Chapter 8

(i, j)-Δ^*-Closed Sets in Bitopological Spaces

8.1 Introduction

The study of bitopological spaces was initiated by (Kelley, 1963). The classical theorems in general topological spaces become particular cases of the analogous theorems for bitopological spaces. All the concepts which are discussed in previous chapters are extended to bitopological spaces. In this chapter, a new class of closed sets, called (i, j)-Δ^*-closed sets are defined and their properties are studied. Further as an application of (i, j)-Δ^*-closed sets, four new spaces namely (i, j)-${}_{\Delta^*}T_{\delta}$-space, (i, j)-${}_{\Delta^*}T_{\delta g^*}$-space, (i, j)-${}_{g\delta}T_{\Delta^*}$-space and (i, j)-${}_{\delta g^\ddagger}T_{\Delta^*}$-space are established and their interrelations are studied. Also (i, j)-Δ^*-continuous functions, (i, j)-Δ^*-irresolute maps and (i, j)-Δ^*-locally closed sets are defined and their interrelations with various (i, j)-closed sets are discussed in this chapter.

8.2 (i, j)-Δ^*-Closed Sets

Definition 8.2.1 A subset A of a bitopological space (X, τ_i, τ_j) is called a **(i, j)-Δ^*-closed set** if τ_j-$\delta cl(A) \subseteq U$ whenever A □ U, U is τ_i-δg-open in (X, τ_i) where $i = 1, 2$ and $i \neq j$.

The family of all (i, j)-Δ^*-closed sets in (X, τ_i, τ_j) is denoted by $D_{\Delta^*}(i, j)$.

Remark 8.2.2 By setting $\tau_i = \tau_j = \tau$ in the definition 8.2.1, a (i, j)-Δ^*-closed set is a Δ^*-closed set in (X, τ).

Remark 8.2.3 In general $\Delta^* C(\tau_i, \tau_j) \neq \Delta^* C(\tau_j, \tau_i)$.

Counter example 8.2.4 Let X = {a, b, c}, τ_1={ϕ , X, {a, b}} and τ_2 ={ϕ , X, {a}, {a, b}, {a, c}}.

Then $D_{\Delta^*}(1,2)$ = {ϕ , X, {c}, {a, c}, {b, c}} and

$D_{\Delta^*}(2,1)$ = {ϕ , X, {b}, {c}, {a, b}, {a, c}, {b, c}}.

Thus $D_{\Delta^*}(1,2) \neq D_{\Delta^*}(2,1)$.

Remark 8.2.5 If $\tau_1 \square \tau_2$ in (X, τ_1, τ_2) then $D_{\Delta^*}(2,1)$ and $D_{\Delta^*}(1,2)$ are independent. That is if $\tau_1 \square \tau_2$ in (X, τ_1, τ_2) then every (2, 1) g-closed set is (1, 2) g-closed.

But this heriditary property is not preserved for Δ^*-closed sets in bitopology which is proved by the following counter example.

Counter example 8.2.6 Let X = {a, b, c}, τ_1={ ϕ , X, {a}}, τ_2 = { ϕ , X, {a}, {b, c}} and τ_3 = { ϕ , X, {a}, {a, b}, {a, c}}.

Then $D_{\Delta^*}(2,1)$ = { ϕ , X};

$D_{\Delta^*}(1,2)$ = { ϕ , X, {a}, {b, c}};

$D_{\Delta^*}(1,3)$ = { ϕ , X, {b, c}} and

$D_{\Delta^*}(3,1)$ = P(X) – {a}.

Therefore $D_{\Delta^*}(2,1) \square D_{\Delta^*}(1,2)$ where $\tau_1 \square \tau_2$ but $D_{\Delta^*}(1,3) \square D_{\Delta^*}(3,1)$ with $\tau_1 \square \tau_3$.

Proposition 8.2.7 If A is a τ_j-δ-closed subset of (X, τ_i, τ_j) then A is a (i, j)-Δ^*-closed set but not conversely.

Proof : Let A be a τ_j-δ-closed subset of (X, τ_i, τ_j). Then τ_j-δcl(A) = A. Let U $\square$ δGO(X, τ_i) such that A $\square$ U. Then τ_j-δcl(A) = A $\square$ U which implies that A is a (i, j)-Δ^*-closed set.

Counter example 8.2.8 Let X = {a, b, c}, τ_1 = {ϕ , X, {a}} and τ_2= {ϕ , X, {a}, {a, b}}.Then the subset {b, c} is (1, 2)-Δ^*-closed but not τ_2-δ-closed in (X, τ_1, τ_2).

Proposition 8.2.9 If A is both τ_i-δg-open and (i, j)-Δ^*-closed then A is τ_j-δ-closed.

Proof : Let A be both τ_i-δg-open and (i, j)-Δ^*-closed. Then we have A □ U and U □ δGO(X, τ_i) which implies that τ_j-δcl(A) □ U. Let U = A. Then we have τ_j-δcl(A) □ A which implies that A is a τ_j-δ-closed set.

Proposition 8.2.10 If A is both τ_i-δg-open and (i, j)-Δ^*-closed then A is τ_j-closed.

Proof : Follows from the above proposition and by the fact that every δ-closed set is closed.

Proposition 8.2.11 Let A be τ_i-δg-open and (i, j)-Δ^*-closed in (X, τ_i, τ_j). Suppose that F is τ_j-δ-closed in (X, τ_i, τ_j). Then (A ∩ F) is a (i, j)-Δ^*-closed in (X, τ_i, τ_j).

Proof : Let A be τ_i-δg-open and (i, j)-Δ^*-closed in (X, τ_i, τ_j). Let F be τ_j-δ-closed. Then by Proposition 8.2.9, A is τ_j-δ-closed. Therefore (A ∩ F) is τ_j-δ-closed. Hence (A ∩ F) is a (i, j)-Δ^*-closed in (X, τ_i, τ_j) by Proposition 8.2.7.

Proposition 8.2.12 Let (X, τ_i, τ_j) be a topological space and A □ X. Then the following results are true.

a) If A is (i, j)-Δ^*-closed then A is (i, j)-πg-closed.

b) If A is (i, j)-Δ^*-closed then A is (i, j)-πgp-closed.

c) If A is (i, j)-Δ^*-closed then A is (i, j)-πgs-closed.

d) If A is (i, j)-Δ^*-closed then A is (i, j)-πgsp-closed.

e) If A is (i, j)-Δ^*-closed then A is (i, j)-πgb-closed.

f) If A is (i, j)-Δ^*-closed then A is (i, j)-πgα-closed.

g) If A is (i, j)-δg^*-closed then A is (i, j)-Δ^*-closed.

Proof : a) Let A be (i, j)-Δ^*-closed. Suppose that A □ U where U is τ_i-πg open. By Theorem 2.2.19, U is τ_i-Δ^*-open. Since A is (i, j)-Δ^*-closed, τ_j-Δ^*cl(A) □ U. But τ_j-πgcl(A) □ τ_j-Δ^*cl(A) by the same Theorem 2.2.19. Hence A is (i, j)-closed.

b) The proof is similar to (a) by Theorem 2.2.21

c) The proof is similar to (a) by Theorem 2.2.25

d) The proof is similar to (a) by Theorem 2.2.23

e) The proof is similar to (a) by Theorem 2.2.29

f) The proof is similar to (a) by Theorem 2.2.27

g) The proof is similar to (a) by Theorem 2.2.4

Remark 8.2.13 The converse of the above proposition is not true as seen from the following example.

Counter example 8.2.14 Let X = {a, b, c}, $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{a, b\}\}$. Then the subset {a} is (1, 2)-πg-closed, (1, 2)-πgp-closed, (1, 2)-πgs-closed, (1, 2)-πgsp-closed, (1, 2)-πgb-closed and (1, 2)-πgα-closed as $\pi G\mathcal{C}(X, \tau_2) = \pi GP\mathcal{C}(X, \tau_2) = \pi GS\mathcal{C}(X, \tau_2) = \pi GSP\mathcal{C}(X, \tau_2) = \pi GB\mathcal{C}(X, \tau_2) = \pi G\alpha\mathcal{C}(X, \tau_2) = P(X)$ but not (1, 2)-Δ^*-closed in (X, τ_1, τ_2).

Proposition 8.2.15 Let (X, τ_i, τ_j) be a topological space and A □ X. Then the following results are true.

a) If A is (i, j)-Δ^*-closed then A is (i, j)-gδ-closed.

b) If A is (i, j)-Δ^*-closed then A is (i, j)-$\delta g^{\ddagger}$-closed.

c) If A is (i, j)-Δ^*-closed then A is (i, j)-rg-closed.

d) If A is (i, j)-Δ^*-closed then A is (i, j)-gpr-closed.

e) If A is (i, j)-Δ^*-closed then A is (i, j)-gspr-closed.

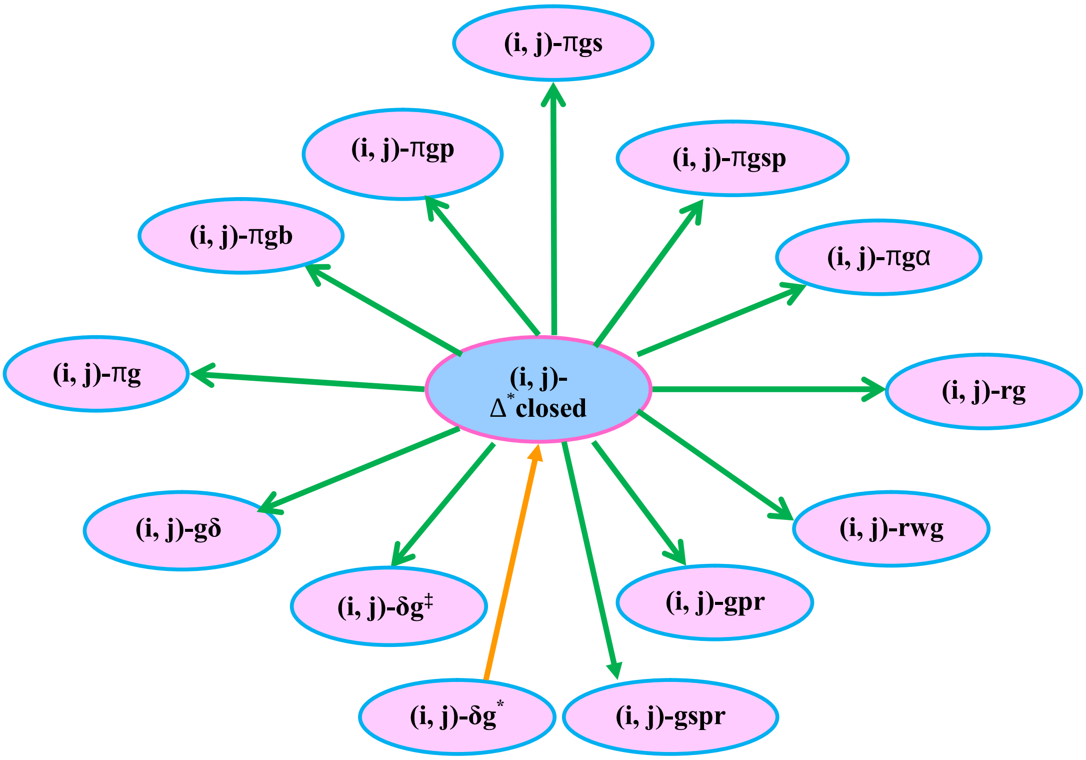
f) If A is (i, j)-Δ^*-closed then A is (i, j)-rwg-closed.

