

## $\gamma$ Generalized Closed Sets in Intuitionistic Fuzzy Topological Spaces

### 2.1 Introduction

Fukutake, Nasef and El-Maghrabi (2003) introduced and established the concept of  $\gamma$  generalized closed sets in topological spaces. After that the set was generalized to fuzzy topological spaces. Benchalli and Jenifer Karnel (2010a) have introduced fuzzy b-open sets in fuzzy topological spaces. Hanafy (2009) studied the properties of intuitionistic fuzzy  $\gamma$  closed sets and intuitionistic fuzzy  $\gamma$  open sets. Now we have generalized the set to intuitionistic fuzzy  $\gamma$  generalized closed sets in intuitionistic fuzzy topological spaces. Some interesting propositions and results on intuitionistic fuzzy  $\gamma$  generalized closed sets, intuitionistic fuzzy  $\gamma$  generalized open sets are discussed and some theoretical applications are provided in this chapter.

### 2.2 Intuitionistic fuzzy $\gamma$ generalized closed sets

In this section we have introduced intuitionistic fuzzy  $\gamma$  generalized closed sets and studied some of their properties.

**Definition 2.2.1:** An IFS  $A$  in an IFTS  $(X, \tau)$  is said to be an *intuitionistic fuzzy  $\gamma$  generalized closed set* (IF $\gamma$ GCS) if  $\gamma\text{cl}(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is an IF $\gamma$ OS in  $(X, \tau)$ .

The complement  $A^c$  of an IF $\gamma$ GCS  $A$  in an IFTS  $(X, \tau)$  is called an *intuitionistic fuzzy  $\gamma$  generalized open set* (IF $\gamma$ GOS) in  $X$ .

The family of all IF $\gamma$ GCSs of an IFTS  $(X, \tau)$  is denoted by IF $\gamma$ GC( $X$ ).

**Example 2.2.2:** Let  $X = \{a, b\}$  and  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  where  $\mu_{G_1}(a) = 0.5_a$ ,  $\mu_{G_1}(b) = 0.6_b$ ,  $\nu_{G_1}(a) = 0.5_a$ ,  $\nu_{G_1}(b) = 0.4_b$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$  where

$\mu_{G_2}(a) = 0.4_a, \mu_{G_2}(b) = 0.3_b, \nu_{G_2}(a) = 0.6_a, \nu_{G_2}(b) = 0.7_b$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on X. Here,

IF $\gamma$ O(X) =  $\{0_-, 1_-, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1] / \text{either } \nu_a < 0.4 \text{ or } \nu_b < 0.3, \nu_a \geq 0.5 \text{ whenever } \nu_b < 0.6, 0.4 \leq \nu_a \leq 0.5 \text{ whenever } \nu_b \leq 0.4, \mu_a \geq 0.5, \mu_b \geq 0.6, 0.5 \leq \nu_a < 0.6 \text{ whenever } \nu_b \geq 0.6, \mu_a \geq 0.4, \mu_b \geq 0.3 \text{ and } \nu_a \geq 0.6 \text{ whenever } \nu_b \geq 0.4, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}$  and

IF $\gamma$ C(X) =  $\{0_-, 1_-, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1] / \text{either } \mu_a < 0.4 \text{ or } \mu_b < 0.3, \mu_a \geq 0.5 \text{ whenever } \mu_b < 0.6, 0.4 \leq \mu_a \leq 0.5 \text{ whenever } \mu_b \leq 0.4, \nu_a \geq 0.5, \nu_b \geq 0.6, 0.5 \leq \mu_a < 0.6 \text{ whenever } \mu_b \geq 0.6, \nu_a \geq 0.4, \nu_b \geq 0.3 \text{ and } \mu_a \geq 0.6 \text{ whenever } \mu_b \geq 0.4, 0 \leq \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}$ .

Let  $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b) \rangle$  be an IFS in  $(X, \tau)$ . Then A is an IF $\gamma$ GCS in  $(X, \tau)$ .

**Proposition 2.2.3:** Every IFCS is an IF $\gamma$ GCS in  $(X, \tau)$  but not conversely in general.

**Proof:** Let A be an IFCS in X. Let  $A \subseteq U$  and U be an IF $\gamma$ OS in X. As  $\gamma\text{cl}(A) \subseteq \text{cl}(A) = A \subseteq U$  by hypothesis, we have  $\gamma\text{cl}(A) \subseteq U$ . Then A is an IF $\gamma$ GCS in  $(X, \tau)$ .

**Example 2.2.4:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on X. Here  $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b) \rangle$  is an IF $\gamma$ GCS but not an IFCS in  $(X, \tau)$ , as  $\text{cl}(A) = G_1^c \neq A$ .

**Proposition 2.2.5:** Every IFRCS is an IF $\gamma$ GCS in  $(X, \tau)$  but not conversely in general.

**Proof:** Let A be an IFRCS in X. Then A is an IFCS as every IFRCS is an IFCS. Therefore by Proposition 2.2.3, A is an IF $\gamma$ GCS in  $(X, \tau)$ .

**Example 2.2.6:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on X. Here  $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b) \rangle$  is an IF $\gamma$ GCS but not an IFRCS in  $(X, \tau)$ , as  $\text{cl}(\text{int}(A)) = \text{cl}(0_-) = 0_- \neq A$ .

**Proposition 2.2.7:** Every IFSCS is an IF $\gamma$ GCS in  $(X, \tau)$  but not conversely in general.

**Proof:** Let  $A$  be an IFSCS in  $X$  and let  $A \subseteq U$  where  $U$  is an IF $\gamma$ OS in  $X$ . As  $\gamma\text{cl}(A) \subseteq \text{scl}(A) = A \subseteq U$  by hypothesis, we have  $\gamma\text{cl}(A) \subseteq U$ . Hence  $A$  is an IF $\gamma$ GCS in  $(X, \tau)$ .

**Example 2.2.8:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b) \rangle$  is an IF $\gamma$ GCS but not an IFSCS in  $(X, \tau)$ , as  $\text{int}(\text{cl}(A)) = \text{int}(G_1^c) = G_2 \not\subseteq A$ .

**Proposition 2.2.9:** Every IFPCS is an IF $\gamma$ GCS in  $(X, \tau)$  but not conversely in general.

**Proof:** Let  $A$  be an IFPCS in  $X$  and let  $A \subseteq U$  where  $U$  is an IF $\gamma$ OS in  $X$ . As  $\gamma\text{cl}(A) \subseteq \text{pcl}(A) = A \subseteq U$  by hypothesis, we have  $\gamma\text{cl}(A) \subseteq U$ . Hence  $A$  is an IF $\gamma$ GCS in  $(X, \tau)$ .

**Example 2.2.10:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$  is an IF $\gamma$ GCS but not an IFPCS in  $(X, \tau)$ , as  $\text{cl}(\text{int}(A)) = \text{cl}(G_2) = G_1^c \not\subseteq A$ .

