On Some Interesting Problems Connected with Bitopological Spaces

BY

Chitra Geetha .D

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE-641 043.

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

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CERTIFIED AS BONAFIDE RESEARCH WORK

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INTRODUCTION

The concept of Bitopological spaces was first introduced by Dr. J.C. Kelly (10) in 1963. He has defined a bitopological space (X, τ_1, τ_2) as a space with two topologies τ_1 and τ_2 defined on it. Here the author has obtained systematic generalizations of Urysohn's lemma, Urysohn's metrization theorem and Tietze's extension theorem.

Every year a number of articles are published dealing with generalisation of the concepts of separation axioms, connectedness, compactness and paracompactness etc to bitopological spaces. The aim of this thesis is to discuss the following two articles on bitopological spaces.

1. Pairwise Locally Semi-Connected spaces. [1]
2. Bitopological preopen sets, Precontinuity and preopen mappings. [8]

In the first chapter of this thesis we deal with preliminary definitions and results. These include the results of J.C. Kelly.

The second chapter studies the two articles mentioned above. In the first section we shall discuss the paper "Pairwise Locally Semi-Connected Spaces" by S.P. Arya and T.M. Nour. In this paper the author generalizing the concept of local semi-connectedness introduce by Sarker and Dasgupta [16] to bitopological spaces. The author has introduced the following definitions.

1. Two non-empty subsets A and B of (X,τ_1,τ_2) are said to be pairwise semi-separated iff $A \cap \tau_1 - scl(B) = \phi = \tau_2 - scl(A) \cap B$ where scl (B) denotes the semiclosure of B.
2. A subset A of (X, τ_1) is said to be pairwise semi-connected iff A cannot be expressed as the union of two pairwise semi separated sets.
3. A bitopological space (X, τ_1,τ_2) is said to τ_1- locally semi-connected with respect to τ_2 if for each $x \in X$ and every τ_1 - semi open set U containing x, there exists a pairwise semi-connected τ_1 - open set G such that $x \in G \subset U$.
4. A space (X, τ_1,τ_2) is said to be pairwise locally semi-connected if it is τ_1 - locally semi - connected with respect to τ_2 and τ_2- locally semi-connected with respect to τ_1.

Using these definitions the following results are obtained.

1. If A is a pairwise semi-connected subset of a bitopological space (X, τ_1,τ_2) such that $A \subset C \cup D$ where C and D are pairwise semi-separated sets in (X, τ_1,τ_2), then either $A \subset C$ or $A \subset D$.
2. The union of any family of pairwise semi-connected sets in (X, τ_1,τ_2) having non-empty intersection is pairwise semi-connected.
3. A pairwise locally semi-connected space (X,τ_1,τ_2) is pairwise semi-connected iff it is pairwise connected.

The author has defined the semi-component of $E \subset X$ in a bitopological space and has obtained a necessary and sufficient condition for the space (X, τ_1, τ_2) to be τ_1 - locally semi-connected with respect to τ_2 interms of semi-components. Further the author has discussed total disconnectedness for bitopological spaces.

In section II, the concepts of Preopen sets, Precontinuity and Preopen mappings in a Bitopological space, introduced by A. Kar and P. Bhattacharyya are studied. The author defines a set A to be (i - j) - preopen in (X, τ_1, τ_2) iff $A \subset \tau_i$ - Int (τ_j - ClA), i, j = 1, 2 ; $i \neq j$. He has obtained the following properties of (i - j) preopen sets.

(1) $A \subset X$ is (i - j) - preopen iff for every τ_j - closed set G, containing A, $A \subset \tau_i$ - Int G, i, j = 1, 2 ; $i \neq j$.

(2) If $\{A_\alpha\}_{\alpha \in \lambda}$ be a family of (i-j) - preopen sets in (X, τ_1, τ_2), then $\cup_{\alpha \in \lambda} A_\alpha$ is (i - j) - preopen i, j = 1, 2; $i \neq j$.

(3) A, $B \subset X$, $A \subset B$ implies $A \subset \tau_i$ - Int (τ_j - Cl B) if A is (i - j) - preopen. i, j = 1, 2 ; $i \neq j$.

(4) If A is (i - j) - preopen and $B \subset X$ is such that $A \subset B \subset \tau_i$ - Int (τ_j - Cl A), then B is (i - j) - preopen i, j = 1, 2 ; $i \neq j$.

(5) $A \subset X$ is (i - j) - preclosed iff τ_i - Cl (τ_j - Int A) $\subset A$, i, j = 1, 2 ; $i \neq j$.

(6) $A \subset X$ is (i - j) - preclosed iff for every τ_j - open set G contained in A, τ_i - Cl $G \subset A$, i, j = 1, 2 ; $i \neq j$.

(7) If A is (i - j) - preclosed and $B \subset X$ is such that τ_i - Cl (τ_i - Int A) $\subset B \subset A$, then B is (i - j) - preclosed i, j = 1, 2 ; $i \neq j$.

According to the author, A mapping $f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*)$ is termed as (i - j) - precontinuous (briefly p.c) iff for $O^* \in \tau_i^*$, $f^{-1}[O^*]$ is (i - j) - preopen in X, i, j = 1, 2 ; $i \neq j$.

The following characterization of percontinuous map is obtained.

Let $f: (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*)$ be a map, then the following statements are equivalent :

(a) f is (i - j) - precontinuous.

(b) For each $x \in X$ and each net $\{x_\alpha, \alpha \in D, \geq\}$, converging to x, the image net $\{f(x_\alpha), \alpha \in D, \geq\}$ is eventually in every τ_i^* - neighbourhood of f(x), whose inverse is τ_j - closed in X.

(c) For each $x \in X$ and each τ_i^* - open set U^* containing f(x), there exists an (i - j) - preopen set $U \subset X$ such that $x \in U$ and $f[U] \subset U^*$.

(d) The inverse image of each τ_i^* - closed set in X^* is (i - j) - preclosed in X.

(e) For each $A \subset X$, f [τ_i - Cl (τ_j - Int A)] $\subset \tau_i^*$ - Cl (f[A]).

(f) For each $A^{*} \subset X^{*}$, τ_i - Cl (τ_j - Int ($If^{-1}[A^{*}]$)) $\subset f^{-1}[\tau_i^{*}$ - Cl $A^{*}]$

The author is able to generalize the properties of continuous map to precontinuous map with regards to product spaces. He has defined a mapping $f : (X, \tau_1, \tau_2) \rightarrow (X^{*}, \tau_1^{*}, \tau_2^{*})$ to be (i - j) - preopen iff for each τ_i - open set A in X, f[A] is (i - j) - preopen in X^{*}. i, j = 1, 2 ; i ≠ j. The following characterization of preopen map is established.

A mapping $f : (X, \tau_1, \tau_2) \rightarrow (X^{*}, \tau_1^{*}, \tau_2^{*})$ is (i - j) - preopen iff

$f^{-1}[\tau_i^{*}$ - Cl V] $\subset \tau_i$ - Cl f^{-1} [V] for each τ_j^{*} - open set,i, j = 1, 2 ; i ≠ j.

Chapter I

CHAPTER - 1

PRELIMINARY DEFINITIONS AND RESULTS

SECTION I

DEFINITION : 1.1.1.

A space X on which are defined two (arbitrary) topologies τ_1 and τ_2 is called a bitopological space and denoted by (X, τ_1, τ_2).

REMARK :

When we consider quasi - metrizable space we get the natural example of bitopological space. If q is a quasi metric on X and q^* is conjugate to q then q and q^* will give two topologies on X.

DEFINITION : 1.1.2

In a space (X, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2 if, for each point x in X, and each τ_1 - closed set B such that $x \notin p$, there are a τ_1 - open set U and a τ_2 - open set V such that $x \in U$, $P \subset V$ and $U \cap V = \phi$. (X, τ_1, τ_2) is, or τ_1 and τ_2 are, pairwise regular if τ_1 is regular with respect to τ_2 and vice versa.

DEFINITION : 1.1.3

A space (X, τ_1, τ_2) is said to be pairwise Hausdorff iff given two distinct points x and y, there are τ_1 - neighbourhood U of x and a τ_2 - neighbourhood V of y such that $U \cap V = \phi$.

DEFINITION : 1.1.4

A space (X, τ_1, τ_2) is said to be pairwise normal if given a τ_1 - closed set A and a τ_2 - closed set B with $A \cap B = \phi$, there exists a τ_2 - open set U and a τ_1 - open set V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$. Equivalently, (X, τ_1, τ_2) is pairwise normal if, given a τ_2 - closed set C and a τ_1 - open set D such that, $C \subset D$, there are a τ_1- open set G and a τ_2 - closed set F such that $C \subset G \subset F \subset D$.

J.C. Kelly has established the following important results :

THEOREM : 1.1.5 (Generalization of Urysohn's lemma)

If (X, τ_1, τ_2) is pairwise normal, then given a τ_2 - closed set F and a τ_1 - closed set H with $F \cap H = \phi$, there exists a real - valued function g on X such that

DEFINITION : 1.2.12

A sequence $< S_n >$ is said to be eventually in a set A if and only if there exists a positive integer M such that $n \geq M \Rightarrow S_n \in A$.

DEFINITION : 1.2.13

A sequence $< S_n >$ in a topological space X is said to be converge to a point S_0 in X if and only if it is eventually in every neighbourhood of S_0.

Chapter II

CHAPTER - 2

PREOPEN SETS, PRECONTINUOUS MAPPINGS IN BITOPOLOGICAL SPACES AND PAIRWISE LOCALLY SEMI-CONNECTED SPACES

SECTION : 1

In this section the article "Pairwise locally semi-connected spaces" by S.P. ARYA is discussed. First we shall start with some definitions.

DEFINITION : 2.1.1 [6]

Two non-empty subsets A and B of (X, τ_1, τ_2) are said to be pairwise semi-separated if and only if $A \cap \tau_1$ - scl (B) = ϕ = τ_2 - scl $(A) \cap B$.

DEFINITION : 2.1.2

If $X = A \cup B$ such that A and B are pairwise semi-separated sets, then A, B form a pairwise semi-separation of X and we write it as $X = A | B$.

DEFINITION : 2.1.3

A subset A of (X, τ_1, τ_2) is said to be pairwise semi-connected if and only if A cannot be expressed as the union of two pairwise semi-separated sets.

RESULT : 2.1.4

A bitopological space (X, τ_1, τ_2) is pairwise semi-connected if and only if X cannot be expressed as the union of two non-empty disjoint sets one of which is τ_1 - semi open and the other is τ_2 - semi open.

PROOF :

Assume that the bitopological space (X, τ_1, τ_2) is pairwise semi-connected.

Claim :

X cannot be expressed as the union of two non-empty disjoint sets one of which is τ_1-semiopen and the other is τ_2 - semi open.

Let $A \neq \phi$ be an τ_1 - semi open and $B \neq \phi$ be an τ_2 - semi open such that $A \cap B = \phi$. Suppose $X = A \cup B$. since A is τ_1 - semi open, $B = X - A$ is τ_1 - semi closed. Since B is τ_2 - semi open, $A = X - B$ is τ_2 - semi closed.

This implies, τ_1 - scl (B) = B = X – A

τ_2 - scl (A) = A = X – B

Therefore, $A \cap \tau_1$ - scl (B) = $A \cap (X - A) = \phi$ and

τ_2 - scl $(A) \cap B = (X - B) \cap B = \phi$

Hence A and B are pairwise semi separated such that $X = A \cup B$.

This implies that X is not pairwise semi-connected.