Proof : The proof follows from the dependency of Δ^*-closedness to the corresponding closed sets in Chapter 2.

Remark 8.2.16 The converse of the above proposition is not true as seen from the following example.

Counter example 8.2.17 Let X = {a, b, c}, $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{a, b\}\}$. Then the subset {a, b} is (1, 2)-gδ-closed, (1, 2)-$\delta g^{\ddagger}$-closed, (1, 2)-rg-closed, (1, 2)-gpr-closed, (1, 2)-gspr-closed and (1, 2)-rwg-closed as $G\delta\mathcal{C}(X, \tau_2) = \delta G^{\ddagger}\mathcal{C}(X, \tau_2) = RG\mathcal{C}(X, \tau_2) = GPR\mathcal{C}(X, \tau_2) = GSPR\mathcal{C}(X, \tau_2) = RWG\mathcal{C}(X, \tau_2) = P(X)$ but not (1, 2)-Δ^*-closed in (X, τ_1, τ_2).

Remark 8.2.18 The above results are depicted by the following diagram.



Remark 8.2.19 The following counter examples show that (i, j)-Δ^*-closed is independent with (i, j)-δg-closed, (i, j)-αg-closed and (i, j)-gp-closed.

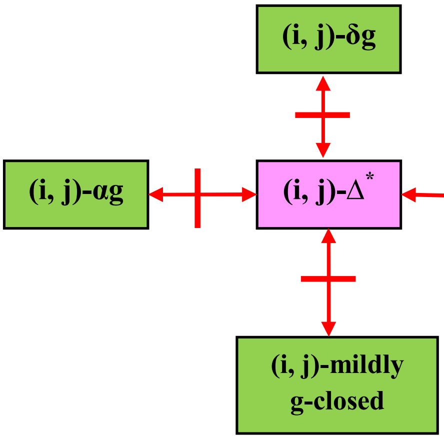
Counter example 8.2.20 Let X = Y = {a, b, c}, τ_1= {ϕ , X, {a}} and τ_2= {ϕ , X, {a, b}}. Then the subset {c} is (1, 2)-δg-closed, (1, 2)-αg-closed and (1, 2)-gp-closed but not (1,2)-Δ^*-closed.

Counter example 8.2.21 Let X = Y = {a, b, c}, τ_1 = {ϕ , X, {a}, {a, b}, {a, c}} and τ_2 = {ϕ , X, {a}}. Then the subset {a, b} is (1, 2)-Δ^*-closed but not (1, 2)-δg-closed, (1, 2)-αg-closed and (1, 2)-gp-closed.

Remark 8.2.22 The following counter example show that (i, j)-Δ^*-closed is independent with (i, j)-mildly g-closed.

Counter example 8.2.23 Let X = Y = {a, b, c}, τ_1={ϕ , X, {a}} and τ_2 = {ϕ , X, {a}, {a, b}}. Then the subset {b} is (2, 1)-mildly g-closed but not (2, 1)-Δ^*-closed. Also the subset {a, c} is (2, 1)-Δ^*-closed but not (2, 1)- mildly g-closed.

Remark 8.2.24 The above results are represented by the following diagram.



8.3 Characterizations of (i, j)-Δ^*-Closed Sets

Proposition 8.3.1 If A is a (i, j)-Δ^*-closed set then τ_i δcl({x})∩A ≠ ϕ holds for each x □ τ_j-δcl(A).

Proof : Let A be (i, j)-Δ^*-closed and suppose that τ_i δcl({x})∩A = ϕ for some x □ τ_jδcl(A). Then A □ X – (τ_jδcl({x})) = B, say. Then B is a τ_i-δ-open set. Since every δ-open is δg-open, B is δg-open in τ_i . Since A is (i, j)-Δ^*-closed we get τ_j-δcl(A) □ B = X – (τ_i-δcl({x})). Then τ_j-δcl(A) ∩ τ_i-δcl({x}) = ϕ which implies that τ_j-δcl(A)∩{x} = ϕ . Hence x □ τ_j-δcl(A) which is a contradiction. Hence τ_i-δcl({x}) ∩ A ≠ ϕ .

The converse of the above Proposition is not true as seen in the following example.

Counter example 8.3.2 Let X = {a, b, c}, τ_1= {ϕ , X, {a}}, τ_2 = { ϕ , X, {a}, {b, c}}. The subset A = {b} in (X, τ_1, τ_2) is not (1, 2)-Δ^*-closed. However τ_1-δcl({x}) ∩ A≠ ϕ holds for each x □ τ_2-δcl(A).

Remark 8.3.3 A (i, j)-Δ^*-closed set need not be a τ_i-Δ^*-closed set or τ_j-Δ^*-closed set as seen from the following examples.

Counter example 8.3.4 Let X = {a, b, c}, τ_1= {ϕ , X, {a, b}} and τ_2 = {ϕ , X, {a}, {b}, {a, b}, {a, c}}. Then $D_{\Delta^*}(1,2)$ = { ϕ , X, {b}, {c}, {b, c}, {a, c}} whereas $\Delta^*C(X,\tau_1)$ ={ϕ , X, {c}, {a, c}, {b, c}} and $\Delta^*C(X,\tau_2)$ ={ϕ , X, {b}, {c}, {a, c}, {b, c}}.

Thus the subset {b} is (1, 2)-Δ^*-closed in (X,τ_1,τ_2) but not a τ_1-Δ^*-closed set.

Counter example 8.3.5 Let X = {a, b, c}, τ_1= {ϕ , X, {a}, {a, b}} and τ_2 = {ϕ , X, {a}, {b, c}}. Then $D_{\Delta^*}(1,2)$ = {ϕ , X, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}} whereas $\Delta^*C(X,\tau_1)$ = {ϕ , {c}, {a, c}, {b, c}} and $\Delta^*C(X,\tau_2)$ = {ϕ , X, {a}, {b, c}}. Thus the subset {c} is (1, 2)-Δ^*-closed in (X,τ_1,τ_2) but not a τ_2-Δ^*-closed set.

Theorem 8.3.6 If A and B are (i, j)-Δ^*-closed sets then (A□B) is also a (i, j)-Δ^*-closed set.

Proof : Suppose that A and B are (i, j)-Δ^*-closed sets. Let U be τ_1-δg-open in (X,τ_i,τ_j) and (A □ B) □ U. Then A □ U and B □ U. Since A and B are (i, j)-Δ^*-closed sets, we have τ_j-δcl(A) □ U and τ_j-δcl(B) □ U.

Therefore τ_j-δcl(A □ B) = τ_j-δcl(A) □ τ_j-δcl(B) □ U. Hence (A □ B) is also a (i, j)-Δ^*-closed set in (X,τ_i,τ_j) .

Remark 8.3.7 The intersection of two (i, j)-Δ^*-closed sets need not be a (i, j)-Δ^*-closed set as seen from the following example.

Counter example 8.3.8 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then $D_{\Delta^*}(1,2) = \{\phi, X, \{b\}, \{b, c\}, \{a, c\}\}$. Thus the subset $\{b, c\}$ and $\{a, c\}$ are (1, 2)-Δ^*-closed sets but their intersection $\{c\}$ is not a (1, 2)-Δ^*-closed set.

Proposition 8.3.9 For each point x of a space (X, τ_i, τ_j), $\{x\}$ is τ_i-δg-closed or $\{x\}^c$ is (i, j)-Δ^*-closed.

Proof : If $\{x\}$ is not τ_i-δg-closed then $\{x\}^c$ is not τ_i-δg-open. Therefore x is the only τ_i-δg-open set containing $\{x\}^c$ which implies that τ_j-δcl($\{x\}^c$) □ X. Hence $\{x\}^c$ is (i, j)-Δ^*-closed.

Proposition 8.3.10 Let A be a (i, j)-Δ^*-closed set in (X, τ_i, τ_j). Then τ_j-δcl(A)$-$A contains no non empty τ_i-δg-closed set.

Proof : Let A be a (i, j)-Δ^*-closed set in (X, τ_i, τ_j). Let F be any τ_i-δg-closed set such that F □τ_j-δcl(A)$-$A. Since τ_j-δcl(A) □ X we have τ_j-δcl(A)$-$A □ $(X-A)$. Hence F □ $(X-A)$ which implies that A □ $X-F$ where $X-F$ is τ_i-δg-open. Since A is (i, j)-Δ^*-closed we have τ_j-δcl(A) □ $X-F$ which implies that F □ $X-\tau_j$-δcl(A). Hence F □ τ_j-δcl(A)$\cap$[$X-\tau_j$-δcl(A)] = ϕ which implies that F = ϕ. Hence τ_j-δcl(A)$-$A contains no non empty τ_i-δg-closed set.

Remark 8.3.11 The converse of the above proposition need not be true as seen from the following example.

Counter example 8.3.12 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$.

Then $\delta GC(X, \tau_1) = \{\phi, X, \{b, c\}\}$;

$\delta GC(X, \tau_2) = \{\phi, X, \{b, c\}\}$ and

$D_{\Delta^*}(1,2) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.

