Bulk Service Queue with Server's Vacation

BY

Kavitha .A

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE
AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE-641 043,
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
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CERTIFIED AS BONAFIDE RESEARCH WORK

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Acknowledgement

ACKNOWLEDGEMENT

The author records her deep gratitude and thanks to **Dr. (Tmt) RAJAMMAL P. DEVADAS, M.A.,M.Sc., Ph.D., (Ohio State), D.Sc.,(Madras)** Vice chancellor of Avinashilingam Institute for Home Science and Higher Education for Women (Deemed University) Coimbatore, for all facilities given to her in course of the preparation of the dissertation.

The author records her deep sense of gratitude to **Dr. (Tmt) SAROJA PRABAKARAN, M.A., Dip. Ed.,(Madras) Ph.D., (Mother Terassa),** Registrar of Avinashilingam Institute for Home Science and Higher Education for Women (Deemed University) Coimbatore, and **Dr.(Tmt) NIRMALA K. MURTHY, B.Sc., (Hons) Annamalai, M.Sc., (Iowa), Ph.D.,(Madras),** Dean, Faculty of Science, for providing the necessary facilities.

The author submits her thanks to **Dr. (Selvi). K.N. MEENAKSHI, M.Sc., Ph.D., (Madras),** Head of the Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women (Deemed University) Coimbatore, for the encouragement extended by her during the entire course.

The author records her deep sense of gratitude and wishes to express her heartfelt thanks to **Tmt. M.I. AFTHAB BEGUM, M.Sc., M.Phil., (Annamalai),** Senior Scale Lecturer, of the Mathematics Department, Avinashilingam Institute for Home Science and Higher Education for Women (Deemed University), Coimbatore, and to **Tmt.K.UDAYACHANDRIKA, M.Sc.,. M.Phil., (Madras),** Senior Scale Lecturer, of the Mathematics Department, Avinashilingam Institute for Home Science and Higher Education for Women (Deemed University), Coimbatore, for their guidance in completing the study.

She records her heartful thanks to other members of the staff and to her friends.

She would like to express her indebtness to her dear parents for their help in the completion of this dissertation.

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CHAPTER I

INTRODUCTION

Waiting lines or queues are familier phenomena which we observe quite frequently in our daily life. We should hope to reduce its inconvenience to bearable levels. Hence Queueing theory has increasingly occupied the attention of researchers. The queueing theory had its origin in 1909 when A.K.ERLANG (1878 - 1929) published his fundamental paper relating to the study of congestion in telephone traffic, who is appropriately called the father of Queueing Theory.

A queueing system can be described by the flow of units for service, forming or joining the queue, if service is not immediately available and leaving the system after being served. By units we mean those demanding service. In practice customers at a bank counter or at a reservation counter, calls arriving at a telephone exchange, vehicular traffic at a traffic intersection, machines for repair before a repairman, planes waiting for take off at a busy airport, computer programmes waiting to be run on a time-sharing bases etc.

are considered as units. The basic characteristic features are (i) the input (ii) the service mechanism (iii) the queue discipline (iv) the number of channels.

(i) INPUT

The input describes the manner in which units arrive (either singly or in a gruop) and join the system. The interval between two consecutive arrivals is called the interarrival time.

(ii) SERVICE MECHANISM

The service mechanism describes the manner in which service is rendered. A unit may be served either singly or in a batch. The time required for servicing a unit (or a group, in case of batch service) is called the service time.

(iii) QUEUE DISCIPLINE

The queue discipline indicates the way in which the units form a queue and are served.
They are,

(a) First come First served (FCFS)
(b) Last come First served (LCFS)
(c) Random ordering before service.

(iv) NUMBER OF SERVICE CHANNELS

In some queueing systems several service channels may be available to provide service. These service channels may be arranged in parallel or in series or a combination of both, depending on the number of servers available and the nature of the service required.

NOTATIONS

A very convinient notation designed by Kendall (1951) to denote the queueing systems has been universally accepted and used. It consists of a three - part descriptor A/B/C, where the first and the second symbols denote the interarrival time and service time distributions respectively, and the third denotes, the number of channels or servers. Some of the familiar notations are the following:

M : for exponential (Markovian) distribution

E_k : for Erlang - k distribution

G : for arbitrary (General) distribution

D : for fixed (Deterministic) interval

1) Thus M/M/1 is a single server Poisson input and exponential service model in which there is no limit on the system capacity.

2) $M/M_{a,b}/1$ is a single server Poisson input and exponential service model with general bulk service of minimum capacity 'a' and maximum capacity 'b'.

DIFFERENT TYPES OF VACATIONS

The non-availability of a server at the system may be termed as server's vacation. In a queueing system, if the queue is empty then the idle time of the server can be utilised to perform additional jobs or for the preventive maintenance work which can be divided into short segments. There are different types of vacations.

1. REPEATED VACATIONS

Any server on completion of a service, will start servicing again if the system has atleast the minimum number of customers required to start the service. Otherwise, the server will withdraw from the system for a vacation. Upon terminating this vacation period, if the server finds less than the required number of customers, he will immediately take another vacation. He will continue in this manner until he finds atleast the minimum number of waiting customers, upon returning from a vacation.

2. EXCEPTIONAL FIRST VACATION

In repeated vacations, the duration of the first vacation and the subsequent vacations have the same distribution. In exceptional first vacation, the first vacation is differently distributed from the subsequent vacations.

3. SINGLE VACATION

The assumptions are same as those of repeated vacations, except that, if the server finds less than the minimum number of customers required for service at the end of a vacation, he stays in the system waiting for the queue size to reach the minimum number.

4. GATED VACATION

When the server returns from a vacation, he accepts only those customers who were waiting when the server returned, deferring the service of subsequent arrivals until after the next vacation. One can imagine that when the server returns from vacation, a gate closes behind the last waiting customer, and the server will serve only those customers in front of the gate before departing on another vacation.

5. LIMITED SERVICE QUEUEING MODELS

In these models an upper bound, say k, is placed on the number of customers that the server will serve per visit to the queue. In one variation of this model, if the server returns from the vacation to find 'j' customers waiting, the server will give service to min (j,k) customers before again going on vacation. In another variation, the server works until either k customers have been served consecutively or the system becomes empty, then goes on another vacation.

In recent years, there have been significant contributions to the theory of vacation models - in congestion. Most of the references on this topic can be found in two excellent review articles by Doshi [3] and Teghem [13].

Yadrin and Naor [14], Romani [12] and Moder and Phillips [8] have studied multiserver queue with variable service capacity. i.e the service capacity of the system as a function of the number of waiting units and the recent history of the system.

Yonatan Levy and Uri Yechiali [15] have studied the queueing systems M/M/S with servers' vacations. In this work, two models, one with repeated vacations and the other with single vacation are analysed. For the first model, formulae for the distribution of the number of busy servers and the mean number of units in the system are derived. It is stated that second model may be analysed similar to the first model.

Some related models of the multiserver queue with break downs in which interarrival times, service times, vacation times and operative times are all exponentially distributed were studied by J.L.Mitrany and B. Avi - Itzhak [7] and M.F.Neuts and D.M.Lucantoni [10]. In their models there are S servers and every server that becomes idle leaves the main system for a random period of time. Recently Nobuko Igaki [11] considered two models of M/M/2 queueing system with vacations where fixed server goes on vacation when there are no customers in the system.

Jayaraman [5] has analysed a single server Erlangian queue and a two server Markovian queue with server's vacation dependent on the queue length. The difference - differential equations are formed and solved using Rouche's Theorem. The waiting time distribution is obtained for both the models.

BULK SERVICE QUEUEING MODELS

Most queueing models assume that customers are served singly. But this assumption is far from truth when we consider those numerous real-world situations in which customers are served in batches. We call such queueing phenomena batch service queues (or bulk service queues). Batch service queues frequently occur, for example, in loading and unloading of cargoes at a seaport, in traffic signal systems, in a shipping department of a small machine shop, and in mass transportation system. There are number of rules according to which batches for bulk service may be formed. The most general bulk service rule is the one introduced by Neuts [9].

Borthakur [2] has obtained the steady state probabilities of the number of customers in the queue for the model $M/M_{a,b}/1$. Medhi [6] has studied the same model and has derived the waiting time distribution.

The multichannel queueing systems with bulk service $M/M_{1,b}/C$ has been analysed by Ghare [4]. For this systems he has obtained the steady state probabilities.