Contradiction to our assumption.

Hence we obtain our claim whereas the converse is obvious.

DEFINITION : 2.1.5 [2]

A bitopological space (X, τ_1, τ_2) is said to be τ_1- locally connected with respect to τ_2 if for each $x \in X$ and every τ_1 - open set U containing x, there exists a pairwise connected τ_1 - open set G such that $x \in G \subset U$.

DEFINITION : 2.1.6

(X, τ_1, τ_2) is said tobe pairwise locally connected if it is τ_1 - locally connected with respect to τ_2 and τ_2 - locally connected with respect to τ_1.

DEFINITION : 2.1.7

Let (X, τ_1, τ_2) be a bitopological space then (X, τ_1, τ_2) is τ_1 - locally semi-connected with respect to τ_2 if for each $x \in X$ and every τ_1 - semi open set U containing x, there exists a pairwise semi-connected τ_1 - open set G such that $x \in G \subset U$.

DEFINITION : 2.1.8

A space (X, τ_1, τ_2) is said to be pairwise locally semi-connected if it is τ_1 - locally semi-connected with respect to τ_2 and τ_2 - locally semi-connected with respect to τ_1.

RESULT : 2.1.9

If a bitopological space is pairwise locally semi-connected then it is pairwise locally connected.

REMARK :

A bitopological space is pairwise locally connected but not pairwise locally semi-connected can be seen from the following example.

EXAMPLE : 2.1.10

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{a, c\}\}$.
Let $a \in X$, τ_1 open sets U containing "a" are X, {a}, {a, b}.

Also for every τ_1 - open set U containing "a" there exists a pairwise connected τ_1- open set G = {a} such that $a \in G \subset U$.
Let $b \in X$, τ_1 - open sets containing "b" are {a, b}, X.

Then for every τ_1 - open set U Containing "b" there exists a pairwise connected τ_1 - open set G = {a, b} such that $b \in G \subset U$.

Let $c \in X$, since the τ_1 - open set U containing "c" is X we have there exists a pairwise connected τ_1 - open set G = X such that $c \in G \subset U$.

Hence (X, τ_1, τ_2) is τ_1 - locally connected with respect to τ_2. - (1).

Let $a \in X$, τ_2 - open sets U containing "a" are X, {a} {a,c}.

Then for every τ_2 - open set U containing "a" there exists a pairwise connected τ_2 - open set G = {a} such that $a \in G \subset U$.

Let $b \in X$, τ_2 - open set U containing b is X.

For this U, there exists a pairwise connected τ_2-open set G = {b} such that $b \in G \subset U$.

Let $c \in X$, τ_2 - open sets U containing "c" are X, {a, c}.

Then for every τ_2 - open set U containing "c" there exists a pairwise connected τ_2 - open set G = {a, c} such that $c \in G \subset U$.

Hence (X, τ_1, τ_2) is τ_2 - locally connected with respect to τ_1. - (2).

By (1) and (2) (X, τ_1, τ_2) is pairwise locally connected.

For, {a, c} is a τ_1 - semi open set and $c \in$ {a, c} but there is no pairwise semi connected τ_1 - open set containing "c" and contained in {a, c}.

Therefore, (X, τ_1, τ_2) is not pairwise semi connected.

REMARK :

The following example shows that if (X, τ_1) and (X, τ_2) are both locally semi-connected spaces, then (X, τ_1, τ_2) need not be a pairwise locally semi-connected space.

EXAMPLE : 2.1.11

Let X = {a, b, c}, τ_1 = {X, ϕ, {a}, {c}, {a, c}, {a, b}

and τ_2 = {X, ϕ, {b}, {b, c}, {a, b}, {c}}

X - τ_1 = {X, ϕ, {b}, {b, c}, {a, b}, {c}}

X - τ_2 = {X, ϕ, {a, c}, {a}, {a, b}, {c}}

τ_1 - semi open sets = {X, ϕ, {a}, {c}, {a,c}, {a, b}}

τ_2 - semi open sets = {X, ϕ, {b}, {b, c}, {a, b}, {c}}

Let $a \in X$, τ_1 - semi open set U containing "a" are X, {a}, {a,c}, {a, b}.

Then for every τ_1 - semi open set U containing "a" there exists a semi connected open set G = {a} such that $a \in G \subset U$.

Let $b \in X$, τ_1 - semi open sets U containing "b" are X, {a, b}.

Then for every τ_1-semi open set U containing "b" there exists a semi connected open set G = {a, b} such that $b \in G \subset U$.

Let $c \in X$, τ_1 - semi open sets U containing "c" are X, {a, c}, {c}.

Then for every τ_1 - semi open set U containing "c" there exists a semi connected open set G = {c} such that $c \in G \subset U$.

Therefore, for each $x \in X$ and every τ_1 - semi open set containing x, there exists a semi connected τ_1 - open set G such that $x \in G \subset U$.

Therefore, (X, τ_1) is locally semi connected.

Let $a \in X, \tau_2$ - semi open sets U containing "a" are {a, b},X.

Then for every τ_2 - semi open set U containing "a" there exists a semi connected τ_2 - open set G = {a, b} such that $a \in G \subset U$

Let $b \in X$, τ_2 - semi open sets U containing "b" are X, {b}, {a, b}, {b, c}.

Then for every τ_2 - semi open sets U containing "b" there exists a semi connected τ_2 - open set G = {b} such that $b \in G \subset U$

Let $c \in X$, τ_2 - semi open sets U containing "c" are X, {c}, {b, c}.

Then for every τ_2 - semi open sets U containing "c" there exists a semi-connected τ_2 - open set G= {c} such that $c \in G \subset U$

Therefore, for each $x \in X$ and every τ_2 - semi open set U containing x, there exists a semi connected τ_2 - open set G such that $x \in G \subset U$

Therefore, (X, τ_2) is locally semi connected.

Let $b \in X$, {a, b} is a τ_1 - semi open set containing "b".

Also, $\{a\} \neq \phi$ is τ_1 - semi open set $\{b\} \neq \phi$ is τ_2- semi open set $\{a\} \cap \{b\} = \phi$.

But $\{a, b\} = \{a\} \cup \{b\}$ is not pairwise semi-connected.

Hence, there is no pairwise semi-connected τ_1 - open set containing "b" and contained in {a, b}.

Therefore, (X, τ_1, τ_2) is not pairwise locally semi-connected.

REMARK :

From the following two examples, we can see that the notations of semi connectedness and locally semi connectedness in a bitopological space are independent.

EXAMPLE : 2.1.12

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi \{a\}, \{a, b\}\}$,

$\tau_2 = \{X, \phi, \{a, b\}$,

$X - \tau_1 = \{X, \phi \{b, c\}, \{c\}\}$

$X - \tau_2 = \{X, \phi, \{c\}\}$

τ_1 - semi open sets = $\{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$

τ_2- semi open sets = $\{X, \phi, \{a, b\}\}$.

X cannot be expressed as the union of two disjoint non-empty sets one of which is τ_1 - semi open set and the other is τ_2 - semi open.

Therefore (X, τ_1, τ_2) is pairwise semi-connected. But, {a,c} is τ_1 - semi open set containing the point "c" and there does not exists a pairwise semi connected τ_1 - open set containing "c" and contained in {a, c}.

Therefore, (X, τ_1, τ_2) is not τ_1 - locally semi connected with respect to τ_1.

Hence (X, τ_1, τ_2) is not pairwise locally semi connected.

EXAMPLE : 2.1.13

Let X = {a, b, c}, τ_1 = {X, ϕ,{a, b}, {c}}

and τ_2 = {X,ϕ, {a}, {c}, {a, c}, {a, b}},

X - τ_1 = {X,ϕ, {c}, {a, b}}

X - τ_2= {X, ϕ, {b, c}, {a, b}, {b}, {c}}

τ_1 - semi open sets = {X,ϕ, {a, b}, {c}}

τ_2 - semi open sets = {X, ϕ, {a}, {c}, {a, c}, {a, b}}.

Let $a \in X$, τ_1 - semi open sets U containing "a" are {a, b}, X. Then for every τ_1 - semi open set U containing "a" there exists a pairwise semi-connected τ_1 - open set G = {a,b} such that $a \in G \subset U$.

Let $b \in X$,τ_1 - semi open set U containing "b" are X, {a, b}, then for every τ_1 - semi open set U containing "b" there exists a pairwise semi-connected τ_1 - open set G = {a, b} such that $b \in G \subset U$.

Let $c \in X$, τ_1 - semi open set U containing "c" are {c}, then for every τ_1 - semi open set U containing "c" there exists a pairwise semi connected τ_1 - open set G = {c} such that $c \in G \subset U$.

Therefore, the space (X, τ_1, τ_2) is τ_1 - locally semi connected with respect to τ_2.

Let $a \in X$, τ_2 - semi open sets U containing "a" are {a}, {a, c}, X, {a, b}, then for every τ_2 - semi open set U containing "a" there exists a pairwise semi connected τ_2 - open set G = {a, b} such that $a \in G \subset U$.

Let $b \in X$, τ_2 - semi open sets U containing "b" are X, {a, b} then for everyτ_2 - semi open set U containing "b" there exists a pairwise semi connected τ_2 - open set G = {a, b} such that $b \in G \subset U$.

Let $c \in X$, τ_2 - semi open sets U containing "c" are X, {c}, {a, c}, then for every τ_2 - semi open set U containing "c" there exists a pairwise semi connectedτ_2 - open set G = {c} such that $c \in G \subset U$.

Therefore, the space (X, τ_1, τ_2) is τ_2 - locally semi connected with respect to τ_1.

Therefore, the space $(X, \tau_1, \tau_2)$is pairwise locally semi connected, but not pairwise semi connected. Because, X can be expressed as the union of two disjoint non-empty subsets.

That is, $X = \{a, b\} \cup \{c\}$

where $\{a, b\}$ is τ_1 - semi open and $\{c\}$ is τ_2 - semi open

Therefore, (X, τ_1, τ_2) is not pairwise semi connected.

THEOREM : 2.1.14

Let A be a pairwise semi-connected subset of a bitopological space (X, τ_1, τ_2). If $A \subset C \cup D$; where C and D are pairwise semi -separated sets in (X, τ_1, τ_2) then either $A \subset C$ or $A \subset D$.

PROOF :

Let A be a pairwise semi-connected subset of a bitopological space(X, τ_1, τ_2)

TO PROVE :

If $A \subset C \cup D$, where C and D are pairwise semi separated sets in (X, τ_1, τ_2), then either $A \subset C$ or $A \subset D$. Since C and D form a pairwise semi-separated sets in (X, τ_1, τ_2), we have

$$C \cap \tau_1\text{-scl}(D) = \phi = \tau_2\text{-scl}(C) \cap D \qquad -(1)$$

Let $V = C \cap A$ and $U = D \cap A$

Since $A \subset C \cup D$, $V \cup U = (C \cap A) \cup (D \cap A)$
$= (C \cup D) \cap A$
$= A$

Claim :

Atleast one of U and V is non-empty. Suppose none of U and V is empty.
That is, $V \neq \phi$ then

$$\begin{aligned} V \cap \tau_1\text{-scl}(V) &= (C \cap A) \cap (\tau_1\text{-scl}(D \cap A)) \\ &\subset (C \cap A) \cap \tau_1\text{-scl}(D) \\ &= (C \cap \tau_1\text{-scl}(D)) \cap A \\ &= \phi \cap A \\ &= \phi \qquad \text{by (1)} \end{aligned}$$

$$\begin{aligned} \tau_2\text{-scl}(V) \cap U &= [\tau_2\text{-scl}(C \cap A)] \cap (D \cap A) \\ &\subset \tau_2\text{-scl}(C) \cap (D \cap A) \\ &= \phi \cap A \\ &= \phi \end{aligned}$$

Hence U and V are pairwise semi separated sets. That is, A can be expressed as the union of two pairwise semi-separated sets.