**Proposition 2.2.11:** Every IF $\alpha$ CS is an IF $\gamma$ GCS in  $(X, \tau)$  but not conversely in general.

**Proof:** Let  $A$  be an IF $\alpha$ CS in  $X$  and let  $A \subseteq U$  where  $U$  is an IF $\gamma$ OS in  $X$ . As  $\gamma\text{cl}(A) \subseteq \alpha\text{cl}(A) = A \subseteq U$  by hypothesis, we have  $\gamma\text{cl}(A) \subseteq U$ . Hence  $A$  is an IF $\gamma$ GCS in  $(X, \tau)$ .

**Example 2.2.12:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b) \rangle$  is an IF $\gamma$ GCS but not an IF $\alpha$ CS in  $(X, \tau)$ , as  $\text{cl}(\text{int}(\text{cl}(A))) = \text{cl}(\text{int}(G_1^c)) = \text{cl}(G_2) = G_1^c \not\subseteq A$ .

**Proposition 2.2.13:** Every IF $\gamma$ CS is an IF $\gamma$ GCS in  $(X, \tau)$  but not conversely in general.

**Proof:** Let  $A$  be an IF $\gamma$ CS in  $X$ , then  $\gamma\text{cl}(A) = A$ . Let  $A \subseteq U$  where  $U$  is an IF $\gamma$ OS in  $X$ . Now as  $\gamma\text{cl}(A) = A \subseteq U$  by hypothesis, we have  $\gamma\text{cl}(A) \subseteq U$ . Hence  $A$  is an IF $\gamma$ GCS in  $(X, \tau)$ .

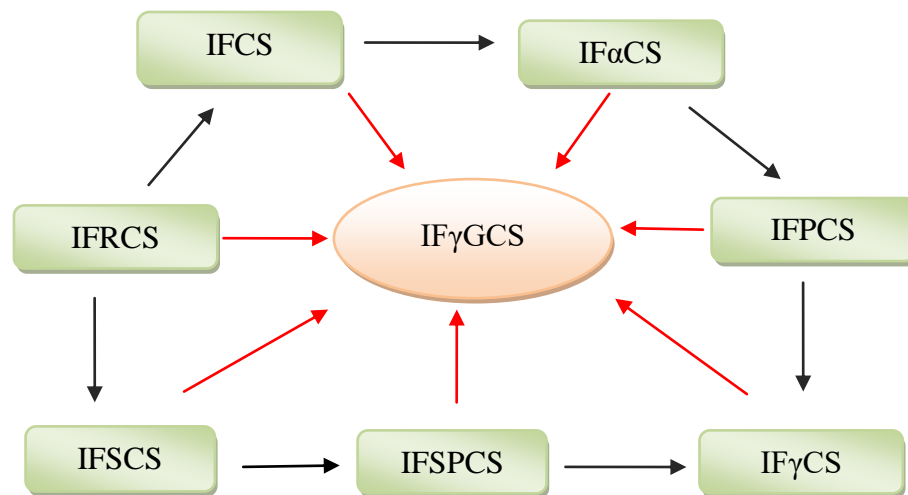
**Example 2.2.14:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.7_b), (0.5_a, 0.3_b) \rangle$  and  $G_2 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$ . Then  $\tau = \{0_{\sim}, G_1, G_2, 1_{\sim}\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.6_a, 0.6_b), (0.4_a, 0.4_b) \rangle$  is an IF $\gamma$ GCS but not an IF $\gamma$ CS in  $(X, \tau)$ , as  $\text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) = 1_{\sim} \cap 1_{\sim} = 1_{\sim} \notin A$ .

**Proposition 2.2.15:** Every IFSPCS is an IF $\gamma$ GCS in  $(X, \tau)$  but not conversely in general.

**Proof:** Let  $A$  be an IFSPCS in  $X$  and let  $A \subseteq U$  where  $U$  is an IF $\gamma$ OS in  $X$ . As  $\gamma\text{cl}(A) \subseteq \text{spcl}(A) = A \subseteq U$  by hypothesis, we have  $\gamma\text{cl}(A) \subseteq U$ . Hence  $A$  is an IF $\gamma$ GCS in  $(X, \tau)$ .

**Example 2.2.16:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.7_b), (0.5_a, 0.3_b) \rangle$  and  $G_2 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$ . Then  $\tau = \{0_{\sim}, G_1, G_2, 1_{\sim}\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.6_a, 0.6_b), (0.4_a, 0.4_b) \rangle$  is an IF $\gamma$ GCS but not an IFSPCS in  $(X, \tau)$ , as there exists no IFPCS  $B$  such that  $\text{int}(B) \subseteq A \subseteq B$  in  $X$ .

In the following diagram, we have provided the relation between various types of intuitionistic fuzzy closedness.



The reverse implications are not true in general in the above diagram.

**Remark 2.2.17:** The union of any two IF $\gamma$ GCSs is not an IF $\gamma$ GCS in general as seen in the following example.

**Example 2.2.18:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.6_a, 0.8_b), (0.4_a, 0.2_b) \rangle$  and  $G_2 = \langle x, (0.5_a, 0.5_b), (0.4_a, 0.4_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ .

Then the IFSs  $A = \langle x, (0.5_a, 0.4_b), (0.4_a, 0.5_b) \rangle$ ,  $B = \langle x, (0.4_a, 0.6_b), (0.5_a, 0.2_b) \rangle$  are IF $\gamma$ GCSs in  $(X, \tau)$ . But  $A \cup B$  is not an IF $\gamma$ GCS in  $X$ , as  $A \cup B = \langle x, (0.5_a, 0.6_b), (0.4_a, 0.2_b) \rangle \subseteq G_1$  where  $G_1$  is an IF $\gamma$ OS in  $X$  and  $\gamma\text{cl}(A \cup B) = 1_- \notin G_1$ .

**Remark 2.2.19:** The intersection of any two IF $\gamma$ GCSs is not an IF $\gamma$ GCS in general as seen in the following example.

**Example 2.2.20:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.7_b), (0.5_a, 0.3_b) \rangle$  and  $G_2 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ .

Then the IFSs  $A = \langle x, (0.5_a, 0.9_b), (0.5_a, 0.1_b) \rangle$ ,  $B = \langle x, (0.5_a, 0.7_b), (0.4_a, 0.3_b) \rangle$  are IF $\gamma$ GCSs in  $(X, \tau)$ . But  $A \cap B$  is not an IF $\gamma$ GCS, as  $A \cap B = \langle x, (0.5_a, 0.7_b), (0.5_a, 0.3_b) \rangle \subseteq G_1$  but  $\gamma\text{cl}(A \cap B) = 1_- \notin G_1$ .

**Proposition 2.2.21:** Let  $(X, \tau)$  be an IFTS. Then for every  $A \in \text{IF}\gamma\text{GC}(X)$  and for every  $B \in \text{IFS}(X)$ ,  $A \subseteq B \subseteq \gamma\text{cl}(A) \Rightarrow B \in \text{IF}\gamma\text{GC}(X)$ .