Recently Mohana Dhas [1] has discussed $M/M_{a,b}/1$ model under the general bulk service rule introduced by Neuts with server's vacation. The mathematical model for the general transient state is constructed interms of difference - differential equations. Using generating function and Rouche's Theorem, the steady state probability distribution of the number of customers in the queue and the waiting time distribution are derived.

Our main interest here is to study the M/M/1 and $M/M_{a,b}/1$ queue under Repeated and Single vacations and obtain the steady state system probabilities in the explicit form.

In section 1 of chapter II the M/M/1 queue with Repeated vacation has been discussed under steady state. The steady state probability distributions are derived and the expected queue length and mean waiting time are also obtained.

In section 2 M/M/1 queue under single vacation is studied and the corresponding results are derived for this model also. For both the models it is found that the Little's formula $\lambda W_q = L_q$ is true.

The objective of chapter III is to discuss the $M/M_{a,b}/1$ queue under vacations. The first section deals with repeated vacation and the second section with single vacation. The steady state probability distributions are derived for both the models. For the queue with repeated vacation the expected queue length is derived. It is checked that when $a = b = 1$ the results coincide with the corresponding results of $M/M/1$ model with vacations.

Chapter II

CHAPTER II

A SINGLE SERVER MARKOVIAN QUEUE WITH SERVER'S VACATION

In this chapter a single server Markovian queue with server's vacation is discussed. In Section 2.1 the model is considered with repeated vacation and in Section 2.2 with server's single vacation, the steady state probabilities, waiting time distribution and expected queue length are derived for both the models.

In both the sections it is assumed that the arrivals are in accordance with a Poisson process with parameter λ and distribution of the service times is exponential with parameter μ. The vacation of the server is an exponentially distributed random variable with mean $1/\alpha$.

SECTION 2.1

M/M/1 WITH REPEATED VACATION

Under the repeated vacation, if the server completes a service and finds the queue is empty, he leaves for a random period time called 'vacation'. On returning to the system after a vacation if he finds no customer in the queue he takes another vacation.

STATE SPACE OF THE SYSTEM

The system under consideration can be studied by a continuous time Markov chain on the state space $\{(n,1)/n \geq 0\} \bigcup \{(n,2)/n \geq 1\}$.

The chain is in the state space :

1) $(n,1)$, $n \geq 0$ when there are 'n' customers in the system and the server is away for vacation.

2) $(n,2)$, $n \geq 1$ when there are 'n' customers in the system and the server is busy.

Define,

$$P_{n,j}(t) = P_r\{\text{at time } t, \text{ the system is in the state } (n,j),\ j = 1,2\} .$$

DIFFERENTIAL - DIFFERENCE EQUATIONS

Then the differential - difference equations are :

$$P_{0,1}'(t) = -\lambda P_{0,1}(t) + \mu P_{1,2}(t)$$

$$P_{1,2}'(t) = -(\lambda + \mu) P_{1,2}(t) + \mu P_{2,2}(t) + \alpha P_{1,1}(t)$$

$$P'_{n,1}(t) = -(\lambda+\alpha) P_{n,1}(t) + \lambda P_{n-1,1}(t), \quad n \geq 1$$

$$P'_{n,2}(t) = -(\lambda+\mu) P_{n,2}(t) + \mu P_{n+1,2}(t) + \lambda P_{n-1,2}(t) + \alpha P_{n,1}(t), \quad n \geq 2$$

STEADY STATE SOLUTION

Assuming the existence of the steady - state, the above set of equations yields the following steady state equations are,

$$0 = -\lambda P_{0,1} + \mu P_{1,2} \qquad \ldots(1)$$

$$0 = -(\lambda+\mu) P_{1,2} + \mu P_{2,2} + \alpha P_{1,1} \qquad \ldots(2)$$

$$0 = -(\lambda+\alpha) P_{n,1} + \lambda P_{n-1,1}, \quad n \geq 1 \qquad \ldots(3)$$

$$0 = -(\lambda+\mu) P_{n,2} + \mu P_{n+1,2} + \lambda P_{n-1,2} + \alpha P_{n,1}, \quad n \geq 2 \qquad \ldots(4)$$

Using (1) and (3) we have

$$P_{1,2} = \frac{\lambda}{\mu} P_{0,1} \qquad \ldots(5)$$

$$P_{n,1} = \frac{\lambda}{\lambda+\alpha} P_{n-1,1}, \quad n \geq 1$$

By iteration we get,

$$P_{n,1} = \left[\frac{\lambda}{\lambda + \alpha}\right]^n P_{0,1} , \quad n \geq 1 \qquad \ldots(6)$$

Now we shall obtain the expression for $P_{n,2}$ with $n \geq 2$

From (4) we have

$$(\lambda + \mu) P_{n,2} - \mu P_{n+1,2} - \lambda P_{n-1,2} = \alpha P_{n,1}, n \geq 2$$

Replacing n by n+1 we have

$$(\lambda + \mu) P_{n+1,2} - \mu P_{n+2,2} - \lambda P_{n,2} = \alpha P_{n+1,1} , \quad n \geq 1$$

Using the shifting operator $E f(n) = f(n+1)$

The above equation becomes :

$$h(E)(P_{n,2}) = \alpha P_{n+1,1} , \quad n \geq 1$$

Where $h(Z) = [(\lambda + \mu)Z - \mu Z^2 - \lambda]$

$h(Z) = 0$ has two roots $1, \dfrac{\lambda}{\mu}$.

Assuming $\dfrac{\lambda}{\mu} < 1$ we have

$$P_{n,2} = A \left(\frac{\lambda}{\mu}\right)^n + \frac{\alpha \left[\frac{\lambda}{\lambda + \alpha}\right]^{n+1} P_{0,1}}{(\lambda + \mu)\left(\frac{\lambda}{\lambda + \alpha}\right) - \mu \left(\frac{\lambda}{\lambda + \alpha}\right)^2 - \lambda} , n \geq 1$$

Simplifying

$$P_{n,2} = A \left(\frac{\lambda}{\mu}\right)^n + \frac{\lambda \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-1} P_{0,1}}{(\mu - \lambda - \alpha)} \qquad \ldots(7)$$

The constant A can be determined by equating the expression for $P_{1,2}$ from (7), (5).

Then
$$\frac{\lambda}{\mu} P_{0,1} = A\frac{\lambda}{\mu} + \frac{\lambda}{\mu - \lambda - \alpha} P_{0,1}$$

$$A = \frac{\lambda + \alpha}{\lambda + \alpha - \mu} P_{0,1}$$

Hence

$$P_{n,2} = \frac{P_{0,1}}{\lambda + \alpha - \mu}\left[(\lambda + \alpha)\left(\frac{\lambda}{\mu}\right)^{n} - \lambda\left(\frac{\lambda}{\lambda + \alpha}\right)^{n-1}\right], \quad n \geq 1 \qquad \ldots\ldots(8)$$

Thus the steady state probabilities are obtained in terms of $P_{0,1}$ and $P_{0,1}$ can be derived from the boundary condition.

The boundary condition implies,

$$\sum_{n=0}^{\infty} [P_{n,1} + P_{n,2}] = 1$$

i.e.
$$\sum_{n=0}^{\infty}\left[\left(\frac{\lambda}{\lambda + \alpha}\right)^{n} + \frac{1}{\lambda + \alpha - \mu}\left((\lambda + \alpha)\left(\frac{\lambda}{\mu}\right)^{n} - \lambda\left(\frac{\lambda}{\lambda + \alpha}\right)^{n-1}\right)\right] P_{0,1} = 1$$

$$P_{0,1}\left[\frac{1}{1 - \frac{\lambda}{\lambda + \alpha}} + \frac{1}{\lambda + \alpha - \mu}\left\{\frac{(\lambda + \alpha)(\lambda/\mu)}{1 - (\lambda/\mu)} - \frac{\lambda}{1 - \frac{\lambda}{\lambda + \alpha}}\right\}\right] = 1$$

$$P_{0,1}\left[\frac{\lambda + \alpha}{\alpha} + \frac{1}{\lambda + \alpha - \mu}\left\{\frac{\lambda(\lambda + \alpha)}{\mu - \lambda} - \frac{\lambda(\lambda + \alpha)}{\alpha}\right\}\right] = 1$$

