



Chapter VII

CHAPTER VII

SEPARATION AXIOMS IN ORDERED FUZZY TOPOLOGICAL SPACES

Definition: 7.1

An ordered fuzzy topological space (X, τ, \leq) is said to be an **upper** (resp. **lower**) FT_1 ordered space iff for each pair of elements $x, y \in X$ with $x \not\leq y$, (By $x \not\leq y$ we mean that the relation $x \leq y$ is false) there exists a decreasing (resp. increasing) τ -open neighbourhood μ of y (resp. x) such that $\mu(x) = 0$ (resp. $\mu(y) = 0$). (X, τ, \leq) is said to be an FT_1 -ordered space iff if it is both an upper and a lower FT_1 -ordered space.

Notation: 7.2

By 1_x and 1_A we denote the characteristic function of $\{x\}$ and $A \subset X$, respectively.

Theorem: 7.3

A fuzzy topological ordered space (X, τ, \leq) is FT_1 -ordered iff, for each $x \in X$, $d(1_x)$ and $i(1_x)$ are τ -closed.

Proof

First, suppose that (X, τ, \leq) is an upper FT_1 -ordered space. Then for a pair $x \not\leq y$, in X , there exists a decreasing τ -open neighbourhood μ of y such that $\mu(x) = 0$. For each $y \in X \setminus \{x\}$, we have that $[1 - i(1_x)](y) > 0$ and

$\mu \leq [1 - i(1_x)]$. i.e. $[1 - i(1_x)]$ is a neighbourhood of each y so that $i(1_x)$ is τ -closed. Similarly $d(1_x)$ is τ -closed.

Conversely, let $d(1_x)$ and $i(1_x)$ be τ -closed for each $x \in X$. Then, for $x \not\leq y$, $[1 - i(1_x)]$ is a decreasing τ -open neighbourhood of y where

$$[1 - i(1_x)](x) = 0, \text{ which means that } (X, \tau, \leq) \text{ is upper FT}_1\text{-ordered.}$$

Definition: 7.4

An ordered fuzzy topological space (X, τ, \leq) is said to be **FT₂-ordered space** iff for each pair $x, y \in X$ with $x \not\leq y$, there is an increasing τ -open neighbourhood μ of x and a decreasing τ -open neighbourhood λ of y such that $\mu \wedge \lambda = 0$.

Definition: 7.5

If (X, τ) is a fuzzy topological space, then $A \subset X$ is called hyperclosed iff $1_A = \overline{1_A}$.

Theorem: 7.6

An ordered fuzzy topological ordered space (X, τ, \leq) is FT₂-ordered iff the graph $G(\leq) = \{(x, y) : x \leq y\} \subseteq X \times X$ of the order \leq is hyper-closed.

Proof

The graph $G(\leq)$ is hyperclosed $\Leftrightarrow 1_{G(\leq)} = \overline{1_{G(\leq)}} \Leftrightarrow 1 - 1_{G(\leq)}$ is τ -open. For $p \not\leq q$ in $\{X \times X - G(\leq)\}$ let $\gamma \in X \times X$ such that $1_\gamma(p, q) = \max\{1_p(p), 1_q(q)\}$ then $\gamma \in \{X \times X - G(\leq)\}$ and for the τ -open fuzzy set $[1 - 1_{G(\leq)}]$, there exists a τ -open fuzzy set $\mu \times \lambda$ in $\{X \times X - G(\leq)\}$ such that

$$\mu \times \lambda \leq 1 - 1_{G(\leq)} \text{ and } \mu \times \lambda(\gamma) = \min\{\mu(p), \lambda(q)\} > 0.$$

Thus, it follows that $\mu(p) > 0$ and $\lambda(q) > 0$. Also, for $x \in X$ we have $0 = \mu \times \lambda(x, x) = \min\{\mu(x), \lambda(x)\}$ i.e., $\mu \wedge \lambda \leq i(\mu) \wedge d(\mu) = 0$ which means that (X, τ, \leq) is FT_2 -ordered.

