subseteq B$, and consequently A_w is the least ideal of X containing A and w .

Theorem: 1.2.6

Let X be a subtraction algebra. For $u, v \in X$, the set

$$X(u, v) = \{x \in X \mid (x - u) - v = 0\}$$

is an ideal of X , and $u, v \in X(u, v)$.

Proof: Let X be a subtraction algebra.

Obviously $0, u, v \in X(u, v) = \{x \in X \mid (x - u) - v = 0\}$.

Let $x, y \in X$ be such that $y \in X(u, v)$ and $x - y \in X(u, v)$.

Then $(y - u) - v = 0$ and $((x - y) - u) - v = 0$. It follows from (P2) and (P12) that

$$0 = ((x - y) - u) - v = ((x - u) - (y - u)) - v$$

$$= ((x - u) - v) - ((y - u) - v)$$

$$= ((x - u) - v) - 0 = (x - u) - v$$

so that $x \in X(u, v)$. Hence $X(u, v)$ is an ideal of X .

Lemma: 1.2.7

Every ideal A of a subtraction algebra X contains the ideal $X(a, b)$ for all $a, b \in A$.

Proof: Let $x \in X(a, b)$. Then $(x - a) - b = 0 \in A$, and hence $x \in A$.

This shows that $X(a, b) \subseteq A$ for all $a, b \in A$.

Theorem: 1.2.8

Every ideal A of a subtraction algebra X can be represented as the union of ideals of the form $X(a, b)$ for all $a, b \in A$, that is, $A = \bigcup_{a, b \in A} X(a, b)$.

Proof: Let A be an ideal of X and $x \in A$. Since $x \in X(x, 0)$, we have

$$A \subseteq \bigcup_{x \in A} X(x, 0) \subseteq \bigcup_{a, b \in A} X(a, b).$$

Now let $x \in \bigcup_{a, b \in A} X(a, b)$. Then there exist $u, v \in A$ such that $x \in X(u, v)$.

It follows from Lemma 1.2.7 that $x \in A$ so that $\bigcup_{a, b \in A} X(a, b) \subseteq A$.

This completes the proof.

SECTION: 1.3

PRIME AND IRREDUCIBLE IDEALS IN SUBTRACTION ALGEBRAS

Theorem: 1.3.1

Let A be an ideal of a subtraction algebra X . For any $w \in X$, the set

$$A_w^\wedge = \{x \in X \mid w \wedge x \in A\}$$

is an ideal of X containing A .

Proof: Since $w \wedge 0 = w - (w - 0) = w - w = 0 \in A$, we have $0 \in A_w^\wedge$.

Let $x, y \in X$ be such that $y \in A_w^\wedge$ and $x - y \in A_w^\wedge$. Then $w \wedge y \in A$ and $w \wedge (x - y) \in A$. Since $(w \wedge x) - (w \wedge y) \leq w \wedge (x - y)$ by (P11), it follows from Lemma 1.2.3 and (SI2) that $w \wedge x \in A$. That is, $x \in A_w^\wedge$. Hence A_w^\wedge is an ideal of X .

Now let $x \in A$. Since $w \wedge x \leq x$ by (P4), we have $w \wedge x \in A$ by Lemma 1.2.3.

Therefore $x \in A_w^\wedge$, and so $A \subseteq A_w^\wedge$. This completes the proof.

Theorem: 1.3.2

Let A be a non-empty subset of a subtraction algebra X such that

(1) $x \in A$ and $y \leq x$ imply $y \in A$.

(2) For $x, y \in A$, there exist $z \in A$ such that $x \leq z$ and $y \leq z$.

Then A is an ideal of X .

Proof: Since A is non-empty, we have $0 \in A$ by (1) and (P2). Let $x, y \in X$ be such that $y \in A$ and $x - y \in A$.

Then by (2), there exist $z \in A$ such that $y \leq z$ and $x - y \leq z$.

It follows from (P2) that

$$x - z = (x - z) - 0 = (x - z) - (y - z) = (x - y) - z = 0.$$

So that $x \leq z$. Since $z \in A$, it follows from (1) that $x \in A$.

Hence A is an ideal of X .

Definition: 1.3.3

Let X be a subtraction algebra. A prime ideal of X is defined to be an ideal P of X such that if $x \wedge y \in P$ then $x \in P$ or $y \in P$.

Theorem: 1.3.4

Let P be an ideal of a subtraction algebra X . Then the following are equivalent.

- (1) P is a prime ideal of X .
 (2) For any ideals A and B of X , $A \wedge B \subset P$ implies $A \subset P$ or $B \subset P$, where

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\}.$$

Proof: Suppose that P is a prime ideal of X such that $A \wedge B \subset P$, where A and B are ideal of X . Assume that $A \not\subset P$ and $B \not\subset P$.

Then there exist $x \in A \setminus P$ and $y \in B \setminus P$, and so $x \wedge y \in A \wedge B \subset P$.

Since P is prime, it follows that $x \in P$ or $y \in P$, which is a contradiction.

Consequently, $A \wedge B \subset P$ implies $A \subset P$ or $B \subset P$.

Conversely, assume that for any ideals A and B of X , $A \wedge B \subset P$ implies $A \subset P$ or $B \subset P$. Let $x, y \in X$ be such that $x \wedge y \in P$.

Now, $(x) = \{a \in X \mid a \leq x\}$ and $(y) = \{b \in X \mid b \leq y\}$ are ideals of X .

Let $a \in (x)$ and $b \in (y)$. Then $a \leq x$ and $b \leq y$. It follows from $a \wedge b \leq a, b$ that $a \wedge b \leq x$ and $a \wedge b \leq y$ so that $a \wedge b \leq x \wedge y$.

Since P is an ideal and $x \wedge y \in P$, by Lemma 1.2.3 we get $a \wedge b \in P$.

Therefore $(x) \wedge (y) \subset P$, which implies that $(x) \subset P$ or $(y) \subset P$ by hypothesis.

In particular, $x \in P$ or $y \in P$, and thus P is a prime ideal of X .

Theorem: 1.3.5

Every prime ideal of a subtraction algebra is a maximal ideal, that is, every prime is not contained in any other proper ideal.

Proof: Let P be a prime ideal of a subtraction algebra X .

Suppose that P is not maximal. Then there exist a proper ideal A of X such that $P \subset A$ and $P \neq A$.

Let $y \in X$ and consider $x \in A \setminus P$.

Then $x \wedge (y-x) = x - (x - (y-x)) = x - x = 0 \in P$, and so $y-x \in P$ because P is a prime ideal and $x \notin P$.

It follows from (I2) that $y \in A$, that is, $X = A$, which contradicts the assumption that A is proper. Hence P is a maximal ideal of X .

Proposition: 1.3.6

If A is a maximal ideal of a subtraction algebra X , then $x-y \in A$ or $y-x \in A$ for all $x, y \in X$.

Proof: Let $x, y \in A$. Then $x-y \in A$, since A is an ideal.

Assume that $x \in A$ and $y \in X \setminus A$. Since $x-y \leq x$, $x-y \in A$.

Similarly, if $x \in X \setminus A$ and $y \in A$, then $y-x \in A$.

Let $x, y \in X \setminus A$ and assume that $y-x \notin A$.

Then the set $Q = \{z \in X \mid z - (y-x) \in A\}$ is the least ideal of X containing A and $y-x$.

Since $y-x \notin A$, we have $A \neq Q$, and so $Q = X$ because A is maximal.

Therefore $x-y \in Q$, that is, $(x-y) - (y-x) \in A$.

Using (P2), (P3), (S3), and (P12), we get

$$\begin{aligned} x-y &= (x-y) - 0 = (x-y) - ((y-x) - y) \\ &= (x-(y-x)) - y = (x-y) - (y-x) \in A. \end{aligned}$$

This completes the proof.

Corollary: 1.3.7

If A is a prime ideal of subtraction algebra X , then $x-y \in A$ or $y-x \in A$ for all $x, y \in X$.

Proof: Obvious

Theorem: 1.3.8

Let A be an ideal of a subtraction algebra X such that $x-y \in A$ or $y-x \in A$ for all $x, y \in X$. Then A is a prime ideal of X .

Proof: Let A be an ideal of X such that $x-y \in A$ or $y-x \in A$ for all $x, y \in X$.

Assume that $x \wedge y \in A$. If $x-y \in A$, then $x - (x-y) = x \wedge y \in A$ and so $x \in A$ by (I2).

If $y - x \in A$, then $y - (y - x) = x - (x - y) = x \wedge y \in A$.

It follows from (I2) that $y \in A$. Hence A is a prime ideal of X .

Note: From above theorems the notion of prime ideals and maximal ideals in a subtraction algebra coincide.

Theorem: 1.3.9

Let A be an ideal of subtraction algebra X . Then the following are equivalent.

- (1) A is a prime ideal.
- (2) A is a maximal ideal.
- (3) $x - y \in A$ or $y - x \in A$ for all $x, y \in X$.

Proof: Obvious

Corollary: 1.3.10 (Extension property of prime ideals)

Let A and B be ideals of a subtraction algebra X such that $A \subset B$. If A is a prime ideal of X , then so in B .

Definition: 1.3.11

An ideal A of a subtraction algebra X is said to be irreducible if for any ideals C and D of X , $A = C \cap D$ implies $A = C$ or $A = D$.

Theorem: 1.3.12

Let A be an ideal of a subtraction algebra X and let $w \in X \setminus A$. Then there exists an irreducible ideal M of X such that $A \subset M$ and $w \notin M$.

Proof: Let $\mathcal{A} = \{I \mid I \text{ is an ideal of } X, A \subset I, w \notin I\}$. Note that any chain of elements in \mathcal{A} has an upper bound.

Thus, by Zorn's Lemma, there exists a maximal element M in \mathcal{A} .

Then $A \subseteq M$ and $w \notin M$.

Let C and D be ideals of X such that $M = C \cap D$.

Assume that $M \neq C$ and $M \neq D$. By the maximality of M , we have $w \in C$ and $w \in D$, that is $w \in C \cap D$.

Hence $M \neq C \cap D$, a contradiction.

Therefore M is an irreducible ideal of X .

Theorem: 1.3.13

Let A be an ideal of a subtraction algebra X . Assume that for any $x, y \in X \setminus A$, there exist $z \in X \setminus A$ such that $z \leq x$ and $z \leq y$. Then A is an irreducible ideal of X .

Proof: Suppose that A is not irreducible ideal of X .

Then there two ideals C and D of X such that $A = C \cap D$, $A \neq C$, and $A \neq D$.

Let $x \in C \setminus A$ and $y \in D \setminus A$.

Using the assumption, there exist $z \in X \setminus A$ such that $z \leq x$ and $z \leq y$.

Since $x \in C$ and $y \in D$, it follows from Lemma 1.2.3 that $z \in C \cap D = A$, which is a contradiction. Hence A is an irreducible ideal of X .

Chapter - II

CHAPTER II

QUOTIENT SUBTRACTION ALGEBRAS AND PSEUDO-VALUTION ON SUBTRACTION ALGEBRAS

SECTION: 2.1

QUOTIENT SUBTRACTION ALGEBRAS

Definition: 2.1.1

Let X be a subtraction algebra and I be an ideal of X . For any x, y in X , we define a relation " \sim " as the following:

$$x \sim y \Leftrightarrow x - y \in I \text{ and } y - x \in I.$$

Proposition: 2.1.2

Let X be a subtraction algebra and I be an ideal of X . The relation " \sim " defined by $x \sim y \Leftrightarrow x - y \in I$ and $y - x \in I$ is an equivalence relation on X .

Proof:

- (1) Since I is an ideal and $0 \in I$, we have $x - x = 0 \in I$. Then we have $x \sim x$. This means that " \sim " is reflexive.
- (2) The symmetry of " \sim " is immediate from the definition of the relation.
- (3) If $x \sim y$ and $y \sim z$ then $x - y, y - x \in I$ and $y - z, z - y \in I$.

Using (P7), we have $(x - z) - (y - z) \leq x - y$.

By virtue of I is an ideal of X , we obtain $x - z \in I$. Furthermore, from (P7), since $(z - y) - (y - x) \leq z - y$, we get $z - x \in I$.

This shows that " \sim " is transitive.

Note:

Denote the equivalence class containing x by $[x] = \{y \in X : y \sim x\}$.