Let A = {a}. Then τ_2-δcl(A) = {a, c} which implies that τ_2-δcl(A) − A = {c}. Hence τ_2-δcl(A)−A does not contain a non empty τ_1-δg-closed set but A is not a (1, 2)-Δ^*-closed set in (X, τ_1, τ_2).

Corollary 8.3.13 Let A be a (i, j)-Δ^*-closed set in (X, τ_i, τ_j). Then A is τ_j-δ-closed if and only if τ_j-δcl(A)−A is τ_j-δg-closed.

Theorem 8.3.14 If A is a (i, j)-Δ^*-closed set in X and A $\subseteq$ B $\subseteq$ τ_j-δcl(A) then B is (i, j)-Δ^*-closed.

Proof : Let A be a (i, j)-Δ^*-closed set in X and A $\subseteq$ B $\subseteq$ τ_j-δcl(A). Let U be τ_i-δg-open in X such that B $\subseteq$ U. Then A $\subseteq$ U. Since A is (i, j)-Δ^*-closed, τ_j-δcl(A) $\subseteq$ U. Since B $\subseteq$ τ_j-δcl(A), τ_j-δcl(B) $\subseteq$ τ_j-δcl(A) $\subseteq$ U. Thus τ_j-δcl(B) $\subseteq$ U. Hence B is (i, j)-Δ^*-closed.

Theorem 8.3.15 If A is a (i, j)-Δ^*-closed set in X and A $\subseteq$ B $\subseteq$ τ_j-δcl(A) then τ_j-δcl(B)−B contains no non empty τ_i-δg-closed.

Proof : Let A be (i, j)-Δ^*-closed and A $\subseteq$ B $\subseteq$ τ_j-δcl(A). Then B is (i, j)-Δ^*-closed (Theorem 8.3.14). Hence by proposition 8.3.10, τ_j-δcl(B)−B contains no non empty τ_j-δg-closed set.

Definition 8.3.16 Let B $\subseteq$ Y $\subseteq$ X. A subset B of Y is said to be **(i, j)-Δ^*-closed relative to Y** if B is (i, j)-Δ^*-closed in the subspace Y.

Theorem 8.3.17 Let B $\subseteq$ A $\subseteq$ X where A is τ_i-δg-open and (i, j)-Δ^*-closed in X. Then B is (i, j)-Δ^*-closed relative to A if and only if B is (i, j)-Δ^*-closed relative to X.

Proof : Suppose that B $\subseteq$ A $\subseteq$ X where A is τ_i-δg-open and (i, j)-Δ^*-closed. Suppose that B is (i, j)-Δ^*-closed relative to A. Let B $\subseteq$ U and U is τ_i-δg-open in X. Since A is τ_i-δg-open, we have (A ∩ U) is τ_i-δg-open in X. Consequently (A ∩ U) is τ_i-δg-open in A. Since B $\subseteq$ A and B $\subseteq$ U we have B $\subseteq$ (A ∩ U). Since B is (i, j)-Δ^*-closed relative to A, τ_j-δcl$_A$(B) □ A ∩ U □ U---------(1). Now since A is τ_i-δ-open, A is τ_i-δg-open in X. Since A □ A and A is (i, j)-Δ^*-closed in X, τ_j-δcl(A) □ A ----------(2). Since B □ A, τ_j-δcl(B) □ τ_j-δcl(A) = A (from (2)). Hence τ_j-δcl(B) ∩ A □ τ_j-δcl(B). i.e., τ_j-δcl$_A$(B) □ τ_j-δcl(B). In general τ_j-δcl(B) □ τ_j-δcl$_A$(B). Hence τ_j-δcl$_A$(B) = τ_j δcl(B)----------(3). From (1) and (3) we have, τ_j-δcl(B) □ U. Therefore B is (i, j)-Δ^*-closed in X.

Conversely, let B be (i, j)-Δ^*-closed relative to X. Let B □ U and U is τ_i-δg-open in A and A is τ_i-δg-open in X, we have U is τ_i-δg-open in X. Since B is (i, j)-Δ^*-closed relative to X, τ_j-δcl(B) □ U. Hence by (3), τ_j-δcl$_A$(B)= τ_j-δcl(B) ∩ A = τ_j-δcl(B) □ U.

Proposition 8.3.18 In a bitopological space (X, τ_i, τ_j), δGO(X,τ_i)   δC(X,τ_j) if and only if every subset of (X, τ_i, τ_j) is (i, j)-Δ^*-closed.

Proof : (Necessity) : Let δGO(X,τ_i)   δC(X,τ_j). Let A be any subset of (X, τ_i, τ_j) and G □ δGO(X, τ_i) such that A   G. Then by the assumption τ_j-δcl(G) = G which implies that τ_j-δcl(A) τ_j-δcl(G) = G. Hence A is (i, j)-Δ^*-closed.

(Sufficiency) : Assume that every subset of (X, τ_i, τ_j) is (i, j)-Δ^*-closed. Let G □ δGO(X, τ_i). By the assumption G is (i, j)-Δ^*-closed. Therefore we have τ_j-δcl(G)   G. Hence τ_j-δcl(G) = G which implies that G is τ_j-δ-closed. Hence δGO(X,τ_i)   δC(X,τ_j).

8.4 (i, j)-Δ^*-Closure Operator

Definition 8.4.1 Let (X, τ_i, τ_j) be a topological space. For each subset A of (X, τ_i, τ_j), the **(i, j)-Δ^*-closure of A** is denoted by (i, j)-Δ^*cl(A) and is defined as follows.

$$(i, j)\text{-}\Delta^*\text{cl}(A) = \bigcap\{F \subseteq X / A \subseteq F \text{ and } F \text{ is a } (i, j)\text{-}\Delta^*\text{closed set}\}.$$

Proposition 8.4.2 Let A and B be any two subsets of (X, τ_i, τ_j). Then the following results are true.

a) (i, j)-Δ^*cl(ϕ) = ϕ and (i, j)-Δ^*-cl(X) = X

b) If A   B then (i, j)-Δ^*cl(A)   (i, j)-Δ^*cl(B)

c) A   (i, j)-Δ^*cl(A) τ_j-δcl(A)

d) If A is (i, j)-Δ^*-closed then (i, j)-Δ^*cl(A) = A

e) (i, j)-Δ^*cl(A ∩ B)   (i, j)-Δ^*cl(A) ∩ (i, j)-Δ^*cl(B)

f) (i, j)-Δ^*cl(A □ B) = (i, j)-Δ^*cl(A) □ (i, j)-Δ^*cl(B)

Proof : a) Since ϕ and X are (i, j)-Δ^*-closed sets in X, the result follows.

b) Let A   B then by the definition of (i, j)-Δ^*closure, (i, j)-Δ^*cl(A)   (i, j)-Δ^*cl(B).

c) By the definition of (i, j)-Δ^*closure, A   (i, j)-Δ^*cl(A).

By Proposition 8.2.7, every τ_j-δ-closed subset of (X, τ_i, τ_j) is (i, j)-Δ^*closed and $(i, j) - \delta cl(A) = \bigcap\{ F \quad X/A \quad F \text{ and } F \text{is } \tau_j - \delta - \text{closed}\}$. Thus (i, j)-$\Delta^*$cl(A) τ_j-δcl(A).

d) By the result c), A   (i, j)-Δ^*cl(A). Since A is (i, j)-Δ^*-closed, by the definition of (i, j)-Δ^*cl(A), (i, j)-Δ^*cl(A)   A. Hence (i, j)-Δ^*cl(A) = A.

e) Since (A ∩ B)   A and (A ∩ B)   B, the proof follows by the result (b).

f) Since A   (A □ B) and B   (A □ B), by the result (b), (i, j)-Δ^*cl(A)   (i, j)-Δ^*cl(A □ B) and (i, j)-Δ^*cl(B)   (i, j)-Δ^*cl(A □ B). Thus (i, j)-Δ^*cl(A) □ (i, j)-Δ^*cl(B)   (i, j)-Δ^*cl(A □ B). ---------(1). To prove the reverse inclusion, we assume that x □ (i, j)-Δ^*cl(A) □ (i, j)-Δ^*cl(B). Then there exist (i, j)-Δ^*-closed sets P and Q such that A   P, x □ P, B   Q and x □ Q. Then (A □ B)   (P □ Q) and x □ (P □ Q). Hence (P □ Q) is a (i, j)-Δ^*-closed set containing (A □ B). Therefore x □ (i, j)-Δ^*cl(A □ B).

Thus (i, j)-Δ^*cl(A □ B)   (i, j)-Δ^*cl(A) □ (i, j)-Δ^*cl(B)---------(2). Hence from (1) and (2), (i, j)-Δ^*cl(A □ B) = (i, j)-Δ^*cl(A) □ (i, j)-Δ^*cl(B).

Proposition 8.4.3 Let x and y be any two points of (X, τ_i, τ_j). If (i, j)-Δ^*cl({x}) = (i, j)-Δ^*cl({y}) and δGC(X, τ_j) = δC(X, τ_j) then x = y.

Proof : Suppose that x ≠ y. By Proposition 8.3.9, {x} is either τ_j-δg-closed or X−{x} is (i, j)-Δ^*-closed.

Case (i) Assume that {x} is τ_j-δg-closed. Since δGC(X, τ_j) = δC(X, τ_j, {x} is τ_j-δ-closed. (Proposition 8.2.7). Then {x} is (i, j)-Δ^*-closed. Hence (i,j)-Δ^*cl({x}) = {x}. Therefore y □ (i, j)-Δ^*cl({x}). Hence (i, j)-Δ^*cl({x}) ≠ (i, j)-Δ^*cl({y}) which is a contradiction. Hence x = y.

Case (ii) Suppose that X−{x} is (i, j)-Δ^*-closed. Since x ≠ y, y □ X−{x} and (i,j)-Δ^*cl({y}) □ X−{x}. Then x □ (i, j)-Δ^*cl({y}). Hence (i, j)-Δ^*cl({x}) ≠ (i, j)-Δ^*cl({y}) which is a contradiction. Hence x = y.

8.5 (i, j)-Δ^*-Open Sets

In this section (i, j)-Δ^*-open sets in bitopological spaces are introduced and their properties are proved.