Therefore, A is not pairwise semi connected which is a contradiction.

Therefore, one of U and V must be empty.

If $V = \phi$ then $C \cap A = \phi$

$$A = V \cup U = (C \cap A) \cup (D \cap A)$$
$$= \phi \cup (D \cap A)$$
$$= D \cap A$$

which implies $A \subset D$

If $U = \phi$ then $D \cap A = \phi$

$$A = V \cup U = (C \cap A) \cup (D \cap A)$$
$$= (C \cap A) \cup \phi$$
$$= C \cap A$$

which implies $A \subset C$.

THEOREM : 2.1.15

The Union E of any family $\{C_\lambda : \lambda \in \Delta\}$ of pairwise semi-connected sets in (X, τ_1, τ_2) having a non-empty intersection is pairwise semi connected.

PROOF :

Let $\{C_\lambda : \lambda \in \Delta\}$ be a family of pairwise semi-connected sets and $\bigcap_{\lambda \in \Delta} C_\lambda \neq \phi$

Claim :

$E = \cup C_\lambda$ is pairwise semi-connected.

Suppose $E = \cup \{C_\lambda\}$ is not pairwise semi - connected then we can write

$\cup_{\lambda \in \Delta} = E = C \cup D$, where C and D are pairwise semi-separated sets.

Therefore, $C \cap \tau_1$ - scl $(D) = \phi = \tau_2$-scl $(C) \cap D$

Let $V = C \cap C_\lambda$; $V = D \cap C_\lambda$, then

$$V \cup U = (C \cap C_\lambda) \cup (D \cap C_\lambda)$$
$$= (C \cup D) \cap C_\lambda = C_\lambda$$

Since C_λ is pairwise semi-connected subset of a bitopological space (X, τ_1, τ_2) as in the proof of last Theorem we can prove that either V or U must be empty if

$V = \phi$ then $C_\lambda = U = D \cap C_\lambda$.

implies $C_\lambda \subset D$.

if $U = \phi$ then $C_\lambda = V = C \cap C_\lambda$.

implies $C_\lambda \subset D$

Therefore, either all C_λ's must be in C or all must be in D. But $\cap C_\lambda \neq \phi$ and $C \cap D = \phi$ which says that $C = \phi$.or $D = \phi$. This is a contradiction to the fact that C and D forming a pairwise semiseparation of E.

Therefore, E is pairwise semi-connected.

THEOREM : 2.1.16

Let (X, τ_1, τ_2) be a pairwise locally semi-connected space. Then it is pairwise semi connected if and only if it is pairwise connected.

PROOF :

Given (X, τ_1, τ_2) be a pairwise locally semi-connected space.

To Prove :

If (X, τ_1, τ_2) is pairwise connected, then it is pairwise semi connected.

Suppose, X is not pairwise semi-connected then X is pairwise semi separated. Then there exists a non-empty proper subset G of X such that $X = G \cup (X - G)$, where G is τ_1 - semi open and τ_2 - semi closed.

Let $x \in X$ then, either $x \in G$ or $x \in X - G$.
If $x \in G$, then there exists a pairwise semi-connected τ_1 - open set O_x such that $x \in O_x \subset G$. If $x \in X - G$., then there exists a pairwise semi-connected.
τ_2 - open set P_x. such that $x \in P_x \subset X - G$.
Let $U = \cup \{O_x : x \in G, O_x \subset G\}$ and
$V = \cup \{P_x : x \in X - G, P_x \subset X - G\}$
Since $G \neq \phi$ and $G \neq x$; $U \neq \phi$ and $V \neq \phi$
Also, $U \cap V = \phi$ and $X = U \cup V$

Which implies that X is not pairwise connected. Therefore, our assumption is wrong and X is pairwise semi-connected.

Trivially, pairwise semi-connectedness implies pairwise connectedness.

THEOREM : 2.1.17

If a subset A of (X, τ_1, τ_2) is pairwise semi-connected, then $(A, \tau_{1A}, \tau_{2A})$ is pairwise semi-connected.

PROOF :

Suppose $(A, \tau_{1A}, \tau_{2A})$ is not pairwise semi-connected, that is, if U and V forms a pairwise semi-separation of A in $(A, \tau_{1A}, \tau_{2A})$, $U \cap \tau_{1A}$ - scl $(V) = \phi = V \cap \tau_2$ - scl (V)

We know that τ_1 - scl $(B) \cap A \subset \tau_{1A}$ - scl (B) for every $B \subset A$.

Therefore, $U \cap \tau_1$ - scl $(V) \cap A \subset U \cap \tau_{1A}$ - scl $(A) = \phi$

Similarly, $V \cap \tau_2$ - scl $(U) \cap A \subset V \cap \tau_2$ - scl $(U) = \phi$
Hence $U \cap \tau_1$ - scl $(V) = \phi = V \cap \tau_2$ - scl (U).
Which shows that U and V form a pairwise semi separation of A in (X, τ_1, τ_2).
Which implies that A is not pairwise semi-connected in (X, τ_1, τ_2). Which is a contradiction.

Therefore, our assumption is wrong.

Hence $(A, \tau_{1A}, \tau_{2A})$ is pairwise connected.

REMARK :

$(A, \tau_{1A}, \tau_{2A})$ is pairwise semi-connected space, but $A \subset X$ is not pairwise semi-connected in X can be seen from the following example.

EXAMPLE : 2.1.18

Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a, b\}, \{c\}, \{a, b, c\}\}$

and $\tau_2 = \{X, \phi, \{b, d\}, \{a, c, d\}, \{d\}$.

$X - \tau_1 = \{X, \phi, \{c, d, \}, \{a, b, d\}, \{d\}\}$

$X - \tau_2 = \{X, \phi, \{a, c\}, \{b\}, \{a, b, c\}\}$

Let $A = \{a, b, c\}$, $\tau_{1A} = \{A, \phi, \{a, b\}, \{c\}\}$ and

$\tau_{2A} = \{A, \phi, \{a, c\}, \{b\}\}$

$A - \tau_{1A} = \{A, \phi, \{c\}, \{a, b\}\}$

$A - \tau_{2A} = \{A, \phi, \{a, c\}, \{b\}$

τ_{1A} - semi open sets $= \{A, \phi, \{a, b\}, \{c\}\}$

τ_{2A} - semi open sets $= \{A, \phi, \{b\}, \{a, c\}\}$

$(A, \tau_{1A}, \tau_{2A})$ cannot be expressed as the union of two disjoint non-empty sets one of which is τ_{1A} semi open and the other is τ_{2A} - semi open. Therefore, $(A, \tau_{1A}, \tau_{2A})$ is pairwise semi-connected.

τ_2 - semi open sets $= \{X, \phi, \{b, d\}, \{a, c, d\}, \{d\}, \{a, b, d\}, \{c, d\}, \{c, b, d\}, \{a, d\}\}$.

τ_2 - semi closed sets $= \{X, \phi, \{a, b\}, \{c\}, \{a, b, c\}, \{a, b, d\} \{c, d\}\}$.

τ_1 - semi closed sets $= \{X, \phi, \{c, d\}, \{a, b, d\}, \{d\}, \{c\}, \{a, b\}\}$

$A \subset X$ is not pairwise semi-connected in X because $A = \{a, b\} \cup \{c\}$ such that
$\{a, b\} \cap \tau_1$ - $scl\,\{c\} = \{a, b\} \cap \{c\} = \phi$ and
$\{c\} \cap \tau_2$ - $scl\,\{a, b\} = \{c\} \cap \{a, b\} = \phi$

DEFINITION : 2.1.19

Let (X, τ_1, τ_2) be a bitopological space and $x \in X$, the semi-component of x, denoted by S.C. (x) is the union of all pairwise semi-connected subsets of X containing x.

DEFINITION : 2.1.20

If $E \subset X$ and $x \in E$, then the union of all pairwise semi-connected sets containing x and contained in E is called a semi-component of E.

REMARK :

A component is not necessarily a semi-component as can be seen from the following example.

EXAMPLE : 2.1.21

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{c\}, \{b, c\}\}$ and $\tau_2 = \{X, \phi, \{b\}\}$

$X - \tau_1 = \{X, \phi, \{a,b\}, \{a\}\}$

$X - \tau_2 = \{X, \phi, \{a, c\}\}$

τ_1 - semi open sets = $\{X, \phi, \{c\}, \{b, c\}, \{a, c\}\}$

τ_2 - semi open sets = $\{X, \phi, \{b\}, \{a, b\}, \{b, c\}\}$

X cannot be expressed as the union of two disjoint opensets one of which is τ_1 - open and the other is τ_2 - open set. Therefore, (X, τ_1, τ_2) is pairwise connected. Therefore, X is a component.

$X = \{c\} \cup \{a, b\}$, where $\{c\}$ is τ_1 - semi open set and $\{a, b\}$ is τ_2 - semi open set. that is, X can be expressed as the union of two disjoint non-empty sets one of which is τ_1 - semiopen and the other is τ_2 - semi open set.

Therefore, (X, τ_1, τ_2) is not pairwise semi connected. Hence X is not a semi component.

THEOREM : 2.1.22

In a bitopological space (X, τ_1, τ_2)

(i) each semi component S.C (x) is a maximal pairwise semi connected set in X.

(ii) the set of all distinct semi - component of points of X form a partition of X.

(iii) each S.C (x) satisfies the equation S.C (x) = τ_1 - scl [S.C (x)] $\cap \tau_2$ - scl [S.C.(x)].

PROOF :

(i) This follows from Definition (2.1.19) and Theorem (2.1.15).

(ii) Let x and y be two distinct points of X semi component of x and y are denoted by S.C (x) and S.C (y) respectively.

If S.C (x) $\cap$ S.C (y) $\neq \phi$ by Theorem (2.1.15), S.C. (x) $\cup$ S.C. (y) is a pairwise semi - connected sets.

But S.C (x) $\subset$ S.C (x) $\cup$ S. C (y).
Therefore, S.C (x) $\cap$ S. C (y) = ϕ.
Which implies that S.C. (x)= τ_1-scl (S.C (x) $\cap$ τ_2 - scl (S.C (X))

Therefore, the set of all distinct semi components of X form a partition of X.

(iii) Let x be a point in X and let S.C (x) be its semi-component suppose $p \in X$ and $p \notin$ S.C (x). Then S.C (x) $\cup$ {p} is not pairwise semi-connected and hence there exists a pairwise semi separation $A|B$ in X such that S.C (x) $\cup$ {p} = $A|B$.

By Theorem (2.1.14), S.C (x) $\subset$A and {p} $\subset$B or S.C (x) $\subset$B and {p} $\subset$A

Thus S.C (x) $\cup$ {p} = S.C (x) | P or S.C (x) $\cup$ {p} = {p} | S.C (x)
Hence p $\notin \tau_1$ - scl [S.C (x)] or p $\notin \tau_2$ - scl [S.C (x)]
This implies p $\notin \tau_1$ - scl [S.C (x)] $\cap$ τ_2 - scl - [(S.C (x)]
Hence, τ_1 - scl [S.C (x)]$\cap$ τ_2 - scl [S.C. (x)] $\subset$ [(S.C. (x)]
But, S.C (x) $\subset\tau_1$ - scl (S.C. (x)) $\cap$ τ_2 - scl [S.C (x)] always
Therefore, S.C (x) = τ_1 - scl [S.C (x) $\cap$ τ_2 - scl [S.C (x)]
Hence the proof.