Simplifying further

$$P_{0,1}\left[\left(\frac{\lambda+\alpha}{\alpha}\right)\left(\frac{\mu}{\mu-\lambda}\right)\right] = 1$$

Hence

$$P_{0,1} = \frac{\alpha\ (\mu-\lambda)}{\mu\ (\lambda+\alpha)} \qquad \ldots(9)$$

EXPECTED QUEUE LENGTH

Now we shall calculate the average number of members in the queue.

$$L_{q_1} = \sum_{n=1}^{\infty} n[P_{n,1}+P_{n+1,2}]$$

$$= \sum_{n=1}^{\infty} n\left[\frac{\lambda}{\lambda+\alpha}\right]^n P_{0,1} + \sum_{n=1}^{\infty} n\frac{P_{0,1}}{\lambda+\alpha-\mu}\left[(\lambda+\alpha)\left(\frac{\lambda}{\mu}\right)^{n+1} - \lambda\left(\frac{\lambda}{\lambda+\alpha}\right)^n\right]$$

Taking $\rho = \frac{\lambda}{\lambda+\alpha}$ and $r = \frac{\lambda}{\mu}$

$$L_{q_1} = P_{0,1}\ \rho\sum_{n=1}^{\infty} n\rho^{n-1} +$$

$$\frac{P_{0,1}}{\lambda+\alpha-\mu}\left\{(\lambda+\alpha)\,r^2\sum_{n=1}^{\infty} nr^{n-1} - \lambda\rho\sum_{n=1}^{\infty} n\rho^{n-1}\right\}$$

$$= P_{0,1}\left[\rho \frac{d}{d\rho}\left(\sum_{n=1}^{\infty} \rho^{n}\right) + \frac{1}{\lambda+\alpha-\mu}\left((\lambda+\alpha)\, r^{2} \frac{d}{dr}\left(\sum_{n=1}^{\infty} r^{n}\right) - \lambda\rho \frac{d}{d\rho}\left[\sum_{n=1}^{\infty} \rho^{n}\right]\right)\right]$$

$$=P_{0,1}\left[\rho \frac{d}{d\rho}\left(\frac{\rho}{1-\rho}\right) + \frac{1}{\lambda+\alpha-\mu}\left((\lambda+\alpha)\, r^{2} \frac{d}{dr}\left(\frac{r}{1-r}\right) - \lambda\rho \frac{d}{d\rho}\left(\frac{\rho}{1-\rho}\right)\right)\right]$$

$$L_{q_1} = P_{0,1}\left[\frac{\lambda}{\alpha^{2}}(\lambda+\alpha) + \frac{\lambda^{2}(\lambda+\alpha)}{\lambda+\alpha-\mu}\left\{\frac{1}{(\mu-\lambda)^{2}} - \frac{1}{\alpha^{2}}\right\}\right] \qquad \ldots(10)$$

$$= \lambda\left[\frac{1}{\alpha} + \frac{\lambda}{\mu(\mu-\lambda)}\right]$$

DISTRIBUTION OF THE WAITING TIME FOR THE SYSTEM

Let T deonte the random variable of the waiting time in the queue for an arriving unit and $\nu(t)$ be the p.d.f of T.

An arriving unit may find the system in any one of the following states :

(i) (0,1)

(ii) (n,1) , $n \geq 1$

(iii) (n,2) , $n \geq 2$

In case of (i), the arriving unit has to wait for the return of the server from vacation.

In case of (ii), the arriving unit has to wait for the server to return from the vacation and for n service completion.

In case of (iii), the arriving unit has to wait for n service completions.

Then the probability density function $\nu(t)$ of T is given by,

$$\nu(t) = P_{0,1}\, \alpha e^{-\alpha t} + \sum_{n=1}^{\infty} P_{n,1} \int_0^t \alpha e^{-\alpha h} \frac{\mu e^{-\mu(t-h)} [\mu(t-h)]^{n-1}}{(n-1)!}\, dh$$

$$+ \sum_{n=1}^{\infty} P_{n,2} \frac{e^{-\mu t} (\mu t)^{n-1} \mu}{(n-1)!}$$

Substituting for $P_{n,1}$ and $P_{n,2}$ we have,

$$\nu(t) = P_{0,1}\Big\{\alpha e^{-\alpha t} + \sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda+\alpha}\right)^{n}$$

$$\int_0^t \alpha e^{-\alpha h}\, \mu e^{-\mu(t-h)} \frac{[\mu(t-h)]^{n-1}}{(n-1)!}\, dh$$

$$+ \frac{1}{\lambda+\alpha-\mu}\sum_{n=1}^{\infty}\left[(\lambda+\alpha)\left(\frac{\lambda}{\mu}\right)^{n} - \lambda\left(\frac{\lambda}{\lambda+\alpha}\right)^{n-1}\right]$$

$$\frac{e^{-\mu t}(\mu t)^{n-1}}{(n-1)!}\,\mu\Big\}$$

$$= P_{0,1}\Big[\alpha e^{-\alpha t} + \frac{\lambda}{\lambda+\alpha}\mu \int_0^t \alpha e^{-\alpha h} e^{-\mu(t-h)} e^{\frac{\lambda}{\lambda+\alpha}[\mu(t-h)]}\, dh.$$

$$+ \frac{\lambda+\alpha}{\lambda+\alpha-\mu}\lambda e^{-\mu t} e^{\lambda t} - \frac{\lambda\mu}{\lambda+\alpha-\mu} e^{-\mu t} e^{\frac{\lambda\mu t}{\lambda+\alpha}}\Big]$$

$$= P_{0,1}\Big[\alpha e^{-\alpha t} - \frac{\lambda\mu}{\lambda+\alpha-\mu} e^{-\alpha t} + \frac{\lambda(\lambda+\alpha)}{\lambda+\alpha-\mu} e^{-(\mu-\lambda)t}\Big]$$

It is checked that $\int_0^{\infty} \nu(t)\, dt = 1$

Now we verify Little's formula: $\lambda W_q = L_q$

$$W_{q_1} = \int_0^{\infty} t\, \nu(t)\, dt$$

$$= P_{0,1} \int_0^\infty t \left[\alpha e^{-\alpha t} - \frac{\lambda \mu}{\lambda + \alpha - \mu} e^{-\alpha t} + \frac{\lambda(\lambda + \alpha)}{\lambda + \alpha - \mu} e^{-(\mu - \lambda)t} \right] dt$$

$$= P_{0,1} \left\{ \frac{1}{\alpha} - \frac{\lambda}{\lambda + \alpha - \mu} \left[\frac{\mu}{\alpha^2} - \frac{\lambda + \alpha}{(\mu - \lambda)^2} \right] \right\}$$

$$= P_{0,1} \left\{ \frac{\alpha(\lambda + \alpha - \mu) - \lambda\mu}{\alpha^2 (\lambda + \alpha - \mu)} + \frac{\lambda(\lambda + \alpha)}{(\lambda + \alpha - \mu)(\mu - \lambda)^2} \right\}$$

$$= P_{0,1} \left\{ \frac{\alpha(\lambda + \alpha) - \mu(\lambda + \alpha)}{\alpha^2 (\lambda + \alpha - \mu)} + \frac{\lambda(\lambda + \alpha)}{(\lambda + \alpha - \mu)(\mu - \lambda)^2} \right\}$$

$$= \frac{P_{0,1} (\lambda + \alpha)}{(\lambda + \alpha - \mu)} \left[\frac{1}{\alpha} - \frac{\mu}{\alpha^2} + \frac{\lambda}{(\mu - \lambda)^2} \right]$$

$$= \frac{P_{0,1} (\lambda + \alpha)}{(\lambda + \alpha - \mu)} \left[\frac{\alpha - \mu}{\alpha^2} + \frac{\lambda}{(\mu - \lambda)^2} \right]$$

$$= \frac{P_{0,1}(\lambda+\alpha)}{(\lambda+\alpha-\mu)}\left[\frac{\alpha-\mu+\lambda}{\alpha^2} - \frac{\lambda}{\alpha^2} + \frac{\lambda}{(\mu-\lambda)^2}\right]$$

$$= P_{0,1}\left[\frac{\lambda+\alpha}{\alpha^2} - \frac{\lambda(\lambda+\alpha)}{\alpha^2(\lambda+\alpha-\mu)}\right.$$

$$\left. + \frac{\lambda(\lambda+\alpha)}{(\lambda+\alpha-\mu)(\mu-\lambda)^2}\right]$$

$$= P_{0,1}\left\{\frac{(\lambda+\alpha)}{\alpha^2} + \frac{\lambda(\lambda+\alpha)}{\lambda+\alpha-\mu}\right.$$

$$\left.\left[\frac{1}{(\mu-\lambda)^2} - \frac{1}{\alpha^2}\right]\right\}$$

$$W_{q_1} = \frac{1}{\lambda} L_{q_1}.$$

SECTION 2.2

M/M/1 WITH SINGLE VACATION

In this model, the server leaves for vacation after completing his service and nobody is waiting in the queue, if the server finds empty queue at the end of a vacation, he stays in the system until the arrival of a customer.