Definition 8.5.1 A subset A of a bitopological space (X, τ_i, τ_j) is called a **(i, j)-Δ^*-open set** if and only if A^c is a (i, j)-Δ^*-closed.

Proposition 8.5.2 In a bitopological space (X, τ_i, τ_j), the following statements are true.

a) Every τ_j-δ-open is (i, j)-Δ^*-open.

b) Every (i, j)-δg^*-open is (i, j)-Δ^*-open.

c) Every (i, j)-Δ^*-open is (i, j)-πg-open.

d) Every (i, j)-Δ^*-open is (i, j)-πgp-open.

e) Every (i, j)-Δ^*-open is (i, j)-πgsp-open.

f) Every (i, j)-Δ^*-open is (i, j)-πgs-open.

g) Every (i, j)-Δ^*-open is (i, j)-πgα-open.

h) Every (i, j)-Δ^*-open is (i, j)-πgb-open.

Proof : Follows from Proposition 8.2.7 and Proposition 8.2.12.

Theorem 8.5.3 A set A is (i, j)-Δ^*-open if and only if G □ τ_j-δInt(A) whenever G is τ_i-δg-closed and G □ A.

Proof : Assume that A is (i, j)-Δ^*-open. Then A^c is (i, j)-Δ^*-closed. Let G be a τ_i-δg-closed set contained in A. Then G^c is τ_i-δg-open set containing A^c. Since A^c is (i, j)-Δ^*-closed, τ_j-δcl(A^c) □ G^c , equivalently G □ τ_j-δInt(A).

Conversely, assume that G □ τ_j-δInt(A) whenever G □ A where G is τ_i-δg-closed. Let A^c □ F where F is τ_i-δ-open. By criteria, F^c □ τ_i-δInt(A). This implies that $\tau_j\delta$cl(A^c) □ F. Thus A^c is (i, j)-Δ^*-closed and hence A is (i, j)-Δ^*-open.

Proposition 8.5.4 If A and B are (i, j)-Δ^*-open sets then (A ∩ B) is also (i, j)-Δ^*-open.

Proof : The proof follows from Proposition 8.3.6

Remark 8.5.5 If A and B are (i, j)-Δ^*-open sets in (X, τ_i, τ_j) then (A □ B) is need not be a (i, j)-Δ^*-open set.

Counter example 8.5.6 Let X = {a, b, c}, τ_1={ϕ , X, {a}, {a, b}, {a, c}} and τ_2 = {ϕ , X, {a}, {b}, {a, b}}. Then {b} and {c} are (1, 2)-Δ^*-open sets in (X, τ_1, τ_2) but their union {b, c} is not(1, 2)-Δ^*-open in (X, τ_1, τ_2).

Definition 8.5.7 Two subsets A and B are said to be **separated (i, j) -Δ^*-open sets** if the following conditions hold good.

i) $[\tau_i\text{-}\delta cl(A)] \cap B = \phi$ ii) $A \cap [\tau_i\text{-}\delta cl(B)] = \phi$

iii) $[\tau_j\text{-}\delta cl(A)] \cap B = \phi$ iv) $A \cap [\tau_j\text{-}\delta cl(B)] = \phi$

Remark 8.5.8 The following theorem gives a condition for the finite union of (i, j) -Δ^*-open sets to be (i, j) -Δ^*-open set.

Theorem 8.5.9 If A and B are separated (i, j) -Δ^*-open sets then A B is (i, j) -Δ^*-open set

Proof : Suppose A and B are separated (i, j) -Δ^*-open sets. Let F be τ_i-δg-closed such that F □ (A B).

Then by above definition we have $[\tau_i\text{-}\delta cl(A)] \cap B = A \cap [\tau_i\text{-}\delta cl(B)] = \phi$ and $[\tau_j\text{-}\delta cl(A)] \cap B = A \cap [\tau_j\text{-}\delta cl(B)] = \phi$.

Then $F \cap (\tau_j\text{-}\delta cl(A) \quad (A \quad B) \cap (\tau_j\text{-}\delta cl(A)) = A \cap [\tau_j\text{-}\delta cl(A)] \quad \phi \quad A$.

Similarly $F \cap [\tau_i\text{-}\delta cl(B)] \quad B$. Now, since F is τ_i-δg-closed by Proposition 2.3.6, $F \cap (\tau_i\text{-}\delta cl(A)$ and $F \cap (\tau_i\text{-}\delta cl(B)$ are τ_i-δg-closed sets.

Since A and B are (i, j)- Δ^*-open sets we have by Theorem 8.5.3, $F \cap (\tau_i\text{-}\delta cl(A) \quad (\tau_j\text{-}\delta int(A))$ and $F \cap (\tau_i\text{-}\delta cl(B) \quad (\tau_j\text{-}\delta int(B))$..

Now $F \quad F \cap (A \quad B) \quad [F \cap (\tau_i\text{-}\delta cl(A)] \, [F \cap (\tau_i\text{-}\delta cl(B)] \; \tau_j\text{-}\delta int(A \quad B)$.

Therefore $(A \quad B)$ is (i, j)- Δ^*-open.

Example 8.5.10 In (X, τ_1, τ_2) where X = {a, b, c}, $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Let A = {a} and B = {b}.

Then τ_1-δcl(A) = {a, c} ; τ_1-δcl(B) = {b, c}; τ_2-δcl(A) = {a} and τ_2-δcl(B) = {b, c}. Therefore A and B are separated (i, j)-Δ^*-open sets.
Also $(A \quad B) = \{a, b\}$ is (1, 2)-Δ^*-open and (2, 1)- Δ^*-open.

Proposition 8.5.11 If A is (i, j)-Δ^*-open in (X, τ_i, τ_j) and τ_j-δInt(A) B A then B is also (i, j)-Δ^*-open in (X, τ_i, τ_j).

Proof : Let U be a τ_i-δg-closed set of (X, τ_i, τ_j) such that U B. Then U A. Since A is (i, j)-Δ^*-open, U τ_j-δInt(A). Since τ_j-δint(A) B A we have τ_j-δInt(δInt(A)) τ_j-δInt(B) = τ_j-δInt(A). Therefore τ_j-δInt(A) = τ_j-δInt(B) and U τ_j-δInt(B). Hence B is (i, j)-Δ^*-open in (X, τ_i, τ_j).

Theorem 8.5.12 The intersection of a (i, j)-Δ^*-open set and τ_j-δ-open set is always a (i, j)-Δ^*-open set.

Proof : Let A be (i, j)-Δ^*-open and B be τ_j-δ-open. Then B^c is (i, j)-Δ^*-closed. (by Proposition 8.2.7). Therefore B is (i, j)-Δ^*-open. Hence by Proposition 8.5.4, $(A \cap B)$ is (i, j)-Δ^*-open.

Proposition 8.5.13 If a set A is (i, j)- Δ^*-closed in (X, τ_i, τ_j) then τ_j - δcl(A) −A is (i, j)-Δ^*-open.

Proof : Suppose A is (i, j)-Δ^*-closed. Let F be a τ_j-δg-closed set and F τ_jδcl(A) −A. Since A is (i, j)-Δ^*-closed, F τ_jδcl(A) −A does not contain a non empty τ_j-δg-closed (By Proposition 8.3.10). Thus F = ϕ. Therefore F τ_i-δint[τ_j-δcl(A)−A]. Hence τ_j-δcl(A)−A is (i, j)-Δ^*-open.

Theorem 8.5.14 If a set is (i, j)-Δ^*-open in a bitopological space (X, τ_i ,τ_j) then G = X whenever G is τ_i-δg-open and τ_j-δInt(A) A^c G.

Proof : Similar to the proof of 2.5.13

Proposition 8.5.15 For x □ τ_j-δcl(A), x □ (i, j)-Δ^*cl(A) if and only if U ∩ A ≠ ϕ for every
(i, j)- Δ^*-open set U containing x.

Proof : Similar to Theorem 2.5.18

8.6 (i, j)-Δ^*-Separation Axioms

In this section four new spaces namely, (i, j)- ${}_{\Delta^*}T_{\delta}$-space, (i, j)- ${}_{\Delta^*}T_{\delta g^*}$ -space, (i, j)- ${}_{g\delta}T_{\Delta^*}$ -space and (i, j)-${}_{\delta g^\ddagger}T_{\Delta}{}^*$-space in bitopological spaces are established and their interrelations are analysed.

Definition 8.6.1 A bitopological space (X, τ_i, τ_j) is said to be a

1) **(i, j)- ${}_{\Delta^*}T_{\delta}$-space** if every (i, j)-Δ^*-closed set is a τ_j-δ-closed set.

2) **(i, j)- ${}_{\Delta^*}T_{\delta g^*}$ -space** if every (i, j)-Δ^*-closed set is a (i, j)-δg^*-closed set.

3) **(i, j)- ${}_{g\delta}T_{\Delta^*}$ -space** if every (i, j)-gδ-closed set is a (i, j)-Δ^*-closed set.

4) **(i, j)-${}_{\delta g^\ddagger}$ $T_{\Delta}{}^*$-space** if every (i, j)-${}_{\delta g}{}^\ddagger$-closed set is a (i, j)-Δ^*-closed set.

Proposition 8.6.2 Every (i, j)- ${}_{\Delta^*}T_{\delta}$-space is a ${}_{\Delta^*}T_{\delta g^*}$-space but not conversely.

Proof : Let A be (i, j)-Δ^*-closed in (X, τ_i, τ_j). Since (X, τ_i, τ_j) is a ${}_{\Delta^*}T_{\delta}$-space, A is τ_j-δ-closed. Since every τ_j-δ-closed is (i, j)-δg^*-closed, A is (i, j)-δg^*-closed. Hence (X, τ_i, τ_j) is a ${}_{\Delta^*}T_{\delta g^*}$-space.

Counter example 8.6.3 Let X = {a, b, c}, τ_1 = {ϕ , X, {a}} and τ_2 = {ϕ , X, {a}, {b}, {a, b}, {a, c}}. Then $\delta\mathcal{C}$(X, τ_1) = {ϕ , X} ; $\delta\mathcal{C}$(X, τ_2) = {ϕ , X, {b}, {a, c}} ; D_{Δ^*}(1,2) = { ϕ , X, {b}, {a, c}, {b, c}} and $D_{\delta g^*}$(1,2) = { ϕ , X, {b}, {a, c}, {b, c}}.