THEOREM : 2.1.22

Let (X, τ_1, τ_2) be a bitopological space then (X, τ_1, τ_2) is τ_1 - locally semi connected with respect to τ_2 if and only if each semi component of every τ_1 - semi open set is τ_1 - open and either τ_1 - semi closed or τ_2 - semi closed.

PROOF :

Necessity

Let C be a semi component of a τ_1 - semi open set G. If x $\in$ C then x $\in$C$\subset$G and so there exists a pairwise semi connected τ_1 - open set U such that x$\in$U$\subset$G. But C is a maximal pairwise semi connected set contained in G. Hence U$\subset$C and so x$\in$U$\subset$C$\subset$G. Thus C is τ_1 - open.

Now, let p $\notin$C. Then C$\cup$ {p} is not pairwise semi connected, which implies {p} is either τ_1 - semi closed or τ_2 - semi closed.

Sufficiency :

Let G be a τ_1 - semi open set containing a point x. Then by hypothesis, there exists a semi - component C of G such that C is τ_1 - open and τ_2 - semi closed or τ_2 - semi closed. Also X $\in$C$\subset$G. Thus (X, τ_1, τ_2) is τ_1 - locally semi connected with respect to τ_2.

THEOREM : 2.1.23

Let (X, τ_1, τ_2) be a bitopological space. Then (X, τ_1, τ_2) is τ_1 - locally semi - connected with respect to τ_2 if and only if given any point x$\in$X and a τ_1 - semi open set U containing x, there is a τ_1 - open set C containing x and contained in a semi component of U.

PROOF :

Necessity

Let A be the semi - component of U containing x • since x $\in$U, there exists a pairwise semi-connected τ_1 - open set C such that x$\in$C$\subset$U. Since A is a maximal pairwise semi-connected set containing x and contained in U, therefore x $\in$C$\subset$A$\subset$U. But distinct semi-components are disjoint. Thus C is not contained in any other semi component of U.

Sufficiency :

Let A be the semi component of U such that $x \in C \subset A \subset U$. Let $y \in A$. Then $y \in U$ and by hypothesis, there is a τ_1 - open set O containing y and contained in A. Thus $y \in O \subset A$. Hence A is τ_1 - open. Therefore, each $x \in X$ and each τ_1 - semi open set containing x, there is a τ_1 - open semi component A which is pairwise semi connected such that $x \in A \subset U$. Thus (X, τ_1, τ_2) is τ_1 - locally semi connected with respect to τ_2.

DEFINITION : 2.1.24

A function $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is said to be pairwise P if, $f : (X, \tau_1) \to (Y, \tau_1^*)$ and $f: (X, \tau_2) \to (Y, \tau_2^*)$ have the property P, where P is a topological property.

THEOREM : 2.1.25

Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a pairwise semi continuous mapping from X on to Y. Then (Y, τ_1^*, τ_2^*) is pairwise connected if (X, τ_1, τ_2) is pairwise semi connected.

PROOF :

Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a pairwise semi continuous mapping for X on to Y. It is given that (X, τ_1, τ_2) is pairwise semi-connected.

To prove (Y, τ_1^*, τ_2^*) is pairwise connected.

Suppose (Y, τ_1^*, τ_2^*) is not pairwise semi connected. Then there exists a pairwise separation $A | B$ of Y, where A and B are non-empty subjects of Y such that $Y = A \cup B$, $A \in \tau_1^*$ and $B \in \tau_2^*$. By pairwise semi continuity $f^{-1}[A]$ is τ_1 - semi open and $f^{-1}[B\}$ is τ_2 - semi open. Also $X = f^{-1}[A] \cup f^{-1}[B]$.

Thus X is not pairwise semi-connected which is a contradiction. Therefore our assumption is wrong. (Y, τ_1^*, τ_2^*) is pairwise connected.

Similarly we can prove the following theorem.

THEOREM : 2.1.26

The pairwise irresolute image of a pairwise semi-connected space is pairwise semi-connected.

DEFINITION : 2.1.27 [17]

(X, τ_1, τ_2) is pairwise totally disconnected if for each pair of points of X can be separated by a pairwise separation of X, that is given two distinct points x and y of X there is a pairwise separation $X = A | B$ such that $x \in A$, $y \in B$.

DEFINITION : 2.1.28

The space $(X,\tau_1, \tau_2)$is pairwise weakly totally connected if for each pair of points x, y $\in$X, there exists a pairwise separation $X = A|B$ such that either x$\in$A and y$\in$B or Y$\in$A and x$\in$B.

DEFINITION : 2.1.29

A space (X,τ_1, τ_2) is pairwise totally semi-connected if each pair of points of X can be separated by a pairwise semi-separation of X, that is given two distinct point x and y of X, there exists a pairwise semi-separation $X = A|B$ such that x$\in$A, y$\in$B.

DEFINITION : 2.1.30

A space (X,τ_1, τ_2) is pairwise weakly totally semi-disconnected if for each pair of points x and y of X, there exists a pairwise semi-separation $X = A|B$. Such that either x$\in$A and y$\in$B or y$\in$A and x$\in$B.

THEOREM : 2.1.31

Every pairwise weakly totally disconnected space is pairwise weakly totally semi disconnected.

PROOF :

Since any pairwise sets in a bitopological space are pairwise semi-separated. We obtain our result.

REMARK :

The following example shows that the converse of above theorem is not necessarily true.

EXAMPLE : 2.1.32

Let X = {a, b, c}, τ_1 = {X, ϕ, {a}} and τ_2 = {X, ϕ, {c}}, {b, c}}. Then (X, τ_1, τ_2) is pairwise weakly totally semi - disconnected but not pairwise weakly totally disconnected.

THEOREM : 2.1.33

Every pairwise totally disconnected space is pairwise totally semi disconnected.

DEFINITION : 2.1.34

A space (X,τ_1, τ_2) is said to be pairwise weakly semi - τ_2 if for every pair of distinct points of X, atleast one belongs to a τ_1 - semi open set U and the other belong to τ_2 - semi open set V satisfying $U \cap V = \phi$.

If the roles of the points are interchangeable, then (X,τ_1, τ_2) is a pairwise semi τ_2 - space.

THEOREM : 2.1.35

Every pairwise (weakly) totally semi disconnected space is pairwise (weakly) semi τ_2

THEOREM : 2.1.36

The bi-preopen pairwise semi-connected subsets of a pairwise weakly totally semi disconnected space are its point.

PROOF :

Let (X,τ_1, τ_2) be a pairwise weakly totally semi-disconnected space and let Y be a bi-preopen subsets of X which contains more than one point.

Let x, y $\in$ X, with $x \neq y$. Let $A|B$ be a pairwise semi-separation of X such that $x \in A$ and $y \in B$. Then $Y = (Y \cap A)|(Y \cap B)$ is a pairwise semi separation of Y in X.

Thus Y is not pairwise semi connected, which is a contradiction. Therefore, our assumption is wrong and Y contains one point.

THEOREM : 2.1.37

Let (X,τ_1, τ_2) be a pairwise weakly Hausdorff space . If τ_1 - has a base whose members are also τ_2 - semi closed or τ_2 has a base whose members are also τ_1 - semi closed, then the space X is pairwise weakly totally semi-disconnected.

PROOF :

Suppose τ_1 has a base whose sets are also τ_2 - semi closed.

Consider, two distinct points x and y of X since the space is pairwise weakly Hausdorff, one of the point (say x) has τ_1 - open neighbourhood G which does not contain the other point (say y) and y has τ_2 - open neighbourhood H which does not contain x. Then there exists a τ_1 - basic open set B such that B is also τ_2 - semi closed and $x \in B \subset G$. Then $X = B|(X-B)$ is pairwise semi separation of X such that $x \in B$ and $y \in X - B$.

Similarly, for the other case, we obtain a pairwise semi separation $X = A|X-A$ with $x \in X - A$, $y \in A$. Thus (X,τ_1, τ_2) is pairwise weakly totally semi disconnected.

RESULT : 2.1.38

If (X,τ_1, τ_2) is a pairwise Hausdorff space such that τ_1 has a base whose members are also τ_2 - semiclosed and τ_2 - has a base whose members are also τ_1 - sdmi closed, then (X,τ_1, τ_2) is pairwise totally semi disconnected.

SECTION : II

A.S. Mashhour et al [14] introduced preopen sets, precontinuous and a preopen mappings in a single topological space and obtained a number of their properties. In this section these notions of Mashhour et al are generalized in bitopological spaces.

DEFINITION : 2.2.1

In (X, τ_1, τ_2), $A \subset X$ is said to be (i - j) - preopen (briefly (i-j) - p.o) iff

$A \subset \tau_i$ - Int (τ_j - Cl A), i, j = 1, 2 ; i $\neq$ j.

DEFINITION : 2.2.2

In (X, τ_1, τ_2), $A \subset X$ is called bitopological preopen (briefly b.p.o) iff A is (i-j) - preopen. i, j = 1,2 ; i $\neq$ j.

REMARK :

Every τ_i - open set is (i-j) - preopen. i, j = 1, 2 ; i $\neq$ j.

SOME INTERESTING EXAMPLES

EXAMPLE : 2.2.3

Consider (X, τ_1, τ_2), where X = {a, b, c, d},
τ_1 = {X, ϕ, {a}, {b, c}, {a, b, c}} and
τ_2 = { X, ϕ, {a, d},{b, c}}

To prove

{c} $\subset$ X is (i-j) - preopen i, j = 1, 2 ; i $\neq$ j
That is, to prove
{c} $\subset \tau_i$ - Int (τ_j - Cl {c}). for i, j = 1, 2 ; i $\neq$ j.
τ_2 - closed sets = {X, ϕ, {a, d}, {b, c}}
τ_2 - Cl {c} = X $\cap$ {b, c} = {b, c}
τ_1 - Int (τ_2 - Cl {c}) = τ_1 -Int {b, c} = {b, c}
and {c} $\subset$ {b, c}
{c} $\subset \tau_1$ - Int (τ_2 - Cl {c})
Therefore, {c} is (1-2)- preopen.

To prove

{c} $\subset$ X is (2-1) - preopen.
τ_1 - closed sets = { X, ϕ {b, c, d}, {a, d}, {d}}
τ_1 - Cl {c} = X $\cap$ {b, c, d}
= {b, c, d}.
and {c} $\subset$ {b, c, d}
{c} $\subset \tau_2$ - Int (τ_1 - Cl {c})

Therefore, {c} is (2-1) - preopen. That is,

{c} $\subset$ X is (i-j) - preopen. i, j = 1,2 ; $i \neq j$

Hence {c} $\subset$ X is bitopologically preopen.

But {c} is not τ_1 - open in X. i = 1, 2 .

Hence {c} $\subset$ X is (i-j) preopen i, j = 1,2 ; $i \neq j$
but {c} is not τ_i - open in X. i = 1, 2.

REMARK :

Preopeness in bitopological space is not equivalent to preopeness in the individual topologies can be seen from the following two examples.