Conversely, let (X, τ, \leq) be FT_2 -ordered space and let $\gamma = (a, b) \in \{X \times X - G(\leq)\}$, then for $a \not\leq b$, there exists a decreasing τ -open neighbourhood μ of a and an increasing τ -open neighbourhood λ of b such that $\mu \wedge \lambda = 0$.

Consider the fuzzy set $\mu \times \lambda$ such that $\mu \times \lambda(\gamma) = \inf\{\mu(a), \lambda(b)\} > 0$. Take any point $\gamma' = (a', b') \in X \times X$, then $\mu \times \lambda(\gamma') = \inf\{\mu(a'), \lambda(b')\}$. Since $\mu \wedge \lambda = 0$ then $\mu \times \lambda(a', a') = 0$,
i.e. $\mu \times \lambda \leq 1 - 1_{G(\leq)} \quad (\forall (a', b') \in X \times X)$.

Thus, for any point $\gamma \in X \times X - G(\leq)$, one can say that there exists a τ -open fuzzy $\mu \times \lambda$ such that $\mu \times \lambda \leq 1 - 1_{G(\leq)}$ and therefore $1 - 1_{G(\leq)}$ is τ -open and this completes the proof.

Theorem: 7.7

An FT_2 -ordered space is an FT_1 -ordered space.

Definition: 7.8

Let $Y \subset X$, where (X, τ, \leq) is an ordered fuzzy topological space. Then the triple (Y, τ_y, \leq_y) consisting of the subset $Y \subset X$, the induced relative order \leq_y and the relative fuzzy topology $\tau_y = \{ \delta / Y : \delta \in \tau \}$ is called an ordered fuzzy topological subspace of (X, τ, \leq) . The ordered fuzzy topological ordered

subspaces (Y, τ_y, \leq_y) is called τ -compatibly ordered iff for each τ_y -closed increasing (resp. decreasing) fuzzy set λ , there exists a τ -closed increasing (resp. decreasing) fuzzy set λ^* such that $\lambda = \lambda^* / Y$.

Theorem: 7.9

Each τ -compatibly ordered fuzzy topological subspace of FT_2 -ordered space is again FT_2 -ordered.

Definition: 7.10

An ordered fuzzy topological space (X, τ, \leq) is said to be **upper** (resp. **lower**) regularly ordered space iff, the following condition holds: Given a point $x \in X$ and decreasing (resp. increasing) τ -open fuzzy neighbourhood λ of x , there exists a decreasing (resp. increasing) τ -closed fuzzy set ν and a τ -open decreasing (resp. increasing) fuzzy set μ such that $\mu(x) > 0$ and $\mu \leq \nu \leq \lambda$.

(X, τ, \leq) is said to be **regularly ordered space** (simply FR-ordered space) iff it is both lower and upper regularly ordered.

Theorem: 7.11

An ordered fuzzy topological space (X, τ, \leq) is lower (resp. upper) FR-ordered iff the following condition holds: For each point $x \in X$ and an increasing (resp. decreasing) τ -open fuzzy neighbourhood μ of x , there exists an increasing (resp. decreasing) τ -open fuzzy set λ such that $\lambda(x) > 0$ and $\lambda \leq I(\lambda) \leq \mu$ (resp. $\lambda \leq D(\lambda) \leq \mu$).

Lemma: 7.12

If (X, τ, \leq) is FR-ordered space, then so is every τ -compatible ordered subspace.

Proof

Let (Y, τ_y, \leq_y) be a τ -compatible ordered subspace of the upper FR-ordered space (X, τ, \leq) and let $x \in Y$ and μ be a τ_y -open decreasing fuzzy neighbourhood of x in Y . Then, there exists a τ -open decreasing fuzzy set λ^* such that $\lambda = \lambda^* / Y$ with $\lambda^*(x) > 0$. Since (X, τ, \leq) is upper FR-ordered then there exists a τ -open decreasing fuzzy set μ^* such that $\mu^*(x) > 0$ and $\mu^* \leq D(\mu^*) \leq \lambda^*$. By the restrictions of μ^* and $D(\mu^*)$ by Y we have that $\mu \leq D(\mu) \leq \lambda$ and (Y, τ_y, \leq_y) is upper FR-ordered.