Hence (X, τ_1, τ_2) is a (1, 2)- ${}_{\Delta^*}T_{\delta g^*}$-space but not a (1, 2)- ${}_{\Delta^*}T_{\delta}$-space since the subset {b, c} is (1, 2)-Δ^*-closed but not a τ_2–δ-closed set.

Proposition 8.6.4 The following examples show that (i, j)- ${}_{\Delta^*}T_{\delta}$-space is independent with (i, j)-T_b-space and (i, j)-T_c-space.

Counter example 8.6.5 Let X = {a, b, c}, τ_1 = {ϕ , X, {a}} and τ_2 = {ϕ , X, {a}, {b, c}}. Then GO(X, τ_1) = {ϕ , X, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}};

$\delta\mathcal{C}$(X, τ_2) = {ϕ , X, {a}, {b, c}};

D_{Δ^*}(1,2) = {ϕ , X, {a}, {b, c}};

D_{gs}(1,2) = {ϕ , X, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}};

D_{g^*}(1,2) = {ϕ , X, {a}, {b, c}}.

Hence (X, τ_1, τ_2) is a (1, 2)- ${}_{\Delta^*}T_{\delta}$-space but not a (1, 2)-T_b-space as {c} is (1, 2)-gs-closed but not a τ_2-δ-closed. Also (X, τ_1, τ_2) is a (1, 2)- ${}_{\Delta^*}T_{\delta}$-space but not a (1, 2)-T_c-space as {c} is (1, 2)-gs-closed but not a (1, 2)-g^*-closed.

Counter example 8.6.6 Let X = {a, b, c}, τ_1 = {ϕ , X, {a}, {b, c}} and τ_2 = {ϕ , X, {a}, {a, b}, {a, c}}. Then

$GO(X, \tau_2) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$;

$\delta\mathcal{C}(X, \tau_1) = \{\phi, X, \{a\}, \{b, c\}$;

$D_{\Delta^*}(2,1) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$;

$D_{gs}(2,1) = \{\phi, X, \{a\}, \{b, c\}\}$;

$D_{g^*}(2,1) = \{\phi, X, \{a\}, \{b, c\}$.

Hence (X, τ_2, τ_1) is a both (2, 1)-T_b-space as well as (2, 1)-T_c-space but not a (2, 1)-${}_{\Delta^*}T_\delta$-space since {c} is (2, 1)-Δ^*-closed but not τ_1-δ-closed.

Proposition 8.6.7 The following examples show that (i, j)-${}_{\Delta^*}T_\delta$-space is independent with (i, j)- T_d-space.

Counter example 8.6.8 Let X = {a, b, c}, $\tau_1 = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\delta\mathcal{C}(X, \tau_2) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}$;

$D_{\Delta^*}(1,2) = \{\phi, X, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{gs}(1,2) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$;

$D_g(1,2) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.

Hence (X, τ_1, τ_2) is a (1, 2)-${}_{\Delta^*}T_\delta$-space but not a (1, 2)-T_d-space since {a} is (1, 2)-gs-closed but not (1, 2)-g-closed.

Counter example 8.6.9 Let X = {a, b, c}, $\tau_1 = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then $\delta\mathcal{C}(X, \tau_1) = \{\phi, X, \{a\}, \{b, c\}$;

$D_{\Delta^*}(2,1) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$;

$D_{gs}(2,1) = \{\phi, X, \{a\}, \{b, c\}\}$;

$D_g(2,1) = \{\phi, X, \{a\}, \{b, c\}\}$. Hence (X, τ_2, τ_1) is a (2, 1)-T_d-space but not (2, 1)-${}_{\Delta^*}T_\delta$-space since {b} is (2, 1)-Δ^*-closed but not τ_1-δ-closed.

Remark 8.6.10 The following examples show that (i, j)- $_{\Delta^*}T_{\delta}$-space is independent with (i, j)-$_{g\delta}T_{\delta g^*}$ -space.

Counter example 8.6.11 : Let X = {a, b, c}, τ_1 = {ϕ , X, {a}} and τ_2 = {ϕ , X, {a}, {b}, {a, b}}. Then $\delta\mathcal{C}(X, \tau_2)$ = {ϕ , {c}, {a, c}, {b, c}};

$D_{\Delta^*}(1, 2)$ = {ϕ , X, {c}, {a, c}, {b, c}};

$D_{g\delta}(1, 2)$ = {ϕ , X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}};

$D_{\delta g^*}(1,2)$ = {ϕ , X, {c}, {a, c}, {b, c}}.

Hence (X, τ_1, τ_2) is a (1, 2)- $_{\Delta^*}T_{\delta}$-space but not a $_{g\delta}T_{\delta g^*}$ -space as the subset {a} is (1,2)-gδ-closed but not a (1, 2)-δg^*-closed in (X, τ_1, τ_2).

Counter example 8.6.12 Let X = {a, b, c}, τ_1 = {ϕ , X, {a}} and τ_2 = {ϕ , X, {a}, {b}, {a, b}}. Then $\delta\mathcal{C}(X, \tau_1)$ = {ϕ , X} ;

$D_{\Delta^*}(2,1)$ = {ϕ , X, {c}, {a, c}, {b, c}};

$D_{g\delta}(2,1)$ = {ϕ , X, {c}, {a, c}, {b, c}};

$D_{\delta g^*}(2,1)$ = {ϕ , X, {c}, {a, c}, {b, c}}.

Hence (X, τ_2, τ_1) is a (2, 1)-$_{g\delta}T_{\delta g^*}$ -space but not a (2, 1)- $_{\Delta^*}T_{\delta}$-space as the subset {c} is (2,1)-Δ^*-closed but not a τ_1–δ-closed in (X, τ_2, τ_1).

Proposition 8.6.13 If (X, τ_i, τ_j) is both (i, j)- $_{\Delta^*}T_{\delta}$-space and (i, j)- $_{\delta g^*}T_{\delta}$-space then it is a (i, j)- $_{\Delta^*}T_{\delta}$-space.

Proof : Let A be a (i, j)-Δ^*-closed set in (X, τ_i, τ_j). Since the space is a (i, j)- $_{\Delta^*}T_{\delta g^*}$ -space, A is (i, j)-δg^*-closed. Also (X, τ_i, τ_j) is a (i, j)- $_{\delta g^*}T_{\delta}$-space.

Therefore A is a δ-closed in (X, τ_i, τ_j). Hence (X, τ_i, τ_j) is a (i, j)- ${}_{\Delta^*}T_{\delta}$-space.

Example 8.6.14 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$.Then $\delta\mathcal{C}(X, \tau_1) = \{\phi, X\}$; $\delta\mathcal{C}(X, \tau_2) = \{\phi, X, \{c\}, \{b, c\}, \{a, c\}\}$;
$GO(X, \tau_1) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$;
$D_{\Delta^*}(1,2) = \{\phi, X, \{c\}, \{b, c\}, \{a, c\}\}$;
$D_{\delta g^*}(1,2) = \{\phi, X, \{c\}, \{b, c\}, \{a, c\}\}$.

Thus (X, τ_1, τ_2) is both (1, 2)- ${}_{\Delta^*}T_{\delta g^*}$-space and (1, 2)- ${}_{\delta g^*}T_{\delta}$-space and hence it is a (1, 2)- ${}_{\Delta^*}T_{\delta}$-space.

Proposition 8.6.15 If (X, τ_i, τ_j) is a (i, j)-${}_{g\delta}T_{\delta g^*}$-space then it is a (i, j)- ${}_{\Delta^*}T_{\delta g^*}$-space.

Proof : Let A be a (i, j)-Δ^*-closed set in (X, τ_i, τ_j). We know that by proposition 8.2.11, every (i, j)-Δ^*-closed is (i, j)-gδ-closed. Since (X, τ_i, τ_j) is a (i, j)-${}_{g\delta}T_{\delta g^*}$-space, A is (i, j)-$\delta g^*$-closed. Hence (X, τ_i, τ_j) is a (i, j)- ${}_{\Delta^*}T_{\delta g^*}$ -space.

Example 8.6.16 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$.
Then $D_{\Delta^*}(2,1) = \{\phi, X, \{c\}, \{b, c\}, \{a, c\}\}$;
$D_{\delta g^*}(2,1) = \{\phi, X, \{c\}, \{b, c\}, \{a, c\}\}$;
$D_{g\delta}(2,1) = \{\phi, X, \{c\}, \{b, c\}, \{a, c\}\}$.

Thus (X, τ_2, τ_1) is a (2, 1)-${}_{g\delta}T_{\delta g^*}$-space and (2, 1)- ${}_{\Delta^*}T_{\delta g^*}$-space.

Remark 8.6.17 The spaces (i, j)-T_b-space, (i, j)-T_c-space and (i, j)-T_d-space are independent with (i, j)- ${}_{\delta g^\ddagger}T_{\Delta^*}$-space as seen from the following examples.

Counter example 8.6.18 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{a\}\}$.

Then $D_{\Delta^*}(1,2) = \{\phi , X, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{\delta g^\ddagger}(1, 2) = \{\phi , X, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{gs}(1,2) = \{\phi , X, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{g^*}(1,2) = \{\phi , X, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{g}(1,2) = \{\phi , X, \{c\}, \{b, c\}, \{a, c\}\}$.

Hence (X, τ_1, τ_2) is a (1, 2)- $_{\delta g^\ddagger}T_{\Delta}{}^*$-space but not a (1, 2)-$T_b$-space, not a (1, 2)-$T_c$-space and (1, 2)-$T_d$-space as the (1, 2)-gs-closed subset {b} is not (1, 2)-closed, not (1, 2)-g^*-closed and not (1, 2)-g-closed respectively.

Counter example 8.6.19 Let X = {a, b, c}, $\tau_1 = \{\phi , X, \{a, b\}\}$ and $\tau_2 = \{\phi , X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$.Then $D_{\Delta^*}(1,2) = \{\phi , X, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{\delta g^\ddagger}(1, 2) = \{\phi , X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$;

$D_{gs}(1,2) = \{\phi , X, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{g^*}(1,2) = \{\phi , X, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{g}(1,2) = \{\phi , X, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$.