The system under consideration can be studied by continuous time Markov chain with state space

$(0,0) \cup \{(n,1) / n \geq 0\} \cup \{(n,2) / n \geq 2\}$

The chain is in the state space :

(1) (0,0), if no member is in the system and the server is idle.

(2) (n,1) $n \geq 0$, if there are 'n' customers in the system and the server is away for vacation.

(3) (n,2) $n \geq 1$, if there are 'n' customers in the system and the server is busy.

Define,

$$P_{n,j}(t) = P_r \{\text{at time } t, \text{ the system is in the state } (n,j),\ j = 0,1,2\}$$

The differential - difference equations for the M/M/1 model with single vacation are obtained as :

$$P'_{0,0}(t) = -\lambda P_{0,0}(t) + \alpha P_{0,1}(t)$$

$$P'_{0,1}(t) = -(\lambda + \alpha) P_{0,1}(t) + \mu P_{1,2}(t)$$

$$P'_{1,2}(t) = -(\lambda + \mu) P_{1,2}(t) + \mu P_{2,2}(t) + \alpha P_{1,1}(t) + \lambda P_{0,0}(t)$$

$$P'_{n,1}(t) = -(\lambda + \alpha) P_{n,1}(t) + \lambda P_{n-1,1}(t), \quad n \geq 1$$

$$P'_{n,2}(t) = -(\lambda + \mu) P_{n,2}(t) + \lambda P_{n-1,2}(t) + \mu P_{n+1,2}(t) + \alpha P_{n,1}(t), n \geq 1$$

STEADY STATE SOLUTION

Assuming the existence of the steady state, the above set of equations yields the following steady state equations.

$$0 = -\lambda P_{0,0} + \alpha P_{0,1} \qquad \ldots(15)$$

$$0 = -(\lambda + \alpha) P_{0,1} + \mu P_{1,2} \qquad \ldots(16)$$

$$0 = -(\lambda + \mu) P_{1,2} + \mu P_{2,2} + \alpha P_{1,1} + \lambda P_{0,0} \qquad \ldots(17)$$

$$0 = -(\lambda + \alpha) P_{n,1} + \lambda P_{n-1,1} , \quad n \geq 1 \qquad \ldots(18)$$

$$0 = -(\lambda + \mu) P_{n,2} + \mu P_{n+1,2} + \lambda P_{n-1,2} + \alpha P_{n,1} , \quad n \geq 2 \qquad \ldots(19)$$

Using (15), (16) and (17) we have

$$P_{0,1} = \frac{\lambda}{\alpha} P_{0,0} \qquad \ldots(20)$$

$$P_{1,2} = \frac{\lambda+\alpha}{\mu} P_{0,1} \qquad \ldots(21)$$

$$P_{n,1} = \frac{\lambda}{\lambda+\alpha} P_{n-1,1} , \quad n \geq 1$$

By iteration we find

$$P_{n,1} = \left[\frac{\lambda}{\lambda+\alpha}\right]^{n} P_{0,1} , \quad n \geq 1 \qquad \ldots(22)$$

$$P_{n,1} = \left[\frac{\lambda}{\lambda+\alpha}\right]^{n} \frac{\lambda}{\alpha} P_{0,0} , \quad n \geq 1 \qquad \ldots(23)$$

Now we shall derive the expressions for $P_{n,2}$ with $n \geq 2$

From (19) we have

$$(\lambda + \mu) P_{n+1,2} - \mu P_{n+2,2} - \lambda P_{n,2} = \alpha P_{n+1,1} , \quad n \geq 1$$

Using the shifting operator E, the equation becomes

$$h(E)\ (P_{n,2}) = \alpha P_{n+1,1} = \lambda\left[\frac{\lambda}{\lambda+\alpha}\right]^{n+1} P_{0,0},\quad n \geq 1$$

Where $h(Z) = [\ (\lambda+\mu)Z - \mu Z^2 - \lambda\ ]$

The roots of $h(Z) = 0$ are $1, \frac{\lambda}{\mu}$. Assuming $\frac{\lambda}{\mu} < 1$,

$$P_{n,2} = A\left(\frac{\lambda}{\mu}\right)^n + \frac{\lambda\left[\frac{\lambda}{\lambda+\alpha}\right]^{n+1} P_{0,0}}{(\lambda+\mu)\left(\frac{\lambda}{\lambda+\alpha}\right) - \mu\left(\frac{\lambda}{\lambda+\alpha}\right)^2 - \lambda},\quad n \geq 1$$

$$= A\left(\frac{\lambda}{\mu}\right)^n + \frac{\lambda^2\left[\frac{\lambda}{\lambda+\alpha}\right]^{n-1} P_{0,0}}{\alpha\,[\mu-\lambda-\alpha]},\quad n \geq 1 \qquad \ldots(24)$$

Using (24) and (21)

$$\left(\frac{\lambda+\alpha}{\mu}\right)\left(\frac{\lambda}{\alpha}\right)P_{0,0} = A\,\frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu-\lambda-\alpha}\right)\left(\frac{\lambda}{\alpha}\right) P_{0,0}$$

$$A = \frac{\mu}{\lambda}\left[\frac{(\lambda+\alpha)^2 - \alpha\mu}{\mu\ (\lambda+\alpha-\mu)}\right]\frac{\lambda}{\alpha}\,P_{0,0}$$

Thus

$$P_{n,2} = \frac{\lambda}{\alpha}\,\frac{P_{0,0}}{(\lambda+\alpha-\mu)}\left[\frac{(\lambda+\alpha)^2-\alpha\mu}{\mu}\left[\frac{\lambda}{\mu}\right]^{n-1} - \lambda\left[\frac{\lambda}{\lambda+\alpha}\right]^{n-1}\right],\quad n \geq 1 \qquad \ldots(25)$$

Now the steady state probability $P_{0,0}$ can be derived from the boundary condition.

The boundary condition implies,

$$P_{0,0} + \sum_{n=0}^{\infty} [P_{n,1} + P_{n,2}] = 1$$

$$P_{0,0} + \sum_{0} \left(\frac{\lambda}{\lambda + \alpha}\right)^{n} \frac{\lambda}{\lambda \ \alpha} \cdot P_{0\ 0} + \sum_{n=1}^{\infty} \left[A\left(\frac{\lambda}{\mu}\right)^{n} + \frac{\lambda^{2}\left(\frac{\lambda}{\lambda+\alpha}\right)^{n-1}}{\alpha(\mu-\lambda-\alpha)} P_{0\ 0} \right] = 1$$

Simplifying

$$P_{0\ 0}\left[1 + \frac{\lambda}{\alpha} \frac{1}{\lambda+\alpha-\mu} \left(\frac{(\lambda+\alpha)^{2} - \alpha\mu}{\mu - \lambda} - \frac{(\lambda+\alpha)\ (\mu-\lambda)}{\alpha} \right) \right] = 1$$

We shall now calculate the expected queue length L_{q2}

$$L_{q2} = \sum_{n=1}^{\infty} nP_{n\ 1} + \sum nP_{n+1\ \ 2}$$

$$= \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\lambda+\alpha}\right)^{n} \frac{\lambda}{\alpha} P_{0,\,0} + \sum_{n=1}^{\infty} n \left[A \left(\frac{\lambda}{\mu}\right)^{n+1} + K\left(\frac{\lambda}{\lambda+\alpha}\right)^{n} \right]$$

where $K = \frac{\lambda^{2}}{\alpha(\mu-\lambda-\alpha)} P_{0,0}$

Simplifying we have

$$L_{q2} = \frac{A\ \lambda^{2}}{(\mu-\lambda)^{2}} + \frac{\lambda^{2}}{\alpha^{3}} P_{0,0} \frac{(\lambda+\alpha)\ (\mu-\alpha)}{(\mu - \lambda - \alpha)}$$

DISTRIBUTION OF THE WAITING TIME FOR THE SYSTEM

Let the random variable T denote the waiting, time in the queue for an arriving unit in the steady state. An arriving unit may find the system in any of the following states :

(i) $(0,1)$

(ii) $(n,1)$, $n \geq 1$

(iii) $(n,2)$, $n \geq 1$

In case of (i), the arriving unit has to wait for the return of the server from vacation.