EXAMPLE : 2.2.4

Consider (X, τ_1, τ_2) where X = {a, b, c, d}

τ_1 = {X, ϕ , {b}, {b, c} {a, b, c}} and

τ_2 = {X, ϕ,{d}, {a, d}, {c, d}, {a, c, d}}

X - τ_1 = {X, ϕ, {a, c, d}, {c, d}, {a, d}, {d}}

X - τ_2 = {X, ϕ, {a, b, c}, {b, c}, {a, b}, {b}}

To prove

{c} $\subset$ X is (i - j) - preopen. i, j = 1, 2 ; $i \neq j$

That is to show

{c} $\subset \tau_1$ - Int (τ_2 - Cl {c})

τ_2 - Cl {c} = X $\cap$ {a, b, c} $\cap$ {b, c}

= {b, c}

τ_1 - Int (τ_2 - Cl {c}) =τ_1 - Int {b, c}

= {b, c} and

{c} $\subset$ {b, c}

Therefore {c} $\subset \tau_1$ - Int (τ_2 - C l {c})

Therefore, {c} is (1-2) - preopen.

To prove

{c} is (2-1) - preopen.

τ_1 - C l {c} = X $\cap$ {a, c, d} $\cap${c, d}

= {c, d}

τ_2 - Int (τ_1 - Cl {c}) = τ_2 -Int {c, d}

= {c, d}

and {c} $\subset$ {c, d}

{c} $\subset \tau_2$ - Int (τ_1 - Cl {c})

Therefore {c} is - (2-1) - preopen.

Hence {c} is bitopologically preopen.

But {c} is not τ_i - preopen. i = 1, 2., because

{c} $\not\subset$ τ_i - Int (τ_i - C l {c}) for i = 1,2.

EXAMPLE : 2.2.5

Consider (X, τ_1, τ_2) where X = {a, b, c, d}

τ_1 = { X, ϕ, {a, d}, {b, c}} and

τ_2 = { X, ϕ, {b, d}, {a, c}}

X - τ_1 = {X, ϕ, {b, c}, {a, d}}

X - τ_2 = {X, ϕ, {a, c}, {b, d}}

Since {b} $\subset$ τ_i - Int (τ_i - Cl {b}) for i = 1, 2

We have {b} $\subset$ X is τ_i - preopen, i = 1, 2.

But {b} $\subset$ X is not (i-j) - preopen because

{b} $\not\subset$ τ_i - Int (τ_j - Cl {b}), for i = 1, 2 ; i $\neq$ j.

and hence it is not bitopologically preopen.

The characterization of (i-j) - preopen sets can be seen from the following theorem.

THEOREM : 2.2.6

In (X, τ_1, τ_2), A$\subset$X is (i-j) - preopen iff for every τ_j - closed set G containing A,

A $\subset$ τ_i -Int G, i, j = 1, 2 ; i $\neq$ j.

PROOF :

Assume A$\subset$X is (i-j) - preopen

To prove

For every τ_j - closed set G containing A, A$\subset$ τ_i - Int G, i, j = 1, 2 ; i $\neq$ j.

Let G be any τ_j - closed set containing A. Then τ_j - Cl A$\subset$G. Since A is (i-j) - preopen we have A$\subset$ τ_i - Int (τ_j - Cl A) for i, j = 1, 2; i $\neq$ j

But τ_j - Cl A$\subset$G

Therefore A$\subset$ τ_i - Int G.

Conversely

Let A$\subset$X be such that for every τ_j - closed set G containing A, A$\subset$ τ_i - Int G. Since τ_j - Cl A is a τ_j - closed set G containing A.

A $\subset$ τ_i - Int (τ_j - Cl A)

Therefore, A is (i-j) - preopen for i, j = 1, 2; i $\neq$ j.

DEFINITION : 2.2.7

In (X, τ_1, τ_2), $A \subset X$ is said to be (i-j) - preclosed iff (X-A) is (i-j) - pre open in X, $i, j = 1, 2 ; i \neq j$.

SOME IMPORTANT PROPERTIES OF PREOPEN SETS

THEOREM : 2.2.8

In (X, τ_1, τ_2), if A is (i-j) preopen and B is biopen then $A \cap B$ is (i-j) - preopen $i, j = 1, 2 ; i \neq j$.

PROOF :

Given A is (i-j) - preopen and B is biopen.

To prove

$A \cap B$ is (i-j) - preopen. $i, j = 1, 2; i \neq j$.

Since A is (i-j) - preopen we have
$A \subset \tau_i$ - Int (τ_j Cl A), for $i, j = 1,2 ; i \neq j$.
Since B is biopen we have
$B = \tau_i$ - Int B.
For $i, j = 1, 2; i \neq j$, we have

$$A \cap B \subset \tau_i\text{-Int}(\tau_j\text{-Cl } A) \cap \tau_i\text{-Int } B$$
$$= \tau_i\text{-Int}(\tau_j\text{-Cl } A \cap B)$$
$$\subset \tau_i\text{-Int}(\tau_j\text{-Cl}(A \cap B))$$

Therefore $A \cap B$ is (i-j) - preopen, for $i, j = 1, 2; i \neq j$.

THEOREM : 2.2.9

If $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of (i-j) preopen sets in (X, τ_1, τ_2), then $\cup_{\alpha \in \Lambda} A_\alpha$ is (i-j) - preopen $i, j = 1,2 ; i \neq j$.

PROOF :

Given $\{A_\alpha\}_{\alpha \in \Lambda}$ is a family of (i-j) - preopen sets.

To prove

$$\cup_{\alpha \in \Lambda} A_\alpha \subset \tau_i\text{-Int}[\tau_j\text{-Cl}(\cup_{\alpha \in \Lambda} A_\alpha)]$$

The proof follows from the following two facts.

(i) $\cup_{\alpha \in \Lambda} \tau_i$ - Int $A_\alpha \subset \tau_i$ - Int $(\cup_{\alpha \in \Lambda} A_\alpha)$

(ii) $\cup_{\alpha \in \Lambda} \tau_j$ - Cl $A_\alpha \subset \tau_j$ - Cl $(\cup_{\alpha \in \Lambda} A_\alpha)$

Now, for, $i, j = 1, 2$ ' $i \neq j$

$$\cup_{\alpha \in \Lambda} A_\alpha \subset \cup_{\alpha \in \Lambda}(\tau_i\text{-int}(\tau_j\text{-Cl } A_\alpha))$$
$$\subset \tau_i\text{-Int}[\cup_{\alpha \in \Lambda} \tau_j\text{-Cl } A_\alpha)] \qquad \text{by (i)}$$

$\subset \tau_i$ - Int (τ_j - Cl ($\cup_{\alpha\in\Lambda} A_\alpha$)] by (ii)

==> $\cup_{\alpha\in\Lambda} A_\alpha$ is (i - j) - preopen.

THEOREM : 2.2.10

If $A \subset Y \subset X$ in (X, τ_1, τ_2) and A is (i-j) - preopen in X, then A is so in $(Y, \tau_{1y}, \tau_{2y})$ i, j = 1, 2 ; $i \neq j$.

PROOF :

Given $A \subset Y \subset X$ in (X, τ_1, τ_2) and A is (i-j) - preopen in X.

To prove

A is (i-j) - preopen in $(Y, \tau_{1y}, \tau_{2y})$ i, j = 1,2 ; $i \neq j$.

Since A is (i - j) - preopen in X. we have

$A \subset \tau_i$ - Int (τ_j - Cl A), for i, j = 1, 2 ; $i \neq j$.

==> $A \cap Y \subset (\tau_i$ - Int(τ_j - Cl A)) $\cap Y$. Since $A \subset Y$.

Also, (τ_i-Int (τ_j - Cl A)) $\cap Y$ is τ_i - open in Y.

Hence τ_{iy} - Int (τ_{jy} - Cl A) = τ_{iy} - Int [τ_j - Cl A)$\cap$Y]

$\supset \tau_{iy}$ - Int [τ_i - Int (τ_j-Cl A)$\cap$Y]

$\supset \tau_i$ - Int (τ_j - Cl A) $\cap$ Y

$\supset$ A

Therefore A is (i-j) preopen in Y.

REMARK :

In the above theorem the (i-j) - preopeness of A is only the sufficient condition but not necessary. Not even it is necessary if Y is restricted to (i-j) - preopen in X. This is clear from the following example.

EXAMPLE : 2.2.11.

Let (X, τ_1, τ_2) where X = {a, b, c, d}

τ_1 = {X, ϕ, {a} {b,c}, {a, b, c}

τ_2 = {X, ϕ, {b} {a, c, d}

Y = {a, b, d}

τ_{1Y} = {Y, ϕ, {a}, {b}, {a,b}}

τ_{2Y} = {Y, ϕ, {b}, {a, d}}

Y- τ_{1Y} = {Y, ϕ, {b, d} {a, d}, {d}}

Y - τ_{2Y} = {Y, ϕ, {a, d}, {b}}

If A = {b} then

{b} $\subset \tau_{iy}$ - Int (τ_{jy} - Cl {b})

Therefore A is (i - j) - preopen in Y., i, j = 1, 2 ; $i \neq j$.

$Y = \{a, b, d\}$

$Y \subset \tau_i$ - Int (τ_j - Cl Y), for i, j = 1, 2 ; i - j.

Therefore, Y IS (i - j) - preopen in X, i,j = 1, 2 ; $i \neq j$.

But τ_1 - Int (τ_2 - Cl A) = ϕ

Therefore A is not (1 - 2) - preopen in X.

THEOREM : 2.2.12

If $A \subset X$ in (X, τ_1, τ_2) and Y is τ_i- open, then A is (i - j) - preopeness in $(Y, \tau_{1y}, \tau_{2y})$ iff A is so in X, i, j = 1, 2 ; $i \neq j$.

PROOF :

Given $A \subset Y \subset X$ in (X, τ_1, τ_2) and Y is τ_i - open. Assume A is (i-j) - preopen in Y.

then, $A \subset \tau_{iy}$ - Int [τ_{jy} - Cl A] = τ_{iy} - Int [(τ_j - Cl A)$\cap$Y]

$A \subset \tau_{iy}$ - Int (τ_{jy} - Cl A) = τ_{iy} - Int (τ_j - Cl A) $\cap$(τ_i-Int Y) Since Y is τ_i - open.

= τ_i - Int (τ_j - Cl A)

=> A is (i-j) - preopen in X, for i,j = 1, 2 ; $i \neq j$

Converse part of the theorem follows from the Theorem 2.2.10.

THEOREM : 2.2.13

In (X, τ_1, τ_2) the following are true.

(1) A, $B \subset X$, $A \subset B$ implies $A \subset \tau_i$ - Int (τ_j - Cl B) if A is (i - j) - preopen i, j = 1, 2 ; $i \neq j$.

(2) If A is (i-j) preopen and $B \subset X$ is such that $A \subset B \subset \tau_i$ - Int (τ_j-Cl A), i, j = 1, 2 ; $i \neq j$.

(3) $A \subset X$ is (i - j) - preclosed iff τ_i - Cl (τ_j - Int A) $\subset A$, i, j = 1, 2 ; $i \neq j$.

(4) $A \subset X$ is (i-j - preclosed iff for every τ_j - open set G contained in A,τ_i - Cl $G \subset A$, i, j = 1, 2 ; $i \neq j$.

(5) If A is (i-j) - preclosed and $B \subset X$ is such that τ_i - Cl (τ_j - Int A) $\subset B \subset A$, then B is (i-j) - preclosed i, j = 1, 2 ; $i \neq j$.

PROOF :

(1) Since A is (i-j) - preopen.

$A \subset \tau_i$ - Int (τ_j-Cl A)

$\subset \tau_i$ - Int (τ_j-- Cl B) since $A \subset B$.