Proposition: 2.1.3

Let X be a subtraction algebra and I be an ideal of X . Let " \sim " be the relation defined as $x \sim y \Leftrightarrow x - y \in I$ and $y - x \in I$. Then

- (1) $0 \in [x] \Leftrightarrow x \in I$,
- (2) $(x - y) - y \in [x - y]$,
- (3) If $y \in [x]$, then $x - (x - y) \in [x]$,
- (4) $(x - y) - (y - x) \in [x]$,
- (5) $x - (x - (x - y)) \in [x - y]$,
- (6) If $y \in [x]$, then $(x - z) - (y - z) \in [x - y]$.

Proof:

(1) $0 \in [x] \Leftrightarrow 0 \sim x \Leftrightarrow 0 - x = 0 \in I$ and $x - 0 = x \in I$.

(2) By (P1), trivial

(3) Since $(x - (x - y)) - y = 0 \in I$ and

$$\begin{aligned} y - (x - (x - y)) &= y - (y - (y - x)) \\ &= y - x \in I, \end{aligned}$$

We have $x - (x - y) \sim y$ and hence $x - (x - y) \in [x]$.

(4) By (P5), trivial.

(5) By (P6), trivial.

(6) We have

$$\begin{aligned} ((x - z) - (y - z)) - (x - y) &= ((x - z) - (x - y)) - (y - z) = ((x - (x - y)) - z) - (y - z) \\ &= ((y - (y - x)) - z) - (y - z) = ((y - z) - (y - x)) - (y - z) \\ &= 0 - (y - x) = 0 \in I. \end{aligned}$$

Furthermore, since

$$\begin{aligned} (x - y) - ((x - z) - (y - z)) &= (x - y) - ((x - y) - z) \\ &= z - (z - (x - y)) \leq (x - y) \end{aligned}$$

and $(x - y) \in I$ we have $(x - y) - ((x - z) - (y - z)) \in I$.

Then we obtain $(x - z) - (y - z) \sim (x - y)$.

Lemma: 2.1.4

Let X be a subtraction algebra and I be an ideal of X . The relation " \sim " defined as

$$x \sim y \Leftrightarrow x - y \in I \text{ and } y - x \in I$$

is an congruence relation on X .

Proof: We have to show that $x - u \sim y - v$ if $x \sim y$ and $u \sim v$.

Using (P7) and Lemma 1.2.3., since $(x - u) - (y - u) \leq x - y \in I$ and

$(y - u) - (x - u) \leq y - x \in I$, we obtain $(x - u) - (y - u) \in I$ and $(y - u) - (x - u) \in I$,

respectively.

This shows that $x - u \sim y - u$. From (P4), we have $y - (y - v) \leq v$ and by (P9) we obtain

$$(y - (y - v)) - u \leq v - u.$$

In the view of Lemma 1.2.3 and by (S3), we get $(y - (y - v)) - u = (y - u) - (y - v) \in I$.

By the same manner, we get $(y - v) - (y - u) \in I$ and so it is obtained that $y - u \sim y - v$.

By the transitivity of " \sim ", we obtain $x - u \sim y - v$.

Corollary: 2.1.5

Let X be a subtraction algebra and I be an ideal of X . Let " \sim " be the relation defined as $x \sim y \Leftrightarrow x - y \in I$ and $y - x \in I$. Then

(1) If $x \sim y$, then $x - z \sim y - z$,

(2) If $x \sim y$, then $z - x \sim z - y$.

Proof: By the above lemma (1) and (2) can be seen easily.

Proposition: 2.1.6

Let X be a subtraction algebra and I be an ideal of X then $I = [0]$.

Proof: If $x \in I$ then $x - 0 = x \in I$ and $0 - x = 0 \in I$, i.e. $x \sim 0$. Hence $x \in [0]$.

Conversely, let $x \in [0]$. Then $x = x - 0 \in I$ and so $x \in I$.

Hence $I = [0]$.

Definition: 2.1.7 [6]

Let X be a subtraction algebra and I be an ideal of X . Let " \sim " be an equivalence relation defined as $x \sim y \Leftrightarrow x - y \in I$ and $y - x \in I$. The set of all equivalence classes in X is denoted by X/I and it is called the *quotient set* of X by I .

Theorem: 2.1.8

Let X be a subtraction algebra and I be an ideal of X . Then X/I is a subtraction algebra with the operation "-" given by $[x] - [y] = [x - y]$ for all $x, y \in X$.

Proof: Since \sim is a congruence relation on X , the operation "-" is well defined.

$$\text{By (S1), } [x] - ([y] - [x]) = [x] - [y - x] = [x - (y - x)] = [x] \quad (1)$$

$$\begin{aligned} \text{By (S2), } [x] - ([x] - [y]) &= [x] - [x - y] = [x - (x - y)] = [y - (y - x)] = [y] - [y - x] \\ &= [y] - ([y] - [x]) \end{aligned} \quad (2)$$

$$\begin{aligned} \text{By (S3), } ([x] - [y]) - [z] &= [x - y] - [z] = [(x - y) - z] = [(x - z) - y] = [x - z] - [y] \\ &= ([x] - [z]) - [y] \end{aligned} \quad (3)$$

By (1), (2) and (3) we get X/I is a subtraction algebra.

Theorem: 2.1.9 [6, 27]

If I and J be any two ideals of subtraction algebra of X and $I \subset J$, then the following conditions are satisfied:

- (a) I is an ideal of subalgebra J
- (b) J/I is an ideal of quotient algebra of X/I .

Proof:

- (a) For all $x, y \in J$, let $y \in I$ and $x - y \in I$. From (P4) we have $x - (x - y) \leq y \in I$ and from Lemma 1.2.3 we have $x - (x - y) \in I \Rightarrow x \in I$.

(b) $J/I = \{[x]: x \in J\} \subset \{[x]: x \in X\} = X/I$.

From hypothesis, since J is an ideal of X we have $0 \in J$ and this implies $[0] \in J/I$.

Let $[x], [y] \in X$ be such that $[y] \in J/I$ and $[x] - [y] \in J/I$.

Then $y \in J$ and $x - y \in J$. From hypothesis, since J is an ideal of $X, x \in J$.

Then $[x] \in J/I$. This completes the proof.

Definition: 2.1.10

Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras. Then the mapping $f: (X, -) \rightarrow (Y, \ominus)$ is called

(1) a *homomorphism* if $f(x - y) = f(x) \ominus f(y) \forall x, y \in X$.

(2) a *monomorphism* if f is one to one homomorphism.

(3) an *epimorphism* if f is onto homomorphism.

(4) an *isomorphism* if f is monomorphism and epimorphism.

In this case X is isomorphic to Y , denoted by $X \cong Y$. In case $X = Y$, a homomorphism is called an *endomorphism* and an isomorphism is referred as an *automorphism*.

Definition: 2.1.11

A function f of a semilattice (\wedge -semilattice) X into itself is a dual closure if f is monotone, non expansive (i.e. $f(x) \leq x \forall x \in X$) and idempotent (i.e. $f \circ f = f$).

Note:

The set of all homomorphisms from X to Y is denoted by $Hom(X, Y)$. This set is always nonempty since it contains the zero homomorphism $0: X \rightarrow Y$ which sends every element of X to 0 .

Definition: 2.1.12 [27]

For any $f \in Hom(X, Y)$ and any nonempty subset $B \subseteq Y$, the set $\{x \in X : f(x) \in B\}$ is denoted by $f^{-1}(B)$, called the inverse image of B under f . In particular, $f^{-1}(\{0'\})$ is called kernel of f , where $f^{-1}(\{0'\}) = \{x \in X : f(x) = 0'\}$. We will simply write $\ker(f)$ instead of $f^{-1}(\{0'\})$.

Definition: 2.1.13

Let X and Y be two subtraction algebras and $f : X \rightarrow Y$ be a mapping. If for any $x, y \in X, x \leq y$ implies $f(x) \leq f(y)$ then f is called *isotone*.

Theorem: 2.1.14

Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras and $f : X \rightarrow Y$ be a homomorphism. Then

- (a) $f(0) = 0'$,
- (b) f is isotone.

Proof:

- (a) $f(0) = f(0 - 0) = f(0) \ominus f(0) = 0'$ since Y be a subtraction algebra.
- (b) If $x, y \in X$ and $x \leq y$ then $x - y = 0$, by (a), $f(x) \ominus f(y) = f(x - y) = f(0) = 0'$.
Hence $f(x) \leq f(y)$.

Theorem: 2.1.15

Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras and let B be an ideal of Y . Then for any $f \in Hom(X, Y)$, $f^{-1}(B)$ is an ideal of X .

Proof: Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras and let B be an ideal of Y . From Theorem 2.1.14, $0 \in f^{-1}(B)$.

Assume that $x - y \in f^{-1}(B)$ and $y \in f^{-1}(B)$.

Then $f(x) \ominus f(y) \in B$ and $f(y) \in B$. Since B is an ideal of Y , $f(x) \in B$.

Thus, $x \in f^{-1}(B)$. Hence, $f^{-1}(B)$ is an ideal of X .

Note:

Since $\{0\}$ is an ideal of Y , we have $\ker f$ is an ideal of X .

Theorem: 2.1.16

Let I be an ideal of a subtraction algebra of X . The mapping φ is from X to X/I which is given by $\varphi(x) = [x]$ for all x in X is a homomorphism.

Proof: For any x, y in X , $\varphi(x - y) = [x - y] = [x] - [y] = \varphi(x) - \varphi(y)$.

This completes the proof.

Theorem: 2.1.17

If J^* is an ideal of quotient subtraction algebra of X/I , then $J = \bigcup_{[x] \in J^*} [x]$ is an ideal of X and $I \subseteq J$.

Proof: Since $I = [0] \in J^*$, $0 \in J$ and since for any $x \in I$, $x \in [0] \subset J$ we obtain $I \subseteq J$.

Let $x - y \in J$ and $y \in J$, then $[x] - [y] = [x - y] \in J^*$ and $[y] \in J^*$.

Hence $[x] \in J^*$ and $x \in J$. This means that J is an ideal of X .

Notation:

The set of all ideals of subtraction algebra of X is denoted by $I(X)$. Also the set of all ideals which are containing I on X is denoted by $I(X, I)$.

Theorem: 2.1.18

Let I be an ideal of subtraction algebra of X . There exists an one-to-one and onto mapping f from $I(X, I)$ to $I(X/I)$ defined by, for any $J \in I(X, I)$, $f(J) = J/I$.

Proof: By the above theorem, obviously f is onto.

For any $A, B \in I(X, I)$, let $f(A) = f(B)$ but $A \neq B$.

Then, atleast for any $x \in B$, we have $x \notin A$.

Since $f(A) = f(B)$, $[x] \in f(A)$ and $[x] \in f(B)$.

Then there exists $z \in A$ such that $[z] = [x]$.

This implies $x \sim z$, that is, $x - z \in I$. By symmetry, $z - x \in I$.

Since $I \subseteq A$, $x - z \in A$ and also since $z \in A$, we obtain $x \in A$ by the definition of an ideal, which contradicts $x \notin A$. Therefore, f is one to one.

Theorem: 2.1.19

If J is an ideal of quotient subtraction algebra of X/I , then $f^{-1}(J)$ is an ideal of X and $I \subseteq f^{-1}(J)$.

Proof: We know that mapping given by $f(x) = [x]$, from X to X/I , is a

homomorphism. For any $x, y \in X$, let $y \in f^{-1}(J)$ and $x - y \in f^{-1}(J)$.

Then $f(y) = [y] \in J$ and $f(x - y) = [x - y] = [x] - [y] \in J$.

Since J is an ideal, we have $[x] \in J$. or $x \in f^{-1}(J)$.

Since $[0] \in J$, $0 \in f^{-1}(J)$. Furthermore for all $x \in I$, since $[x] = [0] \in J$, $x \in f^{-1}(J)$.

Hence $I \subseteq f^{-1}(J)$.

Theorem: 2.1.20

Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras and $f : X \rightarrow Y$ be an epimorphism. Then, $X / \ker f \cong Y$.

Proof: Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras and $f : X \rightarrow Y$ be an epimorphism.

Since $\ker f$ is an ideal of X , by Theorem 2.1.15, $X / \ker f$ is a subtraction algebra.

Assume $g : X / \ker f \rightarrow Y$ such that $g([x]) = f(x)$.

If

$[x] = [y]$ then, $x - y, y - x \in \ker f$, so

$$f(x - y) = f(x) \ominus f(y) = 0'$$

$$f(y - x) = f(y) \ominus f(x) = 0'$$

By (P2) we have $f(x) = f(y)$, i.e., $g([x]) = g([y])$.

Hence g is well-defined. For any $y \in Y$, there exists $x \in X$ such that $y = f(x)$ as f is onto. Hence $g([x]) = y$, which means that $g : X / \ker f \rightarrow Y$ is onto.