**Proof:** Let  $B \subseteq U$  and  $U$  be an IF $\gamma$ OS in  $X$ . Then since  $A \subseteq B$ ,  $A \subseteq U$ . By hypothesis  $B \subseteq \gamma\text{cl}(A)$ . Therefore  $\gamma\text{cl}(B) \subseteq \gamma\text{cl}(\gamma\text{cl}(A)) = \gamma\text{cl}(A) \subseteq U$ , since  $A$  is an IF $\gamma$ GCS in  $X$ . Hence  $B \in \text{IF}\gamma\text{GC}(X)$ .

**Proposition 2.2.22:** An IFS  $A$  of an IFTS  $(X, \tau)$  is an IF $\gamma$ GCS if and only if  $A \overset{\bar{q}}{=} F \Rightarrow \gamma\text{cl}(A) \overset{\bar{q}}{=} F$  for every IF $\gamma$ CS  $F$  of  $X$ .

**Proof: Necessity:** Let  $F$  be an IF $\gamma$ CS in  $X$  and  $A \overset{\bar{q}}{=} F$ , then  $A \subseteq F^c$ , where  $F^c$  is an IF $\gamma$ OS. By hypothesis  $\gamma\text{cl}(A) \subseteq F^c$ . Hence by Definition 1.1.15,  $\gamma\text{cl}(A) \overset{\bar{q}}{=} F$ .

**Sufficiency:** Let  $U$  be an IF $\gamma$ OS such that  $A \subseteq U$ . Then  $U^c$  is an IF $\gamma$ CS and  $A \subseteq (U^c)^c$ . This implies that  $A \overset{\bar{q}}{=} U^c$ . By hypothesis,  $\gamma\text{cl}(A) \overset{\bar{q}}{=} U^c$ . Hence  $\gamma\text{cl}(A) \subseteq (U^c)^c = U$ . Hence  $A$  is an IF $\gamma$ GCS in  $X$ .

**Proposition 2.2.23:** If  $A$  is both an  $IF\gamma OS$  and an  $IF\gamma GCS$  in  $(X, \tau)$ , then  $A$  is an  $IF\gamma CS$  in  $(X, \tau)$ .

**Proof:** Since  $A \subseteq A$  and  $A$  is an  $IF\gamma OS$  in  $X$ , by hypothesis  $\gamma cl(A) \subseteq A$ . But  $A \subseteq \gamma cl(A)$ . Therefore  $\gamma cl(A) = A$ . Hence  $A$  is an  $IF\gamma CS$  in  $(X, \tau)$ .

**Proposition 2.2.24:** Let  $A$  be an  $IF\gamma GCS$  in  $(X, \tau)$  and  $p_{(\alpha, \beta)}$  be an  $IFP$  in  $X$  such that  $p_{(\alpha, \beta)} \not\subseteq \gamma cl(A)$ , then  $cl(p_{(\alpha, \beta)}) \not\subseteq A$ .

**Proof:** Let  $A$  be an  $IF\gamma GCS$  and let  $p_{(\alpha, \beta)} \not\subseteq \gamma cl(A)$ . If  $cl(p_{(\alpha, \beta)}) \subseteq A$ , then  $A \subseteq (cl(p_{(\alpha, \beta)}))^c$  where  $(cl(p_{(\alpha, \beta)}))^c$  is an  $IFOS$  and hence it is an  $IF\gamma OS$ . Then by hypothesis,  $\gamma cl(A) \subseteq (cl(p_{(\alpha, \beta)}))^c = int(p_{(\alpha, \beta)})^c \subseteq (p_{(\alpha, \beta)})^c$ . This implies that  $p_{(\alpha, \beta)} \subseteq \gamma cl(A)$ , which is a contradiction to the hypothesis. Hence  $cl(p_{(\alpha, \beta)}) \not\subseteq A$ .

**Proposition 2.2.25:** Let  $F \subseteq A \subseteq X$  where  $A$  is an  $IF\gamma OS$  and an  $IF\gamma GCS$  in  $X$ . Then  $F$  is an  $IF\gamma GCS$  in  $A$  if and only if  $F$  is an  $IF\gamma GCS$  in  $X$ .

**Proof: Necessity:** Let  $U$  be an  $IF\gamma OS$  in  $X$  and  $F \subseteq U$ . Also let  $F$  be an  $IF\gamma GCS$  in  $A$ . Then  $F \subseteq A \cap U$  and  $A \cap U$  is an  $IF\gamma OS$  in  $A$ . Hence gamma closure of  $F$  in  $A$ ,  $\gamma cl_A(F) \subseteq A \cap U$  and by Proposition 2.2.23,  $A$  is an  $IF\gamma CS$ . Therefore  $\gamma cl(A) = A$ . Now gamma closure of  $F$  in  $X$ ,  $\gamma cl(F) \subseteq \gamma cl(F) \cap \gamma cl(A) = \gamma cl(F) \cap A = \gamma cl_A(F) \subseteq A \cap U \subseteq U$ . That is  $\gamma cl(F) \subseteq U$ , whenever  $F \subseteq U$ . Hence  $F$  is an  $IF\gamma GCS$  in  $X$ .

**Sufficiency:** Let  $V$  be an  $IF\gamma OS$  in  $A$ , such that  $F \subseteq V$ . Since  $A$  is an  $IF\gamma OS$  in  $X$ ,  $V$  is an  $IF\gamma OS$  in  $X$ . Therefore  $\gamma cl(F) \subseteq V$ , as  $F$  is an  $IF\gamma GCS$  in  $X$ . Thus,  $\gamma cl_A(F) = \gamma cl(F) \cap A \subseteq V \cap A \subseteq V$ . Hence  $F$  is an  $IF\gamma GCS$  in  $A$ .

**Proposition 2.2.26:** For an  $IFS A$ , the following conditions are equivalent:

- (i)  $A$  is an  $IFOS$  and an  $IF\gamma GCS$
- (ii)  $A$  is an  $IFROS$

**Proof:** (i)  $\Rightarrow$  (ii) Let  $A$  be both an IFOS and an  $IF\gamma GCS$  in  $X$ . Then  $A$  is an  $IF\gamma OS$  and an  $IF\gamma GCS$ . By Proposition 2.2.23  $A$  is an  $IF\gamma CS$  and  $intcl(A) = int(cl(A)) \cap cl(A) = int(cl(A)) \cap cl(int(A)) \subseteq A$ , by hypothesis. Hence  $int(cl(A)) \subseteq A$ . Since  $A$  is an IFOS, it is an IFPOS. Hence  $A \subseteq int(cl(A))$ . Therefore  $A = int(cl(A))$ . Hence  $A$  is an IFROS.

(ii)  $\Rightarrow$  (i) Let  $A$  be an IFROS in  $X$  then  $A = int(cl(A))$ . Since every IFROS is an IFOS we have  $int(cl(A)) \cap cl(int(A)) = A \cap cl(int(A)) = A \cap cl(A) = A \subseteq A$ . Hence  $A$  is an  $IF\gamma CS$  in  $X$  and by Proposition 2.2.13,  $A$  is an  $IF\gamma GCS$  in  $X$ .

