CHAPTER – IV

Bipolar spherical Neutrosophic Cubic Graph and its Application

4.1 Introduction

Compared to fuzzy set and all other versions of fuzzy set, neutrosophic sets can handle imprecise information in a more effective way. A Neutrosophic cubic set, which is the generalization of neutrosophic set are more flexible as well as compatible to the system compared to other existing fuzzy models. On other hand, graph is a very easy way to understand and handle a problem physically in the form of diagrams. We introduce spherical fuzzy neutrosophic cubic graph and single-valued neutrosophic spherical cubic graphs in bipolar setting and discuss some of their properties such as cartesian product, composition, m-join, n-join, m-union, n-union. We also present a numerical example of the defined model which depicts the advantage of the same.

4.2 Bipolar spherical Neutrosophic Cubic Graph

In this section, first we define bipolar spherical neutrosophic cubic set and bipolar spherical neutrosophic cubic graph and its algebraic properties such as degree, order, union, join, composition. Further we discuss some results related with bipolar spherical neutrosophic cubic graph with numerical examples.

Definition 4.2.1: Let X be a non-empty set. An bipolar spherical ***neutrosophic cubic set*** (BSFNCS) defined by

$$A = \{ x, (T_A^{P+} = [T_{\underline{A}}^{P+}, T_A^{\overline{P}+}], I_A^{P+} = [I_{\underline{A}}^{P+}, I_A^{\overline{P}+}], F_A^{P+} = [F_{\underline{A}}^{P+}, F_A^{\overline{P}+}]),$$
$$T_A^{P-} = ([T_{\underline{A}}^{P-}, T_A^{\overline{P}-}], I_A^{P-} = [I_{\underline{A}}^{P-}, I_A^{\overline{P}-}], F_A^{P-} = [F_{\underline{A}}^{P-}, F_A^{\overline{P}-}]), \lambda_A \mid x \in X\}$$

where $0 \le [T_{\underline{A}}^{P+} \le T_A^{\overline{P}+}], [I_{\underline{A}}^{P+} \le I_A^{\overline{P}+}], [F_{\underline{A}}^{P+} \le F_A^{\overline{P}+}] \le 1$,

$$0 \le [T_{\underline{A}}^{P-} \le T_A^{\overline{P}-}], [I_{\underline{A}}^{P-} \le I_A^{\overline{P}-}], [F_{\underline{A}}^{P-} \le F_A^{\overline{P}-}] \le 1$$

are the interval valued neutrosophic spherical sets with $0 \le ((T_A^{\overline{P}+})^2 + (I_A^{\overline{P}+})^2 + (F_A^{\overline{P}+})^2) \le \sqrt{3}$ and $0 \le ((T_{\underline{A}}^{P-})^2 + (I_{\underline{A}}^{P-})^2 + (F_{\underline{A}}^{P-})^2) \le \sqrt{3}$ and $[T_{\underline{A}}^{P+}, T_A^{\overline{P}+}]$ denote the positive truth membership function, $[I_{\underline{A}}^{P+}, I_A^{\overline{P}+}]$ denote the positive indeterminacy membership function, $[F_{\underline{A}}^{P+}, F_A^{\overline{P}+}]$ denote the positive falsity membership function, $[T_{\underline{A}}^{P-}, T_A^{\overline{P}-}]$ denote the negative truth membership function, $[I_{\underline{A}}^{P-}, I_A^{\overline{P}-}]$ denote the negative indeterminacy membership function, $[F_{\underline{A}}^{P-}, F_A^{\overline{P}-}]$ denote the negative falsity membership function and $\lambda_A^{P+} : X \to [0,1]$, $\lambda_A^{P-} : X \to [0,-1]$ denote the fuzzy membership functions respectively.