If $[x] \neq [y]$, then $x - y \notin \ker f$ or $y - x \notin \ker f$.

Suppose $x - y \notin \ker f$, $f(x) \ominus f(y) = f(x - y) \neq 0'$.

Therefore, $f(x) \neq f(y)$. That is, g one-to-one.

Since $g([x] - [y]) = g([x - y]) = f(x - y) = f(x) \ominus f(y) = g([x]) \ominus g([y])$

g is homomorphism. Therefore, $X / \ker f \cong Y$.

Theorem: 2.1.21

Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras and $f : X \rightarrow Y$ be an epimorphism. If J is an ideal of Y , then $X / I \cong Y / J$ where $I = f^{-1}(J)$.

Proof: Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras and $f : X \rightarrow Y$ be an epimorphism. Then $\mu \circ f : X \rightarrow Y / J$ is an epimorphism.

We now prove that $\ker(\mu \circ f) = f^{-1}(J)$.

For any $x \in X$, we have $(\mu \circ f)(x) = \mu(f(x)) = [f(x)]$.

Suppose $y \in f^{-1}(J)$. Then $f(y) \in J$, and so $[f(y)] = J$. That is, $(\mu \circ f)(y) = J$.

Hence $y \in \ker(\mu \circ f)$. Thus, we have $f^{-1}(J) \subseteq \ker(\mu \circ f)$.

Inverse, we assume $x \in \ker(\mu \circ f)$, i.e., $(\mu \circ f)(x) = J$.

Therefore, we have $[f(x)] = J$, and so $f(x) \in J$, i.e., $x \in f^{-1}(J)$.

Thus, by Theorem 2.1.20, $X / I \cong Y / J$.

Lemma: 2.1.22

Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras, $f: X \rightarrow Y$ be an epimorphism and I be an ideal of X . If $\ker f \subseteq I$, then $f^{-1}(f(I)) = I$.

Proof: Obviously, $I \subseteq f^{-1}(f(I))$.

Inverse, we assume $x \in f^{-1}(f(I))$, then $f(x) \in f(I)$.

There exists $y \in I$ such that $f(x) = f(y)$, so $f(x - y) = f(x) \ominus f(y) = 0'$, hence $x - y \in \ker f \subseteq I$. Since $x - y \in I, y \in I$, we have $x \in I$.

Therefore $f^{-1}(f(I)) \subseteq I$.

Theorem: 2.1.23

Let I and J be two ideals of subtraction algebra X , $I \subseteq J$ and $f: X \rightarrow X/I$ and $g: X/I \rightarrow (X/I)/(J/I)$ be epimorphisms. Then $X/J \cong (X/I)/(J/I)$.

Proof: Let $h = g \circ f$. Then $h: X \rightarrow (X/I)/(J/I)$ is an epimorphism.

Hence $X/\ker h \cong (X/I)/(J/I)$.

We must prove that $\ker h = J$. Since $\ker h = \{x \in X : h(x) = J/I\}$, by Theorem 2.1.9,

Lemma 2.1.22 and Theorem 2.1.21 we have $\ker h = h^{-1}(h(J)) = J$.

The proof is complete.

SECTION: 2.2

SUBTRACTION ALGEBRAS WITH PSEUDO-VALUATIONS

Definition: 2.2.1 [22]

A real valued function φ on a subtraction algebra X is called a *weak pseudo-valuation* on X if it satisfies for all $x, y \in X$, the following condition:

$$\varphi(x - y) \leq \varphi(x) + \varphi(y)$$

Example: 2.2.2

Let $X = \{0, a, b, 1\}$ in which "-" is defined by

-	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Then $(X; -)$ is a subtraction algebra.

Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & 1 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$

Then φ is a weak pseudo-valuation on X .

Definition: 2.2.3 [5]

A real-valued function φ on a subtraction algebra X is called a *pseudo-valuation* on a subtraction algebra X if it satisfies for all $x, y \in X$, the following conditions:

- (1) $\varphi(0) = 0$
- (2) $\varphi(x) \leq \varphi(x - y) + \varphi(y)$

A pseudo-valuation φ on X satisfying the following condition:

$$x \neq 0 \Rightarrow \varphi(x) \neq 0 \text{ for all } x \in X$$

is called a *valuation* on X .

Example: 2.2.4

Let $X = \{0, a, b\}$ be a subtraction algebra with the following cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b \\ 0 & 2 & 3 \end{pmatrix}$$

Then φ is a pseudo-valuation on X .

Example: 2.2.5

Consider the subtraction algebra in Example 2.2.2

Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Then φ is a pseudo-valuation on X .

Proposition: 2.2.6

For a pseudo-valuation φ on a subtraction algebra X , we have $\varphi(x) \geq 0$ for all $x \in X$.

Proof: For any $x \in X$, $\varphi(0) = \varphi(0 - x) \geq \varphi(0) - \varphi(x)$, and so $\varphi(x) \geq 0$.

Proposition: 2.2.7

Let S be a subalgebra of X . For any positive real numbers t_1 and t_2 with $t_1 \leq t_2$, let φ_s be real-valued function on X defined by

$$\varphi_s(x) = \begin{cases} t_1 & \text{if } x \in S, \\ t_2 & \text{if } x \notin S \end{cases}$$

for all $x \in X$. Then φ_s is a weak pseudo-valuation on X .

Proof: Obvious

Definition: 2.2.8

Given an element a of a subtraction algebra X , the set

$$A(a) = \{x \in X \mid x \leq a\}$$

is called the *initial section* of X determined by a .

Corollary: 2.2.9

Let X be a subtraction algebra. For any $a \in X$, let φ be a real valued function on X defined by

$$\varphi_a(x) = \begin{cases} t_1 & \text{if } x \in A(a) \\ t_2 & \text{if } x \notin A(a) \end{cases}$$

for all $x \in X$ where t_1 and t_2 are real numbers with $t_2 > t_1 \geq 0$ and $A(a)$ is the *initial section* on X determined by a . Then φ_a is a *weak pseudo-valuation* on X .

Proof: Obvious

Proposition: 2.2.10

For any pseudo-valuation φ on a subtraction algebra X , we have the following inequalities:

(a) φ is order preserving.

$$(b) \varphi(x - y) + \varphi(y - x) \geq 0 \quad \forall x, y \in X$$

$$(c) \varphi(x - y) \leq \varphi(x - z) + \varphi(z - y) \quad \forall x, y, z \in X$$

Proof: Let φ be a pseudo-valuation on a subtraction algebra X . Then

$$\varphi(0) = 0 \tag{1}$$

$$\text{and } \varphi(x) \leq \varphi(x - y) + \varphi(y) \tag{2}$$

(a) Let $x \leq y$. Then $x - y = 0$.

Therefore $\varphi(x) \leq \varphi(x - y) + \varphi(y) = \varphi(0) + \varphi(y)$, which implies $\varphi(x) \leq \varphi(y)$.

(b) Let $x, y \in X$. Then by (2), $\varphi(x - y) \geq \varphi(x) - \varphi(y)$ and $\varphi(y - x) \geq \varphi(y) - \varphi(x)$.

It follows that $\varphi(x - y) + \varphi(y - x) \geq 0$.

(c) Let $x, y, z \in X$. Note that $(x - y) - (z - y) \leq x - z$.

By (a) and (2), we have $\varphi(x - z) \geq \varphi((x - y) - (z - y)) \geq \varphi(x - y) - \varphi(z - y)$.

Hence $\varphi(x - y) \leq \varphi(z - y) + \varphi(x - z) = \varphi(x - z) + \varphi(z - y)$.

Proposition: 2.2.11

For any pseudo-valuation φ on a subtraction algebra X , we have the following inequalities:

$$(1) \varphi(x - y) \leq \varphi(x - y) + \varphi(y - x) \quad \forall x, y \in X$$

$$(2) \varphi(x - y) \leq \varphi(x - y) + \varphi(y) \quad \forall x, y, z \in X$$

Proof: since φ be a pseudo-valuation on a subtraction algebra X . Then

$$(1) \varphi(x - y) \leq \varphi((x - y) + \varphi(y - x)) + \varphi(y - x) = \varphi(x - y) + \varphi(y - x).$$

$$(2) \varphi(x - y) \leq \varphi((x - y) - y) + \varphi(y) = \varphi(x - y) + \varphi(y).$$

Proposition: 2.2.12

Every pseudo-valuation φ on a subtraction algebra X satisfies the following implication:

$$(x - y) - z = 0 \Rightarrow \varphi(x) \leq \varphi(y) + \varphi(z) \text{ for all } x, y, z \in X.$$

Proof: Let φ be a pseudo-valuation on a subtraction algebra X .

Let $x, y, z \in X$ be such that $(x - y) - z = 0$.

Then $\varphi(x - y) \leq \varphi((x - y) - z) + \varphi(z) = \varphi(0) + \varphi(z) = \varphi(z)$ so that $\varphi(x) \leq \varphi(x - y) + \varphi(y) \leq \varphi(y) + \varphi(z)$.

The following corollary can be proved by induction

Corollary: 2.2.13

Let φ be a pseudo-valuation on X . If

$$(\dots((x - a_1) - a_2) - \dots) - a_n = 0,$$

we get $\varphi(x) \leq \sum_{k=1}^n \varphi(a_k)$.

Proposition: 2.2.14

Every real-valued function φ on X satisfying

$$\varphi(0) = 0 \tag{1}$$

$$\text{and } (x - y) - z = 0 \Rightarrow \varphi(x) \leq \varphi(y) + \varphi(z) \quad \forall x, y, z \in X \tag{2}$$

Proof: By (P4), $(x - (x - y)) - y = 0$ for all $x, y \in X$.

$$\Rightarrow \varphi(x) \leq \varphi(x - y) + \varphi(y). \tag{3}$$

By (1) and (3), φ is a pseudo-valuation on X . This completes the proof.

Proposition: 2.2.15

In a subtraction algebra X , every pseudo-valuation is a weak pseudo-valuation.

Proof: Let φ be pseudo-valuation on X .

Since $((x - y) - x) - y = ((x - x) - y) - y$

$$= (0 - y) - y = 0 \quad \text{for all } x, y \in X.$$

Now $0 = \varphi(0) = \varphi(((x - y) - x) - y)$

$$\geq \varphi((x - y) - x) - \varphi(y)$$

$$\geq \varphi(x - y) - \varphi(x) - \varphi(y).$$

Hence $\varphi(x - y) \leq \varphi(x) + \varphi(y)$, and so φ is a weak pseudo-valuation on X .

This completes the proof.

The following example shows that the converse of Theorem 2.1.15 may not be true.

Example: 2.2.16

The weak pseudo-valuation φ in Example 2.2.2 is not a pseudo-valuation on X since $\varphi(a) = 3 \not\leq 2 = \varphi(a - 1) + \varphi(1)$.

Theorem: 2.2.17

If a real-valued function φ on X satisfies the condition

$$\varphi(0) = 0 \tag{1}$$

$$\text{and } \varphi(((x - y) - y) - z) \geq \varphi(x - y) - \varphi(z), \tag{2}$$

for all $x, y, z \in X$, φ is a pseudo-valuation on X .

Proof: Taking $y = 0$ in (2),

$$\varphi(x - z) = \varphi(((x - 0) - 0) - z)$$

$$\geq \varphi(x - 0) - \varphi(z) = \varphi(x) - \varphi(z).$$

Hence φ is a pseudo-valuation on X .

Theorem: 2.2.18

Let a real-valued function φ be a pseudo-valuation on a subtraction algebra X .

Then the set

$$I = \{x \in X / \varphi(x) = 0\}$$

is an ideal of X .

Proof: Obviously, $0 \in I$.

Let $x, y \in X$ be such that $x - y \in I$ and $y \in I$. Then $\varphi(x - y) = 0$ and $\varphi(y) = 0$.

Therefore, $\varphi(x) \leq \varphi(x - y) + \varphi(y) = 0$ so that $\varphi(x) = 0$.

Hence $x \in I$, which shows that I is an ideal of X .

Definition: 2.2.19

By a *pseudo-metric* we mean a positive function $d : X \times X \rightarrow \mathbb{R}$ satisfying the following properties:

- (1) $d(x, x) = 0$,
- (2) $d(x, y) = d(y, x)$,
- (3) $d(x, z) \leq d(x, y) + d(y, z)$. for all $x, y \in X$.

For a real-valued function φ on a subtraction algebra X , define a mapping

$d_\varphi : X \times X \rightarrow \mathbb{R}$ by $d_\varphi(x, y) = \varphi(x - y) + \varphi(y - x)$ for all $(x, y) \in X \times X$.