In case of (ii), arriving unit has to wait for n service completions and for the server to return from the vacation.

In case of (iii), the arriving unit has to wait for n service completions.

The probability density function $\nu(t)$ of T - the waiting time of arriving unit is

$$\nu(t) = P_{0,1}\, \alpha\, e^{-\alpha t} + \sum_{n=1}^{\infty} P_{n,1} \int_0^t \alpha e^{-\alpha h}\, \mu\, \frac{e^{-\mu(t-h)}\,[\mu\,(t-h)]^{n-1}}{\lfloor n-1} \, dh$$

$$+ \sum_{n=1}^{\infty} P_{n,2}\, \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{\lfloor n-1}$$

The Second term becomes.

$$\frac{\lambda^2 \mu}{(\lambda + \alpha)} P_{0,0} \int_0^t e^{-\mu t} \left[\frac{\alpha}{\lambda + \alpha}\right] e^{-h\alpha \frac{(\lambda + \alpha - \mu)}{\lambda + \alpha}}$$

$$= - \frac{\lambda^2 \mu}{\alpha(\lambda + \alpha - \mu)} P_{0,0} \left[e^{-\alpha t} - e^{-\frac{\mu t \alpha}{\lambda + \alpha}} \right]$$

The 3rd term gives

$$\sum_{n=1}^{\infty} \left[A \left(\frac{\lambda}{\mu}\right)^n + K \left(\frac{\lambda}{\lambda+\alpha}\right)^{n-1} \right] \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{\lfloor n-1}$$

where $K = \dfrac{\lambda^2 P_{0,0}}{\alpha(\mu - \lambda - \alpha)}$

$$= A\lambda e^{-(\mu-\lambda)t} + K\mu e^{-\mu t}\left(\frac{\alpha}{\lambda+\alpha}\right)$$

Then

$$\nu(t) = \left[\lambda e^{-\alpha t} - \frac{\lambda^2\mu}{\alpha(\lambda+\alpha-\mu)}\left(e^{-\alpha t} - e^{-\frac{\mu\alpha t}{\lambda+\alpha}}\right)\right] P_{0,0}$$

$$+ A\lambda e^{-(\mu-\lambda)t} + k\mu e^{-\mu t}\frac{\alpha}{\lambda+\alpha}$$

The expected waiting time W_{q_2} is given by

$$W_{q_2} = \int_0^\infty t\,\nu(t)\,dt$$

$$= \frac{\lambda P_{0,0}}{\alpha^2} - \frac{\lambda^2\mu P_{0,0}}{\alpha(\lambda+\alpha-\mu)}\left[\frac{1}{\alpha^2} - \frac{(\lambda+\alpha)^2}{\mu^2\alpha^2}\right]$$

$$+ \frac{A\lambda}{(\mu-\lambda)^2} + K\mu\frac{(\lambda+\alpha)^2}{\mu^2\alpha^2}$$

Substituting for K and simplifying we get

$$W_{q_2} = \frac{A\lambda}{(\mu-\lambda)^2} + \frac{\lambda^2}{\alpha^3}\left[\frac{\mu}{\mu-\lambda-\alpha} + \frac{\alpha}{\lambda}\right] P_{0,0}$$

$$= \frac{A\lambda}{(\mu - \lambda)^2} + \frac{\lambda}{\alpha^3} \frac{(\lambda+\alpha) \quad (\mu - \alpha)}{(\mu - \lambda - \alpha)} P_{0,0}$$

$$= \frac{1}{\lambda} \left[\frac{A\lambda^2}{(\mu-\lambda)^2} + \frac{P_{0,0}\lambda^2 (\lambda + \alpha) \quad (\mu - \alpha)}{\alpha^3 (\mu - \lambda - \alpha)} \right]$$

$$= \frac{1}{\lambda} (L_{q_2})$$

Thus Little formula is verified.

Chapter III

CHAPTER - III

A GENERAL BULK SERVICE QUEUE WITH SERVER'S VACATION

This chapter deals with a single server queue in which the waiting space of the queue is assumed to be infinite. Arrival process is Poisson with parameter λ. The server processes the customers in batches according to the general bulk service rule introduced by Neuts [9]. According to this rule, the service starts only when a minimum 'a' customers are present in the queue, the maximum service capacity being 'b'. Services are thought of as the removal of groups of customers to a different location. The times between the departures and subsequent returns of the server are assumed to have an exponential distribution with parameter 'μ'. All service times are mutually independent of each other and also of the interarrival times of customers. Symbolically this queueing model may be denoted by $M/M_{a,b}/1$.

We consider this model under two types of vacations. In section 1 of this chapter this model is discussed under repeated vacation and section 2 under single vacation.

SECTION 3.1

$M/M_{a,b}/1$ WITH REPEATED VACATION

If the server completes a service and finds less than the quorum of 'a' customers in the queue, he leaves for a random period of time called 'vacation'. On returning to the system, after a vacation, if the server finds less than 'a' waiting customers, immediately takes another vacation.

STATE SPACE OF THE SYSTEM

This queueing model is studied as a Markov process on the state space $\left\{(n\,,\,j)\,,\,n \geq 0\,,\,j = 1\,,\,2\right\}$.

The process is in the state :

1) $(n\,,\,1)\,,\,n \geq 0$ when there are 'n' customers waiting in the queue for service and the server is away for vacation.

2) $(n\,,\,2)\,,\,n \geq 0$ when there are 'n' customers waiting in the queue and the server is busy.

Define,

$$P_{n\,,\,j}(t) = P_r\left\{\text{At time } t\,,\text{ the system is in the state } (n,j)\,,\,n \geq 0\,,\,j = 1\,,\,2\right\}.$$

DIFFERENCE EQUATIONS

The time dependent difference equations for the $M/M_{a,b}/1$ model with repeated vacation are obtained as:

$$P_{0,1}(t+\Delta t) = P_{0,1}(t)\,[1-\lambda\Delta t] + P_{0,2}(t)\,\mu\Delta t + 0(\Delta t)$$

$$P_{0,2}(t+\Delta t) = P_{0,2}(t)\,[1-(\lambda+\mu)\Delta t] + \sum_{n=a}^{b} P_{n,2}(t)\,\mu\Delta t$$

$$+ \sum_{n=a}^{b} P_{n,1}(t)\alpha\,\Delta t + 0(\Delta t)$$

$$P_{n,1}(t+\Delta t) = P_{n,1}(t)\,[1-\lambda\Delta t] + P_{n,2}(t)\,\mu\Delta t$$

$$+ P_{n-1,1}(t)\,\lambda\Delta t + 0(\Delta t),\quad 1 \le n \le a-1$$

$$P_{n,2}(t+\Delta t) = P_{n,2}(t)\,[1-(\lambda+\alpha)\Delta t] + P_{b+n,2}(t)\,\mu\Delta t$$

$$+ P_{n-1,2}(t)\,\lambda\Delta t + P_{b+n,1}(t)\alpha\Delta t + 0(\Delta t),\quad n \ge 1$$

$$P_{n,1}(t+\Delta t) = P_{n,1}(t)\,[1-(\lambda+\alpha)\Delta t] + P_{n-1,1}(t)\,\lambda\Delta t$$

$$+ 0(\Delta t),\quad n \ge a$$

DIFFERENCE - DIFFERENTIAL EQUATIONS

Dividing the above equations by Δt and taking the limit as $\Delta t \to o$, We get the difference - differential equations , differential equations in time and difference equations in the state of the system as follows :

$$P_{0,1}'(t) = -\lambda P_{0,1}(t) + \mu P_{0,2}(t)$$

$$P_{0,2}'(t) = -(\lambda+\mu) P_{0,2}(t) + \mu \sum_{n=a}^{b} P_{n,2}(t) + \alpha \sum_{n=a}^{b} P_{n,1}(t)$$

$$P_{n,1}'(t) = -\lambda P_{n,1}(t) + \mu P_{n,2}(t) + \lambda P_{n-1,1}(t), 1 \le n \le a-1$$

$$P_{n,2}'(t) = -(\lambda+\mu) P_{n,2}(t) + \mu P_{b+n,2}(t) + \lambda P_{n-1,2}(t) + \alpha P_{b+n,1}(t), \quad n \ge 1$$

$$P_{n,1}'(t) = -(\lambda+\alpha) P_{n,1}(t) + \lambda P_{n-1,1}(t), \quad n \ge a$$

Now we shall calculate the probability distribution under the steady state .