Hence (X, τ_1, τ_2) is a (1, 2)-T_b-space, (1, 2)-T_c-space and (1, 2)-T_d-space but not a (1, 2)- $_{\delta g^\ddagger}T_{\Delta}{}^*$-space as the (1, 2)-$\delta g^\ddagger$-closed set {a} is not a (1, 2)-Δ^*-closed set.

Remark 8.6.20 The (i, j)- $_{\Delta^*}T_{\delta g^*}$ -space and (i, j)- $_{\delta g^\ddagger}T_{\Delta}{}^*$-space are independent as seen from the following examples.

Counter example 8.6.21 Let X = {a, b, c}, $\tau_1 = \{\phi , X, \{a, b\}\}$ and $\tau_2 = \{\phi , X, \{a\}, \{b, c\}\}$. Then $D_{\Delta^*}(1,2) = \{\phi , X, \{a\}, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{\delta g^*}(1,2) = \{\phi , X, \{a\}, \{c\}, \{b, c\}, \{a, c\}\}$;

$D_{\delta g^\ddagger}(1, 2) = \{\phi , X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.

Hence (X, τ_1, τ_2) is a (1, 2)-${}_{\Delta^*}T_{\delta g^*}$-space but not a (1, 2)-$\delta g^{\ddagger}\, T_{\Delta}{}^*$-space as the (1, 2)-$\delta g^{\#}$-closed set {b} is not a (1, 2)-Δ^*-closed set in (X, τ_1, τ_2).

Counter example 8.6.22 Let X = {a, b, c}, τ_1 = {ϕ , X, {a}, {a, b}, {a, c}} and τ_2={ϕ , X, {a}, {b, c}}. Then

$D_{\Delta^*}(1,2)$ = {ϕ , X, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}};

$D_{\delta g^*}(1,2)$ = {ϕ , X, {a}, {b, c}};

$D_{\delta g\ddagger}$ (1, 2) = {ϕ , X, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}}.

Hence (X, τ_1, τ_2) is a (1, 2)-$\delta g^{\ddagger}\, T_{\Delta}{}^*$-space but not a (1, 2)-${}_{\Delta^*}T_{\delta g^*}$-space as the (1, 2)-$\Delta^*$-closed set {b} is not a (1, 2)-δg^*-closed set in (X, τ_1, τ_2).

Proposition 8.6.23 If (X, τ_i, τ_j) is a (i, j)-${}_{g\delta}T_{\Delta^*}$-space and a (i, j)-${}_{\Delta^*}T_{\delta}$-space then it is a (i, j)-T_{δ}-space.

Proof : Let A be a (i, j)-gδ-closed in X. Since (X, τ_i, τ_j) is a (i, j)-${}_{g\delta}T_{\Delta^*}$-space, A is (i, j)-Δ^*-closed. Also (X, τ_i, τ_j) is a (i, j)-${}_{\Delta^*}T_{\delta}$-space. Therefore A is τ_j-δ-closed. Hence (X, τ_i, τ_j) is a (i, j)-T_{δ}-space.

Proposition 8.6.24 In (i, j)-${}_{\Delta^*}T_{\delta}$-space, either {x} is τ_j-δ-open or τ_i–δg-closed for each x ∊ X.

Proof : Let (X, τ_i, τ_j) be a (i, j)-${}_{\Delta^*}T_{\delta}$-space. Suppose that {x} is not τ_i–δg-closed. Then $\{x\}^c$ is (i, j)-Δ^*-closed.(Proposition 8.3.9). Since (X, τ_i, τ_j) is a (i, j)-${}_{\Delta^*}T_{\delta}$-space, $\{x\}^c$ is τ_j–δ-closed. Therefore {x} is τ_j-δ-open.

Proposition 8.6.25 For a space (X, τ_i, τ_j) the following are equivalent.

a) X is (i, j)-${}_{\Delta^*}T_{\delta}$-space

b) Every singleton is either τ_i-δg-closed or τ_j-δ-open for $i \neq j$.

Proof : (a) $\Longrightarrow$ (b) : Suppose that $\{x\}$ is not τ_i–δg-closed subset for some x □ X. Then X –$\{x\}$ is not τ_i–δg-open and hence X is the only τ_i–δg-open set containing X−$\{x\}$. Therefore X –$\{x\}$ is (i, j)-Δ^*-closed. Since (X, τ_i , τ_j) is a (i, j)- $_{\Delta^*}T_{\delta}$ -space, X−$\{x\}$ is τ_j -δ-closed. Therefore $\{x\}$ is τ_j -δ-open.

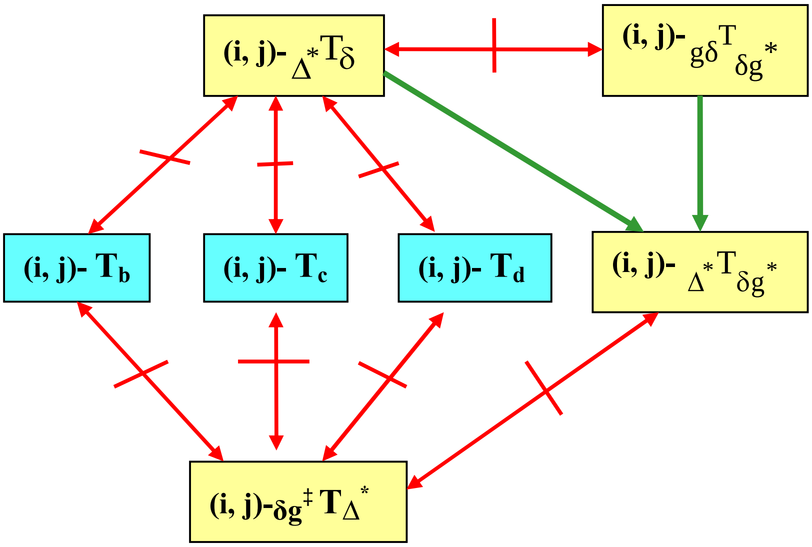
(b) $\Longrightarrow$ (a) : Let A be a (i, j)-Δ^*-closed subset of (X, τ_i , τ_j) and x $\in \tau_j$ − δcl(A). We prove that x □ A.

Case (i) : If $\{x\}$ is τ_i -δg-closed and x □ A then x $\in$ τ_j -δcl(A) \ A. Thus τ_j -δcl(A)−A contains a non empty τ_i –δg-closed set $\{x\}$ which is a contradiction to the fact that A is (i, j)-Δ^*-closed. So x $\in$ A.

Case (ii) : If $\{x\}$ is τ_j -δ-open, since x □ τ_j -δcl(A), then for every τ_j -δ-open set U containing x, we have U ∩ A $\nparallel$ ϕ . But $\{x\}$ is τ_j –δ-open. Therefore $\{x\}$ ∩ A ≠ ϕ . Hence x $\in$ A.

In both cases we have x □ A. Hence A is τ_j -δ-closed.

Remark 8.6.26 The above results are depicted by the following diagram.



8.7 (i, j)-Δ^*-Continuous Functions

In this section (i, j)-Δ^*-continuous functions using Δ^*-closed sets are introduced and their properties are analyzed.

Definition 8.7.1 A map $f : (X, \tau_i, \tau_j) \longrightarrow (Y, \sigma_i, \sigma_j)$ is called a **(i, j)-Δ^*-σ_k-continuous** if the inverse image of every σ_k-closed set in (Y, σ_i, σ_j) is a (i, j)-Δ^*-closed set in (X, τ_i, τ_j) for i, j, k =1, 2 and $i \neq j$.

Remark 8.7.2 If $\tau_i = \tau_j = \tau$ and $\sigma_i = \sigma_j = \sigma$ then the above definition of (i, j)-Δ^*-σ_k-continuous maps coincides with the definition of Δ^*-continuous maps.

Proposition 8.7.3 If $f : (X, \tau_i, \tau_j) \longrightarrow (Y, \sigma_i, \sigma_j)$ is a τ_j-δ-σ_k continuous map then f is (i, j)-Δ^*-σ_k-continuous but not conversely.

Proof : Follows from the fact that every τ_j-δ-closed set is (i, j)-Δ^*-closed. (Proposition 8.2.7).

Counter example 8.7.4 Let X = {a, b, c} = Y, $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$, $\sigma_1 = \{\phi, Y, \{b\}, \{a, b\}\}$ and $\sigma_2 = \{\phi, Y, \{a\}, \{a, c\}\}$. Then

$\delta C(X, \tau_1) = \{\phi, X\}$; $\delta C(X, \tau_2) = \{\phi, X, \{b\}, \{a, c\}\}$ and

$D_{\Delta^*}(1,2) = \{\phi, X, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$.

Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then the inverse image of each σ_k-closed set in Y is (1, 2)-Δ^*-closed in (X, τ_1, τ_2). Therefore f is (1, 2)-Δ^*-σ_k-continuous but not $\tau_2\delta$-$\sigma_1$continuous since the inverse image of σ_1-closed set {c} is not $\tau_2\delta$-closed. Similarly f is not $\tau_2\delta$-$\sigma_2$continuous since the inverse image of σ_2-closed set {b, c} is not $\tau_2\delta$-closed.

Proposition 8.7.5 If a map $f : (X, \tau_i, \tau_j) \longrightarrow (Y, \sigma_i, \sigma_j)$ is a (i, j)-δg^*-σ_k continuous map then f is (i, j)-Δ^*-σ_k-continuous but not conversely.

Proof : Follows from the fact that every (i, j)-δg^*-closed set is (i, j)-Δ^*-closed (Proposition 8.2.12).

Counter example 8.7.6 Let X = {a, b, c} = Y, $\tau_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$, $\sigma_1 = \{\phi, Y, \{a\}, \{a, b\}\}$ and $\sigma_2=\{\phi, Y, \{a\}, \{b\}, \{a, b\}\}$.

Then $D_{\Delta^*}(1,2) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $D_{\delta g^*}(1,2) = \{\phi, X, \{b, c\}$.

Let f : $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be the identity map. Then the inverse image of each σ_k-closed set in Y is (1, 2)-Δ^*-closed in X. Therefore f is (1, 2)-Δ^*-σ_k-continuous but not $\tau_2\delta$-σ_k-continuous as the inverse image of σ_1-closed set {c} is not (1, 2)-τ_2-δg^*-closed.