**Proposition 2.2.27:** For an IFOS  $A$  in  $(X, \tau)$ , the following conditions are equivalent:

- (i)  $A$  is an IFCS
- (ii)  $A$  is an  $IF\gamma GCS$  and an IFQ-set

**Proof:** (i)  $\Rightarrow$  (ii) Since  $A$  is an IFCS, it is an  $IF\gamma GCS$  by Proposition 2.2.3. Now  $int(cl(A)) = int(A) = A = cl(A) = cl(int(A))$ , by hypothesis. Hence  $A$  is an IFQ-set.

(ii)  $\Rightarrow$  (i) Since  $A$  is an IFOS and an  $IF\gamma GCS$ , by Proposition 2.2.26,  $A$  is an IFROS. Therefore  $A = int(cl(A)) = cl(int(A)) = cl(A)$ , by hypothesis. Hence  $A$  is an IFCS in  $X$ .

**Proposition 2.2.28:** If a subset  $A$  of an IFTS  $(X, \tau)$  is intuitionistic fuzzy nowhere dense, then it is an  $IF\gamma GCS$  in  $(X, \tau)$ .

**Proof:** If  $A$  is intuitionistic fuzzy nowhere dense in  $X$ , then  $int(cl(A)) = 0_{\sim}$ . Let  $A \subseteq U$  where  $U$  is an  $IF\gamma OS$  in  $X$ . Now  $\gamma cl(A) \subseteq scl(A) = A \cup int(cl(A)) = A \cup 0_{\sim} = A \subseteq U$  and hence  $A$  is an  $IF\gamma GCS$  in  $(X, \tau)$ .

**Proposition 2.2.29:** Let  $(X, \tau)$  be an IFTS. Then every IFS in  $(X, \tau)$  is an  $IF\gamma GCS$  if and only if  $IF\gamma O(X) = IF\gamma C(X)$ .

**Proof: Necessity:** Suppose that every IFS in  $(X, \tau)$  is an  $IF\gamma GCS$  in  $X$ . Let  $U \in IF\gamma O(X)$ , and by hypothesis,  $\gamma cl(U) \subseteq U \subseteq \gamma cl(U)$ . This implies  $\gamma cl(U) = U$ . Therefore  $U \in IF\gamma C(X)$ . Hence  $IF\gamma O(X) \subseteq IF\gamma C(X) \longrightarrow (i)$ .

Let  $A \in \text{IF}_\gamma\text{C}(X)$ , then  $A^c \in \text{IF}_\gamma\text{O}(X) \subseteq \text{IF}_\gamma\text{C}(X)$ . That is,  $A^c \in \text{IF}_\gamma\text{C}(X)$ . Therefore  $A \in \text{IF}_\gamma\text{O}(X)$ . Hence  $\text{IF}_\gamma\text{C}(X) \subseteq \text{IF}_\gamma\text{O}(X) \longrightarrow$ (ii).

From (i) and (ii),  $\text{IF}_\gamma\text{O}(X) = \text{IF}_\gamma\text{C}(X)$ .

**Sufficiency:** Suppose that  $\text{IF}_\gamma\text{O}(X) = \text{IF}_\gamma\text{C}(X)$ . Let  $A \subseteq U$  and  $U$  be an  $\text{IF}_\gamma\text{OS}$  in  $X$ . Then  $U \in \text{IF}_\gamma\text{O}(X)$  and by hypothesis  $\gamma\text{cl}(A) \subseteq \gamma\text{cl}(U) = U$ , since  $U \in \text{IF}_\gamma\text{C}(X)$ . Therefore  $A$  is an  $\text{IF}_\gamma\text{GCS}$  in  $X$ .

**Proposition 2.2.30:** If  $A$  is an  $\text{IFROS}$  and  $B$  is an  $\text{IF}\alpha\text{CS}$ , then  $A \cap B$  is an  $\text{IF}_\gamma\text{GCS}$  in  $(X, \tau)$ .

**Proof:** Let  $B$  be an  $\text{IF}\alpha\text{CS}$  and  $A$  be an  $\text{IFROS}$ . Then  $\text{cl}(\text{int}(\text{cl}(B))) \subseteq B$  and  $\text{int}(\text{cl}(A)) = A$ . Therefore  $A \cap B \supseteq A \cap \text{cl}(\text{int}(\text{cl}(B))) = \text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(\text{cl}(B))) \supseteq \text{int}(\text{cl}(A)) \cap \text{int}(\text{cl}(B)) = \text{int}(\text{cl}(A \cap B))$ . We have  $\text{int}(\text{cl}(A \cap B)) \subseteq A \cap B$ . Hence  $A \cap B$  is an  $\text{IFSCS}$  and by Proposition 2.2.7,  $A \cap B$  is an  $\text{IF}_\gamma\text{GCS}$  in  $(X, \tau)$ .

**Proposition 2.2.31:** If  $A$  is both an  $\text{IF}\alpha\text{OS}$  and an  $\text{IF}_\gamma\text{GCS}$  in  $(X, \tau)$ , then  $A$  is an  $\text{IF}_\gamma\text{CS}$  in  $(X, \tau)$ .

**Proof:** Let  $A$  be an  $\text{IF}\alpha\text{OS}$  in  $X$ . Then  $A$  is an  $\text{IF}_\gamma\text{OS}$  in  $X$ . As  $A \subseteq A$ , by hypothesis  $\gamma\text{cl}(A) \subseteq A$ . Since  $A \subseteq \gamma\text{cl}(A)$ ,  $\gamma\text{cl}(A) = A$  and  $A$  is an  $\text{IF}_\gamma\text{CS}$  in  $(X, \tau)$ .

**Proposition 2.2.32:** An  $\text{IFS}$   $A$  of  $X$  is an  $\text{IF}_\gamma\text{GCS}$  if  $\gamma\text{cl}(A) \subseteq \ker(A)$ .

**Proof:** Let  $U$  be any  $\text{IF}_\gamma\text{OS}$  such that  $A \subseteq U$ . By hypothesis  $\gamma\text{cl}(A) \subseteq \ker(A)$  and since  $A \subseteq U$ ,  $\ker(A) \subseteq U$ . Therefore  $\gamma\text{cl}(A) \subseteq U$  and hence  $A$  is an  $\text{IF}_\gamma\text{GCS}$ .

**Proposition 2.2.33:** If  $A$  is both an  $\text{IF}_\gamma\text{OS}$  and an  $\text{IF}_\gamma\text{GCS}$  of  $(X, \tau)$ . Then  $A \cap F$  is an  $\text{IF}_\gamma\text{GCS}$  of  $(X, \tau)$  where  $F$  is an  $\text{IFCS}$  of  $X$ .

**Proof:** Suppose that  $A$  is an  $\text{IF}_\gamma\text{OS}$  and an  $\text{IF}_\gamma\text{GCS}$  in  $(X, \tau)$ , then by Proposition 2.2.23,  $A$  is an  $\text{IF}_\gamma\text{CS}$ . But  $F$  is an  $\text{IFCS}$  in  $X$ . Hence  $A \cap F$  is an  $\text{IF}_\gamma\text{CS}$  as every  $\text{IFCS}$  is an  $\text{IF}_\gamma\text{CS}$ . Therefore  $A \cap F$  is an  $\text{IF}_\gamma\text{GCS}$  in  $X$ , by Proposition 2.2.13.