STEADY STATE SOLUTION

In case of steady state , the above difference - differential equations , take the form :

$$0 = -\lambda P_{0,1} + \mu P_{0,2} \quad \text{....(1)}$$

$$0 = -(\lambda+\mu) P_{0,2} + \mu \sum_{n=a}^{b} P_{n,2} + \alpha \sum_{n=a}^{b} P_{n,1} \quad \text{....(2)}$$

$$0 = -\lambda P_{n,1} + \mu P_{n,2} + \lambda P_{n-1,1}, 1 \le n \le a-1 \quad \text{....(3)}$$

$$0 = -(\lambda+\mu) P_{n,2} + \mu P_{b+n,2} + \lambda P_{n-1,2} + \alpha P_{b+n,1}, \quad n \ge 1 \quad \text{....(4)}$$

$$0 = -(\lambda+\alpha)\, P_{n,1} + \lambda P_{n-1,1}, \quad n \geq a \qquad \ldots.(5)$$

Equations (1) and (5) can be rewritten as

$$P_{0,1} = \frac{\mu}{\lambda}\, P_{0,2} \qquad \ldots.(6)$$

$$P_{n,1} = \frac{\lambda}{\lambda+\alpha}\, P_{n-1,1}, \quad n \geq a$$

By iteration we find

$$P_{n,1} = \left[\frac{\lambda}{\lambda+\alpha}\right]^{n-(a-1)} P_{a-1,1}, \quad n \geq a \qquad \ldots.(7)$$

Adding equations of (3) over n = 1 to k, $0 \leq k \leq a-1$ we get

$$P_{k,1} = \frac{\mu}{\lambda} \sum_{n=0}^{k} P_{n,2}, \quad 1 \leq k \leq a-1 \qquad \ldots.(8)$$

Now we shall calculate the expressions for $P_{n,2}$, $n \geq 0$

From (4) we have

$$(\lambda+\mu)\, P_{n,2} - \mu P_{b+n,2} - \lambda P_{n-1,2} = \alpha\, P_{b+n,1}, \quad n \geq 1$$

This is euivalent to

$$(\lambda+\mu)\, P_{n+1,2} - \mu P_{b+n+1,2} - \lambda P_{n,2} = \alpha P_{b+n+1,1}, \quad n \geq 0$$

This equation can be written as

$$h(E)\ \{P_{n,2}\} = \alpha P_{b+n+1,1}, \quad n \geq 0$$

such that the characteristic equation is

$$h(Z) = [\,(\lambda + \mu)Z - \mu Z^{b+1} - \lambda\,] = 0 \qquad \ldots.(9)$$

Taking $f(Z) = (\lambda + \mu)Z$ and $g(Z) = -\mu Z^{b+1} - \lambda$

For $|Z| = 1$, we have $|g(Z)| < |f(Z)|$ and applying Rouche's theorem, there will be only one zero of $h(Z)$, say r_0, inside $|Z| = 1$.

Then

$$P_{n,2} = A r_0^n + \frac{\alpha \left[\frac{\lambda}{\lambda+\alpha}\right]^{n+b-(a-2)} P_{a-1,1}}{(\lambda+\mu)\left(\frac{\lambda}{\lambda+\alpha}\right) - \mu \left(\frac{\lambda}{\lambda+\alpha}\right)^{b+1} - \lambda}, \quad n \geq 0$$

$$= A r_0^n + \frac{\lambda^b \alpha \left[\frac{\lambda}{\lambda+\alpha}\right]^{n-(a-1)} P_{a-1,1}}{(\lambda+\alpha)^b(\lambda+\mu) - \lambda^b \mu - (\lambda+\alpha)^{b+1}}$$

Putting $n = 0$ we get

$$P_{0,2} = A + \frac{\lambda^b \alpha \left[\frac{\lambda}{\lambda+\alpha}\right]^{-(a-1)} P_{a-1,1}}{(\lambda+\alpha)^b(\lambda+\mu) - \lambda^b \mu - (\lambda+\alpha)^{b+1}}$$

This gives

$$A = P_{0,2} - \frac{\lambda^b \alpha \left[\frac{\lambda}{\lambda+\alpha}\right]^{-(a-1)} P_{a-1,1}}{(\lambda+\alpha)^b(\lambda+\mu) - \lambda^b \mu - (\lambda+\alpha)^{b+1}}$$

Hence

$$P_{n\ 2} = \left[P_{0,2} - \frac{\lambda^{b} \alpha \left[\frac{\lambda}{\lambda+\alpha}\right]^{-(a-1)} P_{a-1,1}}{(\lambda+\alpha)^{b}(\lambda+\mu)-\lambda^{b}\mu-(\lambda+\alpha)^{b+1}} \right] r_0^n$$

$$+ \frac{\lambda^{b} \alpha \left[\frac{\lambda}{\lambda+\alpha}\right]^{n-(a-1)} P_{a-1,1}}{(\lambda+\alpha)^{b}(\lambda+\mu)-\lambda^{b}\mu-(\lambda+\alpha)^{b+1}}, \quad n \geq 0$$

....(10)

$$P_{n,2} = [P_{0,2} - B_1 P_{a-1,1}]\, r_0^n + B_1 \left[\frac{\lambda}{\lambda+\alpha}\right]^n P_{a-1,1}, \quad n \geq 0$$

....(11)

Where,

$$B_1 = \frac{\lambda^{b} \alpha \left[\frac{\lambda}{\lambda+\alpha}\right]^{-(a-1)}}{(\lambda+\alpha)^{b}(\lambda+\mu)-\lambda^{b}\mu-(\lambda+\alpha)^{b+1}}$$

....(12)

Substituting $P_{n,2}, P_{n,1}$ for $n \geq a$ in (2) we have

$$0 = -(\lambda+\mu) P_{0,2} + \mu \sum_{n=a}^{b} \left[(P_{0,2} - B_1 P_{a-1,1})\, r_0^n + B_1 \left[\frac{\lambda}{\lambda+\alpha}\right]^n P_{a-1,1} \right] + \alpha \sum_{n=a}^{b} \left[\frac{\lambda}{\lambda+\alpha}\right]^{n-(a-1)} P_{a-1,1}$$

Simplifying

$$P_{0,2} = \frac{P_{a-1,1}}{\lambda+\mu-\mu\left(\frac{r_0^a - r_0^{b+1}}{1-r_0}\right)} \left[B_1\mu\left(\frac{\lambda+\alpha}{\alpha}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^a - \left(\frac{\lambda}{\lambda+\alpha}\right)^{b+1}\right] - \left[\frac{r_0^a - r_0^{b+1}}{1-r_0}\right]\right) + \lambda\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{b-(a+1)}\right]\right] \quad \ldots.(13)$$

$$P_{0,2} = H_1 \; P_{a-1,1} \quad \ldots.(14)$$

Where,

$$H_1 = \frac{1}{\lambda+\mu-\mu\left(\frac{r_0^a - r_0^{b+1}}{1-r_0}\right)} \left[B_1\mu\left(\frac{\lambda+\alpha}{\alpha}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^a - \left(\frac{\lambda}{\lambda+\alpha}\right)^{b+1}\right] - \left[\frac{r_0^a - r_0^{b+1}}{1-r_0}\right]\right) + \lambda\left(1-\left[\frac{\lambda}{\lambda+\alpha}\right]^{b-a+1}\right)\right] \quad \ldots(15)$$

Substituting this in (6) and (13) we get

$$P_{0,1} = \frac{\mu}{\lambda} H_1 \; P_{a-1,1} \quad \ldots.(16)$$

and

$$P_{n,2} = [H_1 \; P_{a-1,1} - B_1 \; P_{a-1,1}] \; r_0^n + B_1 \left[\frac{\lambda}{\lambda+\alpha}\right]^n P_{a-1,1}, \; n \geq 0$$

$$P_{n,2} = P_{a-1,1}\left[(H_1 - B_1)\, r_0^n + \left[\frac{\lambda}{\lambda + \alpha}\right]^n B_1\right], n \geq 0 \qquad \ldots.(17)$$

Now equation (8) becomes

$$P_{k,1} = \frac{\mu}{\lambda} \sum_{n=0}^{k} P_{a-1,1}\left[(H_1 - B_1)\, r_0^n + \left(\frac{\lambda}{\lambda + \alpha}\right)^n B_1\right], 1 \leq k \leq a-1 \qquad \ldots.(18)$$

The only probability $P_{a-1,1}$ can be derived from the normalising condition.