Proposition 8.7.7 If f : $(X, \tau_i, \tau_j) \longrightarrow (Y, \sigma_i, \sigma_j)$ is (i, j)-Δ^*-σ_k-continuous then f is (i, j)-gδ-σ_k-continuous, (i, j)-rg-σ_k-continuous, (i, j)-gpr-σ_k-continuous, (i, j)-rwg-σ_k-continuous, (i, j)-πgp-σ_k-continuous, (i,j)-πgs-σ_k-continuous, (i, j)-πgsp-σ_k-continuous, But the converse is not true.

Proof : Follows from the fact that every (i, j)-Δ^*-closed is (i, j)-gδ-closed, (i, j)-rg-closed, (i, j)-gpr-closed, (i, j)-rwg-closed, (i, j)-πgp-closed, (i, j)-πgs-closed and (i, j)-πgsp-closed. (Proposition 8.2.15 and Proposition 8.2.12).

Counter example 8.7.8 Let X = {a, b, c} = Y, τ_1 = {ϕ , X, {a}}, τ_2 = {ϕ , X, {a, b}}, σ_1={ϕ , Y, {a}, {b, c}} and σ_2={ϕ , Y, {a}, {b}, {a, b}}. Then

$D_{\Delta^*}(1,2)$={ϕ , X, {b, c}};

$D_{g\delta}(1,2) = D_{rg}(1,2) = D_{gpr}(1,2) = D_{rwg}(1,2) = D_{\pi gp}(1,2) = D_{\pi gs}(1,2) = D_{\pi gsp}(1,2) =$

{ϕ , X, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}}.

Let f : $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be the identity map. Then f is(1, 2)-gδ-σ_k-continuous, (1, 2)-rg-σ_k-continuous, (1, 2)-gpr-σ_k-continuous, (1, 2)-rwg-σ_k-continuous, (1, 2)-πgp-σ_k-continuous, (1, 2)-πgs-σ_k-continuous, (1, 2)-πgsp-σ_k-continuous, (1, 2)-πgp-σ_k-continuous and (1, 2)-πgp-σ_k-continuous but not (1, 2)-Δ^*-σ_k-continuous since for the σ_1-closed set {a}, f^{-1}({a}) = {a} is not (1, 2)-Δ^*-closed in (X, τ_1, τ_2).

Proposition 8.7.9 The (i, j)-Δ^*-σ_k-continuous is independent with (i, j)-δg-σ_k-continuous, (i, j)-αg-σ_k-continuous and (i, j)-gp-σ_k-continuous as seen from the following examples.

Counter example 8.7.10 Let X = {a, b, c} = Y, τ_1 = {ϕ , X, {a}}, τ_2 = {ϕ , X, {a, b}}, σ_1 = {ϕ , Y, {a}, {a, b}} and σ_2 = {ϕ , Y, {a}, {b}, {a, b}}. Then

$D_{\Delta^*}(1,2)$ = {ϕ , X, {b, c}};

$D_{\delta g}(1,2)$ = {ϕ , X, {b}, {c}, {a, b}, {b, c}, {a, c}};

$D_{\alpha g}(1,2)$ = {ϕ , X, {b}, {c}, {a, b}, {b, c}, {a, c}} and

$D_{gp}(1,2)$ = {ϕ , X, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}}.

Let f : $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then f is (1, 2)-δg-σ_k-continuous, (1, 2)-αg-σ_k-continuous and (1, 2)-gp-σ_k-continuous but not (1, 2)-Δ^*-σ_k-continuous since for the σ_1-closed set {c}, $f^{-1}(\{c\}) = \{c\}$ is not (1, 2)-Δ^*-closed in (X, τ_1, τ_2).

Counter example 8.7.11 Let X = {a, b, c} = Y, τ_1 = {ϕ , X, {a}, {a, b}, {a, c}}, τ_2 = {ϕ , X, {a}}, σ_1 = {ϕ , Y, {a}, {b}, {a, b}} and σ_2 = {ϕ , Y, {a}, {a, b}}.
Then $D_{\Delta^*}(1,2)$ = {ϕ , X, {b}, {c}, {a, b}, {b, c}, {a, c}};

$D_{\delta g}(1,2)$ = {ϕ , X, {b, c}};

$D_{\alpha g}(1,2)$ = {ϕ , X, {b}, {c}, {b, c}} and

$D_{gp}(1,2)$ = {ϕ , X, {b}, {c}, {b, c}}.

Let f : $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a map such that f(a) = c, f(b) = b and f(c) = c. Then f is (1, 2)-Δ^*-σ_k-continuous but not (1, 2)-δg-σ_k-continuous, (1, 2)-αg-σ_k-continuous and (1, 2)-gp-σ_k-continuous since for the σ_1-closed set {c}, $f^{-1}(\{c\})=\{a, c\}$ is not (1, 2)-δg-closed, (1, 2)-αg-closed and not (1, 2)-gp-closed set in (X, τ_1, τ_2).

Remark 8.7.12 The composition of two (i, j)-Δ^*-σ_k-continuous map need not be a (i, j)-Δ^*-σ_k-continuous map as seen from the following example.

Counter example 8.7.13 Let X = {a, b, c} = Y = Z, τ_1 = {ϕ , X, {a}, {b}, {a, b}}, τ_2 = {ϕ , X, {a}}, σ_1 = {ϕ , Y, {a}, {a, b}}, σ_2 = {ϕ , Y, {a}, {b, c}}, η_1= {ϕ , Z, {a}, {b, c}} and η_2 = {ϕ , Z, {a}, {b}, {a, b}}.
Let f : $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a map such that f(a) = c, f(b) = b and f(c) = c and let g : $(Y, \sigma_1, \sigma_2) \longrightarrow (Z, \eta_1, \eta_2)$ be a map such that g(a) = b, g(b) = b and g(c) = a. Then both f and g are (1, 2)-Δ^*-σ_k-continuous but their composition map $(g \circ f) : (X, \tau_1, \tau_2) \longrightarrow (Z, \eta_1, \eta_2)$ which is defined by $(g \circ f)(a) = a$, $(g \circ f)(b) = b$ and

$(g \circ f)(c) = a$ is not a (1, 2)-Δ^*-σ_k-continuous map since for the η_1-closed set {b, c}, $(g \circ f)^{-1}(\{b, c\}) = \{b\}$ is not (1, 2)-Δ^*-closed in (X, τ_1, τ_2).

Proposition 8.7.14 Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \longrightarrow (Z, \eta_1, \eta_2)$ be two functions such that f is (i, j)-Δ^*-σ_k-continuous and g is (i, j)-Δ^*-η_k-continuous then $(g \circ f)$ is (i, j)-Δ^*-η_k-continuous.

Proof : Obvious.

Proposition 8.7.15 If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-Δ^*-σ_k-continuous then $f\,[(i, j)\text{-}\Delta^*cl(A)] \subseteq \sigma_k cl[f(A)]$.

Proof : For every subset A of X, $\sigma_k cl[f(A)]$ is σ_k-closed in Y and $A \subseteq f^{-1}(\sigma_k\, cl[f(A)]\,)$. Since f is (i, j)-$\Delta^*$-$\sigma_k$-continuous, we have $f^{-1}(\sigma_k cl[f(A)])$ is (i, j)- Δ^*-closed in (X, τ_i, τ_j). Then $A \subseteq f^{-1}(\sigma_k\, cl[f(A)])$.

That is $(i, j)\text{-}\Delta^*cl(A) \subseteq (i, j)\text{-}\Delta^*cl(f^{-1}(\sigma_k cl[f(A)])) = f^{-1}(\sigma_k cl[f(A)])$. Hence $(i, j)\text{-}\Delta^*cl(A) \subseteq f^{-1}(\sigma_k cl[f(A)]$ which implies that $f\,[(i, j)\text{-}\Delta^*cl(A)] \subseteq \sigma_k cl[f(A)]$.

8.8 (i, j)-Δ^*-Irresolute Maps

In this section (i, j)-Δ^*-irresolute maps using Δ^*-closed sets are introduced and their properties are analyzed.

Definition 8.8.1 A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called **(i, j)-Δ^*-irresolute** if the inverse image of every (i, j)-Δ^*-closed set in (Y, σ_1, σ_2) is (i, j)-Δ^*-closed in (X, τ_1, τ_2).

Proposition 8.8.2 A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ (i, j)-Δ^*-irresolute if and only if the inverse image of every (i, j)-Δ^*-open set in Y is (i, j)-Δ^*-open in (X, τ_1, τ_2).

Proof : The proof follows from the definition.

Proposition 8.8.3 Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ be two (i, j)-Δ^*-irresolute maps. Then their composition $g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is a (i, j)-Δ^*-irresolute map.

Proof : The proof follows from the definitions.

Theorem 8.8.4 If f : $(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is bijective, τ_i-δg-open and (i, j)-Δ^*-σ_j continuous, then f is (i, j)-Δ^*-irresolute.

Proof : Let V be (i, j)-Δ^*-closed set in (Y,σ_1,σ_2) and let $f^{-1}(V) \subseteq U$, where U is τ_i- δg -open. Clearly $V \subseteq f(U)$, Since f(U) is τ_i- δg -open and since V is (i, j)-Δ^*-closed set in (Y,σ_1,σ_2), then τ_j - δ cl(V) f(U) and thus $f^{-1}(\tau_j$ - δcl(V)) U. Since f is (i, j)Δ^*-σ_j-continuous and since τ_j - δcl(V) is σ_j-δ-closed and hence σ_j-closed in (Y,σ_1,σ_2), we have τ_j - δcl(f $^{-1}(\tau_j$ δcl(V))) U and hence τ_j - δcl(f $^{-1}$(V)) U. Therefore $f^{-1}(V)$ is (i, j)-Δ^*-closed set in (X,τ_1,τ_2). Hence f is (i, j)-Δ^*-irresolute.

Theorem 8.8.5 If f : $(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is a (i, j)-Δ^*-irresolute and (X,τ_1,τ_2) is a (i, j)-$_{\Delta^*}T_\delta$-space,. Then f is τ_j-δ-irresolute.