**Proposition 2.2.34:** For an  $\text{IF}_\gamma\text{GCS}$   $A$  in an  $\text{IFTS}$   $(X, \tau)$ , the following conditions hold:

- (i) If  $A$  is an  $\text{IFROS}$  then  $\text{scl}(A)$  is an  $\text{IF}_\gamma\text{GCS}$
- (ii) If  $A$  is an  $\text{IFRCS}$  then  $\text{sint}(A)$  is an  $\text{IF}_\gamma\text{GCS}$

**Proof:** (i) Let  $A$  be an IFROS in  $(X, \tau)$ . Then  $\text{int}(\text{cl}(A)) = A$ . By definition we have  $\text{scl}(A) = A \cup \text{int}(\text{cl}(A)) = A$ . Since  $A$  is an IF $\gamma$ GCS in  $X$ ,  $\text{scl}(A)$  is an IF $\gamma$ GCS in  $X$ .

(ii) Let  $A$  be an IFRCS in  $(X, \tau)$ . Then  $\text{cl}(\text{int}(A)) = A$ . By definition we have  $\text{sint}(A) = A \cap \text{cl}(\text{int}(A)) = A$ . Since  $A$  is an IF $\gamma$ GCS in  $X$ ,  $\text{sint}(A)$  is an IF $\gamma$ GCS in  $X$ .

**Proposition 2.2.35:** Let  $(X, \tau)$  be an IFTS, then for every  $A \in \text{IFSPC}(X)$  and for every  $B$  in  $X$ ,  $\text{int}(A) \subseteq B \subseteq A$  implies  $B \in \text{IF}\gamma\text{GC}(X)$ .

**Proof:** Let  $A$  be an IFSPCS in  $X$ . Then there exists an IFPCS, (say)  $C$  such that  $\text{int}(C) \subseteq A \subseteq C$ . By hypothesis,  $B \subseteq A$ . Therefore  $B \subseteq C$ . Since  $\text{int}(C) \subseteq A$ ,  $\text{int}(C) \subseteq \text{int}(A)$  and  $\text{int}(C) \subseteq B$ , by hypothesis. Thus  $\text{int}(C) \subseteq B \subseteq C$  and  $B \in \text{IFSPC}(X)$ . Hence by Proposition 2.2.15,  $B \in \text{IF}\gamma\text{GC}(X)$ .

### 2.3 Intuitionistic fuzzy $\gamma$ generalized open sets

In this section we have discussed and analyzed some of the properties of intuitionistic fuzzy  $\gamma$  generalized open sets and produced many interesting characterization theorems.

**Example 2.3.1:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here the IFS  $A = \langle x, (0.7_a, 0.8_b), (0.3_a, 0.2_b) \rangle$  is an IF $\gamma$ GOS in  $(X, \tau)$ .

**Proposition 2.3.2:** Every IFOS, IFROS, IFSOS, IFPOS, IF $\alpha$ OS, IF $\gamma$ OS, IFSPOS in  $(X, \tau)$  is an IF $\gamma$ GOS in  $(X, \tau)$  but not conversely in general.

**Proof:** Straightforward.

**Example 2.3.3:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.7_a, 0.8_b), (0.3_a, 0.2_b) \rangle$  is an IF $\gamma$ GOS but not an IFOS in  $(X, \tau)$ .

**Example 2.3.4:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.7_a, 0.8_b), (0.3_a, 0.2_b) \rangle$  is an  $IF\gamma$ GOS but not an IFROS in  $(X, \tau)$ .

**Example 2.3.5:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.7_a, 0.8_b), (0.3_a, 0.2_b) \rangle$  is an  $IF\gamma$ GOS but not an IFSOS in  $(X, \tau)$ .

**Example 2.3.6:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.6_a, 0.7_b), (0.4_a, 0.3_b) \rangle$  is an  $IF\gamma$ GOS but not an IFPOS in  $(X, \tau)$ .

**Example 2.3.7:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.7_a, 0.8_b), (0.3_a, 0.2_b) \rangle$  is an  $IF\gamma$ GOS but not an  $IF\alpha$ OS in  $(X, \tau)$ .

**Example 2.3.8:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.7_b), (0.5_a, 0.3_b) \rangle$  and  $G_2 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.4_a, 0.4_b), (0.6_a, 0.6_b) \rangle$  is an  $IF\gamma$ GOS but not an  $IF\gamma$ OS in  $(X, \tau)$ .

**Example 2.3.9:** Let  $X = \{a, b\}$ ,  $G_1 = \langle x, (0.5_a, 0.7_b), (0.5_a, 0.3_b) \rangle$  and  $G_2 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$ . Then  $\tau = \{0_-, G_1, G_2, 1_-\}$  is an IFT on  $X$ . Here  $A = \langle x, (0.4_a, 0.4_b), (0.6_a, 0.6_b) \rangle$  is an  $IF\gamma$ GOS but not an IFSPOS in  $(X, \tau)$ .

**Proposition 2.3.10:** Let  $(X, \tau)$  be an IFTS. Then for every  $A \in IF\gamma GO(X)$  and for every  $B \in IFS(X)$ ,  $\gamma_{int}(A) \subseteq B \subseteq A \Rightarrow B \in IF\gamma GO(X)$ .

**Proof:** Let  $A$  be any  $IF\gamma$ GOS of  $X$  and  $B$  be any IFS of  $X$ . Let  $\gamma_{int}(A) \subseteq B \subseteq A$ . Then  $A^c$  is an  $IF\gamma$ GCS and  $A^c \subseteq B^c \subseteq \gamma_{cl}(A^c)$ . Therefore  $B^c$  is an  $IF\gamma$ GCS by Proposition 2.2.21, which implies  $B$  is an  $IF\gamma$ GOS in  $X$ . Hence  $B \in IF\gamma GO(X)$ .

**Proposition 2.3.11:** An IFS  $A$  of an IFTS  $(X, \tau)$  is an  $IF\gamma$ GOS if and only if  $F \subseteq \gamma_{int}(A)$  whenever  $F$  is an  $IF\alpha$ CS and  $F \subseteq A$ .

**Proof: Necessity:** Suppose  $A$  is an  $IF_\gamma GOS$  in  $X$ . Let  $F$  be an  $IF_\gamma CS$  such that  $F \subseteq A$ . Then  $F^c$  is an  $IF_\gamma OS$  and  $A^c \subseteq F^c$ . By hypothesis  $A^c$  is an  $IF_\gamma GCS$ , we have  $\gamma cl(A^c) \subseteq F^c$ . Therefore  $F \subseteq \gamma int(A)$ .