The normalising condition implies

$$P_{0,1} + \sum_{n=0}^{\infty} P_{n,2} + \sum_{n=1}^{a-1} P_{n,1} + \sum_{n=a}^{\infty} P_{n,1} = 1 .$$

Substituting for $P_{0,1}$, $P_{n,2}$ and $P_{n,1}$ from (16),(17),(18) and (7), we get,

$$\begin{aligned}
&\frac{\mu}{\lambda} H_1 P_{a-1,1} + \sum_{n=0}^{\infty}\left[(H_1 - B_1)\, r_0^n + \left[\frac{\lambda}{\lambda + \alpha}\right]^n B_1\right] P_{a-1,1} \\
&+ \sum_{n=1}^{a-1} \frac{\mu}{\lambda} P_{a-1,1} \sum_{j=0}^{n}\left[(H_1 - B_1)\, r_0^j + \left(\frac{\lambda}{\lambda + \alpha}\right)^j B_1\right] \\
&+ \sum_{n=a}^{\infty}\left[\frac{\lambda}{\lambda + \alpha}\right]^{n-(a-1)} P_{a-1,1} = 1
\end{aligned}$$

On simplification we get

$$P_{a-1,1} = \left[H_1 \frac{\mu}{\lambda} + \frac{H_1 - B_1}{1 - r_0} + B_1\left(\frac{\lambda + \alpha}{\alpha}\right) + \frac{\lambda}{\alpha} \right.$$

$$+ \frac{\mu}{\lambda}(a-1)\left[\frac{H_1 - B_1}{1 - r_0} - B_1\left(\frac{\lambda + \alpha}{\alpha}\right)\right]$$

$$- \frac{\mu}{\lambda}(H_1 - B_1)\; r_0^2 \;\frac{(1-r_0^{a-1})}{(1-r_0)^2}$$

$$\left. - \frac{\mu}{\lambda} B_1 \left(\frac{\lambda}{\alpha}\right)^2 \left(1 - \left(\frac{\lambda}{\lambda + \alpha}\right)^{a-1}\right)\right]^{-1} \qquad \ldots.(19)$$

Now we shall calculate the expected queue Length L_q.

EXPECTED QUEUE LENGTH

$$L_q = \sum_{n=1}^{a-1} nP_{n,1} + \sum_{n=a}^{\infty} n\, P_{n,1} + \sum_{n=1}^{\infty} n\, P_{n,2}$$

$$= \sum_{n=1}^{a-1} n \frac{\mu}{\lambda} \sum_{j=0}^{n} P_{a-1,1} \left[(H_1 - B_1)\, r_0^j + \left(\frac{\lambda}{\lambda + \alpha}\right)^j B_1\right]$$

$$+ \sum_{n=a}^{\infty} n \left[\frac{\lambda}{\lambda + \alpha}\right]^{n-(a-1)} P_{a-1,1}$$

$$+ \sum_{n=1}^{\infty} n \left[(H_1 - B_1)\, r_0^n + \left(\frac{\lambda}{\lambda + \alpha}\right)^n B_1\right] P_{a-1,1} \; .$$

Taking $\rho = \dfrac{\lambda}{\lambda + \alpha}$

$$= \sum_{n=1}^{a-1} n \frac{\mu}{\lambda} P_{a-1,1} \left\{ (H_1 - B_1) \left[\frac{1-r_0^{n+1}}{1-r_0} \right] + B_1 \frac{1-\rho^{n+1}}{1-\rho} \right\}$$

$$+ \rho^{-(a-2)} P_{a-1,1} \sum_{n=a}^{\infty} n \rho^{n-1} + P_{a-1,1} \sum_{n=1}^{\infty} n \left[(H_1 - B_1) r_0^n + B_1 \rho^n \right]$$

$$= \frac{\mu}{\lambda} P_{a-1,1} \left[\frac{H_1 - B_1}{1 - r_0} \left\{ \frac{a(a-1)}{2} - r_0^2 \sum_{n=1}^{a-1} n r_0^{n-1} \right\} + \frac{B_1}{1-\rho} \left\{ \frac{a(a-1)}{2} - \rho^2 \sum_{n=1}^{a-1} n \rho^{n-1} \right\} \right] +$$

$$\rho^{-(a-2)} P_{a-1,1} \frac{d}{d\rho} \left(\frac{\rho^a}{1-\rho} \right)$$

$$+ P_{a-1,1} (H_1 - B_1) r_0 \frac{d}{dr_0} \left(\frac{r_0}{1-r_0} \right) + B_1 P_{a-1,1} \rho \frac{d}{d\rho} \left(\frac{\rho}{1-\rho} \right)$$

Simplifying

$$L_q = P_{a-1,1} \left[\frac{\mu}{\lambda} - \frac{a(a-1)}{2} \left(\frac{H_1 - B_1}{1 - r_0} + \frac{B_1}{1-\rho} \right) \right.$$

$$- \frac{\mu}{\lambda} \frac{H_1 - B_1}{(1-r_0)^3} r_0^2 \left[1 - a r_0^{a-1} + (a-1) r_0^a \right]$$

$$- \frac{\mu}{\lambda} B_1 \frac{\rho^2}{(1-\rho)^2} \left[1 - a \rho^{a-1} + (a-1) \rho^a \right]$$

$$\left. + \frac{\rho \left[a-(a-1)\rho \right]}{(1-\rho)^2} + (H_1 - B_1) \frac{r_0}{(1-r_0)^2} + B_1 \frac{\rho}{(1-\rho)^2} \right]$$

....(20)

In particular when a = b = 1

1. Equation (7) implies

$$P_{n,1} = \left[\frac{\lambda}{\lambda+\alpha}\right]^{n} P_{0,1} \quad , \quad n \geq 0$$

2. Equation (9) becomes $(\lambda + \mu)Z - \mu Z^2 - \lambda = 0$

The root less than 1 of the equation is $r_0 = \dfrac{\lambda}{\mu}$

$$B_1 = \frac{\lambda}{\mu - \lambda - \alpha} \qquad \text{[From (12)]}$$

$$H_1 = \frac{\lambda}{\mu} \qquad \text{[From (15)]}$$

Thus $$P_{n,2} = \frac{P_{0,1}}{\lambda+\alpha-\mu}\left[(\lambda+\alpha)\left(\frac{\lambda}{\mu}\right)^{n+1} - \lambda\left(\frac{\lambda}{\lambda+\alpha}\right)^{n}\right],$$

$n \geq 0$

3. $$P_{a-1,1} = P_{0,1} = \frac{\alpha(\mu-\lambda)}{\mu(\lambda+\alpha)} \qquad \text{[From (19)]}$$

4. Equation (20) implies

$$L_q = \left[\frac{\rho}{(1-\rho)^2} + (H_1 - B_1)\frac{r_0}{(1-r_0)^2} + B_1\frac{\rho}{(1-\rho)^2}\right] P_{0,1}$$

$$= P_{0,1}\left[\frac{\lambda}{\alpha^2}(\lambda+\alpha) + \frac{\lambda^2(\lambda+\alpha)}{\lambda+\alpha-\mu}\left(\left[\frac{1}{\mu-\lambda}\right]^2 - \frac{1}{\alpha^2}\right)\right]$$

The above values $P_{n,1}$, $P_{n,2}$, $P_{0,1}$ and L_q coincide with the corresponding expressions of M/M/1 queue with repeated vacation. This can be seen from the equations (6),(8),(9) and (10) of chapter II.

SECTION 3.2

$M/M_{a,b}/1$ WITH SINGLE VACATION

If the server completes a service and finds less than the minimum number of 'a' customers in the queue, he leaves for a random period of time called 'vacation'. On returning to the system, if the server finds less than 'a' waiting customer at the end of a vacation, he stays in the system itself until the queue size becomes 'a'.