Theorem: 2.2.20

Let X be a subtraction algebra. If a real-valued function φ on X is a pseudo-valuation on X , d_φ is a pseudo-metric on X , and so (X, d_φ) is a pseudo-metric space.

Proof: Obviously, $d_\varphi(x, y) \geq 0$,

$$d_\varphi(x, x) = 0 \text{ and } d_\varphi(x, y) = d_\varphi(y, x) \text{ for all } x, y \in X.$$

Using Proposition 2.2.10, we have

$$\begin{aligned} d_\varphi(x, y) + d_\varphi(y, z) &= [\varphi(x - y) + \varphi(y - x)] + [\varphi(y - z) + \varphi(z - y)] \\ &= [\varphi(x - y) + (y - z)] + [\varphi(z - y) + \varphi(y - x)] \\ &\geq \varphi(x - z) + \varphi(z - x) = \varphi(x - z). \end{aligned}$$

Therefore, (X, d_φ) is a pseudo-metric space.

Proposition: 2.2.21

Let X be a subtraction algebra. Then every pseudo-metric d_φ induced by pseudo-valuation φ satisfying the following inequality.

- (1) $d_\varphi(x, y) \geq d_\varphi(x - a, y - a)$,
- (2) $d_\varphi(x, y) \geq d_\varphi(a - x, a - y)$,
- (3) $d_\varphi(x - y, a - b) \leq d_\varphi(x - y, a - y) + d_\varphi(a - y, a - b)$, for all $x, y, a, b \in X$.

Proof:

- (1) Let $x, y, a \in X$.

Since $((y - a) - (x - a)) - (y - x) = 0$ and $((x - a) - (y - a)) - (x - y) = 0$ from (P7),

it follows from Proposition 2.2.10 (1) that

$$\varphi(y - x) \geq \varphi((y - a) - (x - a)) \text{ and } \varphi(x - y) \geq \varphi((x - a) - (y - a))$$

$$\text{and } d_\varphi(x, y) = \varphi(x - y) + \varphi(y - x)$$

$$\geq \varphi((x - a) - (y - a)) + \varphi((y - a) - (x - a))$$

$$= d_\varphi(x - a, y - a).$$

- (2) It is similar to the proof of (1).
- (3) Using Proposition 2.2.10 (c),

$\varphi((x-y)-(a-b)) \leq \varphi((x-y)-(a-y)) + \varphi((a-y)+(a-b))$ and
 $\varphi((a-b)-(x-y)) \leq \varphi((a-b)-(a-y)) + \varphi((a-y)+(x-y))$ for all $x, y, a, b \in X$.

$$\begin{aligned} \text{Hence } d_\varphi(x-y, a-b) &= \varphi((x-y)-(a-b)) + \varphi((a-b)-(x-y)) \\ &\leq [\varphi((x-y)-(a-y)) + \varphi((a-y)-(a-b))] + \\ &\quad [\varphi((a-b)-(a-y)) + \varphi((a-y)-(x-y))] \\ &= [\varphi((x-y)-(a-y)) + \varphi((a-y)-(x-y))] + \\ &\quad [\varphi((a-b)-(a-y)) + \varphi((a-y)-(a-b))] \\ &= d_\varphi(x-y, a-y) + d_\varphi(a-y, a-b) \end{aligned}$$

for all $x, y, a, b \in X$.

Theorem: 2.2.22

For a real-valued function φ on a subtraction algebra X , if d_φ is a pseudo-metric on X , we have $(X \times X, d_\varphi^*)$ is a pseudo-metric space, where

$$d_\varphi^*((x, y), (a, b)) = \max\{d_\varphi(x, a), d_\varphi(y, b)\}$$

for all $(x, y), (a, b) \in X \times X$.

Proof: Suppose d_φ is a pseudo-metric on X . For any $(x, y), (a, b) \in X \times X$, we have

$$d_\varphi^*((x, y), (x, y)) = \max\{d_\varphi(x, x), d_\varphi(y, y)\} = 0 \quad \text{and}$$

$$\begin{aligned} d_\varphi^*((x, y), (a, b)) &= \max\{d_\varphi(x, a), d_\varphi(y, b)\} \\ &= \max\{d_\varphi(a, x), d_\varphi(b, y)\} \\ &= d_\varphi^*((a, b), (x, y)). \end{aligned}$$

Now let $(x, y), (a, b), (u, v) \in X \times X$. Then

$$\begin{aligned} d_\varphi^*((x, y), (u, v)) + d_\varphi^*((u, v), (a, b)) &= \max\{d_\varphi(x, u), d_\varphi(y, v)\} + \max\{d_\varphi(u, a), d_\varphi(v, b)\} \\ &\geq \max\{d_\varphi(x, u) + d_\varphi(u, a), d_\varphi(y, v) + d_\varphi(v, b)\} \end{aligned}$$

$$\geq \max\{d_\varphi(x, a), d_\varphi(y, b)\} = d_\varphi^*((x, y), (a, b)).$$

Therefore $(X \times X, d_\varphi^*)$ is a pseudo-metric space.

Corollary: 2.2.23

If $\varphi : X \rightarrow \mathbb{R}$ is a pseudo-valuation on a subtraction algebra X , then $(X \times X, d_\varphi^*)$ is a pseudo-metric space.

Theorem: 2.2.24

Let X be a subtraction algebra. If $\varphi : X \rightarrow \mathbb{R}$ is a valuation on X , then (X, d_φ) is a metric space.

Proof: Suppose that φ is a valuation on X .

Then (X, d_φ) is a metric space by Theorem 2.2.10.

Let $x, y \in X$ be such that $d_\varphi(x, y) = 0$ and $d_\varphi(y, x) = 0$.

Then $0 = d_\varphi(x, y) = d_\varphi(x - y) + d_\varphi(y - x)$, and so $\varphi(x - y) = 0$ and $\varphi(y - x) = 0$.

Now since φ is a valuation on X , it follows that $x - y = 0$ and $y - x = 0$.

Therefore we have $x = x - (y - x) = x - 0 = x - (x - y)$

$$= y - (y - x) = y - 0 = y.$$

This implies that (X, d_φ) is a metric space.

Theorem: 2.2.25

Let X be a subtraction algebra. If $\varphi : X \rightarrow \mathbb{R}$ is a valuation on X , $(X \times X, d_\varphi^*)$ is a metric space.

Proof: Note from Corollary 2.2.22 that $(X \times X, d_\varphi^*)$ is a pseudo-metric space.

Let $(x, y), (a, b) \in X \times X$ be such that $d_\varphi^*((x, y), (a, b)) = 0$.

Then $0 = d_\varphi^*((x, y), (a, b)) = \max\{d_\varphi^*(x, a), d_\varphi^*(y, b)\}$, and so $d_\varphi(x, a) = 0 = d_\varphi(y, b)$

since $d_\varphi(x, y) \geq 0$ for all $(x, y) \in X \times X$.

Hence, $0 = d_\varphi(x, a) = \varphi(x - a) + \varphi(a - x)$ and $0 = d_\varphi(y, b) = \varphi(y - b) + \varphi(b - y)$.

Hence it follows that $\varphi(x, a) = \varphi(a - x)$ and $\varphi(y, b) = 0 = \varphi(b - y)$

so that $x - a = 0 = a - x$ and $y - b = 0 = b - y$, which implies $x = a$ and $y = b$.

Hence $(x, y) = (a, b)$ so that $(X \times X, d_\varphi^*)$ is a metric space.

Theorem: 2.2.26

Let X be a subtraction algebra. If $\varphi: X \rightarrow \mathbb{R}$ is a valuation on X , the operation " $-$ " is uniformly continuous.

Proof: For any $\varepsilon > 0$, if $d_\varphi^*((x, y), (a, b)) < \frac{\varepsilon}{2}$, we have $d_\varphi(x, a) < \frac{\varepsilon}{2}$ and $d_\varphi(y, b) < \frac{\varepsilon}{2}$.

Using Proposition 2.2.21, we get

$$\begin{aligned} d_\varphi(x - y, a - b) &\leq d_\varphi(x - y, a - y) + d_\varphi(a - y, a - b) \\ &\leq d_\varphi(x, a) + d_\varphi(y, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, the operation $-: X \times X \rightarrow X$ is uniformly continuous.

Chapter-III

CHAPTER III

ON DERIVATIONS OF SUBTRACTION ALGEBRAS

SECTION: 3.1

DERIVATIONS AND SIMPLE DERIVATIONS OF A SUBTRACTION ALGEBRA

Definition: 3.1.1 [34]

Let X be a subtraction algebra. By a derivation of X we mean a self-map d of X satisfying the identity $d(x - y) = (d(x) - y) \wedge (x - d(y))$ for all $x, y \in X$.

Example: 3.1.2

(1) Let $X = \{0, a, b, 1\}$ in which "-" is defined by

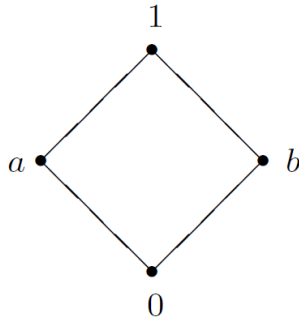
-	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Then $(X; -)$ is a subtraction algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, a, \\ b & \text{if } x = b, 1. \end{cases}$$

Then d is a derivation of the subtraction algebra X .

Figure 1. The Hasse diagram of Example 3.1.2 (1)



(1) Let $X = \{0, a, b\}$ be a subtraction algebra with the following cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, b, \\ b & \text{if } x = a. \end{cases}$$

Then d is a derivation of subtraction algebra X .

Example: 3.1.3

Let X be a subtraction algebra. We define a function d by $d(x) = 0$ for all $x \in X$. Then d is a derivation on X , which is called the zero derivation.

Example: 3.1.4

Let d be the identity function on a subtraction algebra X . Then d is a derivation on X , which is called the identity derivation.

Proposition: 3.1.5

Let d be a derivation on a subtraction algebra X . Then $d(0) = 0$.

Proof: Let d be a derivation on a subtraction algebra of X .

Then $d(0) = d(0 - x) = (d(0) - x) \wedge (0 - d(x))$

$$= (d(0) - x) \wedge 0 = 0$$

Proposition: 3.1.6

Let d be a derivation on a subtraction algebra X . Then $d(x - d(x)) = 0$ for every $x \in X$.

Proof: Let d be a derivation of a subtraction algebra of X and let $x \in X$.

$$\begin{aligned} \text{Then } d(x - d(x)) &= (d(x) - d(x)) \wedge (x - d(d(x))) \\ &= 0 \wedge (x - d(d(x))) = 0. \end{aligned}$$

Proposition: 3.1.7

Let d be a derivation on a subtraction algebra X . Then $d(x) = d(x) \wedge x$.

Proof: Let d be a derivation on X .

$$\begin{aligned} \text{Then } d(x) &= d(x - 0) = (d(x) - 0) \wedge (x - d(0)) \\ &= d(x) \wedge (x - 0) = d(x) \wedge x. \end{aligned}$$

Corollary: 3.1.8

Let d be a derivation of subtraction algebra X . Then we have $d(x) \leq x$. That is, d is a non-expansive map.

Proof: Obvious

Theorem: 3.1.9

Let d be a derivation of subtraction algebra X . If $x \leq y$ for $x, y \in X$, then $d(x) \leq d(y)$.

Proof: Let $x \leq y$ for $x, y \in X$.

Then by (P8), $x = y - w$ for some $w \in X$.

Hence we have $d(x) = d(y - w) = (d(y) - w) \wedge (y - d(w))$

$$\leq d(y) - w \leq d(y).$$

Theorem: 3.1.10

Let d be a derivation of a subtraction algebra X . Then we have $d^2 = d \circ d = d$.

Proof: Let d be a derivation of X .

Then by definition of the derivation d and Proposition 3.1.6, we have

$$\begin{aligned} d^2(x) &= d(d(x)) = d(x \wedge d(x)) = d(x - (x - d(x))) \\ &= d(x - (x - d(x))) \wedge (x - d(x - d(x))) = d(x) \wedge (x - 0) = d(x) \wedge x = d(x) \end{aligned}$$

Corollary: 3.1.11

Let d be a derivation of a subtraction algebra X . Then d is a dual closure operator on X .

Proof: Obvious

Proposition: 3.1.12

Let f is a non-expansive map on a subtraction algebra X , i.e., $f(x) \leq x$ for all $x \in X$. Then $f(x) - y \leq x - f(y)$ for all $x, y \in X$.