(2) Since A is (i-j) - preopen.

$A \subset \tau_i$ - Int (τ_j - Cl A)

Also $A \subset B \subset \tau_i$ - Int (τ_j - Cl A) is given

$B \subset \tau_i$ - Int (τ_j - Cl B)

Therefore B is (i-j) - preopen for i, j = 1, 2; $i \neq j$.

(3) A is (i-j) - pre open iff $(X-A)$ is (i-j) preclosed
iff $(X-A) \subset \tau_i$ - Int (τ_j - Cl $(X-A)$)
iff $(X-A) \subset \tau_i$ - Int ($X-\tau_j$ - Int A)
iff $A \supset X - \tau_i$ - Int ($X - \tau_j$ - Int A))
iff $A \supset \tau_i$ - Cl ($X -(X- \tau_j$- Int A))
iff $A \supset \tau_i$ - Cl (τ_j - Int A)

(4) and (5) can be proved similar to the Theorem 2.2.6 and second result of this Theorem respectively.

DEFINITION : 2.2.14

In (X, τ_1, τ_2), $A \subset X$ is said to be (i-j) - α -open iff $A \subset \tau_i$ - Int (τ_j - Cl (τ_i - Int A)), i, j = 1, 2; $i \neq j$.

DEFINITION : 2.2.15 [3]

In (X, τ_1, τ_2), $A \subset X$ is termed (i-j) - semi open (briefly (i-j) - s.o) iff $O \subset A \subset \tau_j$ - Cl O for some τ_i - open set O, i, j = 1, 2 ; $i \neq j$.

THEOREM : 2.2.16 [3]

In (X, τ_1, τ_2), $A \subset X$ is said to be (i-j) - semi open iff τ_j - Cl A = τ_j -Cl(τ_i - Int A), i, j = 1, 2 $i \neq j$.

THEOREM : 2.2.17

In (X, τ_1, τ_2), $A \subset X$ is (i-j) - α - open iff A is both (i-j) - semi open and (i-j) - preopen in X, i, j = 1,2; $i \neq j$.

PROOF :

Let A be (i - j) - α - open in X

To prove

A is (i - j) - semi open and A is (i-j) - preopen in X. Since A is (i - j) - α -open, we have,

$A \subset \tau_i$ - Int (τ_j - Cl (τ_i - Int A))
Hence $A \subset \tau_j$ - Cl (τ_i - Int A))
So that τ_j - Cl A = τ_j - Cl (τ_i -Int A)
Therefore, A is (i-j) - semi open in X.
Also, A is (i-j) - α - openess of A gives $A \subset \tau_i$ -Int (τ_j - Cl A)
Therefore, A is (i-j) - preopen in X.

Conversely

Let A be both (i-j) - preopen and (i-j) - semi open in X. Then
$A \subset \tau_i$ - Int (τ_j - Cl A) and τ_j - Cl A $=\tau_j$ - Cl (τ_i - Int A).
Hence $A \subset \tau_i$ - Int (τ_j - Cl A) $= \tau_i$ - Int (τ_j - Cl (τ_i - Int A))
So A is (i-j) - α - open.

DEFINITION : 2.2.18

A mapping f : $(X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is termed (i-j) precontinuous (briefly (i-j) - p.c) iff for $O^* \in \tau_i^*$, $f^{-1}[O^*]$ is (i-j) - preopen in X. i, j = 1, 2 ; $i \neq j$.

DEFINITION : 2.2.19

A mapping f : $(X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is bitopologically precontinuous (briefly b.p.c) iff f is (i-j) precontinuous. i, j = 1, 2 ; $i \neq j$.

REMARK :

If f : $(X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is continuous, then. f is obviously bitopologically precontinuous. But the converse is not always true, which can be seen from the following example.

EXAMPLE : 2.2.20

Let f : $(X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ be the identity mapping.
Let $X = X^* = \{a, b, c, d\}$
$\tau_1 = \{X, \phi, \{a\} \{b, c\}, \{a, b, c\}\}$
$\tau_2 = \{X, \phi, \{a, d\}$
$\tau_1^* = \{ X^*, \phi, \{a\}, \{b\}, \{a, b\}\}$
$\tau_2^* = \{ X^*, \phi \{a\}, \{a, c\} \{a, c, d\}, \{a, d\}\}$
$X - \tau_1 = \{X, \phi, \{b, c, d\}, \{a, d\}, \{d\}\}$
$X - \tau_2 = \{X, \phi, \{b, c\}\}$

Then for $O^* \in \tau_i^*$, $f^{-1}[O^*]$ is (i-j) - preopen in X, i, j = 1, 2 ; $i \neq j$.
Therefore f : $(X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is bitopologically precontinuous.
But $f^{-1}[\{b\}] = \{b\} \notin \tau_1$ for $\{b\} \in \tau_1^*$
Therefore f is not continuous.

REMARK :

The notion of bitopologically preopen is not equivalent to precontinuous in individual topological spaces. Which can be seen from the following two examples.

EXAMPLE : 2.2.21

Let (X, τ_1, τ_2) be the bitopological space.

$X \quad = X^* = \{a, b, c, d\}, \tau_i = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$

$\tau_2 \quad = \{X, \phi, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}$

$\tau_1^* \quad = \{X, \phi, \{c\}\}$

$\tau_2^* \quad = \{X, \phi,, \{a\}\}$

$X\text{-}\tau_1 = \{X, \phi,, \{a, c, d\}, \{c, d\}, \{a, d\}, \{d\}\}$

$X - \tau_2 = \{X, \phi,, \{a,b, c\}, \{b, c\}, \{a,b\}, \{b\}\}$

If $f : (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*)$ be the identity mapping then f is bitopologically precontinuous. But $f^{-1}(\{c\})$ is not preopen in (X, τ_1) for $\{c\} \in \tau_1^*$.

So f is not precontinuous on (X, τ_1).

$f^{-1}(\{a\})$ is not preopen in (X, τ_1) for $\{a\} \in \tau_2^*$, and hence f is not precontinuous on (X, τ_2).

EXAMPLE : 2.2.22

Let $X = X^* = \{a, b, c, d\}$

$\tau_1 \quad = \{X, \phi, \{a, d\}, \{b, c\}\}$

$\tau_2 \quad = \{X, \phi, \{b, d\}, \{a, c\}\}$

$\tau_1^* \quad = \{X, \phi^*, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$

$\tau_2^* \quad = \{X, \phi^*, \{b\}\}$

$X\text{-}\tau_1 \quad = \{X, \phi, \{b, c\}, \{a, d\}\}$

$X\text{-} \tau_2 \quad = \{X, \phi, \{a, c\}, \{b, d\}\}$

If $f: (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*)$ be the identity mapping, then f is precontinuous on both (X, τ_1) and (X, τ_2). But $f^{-1}(\{c\})$ is not (1-2) preopen for $\{c\} \in \tau_1^*$, f is not (1-2) precontinuous.

Similarly, for $\{b\} \in \tau_2^*$, $f^{-1}(\{b\})$ is not (2-1) - preopen and hence f is not (2-1) precontinuous so f is not bitopologically precontinuous.

DEFINITION : 2.2.23 [13]

In (X, τ_1, τ_2), a net $\{x_\alpha, \alpha \in D, \geq\}$ is said to coverage to a point $x \in X$, denoted by $\{x_\alpha, \alpha \in D, \geq\} \rightarrow x$, if the net is eventually in every τ_i - neighbourhood of x. i = 1, 2

Characterization of (i-j) - precontinuous are given in the following theorem.

THEOREM : 2.2.24

Let $f: (X, \tau_1, \tau_2) \rightarrow (X^*, \tau_1^*, \tau_2^*)$ then the following statements are equivalent ;

(a) f is (i-j) - precontinuous.

(b) For each $x \in X$ and each net $\{x_\alpha, \alpha \in D, \geq\}$, converging to x,the image net $\{f(x_\alpha), \alpha \in D, \geq\}$ is eventually in every τ_i^* - neighbourhood of f(x), whose inverse is τ_j - closed in X.

(c) For each $x \in X$ and each τ_i*- open set U^* containing f(x), there exists an (i-j) - preopen set $U \subset X$ such that $x \in U$ and $f[U] \subset U^*$.

(d) The inverse image of each τ_i^* - closed set in X^* is (i-j) - preclosed in X.

(e) For each $A \subset X$, $f[\tau_i - Cl(\tau_j - Int A)] \subset \tau_i^* - Cl f\{A]$

(f) For each $A^* \subset X^*$, τ_i-Cl $(\tau_j$ - Int $(f^{-1}[A]) \subset f^{-1}[\tau_i^*$-Cl $A^*]$

PROOF :

To prove (a) => (b)

Assume f is (i-j) - precontinuous.

Let $x \in X$ and M be any τ_i^* - neighbourhood of f(x) such that $f^{-1}[M]$ is τ_j - closed.

Then there exists a τ_i^* - open set U^* such that $f(x) \in U^* \subset M$

Therefore, $x \in f^{-1}[U^*] \subset f^{-1}[M]$

Since f is (i-j) - precontinuous.

$f^{-1}[U^*]$ is (i-j) preopen.

Hence $x \in f^{-1}[U^*] \subset \tau_i$ - Int $(\tau_j$-Cl $(f^{-1}[U^*])$

$\subset \tau_i$ - Int $(\tau_j$ - Cl $(f^{-1}[M]))$

If the net $\{x_\alpha, \alpha \in D, \geq\} \to x$, then

$\{x_\alpha, \alpha \in D, \geq\}$ is eventually in τ_i - Int $(\tau_j$ - Cl$(f^{-1}[M]))$

This implies that there exists $\alpha_0 \in D$ such that $\alpha > \alpha_0$

$f(x_\alpha) \in f[\tau_i$ - Int - $(\tau_j$ - Cl $(f^{-1}[M]))]$

$\subset f(\tau_j$ - Cl $(f^{-1}[M]))$

$\subset f f^{-1}[M]$ since $f^{-1}[M]$ is τ_j-closed.

$\subset M$

Hence $f(x_\alpha)$ is eventually in M.

That is, for every $x \in X$ and each net $\{x_\alpha, \alpha \in D, \geq\}$ Converging to x, the image let $\{f(x_\alpha), \alpha \in D, \geq\}$ is eventually in every τ_i^* - neighbourhood of f(x), whose inverse is τ_j - closed in X.

To prove (b) => (c)

Assume (b) is true.

Let $x \in X$ and U^* be any τ_i^* - open set containing f(x). Suppose (c) is not true. Then for any (i-j) - preopen set U containing x.

$f[U] \not\subset U^*$.

Then, $f[U] \cap [X^* - U^*] \neq \phi$

That is $U^* \cap f^{-1}[X^* - U^*] \neq \phi$ (1)

Let $\mathcal{N}(x)$ be the family of all τ_i-neighbourhoods of x. Then for each $N \in \mathcal{N}(x)$ is τ_i - open and hence (i-j) preopen.

Therefore, from (1) we get

$N \cap f^{-1}[X^* - U^*] \neq \phi$ for all $N \in \mathcal{N}(x)$

Let $x_N \in N \cap f^{-1}[X^* - U^*]$

Now, $\{x_N, N \in \mathcal{N}(x), \subseteq\}$ is a net in X converging to x and $x_N \in f^{-1}[X^*-U^*]$ for all $N \in \mathcal{N}(x)$.

So $f(x_N) \in X^* - U^*$

That is $f(x_N) \notin U^*$ for all $N \in \mathcal{N}(x)$.