STATE SPACE OF THE SYSTEM

This queueing model is studied as a Markov process on the state space $\left\{(n, j), n \geq 0, j = 0,1,2\right\}$.

The process is in the state :

1. $(n,0)$, $n \geq 0$ when there are 'n' customers waiting in the queue for service and the server is idle.

2. $(n,1)$, $n \geq 0$ when there are 'n' customers waiting in the queue for service and the server is away for vacation.

3. $(n,2)$, $n \geq 0$ when there are 'n' customers waiting in the queue for service and the server is busy.

Define ,

$$P_{n,j}(t) = P_r \left\{ \text{At time } t \text{ , the system is in the state } (n , j) , n \geq 0 , j = 0,1,2 \right\} .$$

DIFFERENCE - DIFFERENTIAL EQUATIONS

The time dependent difference equations for the $M/M_{a,b}/1$ model with single vacation are obtained as :

$$P_{0,0}(t + \Delta t) = P_{0,0}(t) \; [1- \lambda \; \Delta t] + P_{0,1}(t)\alpha \; \Delta t + o(\Delta t)$$

$$P_{0,1}(t + \Delta t) = P_{0,1}(t) \; [1-(\lambda + \alpha) \; \Delta t] + P_{0,2}(t) \; \mu \; \Delta t + o(\Delta t)$$

$$P_{0,2}(t + \Delta t) = P_{0,2}(t) \; [1-(\lambda + \mu) \; \Delta t] + \sum_{n=a}^{b} P_{n,2}(t) \; \mu \; \Delta t + \sum_{n=a}^{b} P_{n,1}(t)\alpha\Delta t + P_{a-1,0}(t) \; \lambda \; \Delta t + o(\Delta t)$$

$$P_{n,0}(t + \Delta t) = P_{n,0}(t) \; [1 - \lambda\Delta t] + P_{n,1}(t) \; \alpha \; \Delta t + P_{n-1,0}(t)\lambda \; \Delta t + o(\Delta t) \quad , \quad 1 \leq n \leq a-1$$

$$P_{n,1}(t + \Delta t) = P_{n,1}(t) \; [1-(\lambda + \alpha) \; \Delta t] + P_{n,2}(t) \; \mu \; \Delta t + P_{n-1,1}(t)\lambda \; \Delta t + o(\Delta t) \quad , \quad 1 \leq n \leq a-1$$

$$P_{n,2}(t+\Delta t) = P_{n,2}(t)\,[1-(\lambda+\mu)\,\Delta t] + P_{b+n,2}(t)\,\mu\Delta t$$

$$+ P_{n-1,2}(t)\lambda\,\Delta t + P_{b+n,1}(t)\alpha\,\Delta t + o(\Delta t)\,, \quad n\geq 1$$

$$P_{n,1}(t+\Delta t) = P_{n,1}(t)\,[1-(\lambda+\alpha)\,\Delta t] + P_{n-1,1}(t)\,\lambda\Delta t$$

$$+ o(\Delta t)\,, \quad n \geq a$$

Thus the difference - differential equations are

$$P'_{0,0}(t) = -\lambda P_{0,0}(t) + \alpha\, P_{0,1}(t)$$

$$P'_{0,1}(t) = -(\lambda+\alpha)\, P_{0,1}(t) + \mu\, P_{0,2}(t)$$

$$P'_{0,2}(t) = -(\lambda+\mu)\, P_{0,2}(t) + \mu \sum_{n=a}^{b} P_{n,2}(t)$$

$$+ \alpha \sum_{n=a}^{b} P_{n,1}(t) + \lambda\, P_{a-1,0}(t)$$

$$P'_{n,0}(t) = -\lambda\, P_{n,0}(t) + \alpha P_{n,1}(t) + \lambda P_{n-1,0}(t),\ 1\leq n\leq a-1$$

$$P'_{n,1}(t) = -(\lambda+\alpha)\, P_{n,1}(t) + \mu\, P_{n,2}(t) + \lambda\, P_{n-1,1}(t)\,,$$

$$1\leq n \leq a-1$$

$$P'_{n,2}(t) = -(\lambda+\mu)\, P_{n,2}(t) + \mu\, P_{b+n,2}(t) + \lambda\, P_{n-1,2}(t)$$

$$+ \alpha\, P_{b+n,1}(t)\,, \quad n\geq 1$$

$$P'_{n,1}(t) = -(\lambda+\alpha) P_{n,1}(t) + \lambda P_{n-1,1}(t), \quad n \geq a$$

STEADY STATE SOLUTION

Now we shall calculate the probability distribution under the steady state. In case of the steady state, the above difference - differential equations take the form :

$$0 = -\lambda P_{0,0} + \alpha P_{0,1} \qquad \ldots.(21)$$

$$0 = -(\lambda+\alpha) P_{0,1} + \mu P_{0,2} \qquad \ldots.(22)$$

$$0 = -(\lambda+\mu) P_{0,2} + \mu \sum_{n=a}^{b} P_{n,2} + \alpha \sum_{n=a}^{b} P_{n,1} + \lambda P_{a-1,0} \qquad \ldots.(23)$$

$$0 = -\lambda P_{n,0} + \alpha P_{n,1} + \lambda P_{n-1,0}, \quad 1 \leq n \leq a-1 \qquad \ldots.(24)$$

$$0 = -(\lambda+\alpha) P_{n,1} + \mu P_{n,2} + \lambda P_{n-1,1}, \quad 1 \leq n \leq a-1 \qquad \ldots.(25)$$

$$0 = -(\lambda+\mu) P_{n,2} + \mu P_{b+n,2} + \lambda P_{n-1,2} + \alpha P_{b+n,1}, \quad n \geq 1 \qquad \ldots.(26)$$

$$0 = -(\lambda+\alpha) P_{n,1} + \lambda P_{n-1,1}, \quad n \geq a \qquad \ldots.(27)$$

Equation (21) implies

$$P_{0,0} = \frac{\alpha}{\lambda} P_{0,1} \qquad \ldots.(28)$$

From equation (27) we get

$$P_{n,1} = \left[\frac{\lambda}{\lambda+\alpha}\right]^{n-(a-1)} P_{a-1,1}, \; n \geq a \qquad \ldots.(29)$$

Proceeding as in the case of Repeated vacation equation (26) implies

$$P_{n,2} = \left[\frac{\lambda+\alpha}{\mu} P_{0,1} - B_2 P_{a-1,1}\right] r_0^n + B_2 \left[\frac{\lambda}{\lambda+\alpha}\right]^n P_{a-1,1}, \; n \geq 0 \qquad \ldots.(30)$$

Where $$B_2 = \frac{\lambda^b \alpha \left[\frac{\lambda}{\lambda+\alpha}\right]^{-(a-1)}}{(\lambda+\alpha)^b (\lambda+\mu) - \lambda^b \mu - (\lambda+\alpha)^{b+1}} \qquad \ldots.(31)$$

Using (24) and (21) we get

$$P_{k,0} = \frac{\alpha}{\lambda} \sum_{n=0}^{k} P_{n,1}, \; 1 \leq k \leq a-1 \qquad \ldots.(32)$$

The remaining expressions for $P_{n,1}$ $(1 \leq n \leq a-1)$ can be calculated by iteration using equation (25) and $P_{a-1,1}$ can be derived from the normalizing condition.

In particular when a = b = 1

1. $P_{0,0} = \frac{\alpha}{\lambda} P_{0,1}$ [From (28)]

2. $P_{n,1} = \left[\frac{\lambda}{\lambda + \alpha}\right]^{n} P_{0,1}\ ,\ n \geq 0$ [From (29)]

3. $B_2 = \frac{\lambda}{\mu - \lambda - \alpha}$ [From(31)]

$$\text{Thus } P_{n,2} = \frac{P_{0,1}}{\lambda + \alpha - \mu} \left[\frac{(\lambda + \alpha)^2 - \alpha\mu}{\mu} \left(\frac{\lambda}{\mu}\right)^n - \lambda \left(\frac{\lambda}{\lambda + \alpha}\right)^n \right] , \quad n \geq 0 \qquad \text{[From(30)]}$$

The above values $P_{0,0}$, $P_{n,1}$ and $P_{n,2}$ coincide with the corresponding expressions of M/M/1 queue with single vacation. This can be seen from the equations (20) , (22) and (25) of chapter II.

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