Definition: 7.13

An FR-ordered space which is, also, FT_1 -ordered is called an FT_3 -ordered space.

Theorem: 7.14

An FT_3 -ordered space is an FT_2 -ordered space.

Proof

Let $x \not\leq y$ in X and suppose that (X, τ, \leq) is upper FT_3 -ordered, then $i(1_y)$ and $d(1_y)$ are τ -closed fuzzy sets for each $y \in X$. For a point $x \in X$ and a τ -open decreasing neighbourhood $[1 - i(1_y)]$ of x , by the hypothesis of upper FR-ordered, there exists a decreasing τ -open fuzzy set μ and a decreasing τ -closed fuzzy set ν such that $\mu(x) > 0$ and $\mu \leq \nu \leq [1 - i(1_y)]$. Thus $(1-\nu)$ is an

increasing τ -open fuzzy set containing y and $\mu \wedge (1 - \nu) = 0$. Which means that (X, τ, \leq) is FT_2 -ordered.

Definition: 7.15

A preordered fuzzy topological space X is called **normally preordered** iff the following condition is satisfied: Given a decreasing closed fuzzy set μ and a decreasing open fuzzy set ρ such that $\mu \leq \rho$, there are an open decreasing fuzzy set ρ_1 and a closed decreasing fuzzy set μ_1 such that $\mu \leq \rho_1 \leq \mu_1 \leq \rho$.

If the preorder of a normally preordered space X is a partial order, then X is called a **normally ordered fuzzy topological space**.

If μ is a fuzzy set in a preordered space X , we define

$$D(\mu) = \inf\{\rho : \rho \geq \mu, \rho \text{ closed and decreasing}\}$$

Clearly $D(\mu)$ is the smallest closed decreasing fuzzy set in X which contains μ .

Theorem: 7.16

A preordered fuzzy topological space X is normally preordered iff the following condition holds: Given a decreasing closed fuzzy set μ and a decreasing open fuzzy set ρ with $\mu \leq \rho$, there exists an decreasing open fuzzy set ρ_1 such that $\mu \leq \rho_1 \leq D(\rho_1) \leq \rho$.

Definition: 7.17

Let J_ϕ denote the set of all decreasing functions $\alpha : [0, 1] \rightarrow [0, 1]$. On J_ϕ we consider the equivalence relation \sim defined by

$$\alpha_1 \sim \alpha_2 \text{ iff } \alpha_1(t+) = \alpha_2(t+) \text{ and}$$

$$\alpha_1(t-) = \alpha_2(t-) \text{ for all } t \in [0, 1].$$

The fuzzy unit interval is the set $I_\phi = [0, 1]_\phi$ of all equivalence classes in which J_ϕ is partitioned by \sim . For $\alpha \in J_\phi$, we will denote also by α the equivalence class in which α belongs. For each $t \in (0, 1)$ we define L_t and R_t on I_ϕ by

$$L_t(\alpha) = 1 - \alpha(t-) \text{ and } R_t(\alpha) = \alpha(t+).$$

The fuzzy topology on I_ϕ which is generated by the family $\{L_t, R_t : t \in (0, 1)\}$ is called the usual fuzzy topology on I_ϕ .

By the usual order on I_ϕ we will mean the partial order \geq defined by $\alpha_1 \geq \alpha_2$ iff $\alpha_1(t+) \geq \alpha_2(t+)$ and $\alpha_1(t-) \geq \alpha_2(t-)$ ($t \in [0, 1]$).

Thus I_ϕ with the above fuzzy topology and the preceding order becomes an ordered fuzzy topological space.