Therefore, the image net $\{f(x_N), N \in \mathcal{N}(x), \subseteq\}$ is not eventually in U^*.

Which is a contradiction to our assumption. Hence, for each $x \in X$ and each τ_i^* - open set U^* containing f(x), there exists an (i-j) - preopen set $U \in X$ such that $x \in U$ and $f(U) \subset U^*$.

To prove (c) => (d)

Assume (c) is true.

To prove

The inverse of each τ_i - closed set in X^* is (i-j) - preclosed in X.

Let $U^* \subset X^*$ be τ_i^* - closed set in X^*.

Then to prove $f^{-1}[U^*]$ is (i-j) - preclosed in X.

Let $x \in X - f^{-1}[U^*]$ then

$f(x) \in f[X - f^{-1}[U^*]).$

$\subset X^* - U^*$, where $X^* - U^*$ is τ_i^* - open

By our assumption there exists an (i-j) preopen set $U_x \subset X$ such that $x \in U_x$ and $f(U_x) \subset X^* - U^*$.

$$\text{Hence } x \in U_x \subset f^{-1}f[U_x]$$
$$\subset f^{-1}[X^* - U^*]$$
$$\subset X - f^{-1}[U^*]$$
$$\text{So } X - f^{-1}[U^* = \cup \{U_x; x \in X - f^{-1}[U^*]\}$$

by Theorem 2.2.9 $X - f^{-1}[U^*]$ is (i-j) preopen.

That is $f^{-1}[U^*]$ is (i-j) preclosed in X.

To prove (d) => (e)

Assume the inverse image of each τ_i^* - closed set in X^* is (i-j) - preclosed in X.

To prove

For each $A \subset X$, $f[\tau_i\text{-Cl}(\tau_j - \text{Int}A)] \subset \tau_i^* - \text{Cl}(f[A])$

For any $A \subset X$, $A \subset f^{-1}(f[A])$

$\subset f^{-1}(\tau_i^* - \text{Cl}(f[A])$

Where $f^{-1}(\tau_i^* - \text{Cl}(f[A]))$ is (i-j) - preclosed by our assumption. Since interior and closure respect inclusion it is clear that,

τ_i - Cl (τ_j - IntA) $\subset \tau_i$-Cl (τ_j - Int (f^{-1} [τ_i^* - Cl (f[A])]))

$\subset$ f^{-1} [τ_i^* - Cl (f[A])]

By (3) of Theorem 2.2.13

This implies that

f [τ_i - Cl (τ_j - Int A)] $\subset$ f [f^{-1} [τ_i^* - Cl (f[A])]].

$\subset \tau_i^*$ - Cl (f[A]).

Hence we obtain (e)

To prove (e) => (f)

For any $A^* \subset X^*$, let A = f^{-1} [A^*].

Then for A$\subset$X, we have

f [τ_i-Cl (τ_j - Int A)] $\subset \tau_i^*$ - Cl (f[A]) by (e).

Hence

τ_i - Cl (τ_j - Int A) $\subset$ f^{-1} f [τ_i - Cl (τ_j - Int A)]

$\subset$ f^{-1} [τ_i^* - Cl (f[A])]

that is

τ_i - Cl [τ_j-Int (f^{-1}[A^*])] $\subset$ f^{-1} [τ_i^*- Cl f f^{-1}[A^*])

$\subset$ f^{-1} [τ_i^* - Cl A^*]

Hence for each $A^* \subset X^*$,

τ_i - Cl (τ_j - Int (f^{-1} [A^*]) $\subset$ f^{-1} [τ_i^* - Cl A^*]

To prove (f) =>(a)

That is , if (f) is true then to prove f is (i-j) - precontinuous.

Let $B^* \subset X^*$ be any τ_i^* - open set , then

$A^* = X^* - B^*$ is τ_i^* - closed and

f^{-1} [A^*] = f^{-1} [X^*] $-$ f^{-1} [B^*].

= X $-f^{-1}$ [B^*]

which implies that X $-$ f^{-1} [A^*] = f^{-1} [B^*]

Claim

f^{-1} [B] is (i-j) - preopen.

Now by our assumption for each $A^* \subset X^*$.

τ_i - Cl (τ_j -Int (f^{-1} [A^*])) $\subset$ f^{-1} [τ_i^* - Cl A^*]

= f^{-1} [A^*]

That is

$X - (\tau_i - Cl(\tau_j - Int(f^{-1}[A]))) \supset X - f^{-1}[A^*]$

which implies that

$\tau_i - Int(\tau_j - Cl(X - f^{-1}[A^*]))) \supset X - f^{-1}[A^*]$.

by relations connecting complimentation, closure and interior operator.

That is $\tau_i - Int[\tau_j - Cl(f^{-1}[B^*])) \supset f^{-1}[B^*]$

So that $f^{-1}[B^*]$ is (i-j) - preopen.

Hence f is (i-j) - precontinuous.

DEFINITION : 2.2.25

In (X, τ_1, τ_2), a net $\{x_\alpha, \alpha \in D, \geq\}$ is said to bitopologically preconverges to a point $x \in X$, denoted by $\{x_\alpha, \alpha \in D, \geq\} \xrightarrow{p} x$ if the net is eventually in every (i-j) preopen set containing x, i,j = 1, 2 ; $i \neq j$

In the next theorem we shall see that how (i-j) precontinuous establishes an interesting relation between preconvergence of a net and convergence of its image net.

THEOREM : 2.2.26

If a mapping f $(X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is (i-j) - precontinuous then for each $x \in X$ and each net$\{x_\alpha, \alpha \in D, \geq\}$ in X preconverging to x, the image net $\{f(x_\alpha), \alpha \in D, \geq\}$ converges to f(x).

PROOF

Let f be (i-j)- precontinuous

Let $x \in X$ and $\{x_\alpha, \alpha \in D, \geq\}$ be a net in X such that $\{x_\alpha, \alpha \in D, \geq\} \xrightarrow{p} x$.

Let U^* be any τ_i^* - open neighbourhood of f (x), i=1,2

Then $f^{-1}[U^*]$ is an (i-j)-preopen set containing x.Since$\{x_\alpha, \alpha \in D, \geq\} \xrightarrow{p} x$,

there exists $\alpha_o \in D$

such that $x_\alpha \in f^{-1}[U^*]$ for all $\alpha \geq \alpha_o$.

Therefore $f(x_\alpha) \in f f^{1}[U^*] \subset U^*$ for all $\alpha \geq \alpha_o$

Hence the image net $\{f(x_\alpha), \alpha \in D, \geq\} \to f(x)$.

THEOREM : 2.2.27

Let $(X_\alpha, \tau_{1\alpha}, \tau_{2\alpha}): \alpha \in \Lambda \}$ be a family of bitopolgical spaces $X = \Pi X_\alpha$, the product space and τ_1, τ_2 the product topologies generated by $\tau_{1\alpha}$'s and $\tau_{2\alpha}$'s respectively. If $A = \Pi_{k=1}^{n} A_{\alpha_k} \times \Pi_{\alpha \neq \alpha_k} X_\alpha$ is a non-empty subset of X, n being a positive integer, then A_{α_k} is (i-j)-preopen for each K $(1 \leq k \leq n)$ iff A is (i-j) - preopen in X. i,j = 1, 2 ; $i \neq j$.

PROOF :

Let A_{α_k} be (i-j) - preopen, for K = 1, 2.......n.

Then $A_{\alpha_k} \subset \tau_{i\alpha_k}$ - Int $(\tau_{j\alpha_k})$, K = 1, 2 .....n.

So, τ_i - Int $(\tau_j$ - Cl A) $= \tau_i$-Int $(\tau_j$-Cl $(\Pi_{k=1}^{n} A_{\alpha_k} \times \Pi_{\alpha \neq \alpha_k} X_\alpha))$

$= \tau_i$-Int$(\Pi_{k=1}^{n} \tau_{j\alpha_k}$-Cl $A_{\alpha_k} \times \Pi_{\alpha \neq \alpha_k} X_\alpha)$ [9]

$= \Pi_{k=1}^{n} \tau_{i\alpha_k}$- Int $(\tau_{j\alpha_k}$- Cl $A_{\alpha_k} \times \Pi_{\alpha \neq \alpha_k} X_\alpha)$

$\supset \Pi_{k=1}^{n} A_{\alpha_k} \times \Pi_{\alpha \neq \alpha_k} X_\alpha$

$= A$

Hence A is (i-j) - preopen in X.

Conversely

Let A be (i-j) - preopen in X. Then $A \subset \tau_i$ - Int (X - Cl A).

Hence

$\Pi_{k=1}^{n} A_{\alpha_k} \times \Pi_{\alpha \neq \alpha_k} X_\alpha \subset \tau_i$ -Int$(\tau_j$ -Cl$(\Pi_{k=1}^{n} A_{\alpha_k} \times \Pi_{\alpha \neq \alpha_k} X_\alpha))$

$= \Pi_{k=1}^{n} \tau_{i\alpha_k}$-Int$(\tau_{j\alpha_k}$-Cl $A_k) \times \Pi_{\alpha \neq \alpha_k} X_\alpha$

So $A_{\alpha_k} \subset \tau_{i\alpha_k}$-Int$((\tau_{j\alpha_k}$-Cl $A_k)$

Therefore A_{α_k} is (i-j) - preopen., k = 1, 2 ....n

THEOREM : 2.2.28

Let $(X_\alpha, \tau_{1\alpha}, \tau_{2\alpha}): \alpha \in \Lambda \}$ and $\{(X^*_\alpha, \tau^*_{1\alpha}, \tau^*_{2\alpha}): \alpha \in \Lambda\}$ be two arbitary families of bitopological spaces with same set of indices. For each $\alpha \in \Lambda$, let $f_\alpha : (X_\alpha, \tau_{1\alpha}, \tau_{2\alpha}) \longrightarrow (X^*_\alpha, \tau^*_{1\alpha}, \tau^*_{2\alpha})$ be given and define $f : (X, \tau_1, \tau_2) \to (X^*, \tau^*_1, \tau^*_2)$ by $f(x) = f(\langle x_\alpha \rangle) = \langle f_\alpha(x_\alpha) \rangle$, where $X = \pi X_\alpha$, $X^* = \pi X^*_\alpha$ and $\tau_1, \tau_2, \tau^*_1, \tau^*_2$, are the product topologies generated by $\tau_{1\alpha}$'s, $\tau^*_{2\alpha}$'s, $\tau_{1\alpha}$'sand $\tau^*_{2\alpha}$'s respectively. Then f is bitopologically precontinuous iff f_α is so for each $\alpha \in \Lambda$

PROOF :

Let f be bitopologically precontinuous. Then f is (i-j) precontinuous, i,j = 1,2 ' $i \neq j$. Let $\beta \in \Lambda$ be arbitrary and $U_\beta{}^*$ be any $\tau_{i\beta}{}^*$ - open set in $X_\beta{}^*$. Then $U_\beta \times \pi_{\alpha \neq \beta} X_\alpha{}^*$ is $\tau_i{}^*$ open in X^*. So,

$$f^{-1}[U^*_\beta \times \pi_{\alpha \neq \beta} X_\alpha{}^*] = f_\beta{}^{-1}[U_\beta{}^*] X \pi_{\alpha \neq \beta} f^{-1}[X_\alpha{}^*]$$

$$= f_\beta{}^{-1}[U_\beta{}^*] X \pi_{\alpha \neq \beta} X_\alpha \text{ is (i-j) - preopen in X, i,j = 1,2; } i \neq j.$$

Hence by Theorem 2.2.27 $f_\beta{}^{-1}[U_\beta{}^*]$ is (i-j) - preopen in X_β so that f_β is (i-j) - precontinuous i,j = 1,2; $i \neq j$. Hence f_β is bitopologically precontinuous since $\beta \in \Lambda$ is arbitrary, f_α is bitopologically precontinuous for each $\alpha \in \Lambda$.