Proof: Suppose that f is a non-expansive map on a subtraction algebra X and $x, y \in X$.

Then $f(x) \leq x$ and $f(y) \leq y$.

Hence $f(x) - y \leq x - y$ and $x - y \leq x - f(y)$ by (P9).

It follows that $f(x) - y \leq x - f(y)$.

Theorem: 3.1.13

Let d be a map on a subtraction algebra X . Then the following are equivalent:

- (1) d is a derivation of X ;
- (2) $d(x - y) = d(x) - y$ for all $x, y \in X$.

Proof: Suppose that d is a derivation of X .

Then d is non-expansive by Corollary 3.1.8.

Hence for any $x, y \in X$, $d(x) - y \leq x - d(y)$ by Proposition 3.1.12 and

$$d(x - y) = (d(x) - y) \wedge (x - d(y)) = d(x) - y.$$

Suppose that d is a map satisfying $d(x - y) = d(x) - y$ for all $x, y \in X$.

Then $d(0) = d(0 - d(0)) = d(0) - d(0) = 0$.

Hence, $0 = d(0) = d(x - x) = d(x) - x$ for any $x \in X$.

It follows that $d(x) \leq x$ for any $x \in X$. That is, d is non-expansive.

Hence by Proposition 3.1.12, $d(x) - y \leq x - d(y)$ and

$$d(x - y) = d(x) - y = (d(x) - y) \wedge (x - d(y)) \text{ for any } x, y \in X.$$

Theorem: 3.1.14

Let X be a subtraction algebra. The every derivation of X is an homomorphism.

Proof: Suppose that d is a derivation of X and $x, y \in X$. Then $d(y) \leq y$.

It implies $d(x - y) = d(x) - y \leq d(x) - d(y)$ by (P9). Also we have

$$(d(x) - d(y)) - (d(x) - y) = (dd(x) - d(y)) - (d(x) - y) \text{ (by Theorem 3.1.10)}$$

$$= (dd(x) - (d(x) - y)) - d(y) \text{ (by (S3))}$$

$$= d(d(x) - (d(x) - y)) - d(y) \text{ (by Theorem 3.1.13)}$$

$$= d(y - (y - d(x))) - d(y) \text{ (by (S2))}$$

$$\leq d(y) - d(y) = 0 \text{ (by (P3), Theorem 3.1.9 and (P9))}$$

It follows that $(d(x) - d(y)) - (d(x) - y) = 0$ and $d(x) - d(y) \leq d(x) - y = d(x - y)$.

Hence $d(x) - d(y) = d(x - y)$.

In general, the converse of Theorem 3.1.14 is not true which can be seen from the following example.

Example: 3.1.15

Let $X = \{0, a, b, 1\}$ be the subtraction algebra of Example 3.1.2 (1).

Define a map $f : X \rightarrow X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a, \\ 1 & \text{if } x = b, 1. \end{cases}$$

Then f is a endomorphism of X which is not a derivation because of $f(b-a) = f(b) = 1 \neq b = 1-a = f(b) - a$.

Definition: 3.1.16

Let X be a subtraction algebra. Then, for each $a \in X$, a map $d_a : X \rightarrow X$ is defined by $d_a(x) = x - a$ for all $x \in X$.

Proposition: 3.1.17

Let X be a subtraction algebra. Then for each $a \in X$, the map d_a is a derivation of X .

Proof: Suppose that d_a is the map defined by $d_a(x) = x - a$ for each $x \in X$.

Then for any $x, y \in X$, we have $d_a(x - y) = (x - y) - a = (x - a) - y = d_a(x) - y$ by (S3).

Hence d_a is a derivation of X by Theorem 3.1.13.

Note:

The derivation d_a of Proposition 3.1.17 is called a *simple derivation*.

Proposition: 3.1.18

Let d be a derivation of a subtraction algebra X . Then for each $x \in X$, there exists a unique $\hat{x} \in [0, x]$ such that $d(x) = x - \hat{x}$ and $d(\hat{x}) = 0$.

Proof: Suppose that d is a derivation on X and $x \in X$.

Then $d(x) \leq x$ since d is non-expansive.

Let $\hat{x} = x - d(x)$. Then $\hat{x} \in [0, x]$ and $d(\hat{x}) = 0$ by Proposition 3.1.6, and

$$x - \hat{x} = x - (x - d(x)) = x \wedge d(x) = d(x).$$

If $x - \hat{x} = d(x) = x - w'$ for some $w' \in [0, x]$, then

$$\hat{x} - w' = (x \wedge \hat{x}) - w' = (x - (x - \hat{x})) - w' = (x - w') - (x - \hat{x}) = 0.$$

It follows that $\hat{x} \leq w'$. Similarly, $w' \leq \hat{x}$ can be proved.

Hence $\hat{x} = w'$, and \hat{x} is the unique element in $[0, x]$ such that $d(x) = x - \hat{x}$.

Lemma: 3.1.19

Let d be a derivation of a subtraction algebra X . Then $\ker(d) = \{\hat{x} \mid x \in X\}$ where \hat{x} is a unique element in $[0, x]$ such that $d(x) = x - \hat{x}$ and $d(\hat{x}) = 0$.

Proof: Obviously, $\{\hat{x} \mid x \in X\} \subseteq \ker(d)$ (1)

If $x \in \ker(d)$, then $x = x - 0 = x - d(x) = \hat{x}$.

This implies $\ker(d) \subseteq \{\hat{x} \mid x \in X\}$ (2)

By (1) and (2), $\ker(d) = \{\hat{x} \mid x \in X\}$.

Theorem: 3.1.20

Let d be a derivation of a subtraction algebra X . Then for each interval $[0, a]$ in X ,

$$d(x) = d_{\hat{a}}(x)$$

for all $x \in [0, a]$. That is, the restriction $d|_{[0, a]}: [0, a] \rightarrow X$ of d is a simple derivation $d_{\hat{a}}$, where $\hat{a} \in [0, a]$ is the unique element such that $d(a) = a - \hat{a}$ and $d(\hat{a}) = 0$.

Proof: Suppose that d is a derivation of X and $a \in X$. Then there is a unique $\hat{a} \in [0, a]$ such that $d(a) = a - \hat{a}$, and for any $x \in [0, a]$ we have

$$d(x) = d(a \wedge x) = d(a - (a - x)) = d(a) - (a - x) = (a - \hat{a}) - (a - x)$$

$$= (a - (a - x)) - \hat{a} = (a \wedge x) - \hat{a} = x - \hat{a}.$$

Hence $d(x) = x - \hat{a} = d_{\hat{a}}(x)$ for all $x \in [0, a]$.

Corollary: 3.1.21

Let X be a subtraction algebra with greatest element 1. Then every derivation d of X is a simple derivation $d_{\hat{1}}$.

Proof: Suppose that $1 \in X$ and d is a derivation on X .

Then $X = [0, 1]$ and by Theorem 3.1.20, $d(x) = x - \hat{1} = d_{\hat{1}}(x)$ for all $x \in [0, 1] = X$.

Hence d is the simple derivation $d_{\hat{1}}$.

Note:

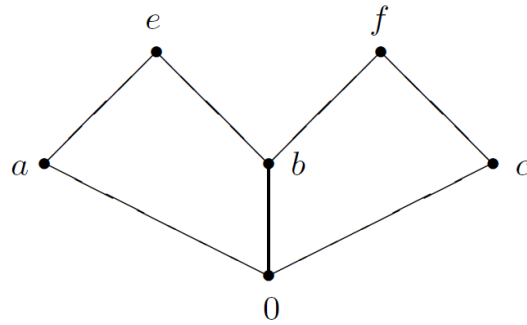
There can be a derivation on a subtraction algebra which is not simple.

Example: 3.1.22

Let $X = \{0, a, b, c, e, f\}$ be a subtraction algebra with "-" defined by

-	0	a	b	c	e	f
0	0	0	0	0	0	0
a	a	0	a	a	0	a
b	b	b	0	b	0	0
c	c	c	c	0	c	0
e	e	b	a	e	0	a
f	f	f	c	b	c	0

Figure 2. The Hasse diagram of Example 3.1.22



Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, a, c, \\ b & \text{if } x = b, e, f. \end{cases}$$

Then d is a derivation of X which is not simple, because there is no $x \in X$ satisfying either $d(e) = b = e - x$ or $d(f) = b = f - x$.

For the interval $A = [0, e]$ and $B = [0, f]$, $\hat{e} = e - d(e) = e - b = a$ and $\hat{f} = c$.

Hence the restrictions $d|_A$ and $d|_B$ are simple, being given by

$$d|_{A(x)} = x - a = d(x) \quad (x \in A) \text{ and}$$

$$d|_{B(x)} = x - c = d(x) \quad (x \in B), \text{ respectively.}$$

SECTION: 3.2

DERIVATIONS AND IDEALS OF SUBTRACTION ALGEBRAS

Proposition: 3.2.1

Let d be a derivation of a subtraction algebra X . Then $\ker d = \{x \in X \mid d(x) = 0\}$ is an ideal of X .

Proof: Let $y \in \ker d$ and $x \in X$ with $x - y \in \ker d$.

Then $d(y) = 0$ implies $d(x) = d(x) - 0 = d(x) - d(y) = d(x - y) = 0$.

Hence $x \in \ker d$.

Proposition: 3.2.2

Let d be a derivation of a subtraction algebra X . If $\ker d = \{0\}$, then d is the identity derivation.

Proof: Let $x \in X$. Then $d(x) \leq x$, and $x - d(x) \in \ker d = \{0\}$ by Proposition 3.1.6.

It implies $x - d(x) = 0$ and $x \leq d(x)$. Hence $d(x) = x$.

Definition: 3.2.3

Let X be a subtraction algebra and A a non-empty subset of X . Then A^* is defined as $A^* = \{x \in X \mid x \wedge a = 0 \text{ for all } a \in A\}$.

Proposition: 3.2.4

Let X be a subtraction algebra and A non-empty subset of X . Then A^* is an ideal of X .

Proof: Let $y \in A^*$ and $x - y \in A^*$ for any $x \in X$.

Then $y \wedge a = 0$ and $(x - y) \wedge a = 0$ for all $a \in A$.

By (P11), we have $x \wedge a = (x \wedge a) - 0 = (x \wedge a) - (y \wedge a) \leq (x - y) \wedge a = 0$ for all $a \in A$.

It implies $x \wedge a = 0$ for all $a \in A$, and $x \in A^*$. Hence A^* is an ideal of X .

Note: In particular, for any singleton subset $A = \{a\}$ of a subtraction algebra X , $\{a\}^* = A^* = \{x \in X \mid x \wedge a = 0\}$ is an ideal of X .

Proposition: 3.2.5

Let X be a subtraction algebra and d_y a simple derivation with $y \in X$. Then $d_y(x) = x$ iff $x \in \{y^*\}$.

Proof: Suppose that $x, y \in X$ and $d_y(x) = x$. Then

$$x \wedge y = x - (x - y) = x - d_y(x) = x - x = 0. \text{ Hence } x \in \{y\}^*.$$

Conversely, suppose that $x \in \{y\}^*$. Then $y - (y - x) = x - (x - y) = x \wedge y = 0$.

Hence $d_y(x) = x - y = (x - y) - (y - x)$ (by (P5))

$$= (x - (y - x)) - (y - (y - x)) \text{ (by (P12))}$$

$$= x - 0 = x.$$

Corollary: 3.2.6

Let X be a subtraction algebra and d_y a simple derivation with respect to $y \in X$. Then $d_y(X) = \{y\}^*$, that is, $\text{Im}(d_y)$ is an ideal of X .

Proof: Let $x \in d_y(X)$. Then $x = d_y(z)$ for some $z \in X$, and by Theorem 3.1.10.

$x = d_y(z) = d_y(d_y(z)) = d_y(x)$. It implies $x \in \{y\}^*$ by Proposition 3.2.5.

Hence $d_y(X) \subseteq \{y\}^*$. (1)

Also, $\{y\}^* \subseteq d_y(X)$ by Proposition 3.2.5 (2)

By (1) and (2), $\{y\}^* = d_y(X)$ which is an ideal of X .

Hence $\text{Im}(d_y)$ is an ideal of X .

Proposition: 3.2.7

Let d be a derivation of a subtraction algebra X . If I is an ideal of X , then we have $d(I) \subseteq I$.

Proof: For all $x \in I$, we have $d(x) \leq x$, and $d(x) = x - w$ for some $w \in X$ by (P8).

Hence by the definition of an ideal, we have $d(x) \in I$.