Proof : Let V be a τ_j-δ-closed set in (Y,σ_1,σ_2). Then V is (i, j)-Δ^*-closed in (Y,σ_1,σ_2). Since f is (i, j)-Δ^*-irresolute, then $f^{-1}(V)$ is (i, j)-Δ^*-closed in (X,τ_1,τ_2). Since X is (i, j)-$_{\Delta^*}T_\delta$-space,, $f^{-1}(V)$ is τ_j-δ-closed in (X,τ_1,τ_2). Hence f is τ_j-δ-irresolute.

Theorem 8.8.6 If f : $(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is a τ_i-δg-irresolute and τ_j-δ-closed, then for every (i, j)-Δ^*-closed set A of X, f(A) is (i, j)-Δ^*-closed set of Y.

Proof: Let A be a (i, j)-Δ^*-closed set of (X,τ_1,τ_2). Suppose that $f(A) \subseteq U$, where U is τ_i-δg-open in (Y,σ_1,σ_2). Then $A \subseteq f^{-1}(U)$ and $f^{-1}(U)$ is τ_i- δg -open, since f is τ_i-δg-irresolute. Since A is (i, j)-Δ^*-closed, $\tau_j - \delta cl(A) \quad f^{-1}(U)$ and hence $f(\tau_j - \delta cl(A)) \quad U$.

Therefore we have $\tau_j - \delta cl(f(A)) \quad \tau_j - \delta cl(f(\tau_j - \delta cl(A))) = f(\tau_j - \delta cl(A)) \quad U$.

Hence f(A) is a (i, j)-Δ^*-closed set in (Y,σ_1,σ_2).

Theorem 8.8.7 Let f : $(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ and g $:(Y,\sigma_1,\sigma_2) \rightarrow (Z,\eta_1,\eta_2)$ be two functions, then

a) If f is (i, j)-Δ^*-irresolute and g is (i, j)-Δ^*-η_k-continuous, then (g∘f) is (i, j)-Δ^*-η_k-continuous.

b) If f is (i, j)-лg-irresolute and g is (i, j)-Δ^*-η_k-continuous, then (g ∘ f) is $(i,\ j) - \pi g - \eta_k$ -continuous.

c) If f is (i, j)-gδ-irresolute and g is (i, j)-Δ^*-η_k-continuous, then (g∘f) is $(i,\ j) - g\delta - \eta_k$ -continuous.

Proof : a) Let V be a η_k-closed in (Z, η_1, η_2). Since g is (i, j)-Δ^*- η_k-continuous, $g^{-1}(V)$ is (i, j)-Δ^*-closed in (Y, σ_1, σ_2). Since f is (i, j)-Δ^*-irresolute, $f^{-1}(g^{-1}(V))$ is (i, j)-Δ^*-closed in (X, τ_1, τ_2). Hence (g ∘ f) is (i, j)-Δ^*-η_k-continuous.

b) Let V be a η_k-closed in (Z, η_1, η_2). Since g is (i, j)-Δ^*-η_k-continuous, $g^{-1}(V)$ is (i, j)-Δ^*-closed in (Y, σ_1, σ_2). Since every (i, j)-Δ^*-closed set is (i, j)-лg-closed and f is (i, j)-лg-irresolute, $f^{-1}(g^{-1}(V))$is (i, j)-лg-closed in (X, τ_1, τ_2). Hence (g ∘ f) is (i, j)-πg-η_k -continuous.

c) Let V be a η_k-closed in (Z, η_1, η_2). Since g is (i, j)-Δ^*-η_k-continuous, $g^{-1}(V)$ is (i, j)-Δ^*-closed in(Y, σ_1, σ_2). Since every (i, j)-Δ^*-closed set is (i, j)-gδ-closed and f is (i, j)-gδ- irresolute, $f^{-1}(g^{-1}(V))$is (i, j)-gδ-closed in(X, τ_1, τ_2). Hence (g ∘ f) is (i, j)-$g\delta$-η_k -continuous.

8.9 (i, j)-Δ^*-Locally Closed Sets

Definition 8.9.1 A subset A of a bitopological space (X, τ_i, τ_j) is said to be a

i) (i, j)-Δ^*-locally closed set if A = G∩F where G is τ_i-Δ^*-open and F is τ_j-Δ^*-closed in (X, τ_1, τ_2).

ii) (i, j)-Δ^*lc* set if A = G∩F where G is τ_i-Δ^*-open and F is τ_j-δ-closed in (X, τ_1, τ_2).

iii) (i, j)-Δ^*lc** set if A = G∩F where G is τ_i-δ-open and F is τ_j-Δ^*-closed in (X, τ_1, τ_2).

Remark 8.9.2 i) The class of all (i, j)-Δ^*lc sets in (X, τ_i, τ_j) is denoted by $(i, j)\text{-}\Delta^*LC(X, \tau_i, \tau_j)$.

ii) The class of all (i, j)-Δ^*lc* sets in (X, τ_i, τ_j) is denoted by $(i, j)\text{-}\Delta^*LC^*(X, \tau_i, \tau_j)$.

iii) The class of all (i, j)-Δ^*lc** sets in (X, τ_i, τ_j) is denoted by $(i, j)\text{-}\Delta^*LC^{**}(X, \tau_i, \tau_j)$.

Example 8.9.3 Let X = { a, b, c}, τ_1= {ϕ , X, {a}} and τ_2= {ϕ , X, {a}, {b, c}}. Then

$\Delta^*O(X, \tau_1) = \{ \phi, X, \{a\}\}$;

$\Delta^*O(X, \tau_2) = \{\phi, X, \{a\}, \{b, c\}\}$;

$\delta O(X, \tau_1) = \{ \phi, X \}$;

$\delta O(X, \tau_2) = \{\phi, X, \{a\}, \{b, c\}\}$;

$\Delta^*LC(X, \tau_1, \tau_2) = \phi, X, \{a\}, \{b, c\}$;

$\Delta^*LC^*(X, \tau_1, \tau_2) = \phi, X, \{a\}, \{b, c\}$ and

$\Delta^*LC^{**}(X, \tau_1, \tau_2) = \phi, X, \{a\}, \{b, c\}$.

Remark 8.9.4 In general every (i, j)-Δ^*-locally closed set in (X, τ_i, τ_j) is not τ_j-δ-closed as seen by the following example.

Example 8.9.5 Let X = {a, b, c}, τ_1= {ϕ , X, {a}, {b, c}} and τ_2= {ϕ , X, {a}, {a, b}, {a, c}}.

Then $\Delta^*O(X, \tau_1)$ = {ϕ , X, {a}, {b ,c}};

$\Delta^*O(X, \tau_2)$ = {ϕ , X, {a}, {b}, {c}, {a, b}, {a, c}};

δ O(X, τ_1) = { ϕ , X, {a}, {b ,c}};

δ O(X, τ_2) = {ϕ , X};

$\Delta^*LC(X, \tau_1, \tau_2) = P(X)$;

$\Delta^*LC^*(X, \tau_1, \tau_2) =$ ϕ , X, {a}, {b, c} and

$\Delta^*LC^{**}(X, \tau_1, \tau_2) = P(X)$.

Here {b, c} is (1, 2)-Δ^*-locally closed set in (X, τ_1, τ_2) but it is not τ_2- δ-closed in (X, τ_1, τ_2).

Remark 8.9.6 Every -Δ^*-locally closed set in (X, τ_1, τ_2) is not τ_2-δ-closed in general as seen from the example 8.9.3

Theorem 8.9.7 In any bitopological space (X, τ_i, τ_j) the following results are true.

a) $A \square \text{(i, j)} - \Delta^*LC^*(X, \tau_i, \tau_j) \Longrightarrow A \square \text{(i, j)} - \Delta^*LC(X, \tau_i, \tau_j)$

b) $A \square \text{(i, j)} - \Delta^*LC^{**}(X, \tau_i, \tau_j) \Longrightarrow A \square \text{(i, j)} - \Delta^*LC(X, \tau_i, \tau_j)$

c) $A \in \tau_2\text{-}\Delta^* C(X, \tau_i, \tau_j) \Rightarrow A \in (i, j)\text{-}\Delta^* LC(X, \tau_i, \tau_j)$

d) $A \in \tau_1\text{-}\Delta^* O(X, \tau_i, \tau_j) \Rightarrow A \in (i, j)\text{-}\Delta^* LC(X, \tau_i, \tau_j)$

Proof : a) Let $A \in (i, j)\text{-}\Delta^* LC^*(X, \tau_i, \tau_j)$. Then A = G∩F where G is τ_i-Δ^*-open and F is τ_j-δ-closed set in X. We know that by Proposition 8.2.7, every τ_j-δ-closed subset of (X, τ_i, τ_j) is (i, j)-Δ^*-closed. Therefore F is τ_j-Δ^*-closed in (X, τ_i, τ_j). Thus A = G∩F, where G is τ_j-Δ^*-closed and F is Δ^*-closed. Hence $A \in (i, j)\text{-}\Delta^* LC(X, \tau_i, \tau_j)$.

Similarly we can prove the results (b), (c) and (d).

Remark 8.9.8 The converse of the above theorem is not true in general which can be seen from the example 8.9.5

Theorem 8.9.9 If (X, τ_i, τ_j) is pairwise δ-door space then every subset of (X, τ_i, τ_j) is both (i, j)- Δ^*-locally closed and (j, i)- Δ^*-locally closed.

Proof : Since (X, τ_i, τ_j) is pairwise δ-door space every subset of (X, τ_i, τ_j) is either τ_i-δ-open or τ_j-δ-closed and τ_j-δ-open or τ_i-δ-closed. Since every τ_i-δ-open (resp., τ_j-δ-closed) subset is τ_i-Δ^*-open (resp., τ_j-Δ^*-closed), we have every subset of (X, τ_i, τ_j) is either τ_i-Δ^*-open or τ_j-Δ^*-closed. Since from

Theorem 8.9.7 (c) and (d) , every τ_i-Δ^*-open and τ_j-Δ^*-closed subset of (X, τ_i, τ_j) is (i, j)- Δ^*-locally closed, we have every subset of X is (i, j)-Δ^*-locally closed in (X, τ_i, τ_j) .

Similarly we can prove every subset of X is (j, i)-Δ^*-locally closed in (X, τ_i, τ_j).