**Sufficiency:** Let  $F$  be an  $IF_\gamma CS$  such that  $F \subseteq A$  and  $F \subseteq \gamma int(A)$ . Then  $(\gamma int(A))^c \subseteq F^c$  and  $A^c \subseteq F^c$ . This implies that  $\gamma cl(A^c) \subseteq F^c$ , where  $F^c$  is an  $IF_\gamma OS$ . Therefore  $A^c$  is an  $IF_\gamma GCS$ . Hence  $A$  is an  $IF_\gamma GOS$  in  $X$ .

**Proposition 2.3.12:** Let  $(X, \tau)$  be an IFTS. Then for every  $A \in IFS(X)$  and for every  $B \in IF_\gamma O(X)$ ,  $B \subseteq A \subseteq int(cl(int(B))) \Rightarrow A \in IF_\gamma GO(X)$ .

**Proof:** Let  $B$  be an  $IF_\gamma OS$  in  $X$ . Then  $B \subseteq cl(int(cl(B)))$ . By hypothesis,  $A \subseteq int(cl(int(B))) \subseteq int(cl(int(cl(int(cl(B)))))) \subseteq int(cl(cl(int(cl(B)))))) = int(cl(int(cl(B))) \subseteq int(cl(cl(A))) \subseteq int(cl(A))$  as  $B \subseteq A$ . Therefore  $A$  is an IFPOS and by Proposition 2.3.2,  $A$  is an  $IF_\gamma GOS$ . Hence  $A \in IF_\gamma GO(X)$ .

**Proposition 2.3.13:** Let  $(X, \tau)$  be an IFTS. Then for every  $A \in IFS(X)$  and for every  $B \in IFRC(X)$ ,  $B \subseteq A \subseteq int(cl(B)) \Rightarrow A \in IF_\gamma GO(X)$ .

**Proof:** Let  $B$  be an IFRC in  $X$ . Then  $B = cl(int(B))$ . By hypothesis,  $A \subseteq int(cl(B)) \subseteq int(cl(cl(int(B)))) = int(cl(int(B))) \subseteq int(cl(int(A)))$  as  $B \subseteq A$ . Therefore  $A$  is an  $IF_\alpha OS$  and by Proposition 2.3.2,  $A$  is an  $IF_\gamma GOS$ . Hence  $A \in IF_\gamma GO(X)$ .

**Proposition 2.3.14:** Let  $(X, \tau)$  be an IFTS then for every  $A \in IFSP O(X)$  and for every IFS  $B$  in  $X$ ,  $A \subseteq B \subseteq cl(A) \Rightarrow B \in IF_\gamma GO(X)$ .

**Proof:** Let  $A$  be an IFSP OS in  $X$ . Then there exists an IFPOS, (say)  $C$  such that  $C \subseteq A \subseteq cl(C)$ . By hypothesis,  $A \subseteq B$ . Therefore  $C \subseteq B$ . Since  $A \subseteq cl(C)$ ,  $cl(A) \subseteq cl(C)$  and  $B \subseteq cl(C)$ , by hypothesis. Thus  $C \subseteq B \subseteq cl(C)$ . This implies  $B$  is an IFSP OS. As every IFSP OS is an  $IF_\gamma GOS$  by Proposition 2.3.2,  $B \in IF_\gamma GO(X)$ .

**Proposition 2.3.15:** If  $A$  is an  $IF_\gamma CS$  and an  $IF_\gamma GOS$  in  $(X, \tau)$ , then  $A$  is an  $IF_\gamma OS$  in  $(X, \tau)$ .

**Proof:** As  $A \supseteq A$ , by hypothesis  $\gamma \text{int}(A) \supseteq A$ . But we have  $A \supseteq \gamma \text{int}(A)$ . This implies  $A = \gamma \text{int}(A)$ . Hence  $A$  is an  $\text{IF}_\gamma\text{OS}$  in  $X$ .

**Proposition 2.3.16:** Let  $(X, \tau)$  be an IFTS. Then for every  $A \in \text{IFS}(X)$  and for every  $B \in \text{IFSO}(X)$ ,  $B \subseteq A \subseteq \text{int}(\text{cl}(B)) \Rightarrow A \in \text{IF}_\gamma\text{GO}(X)$ .

**Proof:** Let  $B$  be an IFSOS in  $X$ . Then  $B \subseteq \text{cl}(\text{int}(B))$ . By hypothesis,  $A \subseteq \text{int}(\text{cl}(B)) \subseteq \text{int}(\text{cl}(\text{cl}(\text{int}(B)))) = \text{int}(\text{cl}(\text{int}(B))) \subseteq \text{int}(\text{cl}(\text{int}(A)))$  as  $B \subseteq A$ . Therefore  $A$  is an  $\text{IF}_\alpha\text{OS}$  and by Proposition 2.3.2,  $A$  is an  $\text{IF}_\gamma\text{GOS}$ . Hence  $A \in \text{IF}_\gamma\text{GO}(X)$ .

**Proposition 2.3.17:** If  $A$  is an  $\text{IFRCS}$  and  $B$  is an  $\text{IF}_\alpha\text{OS}$ , then  $A \cup B$  is an  $\text{IF}_\gamma\text{GOS}$  in  $(X, \tau)$ .

**Proof:** Let  $B$  be an  $\text{IF}_\alpha\text{OS}$  and  $A$  be an  $\text{IFRCS}$  in  $X$ . Then  $B \subseteq \text{int}(\text{cl}(\text{int}(B)))$  and  $\text{cl}(\text{int}(A)) = A$ . Therefore  $A \cup B \subseteq A \cup \text{int}(\text{cl}(\text{int}(B))) = \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(\text{int}(B))) \subseteq \text{cl}(\text{int}(A)) \cup \text{cl}(\text{int}(B)) \subseteq \text{cl}(\text{int}(A \cup B))$ . We have  $A \cup B \subseteq \text{cl}(\text{int}(A \cup B))$ . Therefore  $A \cup B$  is an IFSOS and by Proposition 2.3.2,  $A \cup B$  is an  $\text{IF}_\gamma\text{GOS}$  in  $X$ .

**Proposition 2.3.18:** If an IFS  $A$  of an IFTS  $X$  is both an  $\text{IFCS}$  and an  $\text{IFGOS}$ , then  $A$  is an  $\text{IF}_\gamma\text{GOS}$  in  $(X, \tau)$ .

**Proof:** Suppose  $A$  is both an  $\text{IFCS}$  and an  $\text{IFGOS}$ . Then as  $A \subseteq A$ , by hypothesis  $A \subseteq \text{int}(A)$ . But  $\text{int}(A) \subseteq A$ . Therefore  $\text{int}(A) = A$ . We have  $A$  is an  $\text{IFOS}$ , since every  $\text{IFOS}$  is an  $\text{IF}_\gamma\text{GOS}$ . Hence  $A$  is an  $\text{IF}_\gamma\text{GOS}$  in  $X$ .

**Proposition 2.3.19:** Let  $(X, \tau)$  be an IFTS. Then  $\text{IF}_\gamma\text{O}(X) = \text{IF}_\gamma\text{GO}(X)$  if every IFS in  $(X, \tau)$  is an  $\text{IF}_\gamma\text{CS}$  in  $X$ .