Proposition: 3.2.8

Let d be a derivation of a subtraction algebra X . Then $d(X) = \text{Im}(d)$ is an ideal of X .

Proof: Let $y \in d(X)$ and $x - y \in d(X)$ with $x \in X$.

Then $d(y) = y$ and $d(x - y) = x - y$ by Theorem 3.1.10, there exists $\hat{x} \in [0, x]$

satisfying $d(x) = (x - \hat{x})$ and $d(\hat{x}) = 0$, and $d_{\hat{x}}(z) = d(z)$ for all $z \in [0, x]$ by

Theorem 3.1.17 and Theorem 3.1.19.

Since $x - y \leq x$, we have $d_{\hat{x}}(x - y) = d(x - y) = x - y$.

It implies $x - y \in \{\hat{x}\}^*$ by Proposition 3.2.5, i.e., $(x - y) \wedge \hat{x} = 0$.

Since $\hat{x} \leq x$, we have

$$\begin{aligned}\hat{x} - y &= (x \wedge \hat{x}) - y = (x - (x - \hat{x})) - y = (x - y) - ((x - \hat{x}) - y) \text{ (by (P12))} \\ &= (x - y) - ((x - y) - \hat{x}) = (x - y) \wedge \hat{x} = 0.\end{aligned}$$

Hence $\hat{x} \leq y$ and we have

$$\begin{aligned}\hat{x} = y \wedge \hat{x} &= y - (y - \hat{x}) = y - (d(y) - \hat{x}) = y - d(y - \hat{x}) \\ &= y - (d(y) - d(\hat{x})) = y - (d(y) - 0) = y - y = 0.\end{aligned}$$

This implies $x = x - 0 = x - \hat{x} = d(x) \in d(X)$, and so $d(X)$ is an ideal of X .

Theorem: 3.2.9

Let d be a derivation of a subtraction algebra X . Then there exists a monomorphism $\bar{d}: X/\ker(d) \rightarrow X$ such that $\bar{d}([x]) = d(x)$. Hence $X/\ker(d)$ is isomorphic to $\text{Im}(\bar{d}) = \text{Im}(d)$.

Proof: Suppose that d is a derivation on X .

Then d is a homomorphism of X by Theorem 3.1.14.

Define a map $\bar{d}: X/\ker(d) \rightarrow X$ by $\bar{d}([x]) = d(x)$ for all $[x] \in X/\ker(d)$.

If $[x] = [y]$, then $x \sim_{\ker(d)} y$ implies $x - y, y - x \in \ker(d)$.

Hence $d(x) - d(y) = d(x - y) = 0$ and $d(y) - d(x) = d(x - y) = 0$.

It follows that $d(x) \leq d(y)$ and $d(y) \leq d(x)$. That is, $\bar{d}([x]) = d(x) = d(y) = \bar{d}([y])$.

Therefore \bar{d} is well-defined.

Let $[x], [y] \in X/\ker(d)$. Then we have

$$\bar{d}([x] - [y]) = \bar{d}([x - y]) = d(x - y) = d(x) - d(y) = \bar{d}([x]) - \bar{d}([y]).$$

Hence \bar{d} is a homomorphism.

To show that \bar{d} is a monomorphism, let $d(x) = d(y)$.

Then $d(x - y) = d(x) - d(y) = 0$ and $d(y - x) = d(y) - d(x) = 0$.

Hence $x - y, y - x \in \ker(d)$. It follows that $x \sim_{\ker(d)} y$, and $[x] = [y]$.

Therefore \bar{d} is a monomorphism.

Theorem: 3.2.10

Let X be a subtraction algebra and d a derivation on X . If $\mu: X \rightarrow X$ is the map defined by

$$\mu(x) = \hat{x} = x - d(x)$$

for all $x \in X$, then μ is a derivation with $\ker(\mu) = \text{Im}(d)$.

Proof: Suppose that $\mu: X \rightarrow X$ is the map defined by $\mu(x) = \hat{x} = x - d(x)$ for all $x \in X$. Since $\hat{x} = x - d(x)$ is unique for each $x \in X$, μ is well-defined.

Let $x, y \in X$. Then

$$\begin{aligned} \mu(x - y) &= (x - y) - d(x - y) = (x - y) - (d(x) - d(y)) \\ &= (x - d(x)) - y = \mu(x) - y \end{aligned}$$

Hence μ is a derivation.

If $d(x) \in \text{Im}(d)$, then $\mu(d(x)) = d(x) - d(x) = 0$, and $d(x) \in \text{Ker}(\mu)$, hence

$\text{Im}(d) \subseteq \text{Ker}(\mu)$. If $x \in \text{Ker}(\mu)$, then $0 = \mu(x) = x - d(x)$ and

$x = x - 0 = x - (x - d(x)) = x \wedge d(x) = d(x) \in \text{Im}(d)$, and so $\ker(\mu) \subseteq \text{Im}(d)$.

Hence it follows that $\ker(\mu) = \text{Im}(d)$.

Corollary: 3.2.11

Let X be a subtraction algebra and d a derivation of X . Then the corestriction $\mu^\circ: X \rightarrow \ker(d)$ of μ is an epimorphism.

Proof: By Theorem 3.2.10, $\mu: X \rightarrow X$ is a derivation, hence μ is a homomorphism, and it is clear that $\text{Im}(\mu) = \ker(d)$ by Lemma 3.1.18.

Theorem: 3.2.12

Let X be a subtraction algebra and d a derivation of X . If $\bar{\mu}: X/\text{Im}(d) \rightarrow X$ is the map defined by

$$\bar{\mu}([x]) = \mu(x)$$

for all $[x] \in X/\text{Im}(d)$, then $\bar{\mu}$ is a monomorphism. In particular, $X/\text{Im}(d) \cong \ker(d)$.

Proof: Suppose that $\bar{\mu}: X/\text{Im}(d) \rightarrow X$ is the map defined by $\bar{\mu}([x]) = \mu(x)$ for all $[x] \in X/\text{Im}(d)$.

If $[x] = [y]$, then $x \sim_{\text{Im}(d)} y$, which implies $x - y, y - x \in \text{Im}(d)$, hence $d(x - y) = x - y$ and $d(y - x) = y - x$. It follows that

$$\begin{aligned} \bar{\mu}([x]) - \bar{\mu}([y]) &= \mu(x) - \mu(y) = \mu(x - y) \\ &= (x - y) - d(x - y) = 0, \end{aligned}$$

and $\bar{\mu}([y]) - \bar{\mu}([x]) = 0$ in a similar way.

Hence $\bar{\mu}([x]) = \bar{\mu}([y])$, and $\bar{\mu}$ is well-defined.

Let $[x], [y] \in X/\text{Im}(d)$. Then we have

$$\begin{aligned} \bar{\mu}([x] - [y]) &= \bar{\mu}([x - y]) = \mu(x - y) = \mu(x) - \mu(y) = \bar{\mu}([x]) - \bar{\mu}([y]), \text{ and} \\ \bar{\mu} &\text{ is a homomorphism.} \end{aligned}$$

To show that $\bar{\mu}$ is a monomorphism, let $\bar{\mu}([x]) = \bar{\mu}([y])$. Then $\mu(x) = \mu(y)$, and

$$0 = \mu(x) - \mu(y) = \mu(x - y) = (x - y) - d(x - y),$$

$$0 = \mu(y) - \mu(x) = \mu(y - x) = (y - x) - d(y - x),$$

hence $x - y \leq d(x - y)$ and $y - x \leq d(y - x)$.

Since d is non-expansive, $x - y = d(x - y) \in \text{Im}(d)$ and $y - x = d(y - x) \in \text{Im}(d)$.

Therefore, $x \sim_{\text{Im}(d)} y$. This implies $[x] = [y]$.

Hence $\bar{\mu}$ is a monomorphism. It is clear that $\text{Im}(\bar{\mu}) = \text{Im}(\mu)$, and $\text{Im}(\mu) = \ker(d)$ by Corollary 3.2.11. Hence $X / \text{Im}(d) \cong \ker(d)$.

Note: Now consider the sequence

$$0 \longrightarrow \text{Im}(d) \xrightarrow{i} X \xrightarrow{\mu^\circ} \ker(d) \longrightarrow 0,$$

of homomorphisms of subtraction algebras, where i is the inclusion map. Then $\ker(\mu^\circ) = \text{Im}(i)$ by Corollary 3.2.11 and Theorem 3.2.10.

Proposition: 3.2.13

Let d be a derivation of a subtraction algebra X . Then for each $x \in X$, $x = d(x) \vee \hat{x}$ with $d(x) \in \text{Im}(d)$ and $\hat{x} \in \ker(d)$.

Proof: Let X be a subtraction algebra and $x \in X$.

Then the interval $[0, x]$ is a Boolean algebra with respect to the induced partial order and $\hat{x} = x - d(x)$ is the complement of $d(x)$ in $[0, x]$.

Hence $d(x) \vee \hat{x} = d(x) \vee (x - d(x)) = x$.

Note: Let d be a derivation of a subtraction algebra X . Then $\text{Im}(d)$ and $\ker(d)$ are subtraction algebras. Then $\text{Im}(d) \times \ker(d)$ is also a subtraction algebra with the binary operation "-" defined by $(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$ for all $(x_1, y_1), (x_2, y_2) \in \text{Im}(d) \times \ker(d)$.

Theorem: 3.2.14

Let d be a derivation of a subtraction algebra X . If $\phi = (d, \mu) : X \rightarrow \text{Im}(d) \times \ker(d)$ is the map defined by

$$\phi(x) = (d(x), \mu(x))$$

for all $x \in X$, then ϕ is a monomorphism.

Proof: Suppose that $\phi = (d, \mu): X \rightarrow \text{Im}(d) \times \ker(d)$ is the map defined by

$\phi(x) = (d(x), \mu(x))$ for all $x \in X$. Then for any $x, y \in X$ we have

$$\phi(x - y) = (d(x - y), \mu(x - y)) = (d(x) - d(y), \mu(x) - \mu(y))$$

$$= (d(x), \mu(x)) - (d(y), \mu(y)) = \phi(x) - \phi(y).$$

If $\phi(x) = \phi(y)$, then $(d(x), \mu(x)) = (d(y), \mu(y))$, and by Proposition 3.2.13,

$$x = d(x) \vee \hat{x} = d(x) \vee \mu(x) = d(y) \vee \mu(y) = d(y) \vee \hat{y} = y.$$

Hence ϕ is a monomorphism.

Chapter-IV

CHAPTER IV

MULTIPLIERS OF SUBTRACTION ALGEBRAS AND STATES ON SUBTRACTION ALGEBRAS

SECTION: 4.1

MULTIPLIERS OF SUBTRACTION ALGEBRAS

Definition: 4.1.1 [30]

Let $(X, -, 0)$ be a subtraction algebra. A self-map f is called a *multiplier* if

$$f(x - y) = f(x) - y$$

for all $x, y \in X$.

Example: 4.1.2

Let $X = \{0, a, b\}$ be a subtraction algebra with the following Cayley table

-	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a map $f : X \rightarrow X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b \end{cases}$$

Then f is a multiplier of subtraction algebra X .

Lemma: 4.1.3

Let f be a multiplier in subtraction algebra X . Then $f(0) = 0$.

Proof: Obvious

Proposition: 4.1.4

Let X be a subtraction algebra. A multiplier $f : X \rightarrow X$ is an identity map if it satisfies $f(x) - y = x - f(y)$ for all $x, y \in X$.

Proof: Suppose that f satisfy the identity $f(x) - y = x - f(y)$ for all $x, y \in X$.

Then $f(x) = f(x - 0) = f(x) - 0 = x - f(0) = x - 0 = x$.

Thus f is an identity map.

Proposition: 4.1.5

Let f be a multiplier of a subtraction algebra X . Then f is idempotent, that is,
 $f^2 = f \circ f = f$.

Proof: Obvious

In general, every multiplier of X need not identity. However, in the following theorem, we give a set of conditions which are equivalent to be an identity multiplier.

Theorem: 4.1.6

Let X be a subtraction algebra. A multiplier f of X is an identity map if and only if the following conditions are satisfied for all $x, y \in X$,

$$(1) f(x - y) = f(x) - f(y),$$

$$(2) x - f^2(y) = f(x) - f(y).$$

Proof: The conditions for necessary is trivial.

For sufficiently, assume that (1) and (2) hold. Then for $x, y \in X$, we get

$$x - f(y) = x - f^2(y) = f(x) - f(y) = f(x - y).$$

Also, by the definition of the multiplier, $f(x - y) = f(x) - y$.

$$\text{Hence } f(x - y) = f(x) - y = x - f(y).$$

By Proposition 4.1.4, f is an identity multiplier of X .