Theorem: 7.18

Let X be a preordered fuzzy topological space. Then X is normally preordered iff the following condition is satisfied: Given a closed decreasing fuzzy set μ in X and an open decreasing fuzzy set ρ with $\mu \leq \rho$ there exists an increasing continuous function $f : X \rightarrow [0, 1]_\phi$ such that, for each $x \in X$,

$$\mu(x) \leq 1 - f(x)(0+) \leq 1 - f(x)(1-) \leq \rho(x).$$

Proof

Suppose that the condition is satisfied and let μ be closed decreasing and ρ open decreasing with $\mu \leq \rho$. Let f have the properties mentioned in the statement of the Theorem. Let $x \in X$. Since $f(x)$ is decreasing, we have

$$f(x) (0+) \geq f(x)(\frac{1}{2}-) \geq f(x)(\frac{1}{2}+) \geq f(x) (1-)$$

Thus

$$\mu(x) \leq 1-f(x)(\frac{1}{2}-) \leq 1 - f(x)(\frac{1}{2}+) \leq \rho(x).$$

But

$$1- f(x)(\frac{1}{2}-) = L_{\frac{1}{2}}(f(x)) = f^{-1}(L_{\frac{1}{2}})(x)$$

and

$$1- f(x)(\frac{1}{2}+) = 1- R_{\frac{1}{2}}(f(x)) = f^{-1}(1-R_{\frac{1}{2}})$$

Thus

$$\mu \leq f^{-1}(L_{\frac{1}{2}}) \leq f^{-1}(1- R_{\frac{1}{2}}) \leq \rho.$$

Note that, since $L_{\frac{1}{2}}$ and $1 - R_{\frac{1}{2}}$ are decreasing fuzzy sets in $[0, 1]_{\phi}$ and since f is increasing, the fuzzy sets $f^{-1}(L_{\frac{1}{2}})$ and $f^{-1}(1-R_{\frac{1}{2}})$ are decreasing (Theorem 6.1.9). Also, since f is continuous, the last two fuzzy sets are open and closed, respectively, in X . This proves that X is normally preordered.

Conversely, suppose that X is normally preordered and let μ be closed, ρ open, both decreasing with $\mu \leq \rho$. Let λ_0 be the open decreasing fuzzy set in X with $\mu \leq \lambda_0 \leq D(\lambda_0) \leq \rho$. For each integer $n \geq 0$. Let

$$A_n = \{ \frac{m}{2^n} ; m = 0, 1, \dots, 2^n \}$$

and

$$A = \bigcup_{n=1}^{\infty} A_n.$$

For each $r \in A$ we will define an open decreasing fuzzy set ρ_r such that the following three conditions hold:

- (1) $\rho_0 = \lambda_0$ and $\rho_1 = \rho$;
- (2) $\mu \leq \rho_r \leq \rho$ for each $r \in A$;

(3) If $r < s$, then $D(\rho_r) \leq \rho_s$.

In fact, suppose that we have already define ρ_r , for each $r \in A_n$ so that (1), (2), (3) are satisfied for $r, s \in A_n$. We will define ρ_r for each $r \in A_{n+1}$ which is not in A_n . Indeed, let r be such an element of A_{n+1} . Then $r = m / 2^{n+1}$ with m an odd integer, $m = 2k + 1$. Since

$$D(\rho_{k/2^n}) \leq \rho_{(k+1)/2^n}.$$

there exists an open decreasing set λ_1 with

$$D(\rho_{k/2^n}) \leq \lambda_1 \leq D(\lambda_1) \leq \rho_{(k+1)/2^n}.$$

Define $\rho_r = \lambda_1$. In this way we define the family $\{\rho_r : r \in A_{n+1}\}$ which clearly satisfies (1), (2) and (3).

Inductively now we can define a family

$\{\rho_r : r \in A\}$ of open decreasing fuzzy sets satisfying (1), (2) and (3). Next

we define a family $\{\sigma_t : t \in [0, 1]\}$ by taking $\sigma_t = \inf\{\rho_r : r \in A, r \geq t\}$.