Conversely,

Let f_α be bitopologically precontinuous for each $\alpha \in \Lambda$. Then f_α is (i-j) - precontinuous for each $\alpha \in \Lambda$, i,j = 1,2; $i \neq j$ Let $U^* \subset X^*$ be any basis $\tau_i{}^*$ - open set. Then $U^* = \pi_{k=1}^n U_{\alpha_k}{}^*, X \pi_{\alpha \neq \alpha_k} X^*_\alpha$ where $U_{\alpha_k}{}^*$ is $\tau_{i\alpha_k}{}^*$ in $X_{\alpha_k}{}^*$. K = 1,2......n.

Now $f^{-1}[U^*] = f^{-1}[\pi_{k=1}^n U_{\alpha_k}{}^*, X \pi_{\alpha \neq \alpha_k} X^*_\alpha]$

$$= \pi_{k=1}^n f^{-1}_{\alpha_k} [U_{\alpha_k}{}^*] X \pi_{\alpha \neq \alpha_k} f^{-1}{}_\alpha [X^*_\alpha]$$

where $f_{\alpha_k}{}^{-1}[U_{\alpha_k}{}^*]$ is (i-j) - preopen in each X_{α_k} $(1 \leq k \leq n)$; i, j = 1, 2 ; $i \neq j$.

Hence by Theorem 2.2.27, $f^{-1}[U^*]$ is (i-j) - preopen in X, i, j = 1, 2 ; $i \neq j$.

Let O^* be any $\tau_i{}^*$ - open set in X^*

Then $O^* = \cup_\beta U_\beta{}^*$ where $U_\beta{}^*$'s are basis $\tau_i{}^*$ - open sets in X^*.

So $f^{-1}[O^*] = f^{-1}[\cup_\beta U_\beta{}^*]$

$$= \cup_\beta f^{-1}[U_\beta{}^*]$$

Since $f^{-1}[U_\beta{}^*]$ is (i-j) - preopen for each β by Theorem 2.2.9 $f^{-1}[O^*]$ is (i-j)- preopen., i,j = 1,2; $i \neq j$. Hence f is bitopologically precontinuous.

THEOREM : 2.2.29

Let $(X_\alpha, \tau_{1\alpha}, \tau_{2\alpha}): \alpha \in \Lambda \}$ be an arbitrary family of bitopological spaces. For each $\alpha \in \Lambda$, let $f_\alpha : (X_\alpha, \tau_{1\alpha}, \tau_{2\alpha}) \longrightarrow (X^*_\alpha, \tau_{1\alpha}{}^*, \tau_{2\alpha}{}^*)$ be given. Define $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ by $f(x) = \langle f_\alpha(x) \rangle$, where $X^* = \pi X_\alpha$ and $\tau_1{}^*, \tau_2{}^*$ are the product topologies generated by $\tau_{1\alpha}$'s, $\tau_{2\alpha}$'s respectively. Then f_α is bitopologically precontinuous for each α, if f is so.

PROOF :

Let $\beta \in \Lambda$ be arbitrary and $\dot{U}_\beta$ be any $\tau_{i\beta}$ - open set in X_β. Then $U_\beta X \pi_{\alpha \neq \beta} X_\alpha$ is $\tau_i{}^*$ - open in X^*. The verification of the relation

$f_\beta^{-1}[U_\beta] = f^{-1}[U_\beta X \pi_{\alpha\neq\beta} X_\alpha]$ and the hypothesis that f is bitopologically precontinuous give that $f_\beta^{-1}[U_\beta]$ is (i-j) - preopen., $i,j = 1,2$; $i \neq j$.

Hence f_β is bitopologically precontinuous. Since $\beta \in \Lambda$. is arbitrary, f_α is bitopologically precontinuous for each $\alpha \in \Lambda$.

REMARK :

The converse of Theorem 2.2.29 is in general, flase as shown by the following example.

EXAMPLE : 2.2.30

Let $X = X_\alpha = X_\beta = [0, 1]$ = the closed unit interval on the real line and (X, τ_1, τ_2), $(X_\alpha, \tau_{1\alpha}, \tau_{2\alpha})$, $(X_\beta, \tau_{1\beta}, \tau_{2\beta})$ be the bitopological spaces, where $\tau_1 = \tau_2$ = cofinite topology. on X which $\tau_{1\alpha} = \tau_{2\alpha} = \tau_{1\beta} = \tau_{2\beta}$ = the relativised usual topology on [0, 1].

Let $f_\alpha : (X, \tau_1, \tau_2) \to (X_\alpha, \tau_{1\alpha}, \tau_{2\alpha})$, $f_\beta : (X, \tau_1, \tau_2) \to (X_\beta, \tau_{1\beta}, \tau_{2\beta})$ be two mappings defined by $f_\alpha(x) = 1$, $0 \leq x \leq 1/2$, $f_\alpha(x) = 0$, $1/2 \leq x \leq 1$ and $f_\beta(x) = 1$, $0 \leq x < 1/2$, $f_\beta(x) = 0$, $1/2 \leq x \leq 1$.

Clearly, f_α, f_β are (i-j)- precontinuous. But $f : (X, \tau_1, \tau_2) \to (X_\alpha \times X_\beta, P_1, P_2)$, where P_1, P_2 are the product topologies generated by $\tau_{1\alpha}$, $\tau_{1\beta}$ and $\tau_{2\alpha}$, $\tau_{2\beta}$ respectively, is not (i-j) - precontinuous. For $f^{-1}[S_{1/2}(1, 0)] = \{1/2\}$ is not an (i-j) - preopen set in X, though $S_{1/2}(1,0)$ is P_i - open in $X_\alpha \times X_\beta$ where $S_{1/2}(1,0)$ denotes an open sphere with centre (0,1) and radius 1/2.

BITOPOLOGICAL PREOPEN MAPPING

DEFINITION : 2.2.31

A mapping $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is said tobe (i-j) - preopen iff for each τ_i - open set A in X, f [A] is (i-j) - preopen in X^* ; $i, j = 1,2$; $i \neq j$.

DEFINITION : 2.2.32

A mapping $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is termed bitopologically preopen (briefly b.p.o) iff f is (i-j) - preopen in X^* ; $i,j = 1,2$; $i \neq j$.

THEOREM : 2.2.33

A mapping $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is (i-j) - preopen iff $f^{-1}[\tau_i^* - Cl\, V] \subset \tau_i - Cl\, f^{-1}[V]$ for each τ_j^* - open set, $i,j = 1,2$; $i \neq j$.

PROOF :

Let $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ be (i-j) - preopen. Suppose $x \in f^{-1}[\tau_i^* - Cl\, V]$. Then $f(x) \in \tau_i^* - Cl\, V$.

Let U be any τ_i - open set in X containing x. Then $f(x) \in f[U]$ so that

$f(x) \in f[U] \cap \tau_i^* - Cl\, V$

$\subset [\tau_i^* - Int\, (\tau_j^* - Cl\, f[U])) \cap \tau_i^* - Cl\, V.$

(by (i-j) - preopens of f [U]).

Now, τ_i^* - Int (τ_j - Cl f [U]) is a τ_i^* - open set containing f(x).

Since $f(x) \in \tau_i^* - Cl\, V$,

$V \cap (\tau_i^* - Int\, (\tau_j^* - Cl\, f[U]) \neq \phi$

which implies that $V \cap \tau_j^* - Cl\, f[U] \neq \phi$

Let $y \in V \cap \tau_j^* - Cl\, f[U]$. Then

$y \in \tau_j^* - Cl\, f[U]$ and V is a τ_j^* - open set containing y. So $V \cap f[U] \neq \phi$

which implies that $f^{-1}[V] \cap U \neq \phi$

Hence $x \in \tau_i - Cl\, f^{-1}[V]$.

Conversely,

Let $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ be such that $f^{-1}[\tau_i^* - Cl V] \subset [\tau_i - Cl\, f^{-1}[V]$ for each τ_j^* - open set V. If possible let f be not (i-j) - preopen. Then there exists at least one τ_i - open set U in X such that

$f[U] \not\subset \tau_i^* - Int\, [\tau_j^* - Cl\, f[U])$

Hence

$f[U] \cap (X^* - \tau_i^* - Int\, (\tau_j^* - Cl\, f[U])) \neq \phi$

Let $V = X^* - \tau_j^* - Clf[U]$. Then V is τ_j^* - open in X^*.

$V \cap f[U] = \phi$ and

$\tau_i^* - Cl\, V = \tau_i^* - Cl\, (X^* - \tau_j^* - Cl\, f[U])$

$= X^* - \tau_i^* - Int\, (\tau_j^* - Cl\, f[U])$

So $f[U] \cap \tau_i^* - Cl\, V \neq \phi$

$\Longrightarrow U\, f^{-1}[\tau_i^* - Cl\, V] \neq \phi$ and since $U \cap f^{-1}[\tau_i^* - Cl\, V] \subset U \cap \tau_i - Cl\, f^{-1}[V]$,

it follows that $U \cap \tau_i - Clf^{-1}[V] \neq \phi$

Let $y \in U \cap \tau_i - Cl\, f^{-1}[V]$.

Then $y \in \tau_i - Cl\, f^{-1}[V]$ and since U is a τ_i - open set containing y, $U \cap f^{-1}[V] \neq \phi$, which contradicts the fact that $f[U] \cap V = \phi$

Hence f is (i-j) - preopen.

THEOREM : 2.2.34

A mapping $f:(X, \tau_1, \tau_2) \to (X, \tau_1^*, \tau_2^*)$ is (i-j) - preopen iff for any subset S of X^* and for any τ_i - closed set A of X, containing $f^{-1}[S]$, there exists an (i-j)- preclosed set B of X^*, containing S. such that $f^{-1}[B] \subset A$, i,j = 1, 2 ; $i \neq j$.

PROOF :

Let f be (i-j) - preopen. Let $S \subset X^*$ and A be a τ_1 - closed of X, containing $f^{-1}(S)$.

Let $B = X^* - f[X-A]$. Since $f^{-1}[S] \subset A$,

$f[X-A] \subset X^* - S$ and hence $S \subset B$.

Since f is (i-j) - preopen and X–A is τ_i - open in X, it follows that f [X–A] is (i-j) - preopen in X^* and so B is (i-j) - preclosed in X^*,

Also $f^{-1}[B] = X - f^{-1}[f[X-A]]$

$\subset X - (X - A)$

$= A.$

Conversely

Let U be τ_i - open in X and let

$S = X^* - f[U]$. Then

$f^{-1}[X^* - f[U]] \subset X - U$ and X– U is a τ_i - closed set containing $f^{-1}[S]$. So, by the given condition there is an (i-j) - preclosed set B of X^* such that $S \subset B$ and $f^{-1}[B] \subset X-U$

whence $X^* - B$ f [U] and $f^{-1}[B] \cap U = \phi$

The second relation gives $B \cap f[U] = \phi$

That is, $f[U] \subset X^* - B$. Thus

$X^* - B \subset f[U]$

$\subset X^* - B$

=> $f[V] = X^* - B$.

So, f [U] is (i-j) - preopen and hence f is an (i-j) - preopen mapping, $i,j = 1,2 ; i \neq j$.

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