Definition: 4.1.7

Let X be a subtraction algebra. A non-empty set F of X is called a *normal ideal* if it satisfies the following conditions:

$$(1) 0 \in F,$$

$$(2) x \in F \text{ and } y \in F \text{ imply } x - y \in F.$$

Example: 4.1.8

Let $X = \{0, a, b, 1\}$ in which "-" is defined by

-	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Then $(X; -, 0)$ is a subtraction algebra. Now consider $F = \{0, a\}$.

Then F is a normal ideal of X .

Proposition: 4.1.9

Let X be a subtraction algebra. For any $a \in X$, $S_a = \{x - a \mid x \in X\}$ is a subalgebra of X .

Proof: Let $x - a, y - a \in S_a$. Then $(x - a) - (y - a) = (x - (y - a)) - a \in S_a$.

Therefore S_a is a subalgebra of X .

Theorem: 4.1.10

Let X be a subtraction algebra. For any $a \in X$, S_a is a normal ideal of X .

Proof: Clearly, $0 - a = 0 \in S_a$. Let $r \in X$ and $b \in S_a$.

Then $b = x - a$ for some $x \in X$. Hence $b - r = (x - a) - r = (x - r) - a \in S_a$.

Therefore S_a is a normal ideal of X .

Proposition: 4.1.11

Let X be a subtraction algebra. For $u, v \in X$, the set

$$X(u, v) = \{x \mid (x - u) - v = 0\}$$

is a subalgebra of X .

Proof: Let $x, y \in X(u, v)$. Then $(x - u) - v = 0$ and $(y - u) - v = 0$.

Hence $((x - y) - u) - v = ((x - u) - y) - v = ((x - u) - v) - y = 0 - y = 0$,

which implies $x - y \in X(u, v)$. This completes the proof.

Proposition: 4.1.12

Let X be a subtraction algebra. For $u, v \in X$, the set

$$X(u, v) = \{x \mid (x - u) - v = 0\}.$$

is a normal ideal of X , and $u, v \in X(u, v)$.

Proof: Obviously, $0, u, v \in X(u, v)$. Let $x, r \in X$ be such that $x \in X(u, v)$.

Then $(x - u) - v = 0$, and so

$$\begin{aligned} ((x - r) - u) - v &= ((x - u) - r) - v \\ &= ((x - u) - v) - r = 0 - r = 0. \end{aligned}$$

This implies $x - r \in X(u, v)$. This completes the proof.

Proposition: 4.1.13

Let F is a normal ideal of X . For any $w \in X$, the set

$$F_w = \{x \mid x - w \in F\}$$

is a subalgebra of X .

Proof: Let $x, y \in F_w$. Then $x - w, y - w \in F$.

Therefore, $(x - y) - w = (x - w) - (y - w) \in F$, which implies $x - y \in F_w$.

This completes the proof.

Proposition: 4.1.14

If F is a normal ideal of X , the set F_w is a normal ideal containing F and w .

Proof: Let $w \in X$. Since $0 - w = 0 \in F$, we have $0 \in F_w$.

Let $x, r \in X$ be such that $x \in F_w$. Then $x - w \in F$.

Therefore, $(x - r) - w = (x - w) - r \in F$, which implies $x - r \in F_w$.

Obviously, F_w contains F and w . This completes the proof.

Proposition: 4.1.15

Let X_1 and X_2 be two subtraction algebras. Define a map $f : X_1 \times X_2 \rightarrow X_1 \times X_2$ by $f(x, y) = (x, 0)$ for all $(x, y) \in X_1 \times X_2$. Then f is a multiplier of $X_1 \times X_2$ with respect to the point-wise operation given by $(a, b) - (c, d) = (a - c, b - d) \quad \forall a, c \in X_1$ and $b, d \in X_2$.

Proof: Let $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. Then we have

$$\begin{aligned} f((x_1, y_1) - (x_2, y_2)) &= f(x_1 - x_2, y_1 - y_2) = (x_1 - x_2, 0) = (x_1 - x_2, 0 - y_2) \\ &= (x_1, 0) - (x_2, y_2) = f(x_1, y_1) - (x_2, y_2). \end{aligned}$$

Therefore f is a multiplier of the direct product $X_1 \times X_2$.

Theorem: 4.1.16

If F_1 and F_2 are normal ideals of X_1 and X_2 respectively, then $F_1 \times F_2$ is a normal ideal of the product algebra $X_1 \times X_2$.

Proof: Let F_1 and F_2 be normal ideals of X_1 and X_2 respectively.

Since $0 \in F_1$ and $0 \in F_2$, we have $(0, 0) \in F_1 \times F_2$.

Let $(x, y) \in X_1 \times X_2$ and $(x_1, y_1) \in F_1 \times F_2$.

Also, since F_1 and F_2 are normal ideas of X_1 and X_2 respectively, we get $x_1 - x \in F_1$ and $y_1 - y \in F_2$.

Hence $(x_1, y_1) - (x, y) = (x_1 - x, y_1 - y) \in F_1 \times F_2$.

Therefore, $F_1 \times F_2$ is a normal ideal of $X_1 \times X_2$.

Theorems: 4.1.17

Let f be a multiplier of subtraction X . For any normal ideal F of X , both $f(F)$ and $f^{-1}(F)$ are normal ideals of X .

Proof: Clearly, $0 = f(0)$. Let $x \in X$ and $a \in f(F)$. Then $a = f(s)$ for some $s \in F$.

Now $a - x = f(s) - x = f(s - x) \in f(F)$ because $s - x \in F$.

Therefore $f(F)$ is a normal ideal of X .

Since F is a normal ideal of X , we obtain $f(0) = 0 \in F$. Hence $0 = f^{-1}(F)$.

Let $x \in X$ and $a \in f^{-1}(F)$. Then $f(a) \in F$.

Since F is a normal ideal, we get $f(a-x) = f(a) - x \in F$. Hence $a-x \in f^{-1}(F)$.

Therefore $f^{-1}(F)$ is a normal ideal of X .

Definition: 4.1.18

Let f be a multiplier of a subtraction algebra X . Define the kernel of the multiplier f by

$$\ker f = \{x \in X \mid f(x) = 0\}.$$

Proposition: 4.1.19

For any multiplier f of a subtraction algebra X , $\ker f$ is a normal ideal of X .

Proof: Clearly, $0 \in \ker f$.

Let $a \in \ker f$ and $x \in X$.

Then $f(a-x) = f(a) - x = 0 - x = 0$.

Hence $a-x \in \ker f$, which implies that $\ker f$ is a normal ideal of X .

Definition: 4.1.20

Let f be a multiplier of a subtraction algebra. An element $a \in X$ is called a *fixed element* if $f(a) = a$.

Notation:

Let us denote the set of all fixed elements of X by $Fix_f(X) = \{x \in X \mid f(x) = x\}$ and the image of X under the multiplier f by $Im(f)$.

Lemma: 4.1.21

Let f be a multiplier of subtraction algebra X . Then $Im(f) = Fix_f(X)$.

Proof: Let $x \in \text{Fix}_f(X)$. Then $x = f(x) \in \text{Im}(f)$. Hence $\text{Fix}_f(X) \subseteq \text{Im}(f)$.

Now let $a \in \text{Im}(f)$. Then we get $a = f(b)$ for some $b \in X$.

Thus $f(a) = f(f(b)) = f(b) = a$, which implies $\text{Im}(f) \subseteq \text{Fix}_f(X)$.

Therefore, $\text{Im}(f) = \text{Fix}_f(X)$. This completes the proof.

Theorem: 4.1.22

Let f be a multiplier of a subtraction algebra X . Then we have

- (1) $\text{Fix}_f(X)$ is a normal ideal of X .
- (2) $\text{Im}(f)$ is a normal ideal of X .

Proof:

- (1) Since $f(0) = 0$, we have $0 \in \text{Fix}_f(X)$.

Let $x \in X$ and $a \in \text{Fix}_f(x)$. Then $f(a) = a$. Now $f(a-x) = f(a) - x = a - x$.

Hence $a - x \in \text{Fix}_f(X)$.

Therefore, $\text{Fix}_f(X)$ is a normal ideal of X .

- (2) Obviously, $0 = f(0)$.

Let $x \in X$ and $a \in \text{Im}(f)$. Then $a = f(b)$ for some $b \in X$.

Now $a - x = f(b) - x = f(b-x) \in f(X)$.

Therefore, $\text{Im}(f)$ is a normal ideal of X .

Note:

The composition of two multipliers f and g of a subtraction algebra X is a multiplier of X where $(f \circ g)(x) = f(g(x))$ for all $x \in X$.

Theorem: 4.1.23

Let f and g be two multipliers of X such that $f \circ g = g \circ f$. Then the following conditions are equivalent.

- (1) $f = g$.
- (2) $f(X) = g(X)$.
- (3) $Fix_f(X) = Fix_g(X)$.

Proof:

(1) \Rightarrow (2): Obvious.

(2) \Rightarrow (3): Assume that $f(X) = g(X)$.

Let $x \in Fix_f(X)$. Then $x = f(x) \in f(X) = g(X)$.

Hence $x = g(y)$ for some $y \in X$.

Now $g(x) = g(g(y)) = g^2(y) = g(y) = x$. Thus $x \in Fix_g(X)$.

Therefore, $Fix_f \subseteq Fix_g(X)$.

Similarly, we can obtain $Fix_g(X) = Fix_f(X)$. Thus $Fix_f(X) = Fix_g(X)$.

(3) \Rightarrow (1): Assume that $Fix_f(X) = Fix_g(X)$.

Let $x \in X$. Since $f(x) \in Fix_f(X) = Fix_g(X)$. We have $g(f(x)) = f(x)$.

Also, we obtain $g(x) \in Fix_g(X) = Fix_f(X)$. Hence we get $f(g(x)) = g(x)$.

Thus we have $f(x) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = g(x)$.

Therefore, f and g are equal in the sense of mappings.

Definition: 4.1.24

Let X be a subtraction algebra. An equivalence relation θ on X is called a *weak congruence* if $(x, y) \in \theta$ implies $(x - a, y - a) \in \theta$ for any $a \in X$.

Note:

Clearly, every congruence on X is a weak congruence on X . In the following theorem, we have an example for a weak congruence in terms of multipliers.

Theorem: 4.1.25

Let f be a multiplier of a subtraction algebra X . Define a binary operation θ_f on X as follows:

$$(x, y) \in \theta_f \text{ if and only if } f(x) = f(y) \text{ for all } x, y \in X.$$

Then θ_f is a weak congruence on X .

Proof: Clearly, θ_f is an equivalence relation on X .

Let $(x, y) \in \theta_f$. Then we have $f(x) = f(y)$.

Now for any $a \in X$, we have $f(x - a) = f(x) - a = f(y) - a = f(y - a)$.

Hence $(x - a, y - a) \in \theta_f$.

Lemma: 4.1.26

Let f be a multiplier of a subtraction algebra X . Then

(1) $f(x) = x$ for all $x \in f(X)$.

(2) If $(x, y) \in \theta_f$ and $x, y \in f(X)$, $x = y$.

Proof:

(1) Let $x \in f(X)$. Then $x = f(a)$ for some $a \in X$.

$$\text{Now } f(x) = f^2(a) = f(f(a)) = f(a) = x.$$

(2) Let $(x, y) \in \theta_f$ and $x, y \in f(X)$.

$$\text{Then by (1), } x = f(x) = f(y) = y.$$

Theorem: 4.1.27

Let X be a subtraction algebra and F be a normal ideal of X . Then there exists multiplier f of X such that $f(X) = F$ if and only if $F \cap \theta_f(x)$ is a single-ton set for all $x \in X$, where θ_f is the congruence class of x with respect to θ_f .

Proof: Let f be a multiplier of a subtraction algebra X such that $f(X) = F$.

Then clearly θ_f is a weak congruence on X .

Let $x \in X$ be an arbitrary element. Since $f(x) = f^2(x)$, we get $(x, f(x)) \in \theta_f$.

Hence $f(x) \in \theta_f(x)$. Also, $f(x) \in f(X) = F$, which implies $f(x) \in F \cap \theta_f(x)$.

Therefore $F \cap \theta_f(x)$ is non-empty.

Let a, b be two element of $F \cap \theta_f(x)$. Then by Lemma 4.1.26, we get $a = b$.

Hence $F \cap \theta_f(x)$ is a single-ton set.