Clearly each σ_t is decreasing and $\sigma_r = \rho_r$ if $r \in A$. If $s < t$, then $D(\sigma_s) \leq \sigma_t^0$, where σ_t^0 denotes the interior of the set σ_t .

In fact, let $r_1, r_2 \in A$ with $s < r_1 < r_2 < t$. Then $\sigma_s \leq \rho_{r_1} \leq \rho_{r_2} \leq \sigma_t^0$ and thus

$$D(\sigma_s) \leq D(\rho_{r_1}) \leq \rho_{r_2} \leq \sigma_t^0.$$

We now define $f : X \rightarrow [0, 1]_\phi$ by

$$f(x)(t) = 1 - \sigma_t(x) \quad (x \in X, t \in [0, 1]).$$

It is easy to see that, for each $x \in X$, $f(x) \in [0, 1]_\phi$.

Also, for $t \in (0, 1)$, we have

$$1 - \mu(x) \geq 1 - \sigma_t(x) = f(x)(t) \geq 1 - \rho(x).$$

Hence

$$\mu(x) \leq 1 - f(x)(0+) \leq 1 - f(x)(t) \leq 1 - f(x)(1-) \leq \rho(x)$$

The function f is increasing since each σ_t is decreasing. Finally f is continuous, In fact

$$\begin{aligned} f^{-1}(L_t)(x) &= L_t(f(x)) = 1 - f(x)(t-) \\ &= 1 - \inf_{s < t} f(x)(s) \\ &= 1 - \inf_{s < t} (1 - \sigma_s(x)) \\ &= \sup_{s < t} \sigma_s(x) = \sup_{s < t} \sigma_s^0(x) \end{aligned}$$

and thus $f^{-1}(L_t) = \sup_{s < t} \sigma_s^0$ is open. Similarly,

$$\begin{aligned} f^{-1}(R_t)(x) &= R_t(f(x)) = f(x)(t+) \\ &= \sup_{s > t} f(x)(s) = \sup_{s > t} (1 - \sigma_s(x)) \\ &= 1 - \inf_{s > t} \sigma_s(x) = 1 - \inf_{s > t} D(\sigma_s)(x) \end{aligned}$$

and thus $f^{-1}(R_t) = 1 - \inf_{s > t} D(\sigma_s)$ is open.

Theorem: 7.19

An FN-ordered space is an FR-ordered space.

Proof

Let $x \in X$ be a point, μ be a decreasing τ -closed fuzzy set and ρ be a decreasing τ -open neighbourhood of x with $\mu \leq \rho$. Fuzzy normality of (X, τ, \leq) implies that there exists a decreasing τ -open fuzzy set λ such that $\mu \leq \lambda \leq D(\lambda) \leq \rho$.

So that (X, τ, \leq) is FR-ordered.

Definition: 7.20

An FN-ordered space which is also, FT_1 -ordered is called an FT_4 -ordered space.

Lemma: 7.21

An FT_4 -ordered space is an FT_3 -ordered space.

Definition: 7.22

An ordered fuzzy topological space (X, τ, \leq) is said to be **fuzzy perfectly normal (FPN) ordered space** iff it is fuzzy normal and every decreasing (resp. increasing) τ -closed fuzzy set is a countable intersection of decreasing (resp. increasing) τ -open fuzzy sets.

Theorem: 7.23

An ordered fuzzy topological space (X, τ, \leq) is called an FPN-ordered space iff for every decreasing (resp. increasing) τ -closed fuzzy set μ and a decreasing (resp. increasing) τ -open fuzzy set ρ such that $\mu \leq \rho$, there exists an order preserving continuous function $f : X \rightarrow [0, 1]_{\phi}$ such that, for each $x \in X$,

$$\mu(x) = 1 - f(x)(0+) \leq 1 - f(x)(1-) = \rho(x)$$

Proof

The proof is trivial as a consequence of theorem 7.16 and 7.18