**Proof:** Assume that every IFS in  $(X, \tau)$  is an  $\text{IF}_\gamma\text{CS}$  in  $X$ . Let  $A \in \text{IF}_\gamma\text{GO}(X)$ . Then  $A$  is also an  $\text{IF}_\gamma\text{CS}$ , by hypothesis. Therefore by Proposition 2.3.15  $A$  is an  $\text{IF}_\gamma\text{OS}$ . Therefore  $A \in \text{IF}_\gamma\text{O}(X)$ . Hence  $\text{IF}_\gamma\text{GO}(X) \subseteq \text{IF}_\gamma\text{O}(X) \longrightarrow$  (i)

Let  $A \in \text{IF}_\gamma\text{O}(X)$  then by Proposition 2.3.2,  $A \in \text{IF}_\gamma\text{GO}(X)$ . Hence  $\text{IF}_\gamma\text{O}(X) \subseteq \text{IF}_\gamma\text{GO}(X) \longrightarrow$  (ii).

Therefore from (i) and (ii)  $\text{IF}_\gamma\text{O}(X) = \text{IF}_\gamma\text{GO}(X)$ .

## 2.4 Theoretical applications of intuitionistic fuzzy $\gamma$ generalized closed sets

In this section we have investigated many theoretical applications of intuitionistic fuzzy  $\gamma$  generalized closed sets by defining new spaces and obtained many interesting propositions.

**Definition 2.4.1:** An IFTS  $(X, \tau)$  is an *intuitionistic fuzzy  $\gamma$  generalized c  $T_{1/2}$*  ( $IF\gamma_c T_{1/2}$ ) *space* if every  $IF\gamma GCS$  is an  $IFCS$  in  $(X, \tau)$ .

**Definition 2.4.2:** An IFTS  $(X, \tau)$  is called an *intuitionistic fuzzy  $\gamma$  generalized  $\gamma T_{1/2}$*  ( $IF\gamma_\gamma T_{1/2}$ ) *space* if every  $IF\gamma GCS$  in  $(X, \tau)$  is an  $IF\gamma CS$  in  $(X, \tau)$ .

**Example 2.4.3:** Let  $X = \{a, b\}$  and  $G = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$ . Then  $\tau = \{0_\sim, G, 1_\sim\}$  is an IFT on  $X$ . Therefore the space  $(X, \tau)$  is an  $IF\gamma_\gamma T_{1/2}$  space, as every  $IF\gamma GCS$  is an  $IF\gamma CS$  in  $(X, \tau)$ .

**Definition 2.4.4:** An IFTS  $(X, \tau)$  is an *intuitionistic fuzzy  $\gamma$  generalized pre  $T_{1/2}$*  ( $IF\gamma_p T_{1/2}$ ) *space* if every  $IF\gamma GCS$  in  $(X, \tau)$  is an  $IFPCS$  in  $(X, \tau)$ .

**Example 2.4.5:** Let  $X = \{a, b\}$  and  $G = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$ . Then  $\tau = \{0_\sim, G, 1_\sim\}$  is an IFT on  $X$ . Therefore the space  $(X, \tau)$  is an  $IF\gamma_p T_{1/2}$  space, as every  $IF\gamma GCS$  is an  $IFPCS$  in  $(X, \tau)$ .

**Proposition 2.4.6:** Every  $IF\gamma_p T_{1/2}$  space is an  $IF\gamma_\gamma T_{1/2}$  space but not conversely in general.

**Proof:** Let  $(X, \tau)$  be an  $IF\gamma_p T_{1/2}$  space and let  $A$  be an  $IF\gamma GCS$  in  $X$ . By hypothesis,  $A$  is an  $IFPCS$  in  $X$ . Since every  $IFPCS$  is an  $IF\gamma CS$ ,  $A$  is an  $IF\gamma CS$  in  $X$ . Hence  $(X, \tau)$  is an  $IF\gamma_\gamma T_{1/2}$  space.

**Example 2.4.7:** Let  $X = \{a, b\}$  and  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$  is an IFT on  $X$ . Here  $(X, \tau)$  is an  $IF\gamma_\gamma T_{1/2}$  space, but not an  $IF\gamma_p T_{1/2}$  space, as  $A = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$  is an  $IF\gamma GCS$  but not an  $IFPCS$  in  $(X, \tau)$ , since  $cl(int(A)) = cl(G_2) = G_1^c \not\subseteq A$ .

**Proposition 2.4.8:** Every  $IF\gamma_c T_{1/2}$  space is an  $IF\gamma_\gamma T_{1/2}$  space but not conversely in general.

**Proof:** Let  $(X, \tau)$  be an  $\text{IF}\gamma_c T_{1/2}$  space and let  $A$  be an  $\text{IF}\gamma\text{GCS}$  in  $X$ . By hypothesis,  $A$  is an  $\text{IFCS}$  in  $X$ . Since every  $\text{IFCS}$  is an  $\text{IF}\gamma\text{CS}$ ,  $A$  is an  $\text{IF}\gamma\text{CS}$  in  $X$ . Hence  $(X, \tau)$  is an  $\text{IF}\gamma T_{1/2}$  space.

**Example 2.4.9:** Let  $X = \{a, b\}$  and  $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b) \rangle$  and  $G_2 = \langle x, (0.4_a, 0.3_b), (0.6_a, 0.7_b) \rangle$ . Then  $\tau = \{0, G_1, G_2, 1\}$  is an  $\text{IFT}$  on  $X$ . Here  $(X, \tau)$  is an  $\text{IF}\gamma T_{1/2}$  space, but not an  $\text{IF}\gamma_c T_{1/2}$  space, since the  $\text{IFS}$   $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b) \rangle$  is an  $\text{IF}\gamma\text{GCS}$  but not an  $\text{IFCS}$  in  $(X, \tau)$ , as  $\text{cl}(A) = G_1^c \neq A$ .

**Proposition 2.4.10:** An  $\text{IFTS}$   $(X, \tau)$  is an  $\text{IF}\gamma_c T_{1/2}$  space iff  $\text{IF}\gamma\text{GO}(X) = \text{IFO}(X)$ .

**Proof: Necessity:** Let  $A$  be an  $\text{IF}\gamma\text{GOS}$  in  $(X, \tau)$ , then  $A^c$  is an  $\text{IF}\gamma\text{GCS}$  in  $(X, \tau)$ . By hypothesis,  $A^c$  is an  $\text{IFCS}$  in  $(X, \tau)$ . Hence  $A$  is an  $\text{IFOS}$  in  $(X, \tau)$ . Thus  $\text{IF}\gamma\text{GO}(X) = \text{IFO}(X)$ .

**Sufficiency:** Let  $A$  be an  $\text{IF}\gamma\text{GCS}$  in  $(X, \tau)$  then  $A^c$  is an  $\text{IF}\gamma\text{GOS}$  in  $(X, \tau)$ . By hypothesis  $A^c$  is an  $\text{IFOS}$  in  $(X, \tau)$ . Therefore  $A$  is an  $\text{IFCS}$  in  $(X, \tau)$ . Hence  $(X, \tau)$  is an  $\text{IF}\gamma_c T_{1/2}$  space.