Conversely, assume that $F \cap \theta_f(x)$ is a single-ton set for all $x \in X$.

Let x_0 be the single element of $F \cap \theta_f(x)$.

Now define a self map as follows, $f : X \rightarrow X$ by $f(x) = x_0$ for all $x \in X$.

By the definition of the map f , we get $f(a) \in F$ and $f(f(a)) = f(a)$.

Since F is normal, we get $f(a) - b \in F$, and so

$$\begin{aligned} f(f(a)) = f(a) &\Rightarrow (f(a), a) \in \theta_f \\ &\Rightarrow (f(a - b), a - b) \in \theta_f \\ &\Rightarrow f(a) - b \in \theta_f(a - b) \\ &\Rightarrow f(a) - b \in F \cap \theta_f(a - b) \quad (f(a) - b \in F) \end{aligned}$$

Since $f(a - b) \in F \cap \theta_f(a - b)$ and $F \cap \theta_f(a - b)$ is a single-ton set, we get

$$f(a - b) = f(a) - b.$$

Therefore f is a multiplier of X .

SECTION: 4.2

STATES ON SUBTRACTION ALGEBRAS

Definition: 4.2.1 [31]

Let X be a subtraction algebra. A *Bosbach state* on X is a function $s : X \rightarrow [0,1]$ such that the following axioms hold:

- (1) $s(0) = 0$.
- (2) $s(x) + s(y - x) = s(y) + s(x - y)$ for all $x, y \in X$.

For a Bosbach state $s : X \rightarrow [0,1]$, define a set $\ker(s)$ by

$$\ker(s) = \{x \in X \mid s(x) = 0\}.$$

A function $s : X \rightarrow [0,1]$ is called *trivial Bosbach state* on X if $s(x) = 0$ for all $x \in X$.

Also, the function

$$s(x) = \begin{cases} 0 & \text{if } x = 0, \\ t & \text{otherwise.} \end{cases}$$

is called a *constant Bosbach state* for some $t \in [0,1]$.

Example: 4.2.2

Let $X = \{0, a, b\}$ be a set in which "-" is defined by

-	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Then $(X, -)$ is a subtraction algebra.

Define a function $s : X \rightarrow [0,1]$ by

$$s(0) = 0, \quad s(a) = 0.3, \quad s(b) = 0.7.$$

Then s is a Bosbach state on X .

Example: 4.2.3

Let $X = \{0, x, y, 1\}$ be a set in which "-" is defined by

-	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

Then $(X, -)$ is a subtraction algebra.

Define a function $s : X \rightarrow [0,1]$ by

$$s(0) = 0, s(x) = 0.3, s(y) = 0.5, s(1) = 0.8.$$

Then s is a Bosbach state on X .

Proposition: 4.2.4

If $s : X \rightarrow [0,1]$ is a Bosbach state on a subtraction algebra X , $s_a : X \rightarrow [0,1]$, $s_a(x) = s(x-a)$ for every $x \in X$ is also a Bosbach state on X .

Proof: Let X be a subtraction algebra and $a \in X$.

Then we have $s_a(0) = s(0-a) = s(0) = 0$. Also, we have for all $x, y \in X$,

$$\begin{aligned} s_a(x) + s_a(y-x) &= s(x-a) + s((y-x)-a) = s(x-a) + s((y-a)+(x-a)) \\ &= s(y-a) + s((x-a)+(y-a)) = s(y-a) + s((x-y)-a) \\ &= s_a(y) + s_a(x-y). \end{aligned}$$

This completes the proof.

Proposition: 4.2.5

Let $s : X \rightarrow [0,1]$ be a Bosbach state on a subtraction algebra X . Then If $x \leq y$, then $s(x) \leq s(y)$ and $s(x) + s(y-x) = s(y)$.

Proof: Let s be a Bosbach state on X and $x \leq y$. Then $x-y=0$.

Since $s(x) + s(y-x) = s(y) + s(x-y)$, we have

$$0 \leq s(y-x) = s(x-y) + s(y) - s(x) = 0 + s(y) - s(x) = s(y) - s(x),$$

and so $s(y) \geq s(x)$.

Proposition: 4.2.6

Let $s : X \rightarrow [0,1]$ be a Bosbach state on a subtraction algebra X . Then $\ker(s)$ is a subalgebra of X .

Proof: Let $x, y \in \ker(s)$. Then $s(x) = s(y) = 0$.

Since $x - y \leq x$, we have $s(x - y) \leq s(x)$ by Proposition 4.2.5, and so $0 \leq s(x - y) \leq s(x) = 0$. This implies $s(x - y) = 0$, i.e., $x - y \in \ker(s)$.

Proposition: 4.2.7

Let $s : X \rightarrow [0,1]$ be a Bosbach state on a subtraction algebra X . Then $\ker(s)$ is an ideal of X .

Proof: Since $s(0) = 0$, we have $0 \in \ker(s)$.

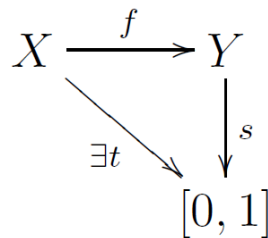
Let $x - y \in \ker(s)$ and $y \in \ker(s)$. So, $s(x - y) = s(y) = 0$.

Since $y - x \leq y$, we have $s(y - x) \leq s(y)$ by Proposition 4.2.5 which implies $0 \leq s(x - y) \leq s(y) = 0$. Hence $s(y - x) = 0$.

Now $s(x) + s(y - x) = s(y) + s(x - y)$, implies $s(x) = 0$. i.e., $x \in \ker(s)$.

Theorem: 4.2.8

Let X and Y be subtraction algebras. Let $f : X \rightarrow Y$ be a homomorphism $s : Y \rightarrow [0,1]$ be a Bosbach state on Y . Then there exists an unique Bosbach state $t : X \rightarrow [0,1]$ such that the following diagram is commutative (i.e., $t = s \circ f$).



Proof: Let X and Y be subtraction algebras and $f : X \rightarrow Y$ be a homomorphism.

Let $s : Y \rightarrow [0,1]$ be a Bosbach state on Y .

Let $t = s \circ f$, where $t : X \rightarrow [0,1]$. Then t is well-defined.

To prove: t is a Bosbach state.

$$t(0) = (s \circ f)(0) = s(f(0)) = s(0) = 0. \quad (1)$$

Now, let $x, y \in X$. Then

$$\begin{aligned} t(x) + t(y - x) &= (s \circ f)(x) + (s \circ f)(y - x) = s(f(x)) + s(f(y) - f(x)) \\ &= s(f(y)) + s(f(x) - f(y)) = s(f(y)) + s(f(x - y)) \\ &= (s \circ f)(y) + (s \circ f)(x - y) = t(y) + t(x - y). \end{aligned} \quad (2)$$

By (1) and (2) Hence t is a Bosbach state.

Finally, we prove that t is unique.

Suppose that there exist an another Bosbach state $r : X \rightarrow [0,1]$ such that $r = s \circ f$.

Then $r(x) = (s \circ t)(x) = t(x)$ for all $x \in X$. This implies $r = t$.

Hence the proof.

Theorem: 4.2.9

Let $f : X \rightarrow Y$ be a bijection homomorphism and $s : X \rightarrow [0,1]$ be a Bosbach state on X . Then there exists an unique Bosbach state $t : Y \rightarrow [0,1]$ such that $s = t \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ s \downarrow & & \swarrow \exists t \\ & & [0, 1] \end{array}$$

Proof: Let X and Y be subtraction algebras and $f : X \rightarrow Y$ be a bijection homomorphism. Let $s : X \rightarrow [0,1]$ be a Bosbach state on X .

Let $y \in Y$. Since f is onto, there exists $x \in X$ such that $f(x) = y$.

Put $t(y) = s(x)$. Then $t(y) = t(f(x)) = (t \circ f)(x) = s(x)$, and so $t \circ f = s$.

Since f is one to one, then $\ker(f) = 0$, and so $t(0) = t(f(0)) = (t \circ f)(0) = s(0) = 0$.

Let $y_1, y_2 \in Y$. Then there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Then we have

$$\begin{aligned}
 t(y_1) + t(y_2 - y_1) &= t(f(x_1)) + t(f(x_2) - f(x_1)) = t(f(x_1)) + t(f(x_2 - x_1)) \\
 &= (t \circ f)(x_1) + (t \circ f)(x_2 - x_1) = s(x_1) + s(x_2 - x_1) \\
 &= s(x_2) + s(x_1 - x_2) = (t \circ f)(x_2) + (t \circ f)(x_1 - x_2) \\
 &= (t(f(x_2)) + t(f(x_1 - x_2))) = t(f(x_2)) + t(f(x_1) - f(x_2)) \\
 &= t(y_2) + t(y_1 - y_2).
 \end{aligned}$$

Hence t is a Bosbach state.

Suppose that there exists an another Bosbach state $r : X \rightarrow [0,1]$ such that $s = r \circ f$.

Let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$.

Now, $r(y) = r(f(x)) = (r \circ f)(x) = s(x)$, but by definition of t , we get

$r(y) = s(x) = t(y)$. Hence we have $r = t$.

Note:

Introduce a relation " \leq " on X/I by $[x] \leq [y]$ iff $[x] - [y] = [0] = 0$.

Proposition: 4.2.10

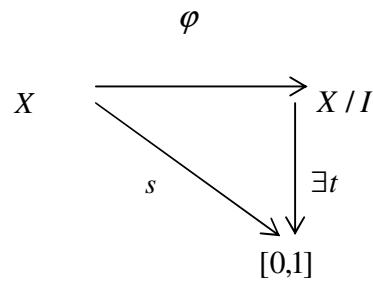
Let $s : X \rightarrow [0,1]$ be a Bosbach state on X . Then

- (1) $[x] \leq [y] \Leftrightarrow s(x - y) = 0$.
- (2) $[x] = [y] \Leftrightarrow s(x - y) = s(y - x) = 0$.

Proof: Proof follows from Proposition 4.2.5.

Theorem: 4.2.11

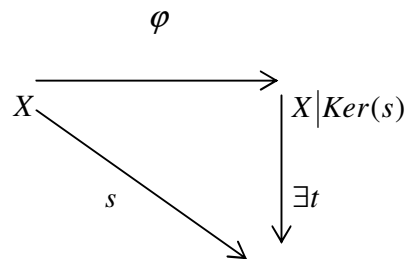
Let I be an ideal of a subtraction algebra X and $s: X \rightarrow [0,1]$ be a Bosbach state on X . Then there exists an unique Bosbach state $t: X/I \rightarrow [0,1]$ such that the following diagram is commutative (i.e., $s = t \circ \varphi$), infact, " \sim " defined by $x \sim y \Rightarrow x - y \in I$ and $y - x \in I$ is a congruence relation induced by ideal I , where $\varphi: X \rightarrow X/I$ is defined by $\varphi(x) = [x]$.



Proof: Obvious.

Corollary: 4.2.12

Let X be a subtraction algebra and $s: X \rightarrow [0,1]$ be a Bosbach state on X . Then there exists an unique Bosbach state $t: X/\ker(s) \rightarrow [0,1]$ such that the following diagram is commutative (i.e., $s = t \circ \varphi$), infact, \sim is a congruence relation induced by ideal $\ker(s)$.



[0,1]

Proof: Obvious.

Summary and Conclusion

SUMMARY AND CONCLUSION

In 1998, Schein, B.M. [32] considered systems of the form (Φ, \circ, \setminus) , where Φ is a set of functions closed under the composition " \circ " of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction " \setminus " (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of Abbott, J.C. [1]).

In this thesis, we have made an attempt to give a discussion about the ideal theory of subtraction algebras, derivations of subtraction algebras, homomorphism of subtraction algebras etc.

In the first chapter, preliminary definitions and properties of subtraction algebras are presented. Also, prime and irreducible ideals in subtraction algebras due to Jun, Y.B. et al. [14, 15] are discussed.

In chapter II, the concept of quotient subtraction algebras due to Çeven, Y. and Küçükkoç, Ş. [6] and pseudo-valuation of subtraction algebras due to Kim, K.H. and Yon, Y.H. [22] and their properties are investigated.

In chapter III, we have made a detailed study of derivations of subtraction algebras due to Yon, Y.H. and Kim, K.H. [34].

In chapter IV, properties of multipliers of subtraction algebras due to Sang Deok Lee and Kim, K.H. [30] and states on subtraction algebras due to Sang Moon Lee and Kim, K.H. [31] are discussed.

We hope that a deep study of these concepts will lead to many interesting open problems yielding very good scope for further research.

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