**Proposition 2.4.11:** An  $\text{IFTS}$   $(X, \tau)$  is an  $\text{IF}\gamma T_{1/2}$  space iff  $\text{IF}\gamma\text{O}(X) = \text{IF}\gamma\text{GO}(X)$ .

**Proof: Necessity:** Let  $A$  be an  $\text{IF}\gamma\text{GOS}$  in  $(X, \tau)$  then  $A^c$  is an  $\text{IF}\gamma\text{GCS}$  in  $(X, \tau)$ . By hypothesis  $A^c$  is an  $\text{IF}\gamma\text{CS}$  in  $(X, \tau)$ . Hence  $A$  is an  $\text{IF}\gamma\text{OS}$  in  $(X, \tau)$ . Thus  $\text{IF}\gamma\text{O}(X) = \text{IF}\gamma\text{GO}(X)$ .

**Sufficiency:** Let  $A$  be an  $\text{IF}\gamma\text{GCS}$  in  $(X, \tau)$  then  $A^c$  is an  $\text{IF}\gamma\text{GOS}$  in  $(X, \tau)$ . By hypothesis  $A^c$  is an  $\text{IF}\gamma\text{OS}$  in  $(X, \tau)$ . Therefore  $A$  is an  $\text{IF}\gamma\text{CS}$  in  $(X, \tau)$ . Hence  $(X, \tau)$  is an  $\text{IF}\gamma T_{1/2}$  space.

**Proposition 2.4.12:** Let  $(X, \tau)$  be an  $\text{IF}\gamma T_{1/2}$  space. An  $\text{IFS}$   $A$  is an  $\text{IF}\gamma\text{GOS}$  in  $X$  if and only if  $A$  is an  $\text{IF}\gamma\text{N}$  of  $p_{(\alpha,\beta)}$  for each  $p_{(\alpha,\beta)} \in A$ .

**Proof: Necessity:** Let  $p_{(\alpha,\beta)} \in A$ . Let  $A$  be an  $IF_\gamma GOS$  in  $X$ . Since  $X$  is an  $IF_\gamma T_{1/2}$  space,  $A$  is an  $IF_\gamma OS$  in  $X$ . Then clearly  $A$  is an  $IF_\gamma N$  of  $p_{(\alpha,\beta)}$  as  $p_{(\alpha,\beta)} \in A \subseteq A$ .

**Sufficiency:** Let  $p_{(\alpha,\beta)} \in A$ . Since  $A$  is an  $IF_\gamma N$  of  $p_{(\alpha,\beta)}$ , there is an  $IF_\gamma OS$   $B$  in  $X$  such that  $p_{(\alpha,\beta)} \in B \subseteq A$ . Now  $A = \bigcup_{p_{(\alpha,\beta)} \in A} \{p_{(\alpha,\beta)}\} \subseteq \bigcup_{p_{(\alpha,\beta)} \in A} B \subseteq A$ . This implies that  $A = \bigcup_{p_{(\alpha,\beta)} \in A} B$

Since each  $B$  is an  $IF_\gamma OS$ ,  $A$  is an  $IF_\gamma OS$  and hence  $A$  is an  $IF_\gamma GOS$  in  $X$ .

**Proposition 2.4.13:** For any IFS  $A$  in  $(X, \tau)$  where  $X$  is an  $IF_\gamma T_{1/2}$  space,  $A \in IF_\gamma GO(X)$  iff for every IFP  $p_{(\alpha,\beta)} \in A$ , there exist an  $IF_\gamma GOS$   $B$  in  $X$  such that  $p_{(\alpha,\beta)} \in B \subseteq A$ .

**Proof: Necessity:** If  $A \in IF_\gamma GO(X)$ , then we can take  $B = A$  so that  $p_{(\alpha,\beta)} \in B \subseteq A$  for every IFP  $p_{(\alpha,\beta)} \in A$ .

**Sufficiency:** Let  $A$  be an IFS in  $(X, \tau)$  and assume that there exists  $B \in IF_\gamma GO(X)$  such that  $p_{(\alpha,\beta)} \in B \subseteq A$ . Since  $X$  is an  $IF_\gamma T_{1/2}$  space,  $B$  is an  $IF_\gamma OS$ . Then  $A = \bigcup_{p_{(\alpha,\beta)} \in A} \{p_{(\alpha,\beta)}\} \subseteq \bigcup_{p_{(\alpha,\beta)} \in A} B \subseteq A$ . Therefore  $A = \bigcup_{p_{(\alpha,\beta)} \in A} B$ , which is an  $IF_\gamma OS$ . Hence  $A$  is an  $IF_\gamma GOS$  in  $X$ .

**Proposition 2.4.14:** Let  $(X, \tau)$  be an  $IF_\gamma T_{1/2}$  space. Then

- (i) Any union of  $IF_\gamma GCS$ s is an  $IF_\gamma GCS$  in  $X$
- (ii) Any intersection of  $IF_\gamma GOS$ s is an  $IF_\gamma GOS$  in  $X$

**Proof:** (i) Let  $\{A_i\}$ ,  $i \in J$  be the collection of  $IF_\gamma GCS$ s in  $X$ . Since  $(X, \tau)$  is an  $IF_\gamma T_{1/2}$  space, every  $IF_\gamma GCS$  is an  $IF_\gamma CS$  and hence each  $A_i$ ,  $i \in J$  is an  $IF_\gamma CS$  in  $(X, \tau)$ . But any union of  $IF_\gamma CS$  is an  $IF_\gamma CS$ ,  $\cup A_i$  for every  $i \in J$  is an  $IF_\gamma CS$ . Since every  $IF_\gamma CS$  is an  $IF_\gamma GCS$ ,  $\cup A_i$  is also an  $IF_\gamma GCS$  in  $X$ .

(ii) can be proved by taking complement in (i).

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**Proposition 2.4.15:** If  $A$  is both an IFOS and an  $IF_\gamma GCS$  in  $X$  and if  $X$  is an  $IF_\gamma T_{1/2}$  space, then

- (i)  $A$  is an IFROS in  $X$ ,
- (ii)  $A$  is an IFRCS in  $X$ ,
- (iii)  $A$  is an IFQ- set in  $X$ .

**Proof:** Let  $A$  be an  $IF_\gamma GCS$  in  $X$ , then  $A$  is an IFCS in  $X$ , as  $X$  is an  $IF_\gamma T_{1/2}$  space. Now

(i)  $\text{int}(\text{cl}(A)) = \text{int}(A) = A$  and therefore  $A$  is an IFROS in  $X$ ,

(ii)  $\text{cl}(\text{int}(A)) = \text{cl}(A) = A$  and therefore  $A$  is an IFRCS in  $X$  and

(iii) from (i) and (ii)  $\text{int}(\text{cl}(A)) = \text{cl}(\text{int}(A))$ . Hence  $A$  is an IFQ-set in  $X$ .