Definition 4.2.2: Let $G^* = (V, E)$ be a graph and $G(P, Q)$ is an bipolar spherical ***neutrosophic cubic graph*** (BSFNCG) of G^*, if

$$P = (A, \lambda) = \{V, (T_A^{P+}, I_A^{P+}, F_A^{P+}), (T_A^{P-}, I_A^{P-}, F_A^{P-}), \lambda_A\}$$

is the BSFNCS representation of vertex set V and

$$Q = (B, \mu) = \{E, (T_B^{P+}, I_B^{P+}, F_B^{P+}), (T_B^{P-}, I_B^{P-}, F_B^{P-}), \mu_B\}$$

is the BSFNCS representation of edge set E such that

1. $T_B^{P+}(u_i v_i) \le r\min\{T_A^{P+}(u_i), T_A^{P+}(v_i)\}$, $T_\mu^{P+}(u_i v_i) \ge r\max\{T_\lambda^{P+}(u_i), T_\lambda^{P+}(v_i)\}$
   $T_B^{P-}(u_i v_i) \ge r\max\{T_A^{P-}(u_i), T_A^{P-}(v_i)\}$, $T_\mu^{P-}(u_i v_i) \ge r\min\{T_\lambda^{P-}(u_i), T_\lambda^{P-}(v_i)\}$

2. $I_B^{P+}(u_i v_i) \le r\min\{I_A^{P+}(u_i), I_A^{P+}(v_i)\}$, $I_\mu^{P+}(u_i v_i) \ge r\max\{I_\lambda^{P+}(u_i), I_\lambda^{P+}(v_i)\}$
   $I_B^{P-}(u_i v_i) \ge r\max\{I_A^{P-}(u_i), I_A^{P-}(v_i)\}$, $I_\mu^{P-}(u_i v_i) \ge r\min\{I_\lambda^{P-}(u_i), I_\lambda^{P-}(v_i)\}$

3. $F_B^{P+}(u_iv_i) \le r\max\{F_A^{P+}(u_i), F_A^{P+}(v_i)\}$, $F_\mu^{P+}(u_iv_i) \ge r\min\{F_\lambda^{P+}(u_i), F_\lambda^{P+}(v_i)\}$
$F_B^{P-}(u_iv_i) \ge r\min\{F_A^{P-}(u_i), F_A^{P-}(v_i)\}$, $F_\mu^{P-}(u_iv_i) \ge r\max\{F_\lambda^{P-}(u_i), F_\lambda^{P-}(v_i)\}$

Let $G^* = (V, E)$ be a graph and $G(P,Q)$ is an bipolar spherical Neutrosophic Cubic Graph (BSFNCG) of G^*, if

$$P = (A,\lambda) = \{V, (T_A^{P+}, I_A^{P+}, F_A^{P+}), (T_A^{P-}, I_A^{P-}, F_A^{P-}), \lambda_A\}$$

is the BSFNCG representation of vertex set *V* and

$$Q = (B,\mu) = \{E, (T_B^{P+}, I_B^{P+}, F_B^{P+}), (T_B^{P-}, I_B^{P-}, F_B^{P-}), \mu_B\}$$

is the BSFNCG representation of edge set *E* and λ and μ are bipolar spherical neutrosophic cubic sets.

Example 4.2.3: Let $G^* = (V, E)$ be a graph where $V = \{a,b,c,d\}$ and $E = \{ab, ac, ad, bc, bd, cd\}$ where *P* and *Q* are as follows:

$$P = \left\langle \begin{array}{l} \begin{Bmatrix} a, ([0.3,0.5],0.2), ([0.8,0.9],0.5), ([0.2,0.4],0.5), \\ ([-0.4,-0.3],-0.2), ([-0.6,-0.5],-0.8), ([-0.9,-0.8],-0.6) \end{Bmatrix} \\ \begin{Bmatrix} b, ([0.9,0.1],0.7), ([0.6,0.8],0.1), ([0.4,0.7],0.1), \\ ([-0.8,-0.7],-0.5), ([-0.5,-0.2],-0.1), ([-0.3,-0.2],-0.1) \end{Bmatrix} \\ \begin{Bmatrix} c, ([0.3,0.6],0.8), ([0.4,0.6],0.7), ([0.5,0.6],0.4), \\ ([-0.5,-0.4],-0.1), ([-0.6,-0.3],-0.1), ([-0.7,-0.6],-0.3) \end{Bmatrix} \\ \begin{Bmatrix} d, ([0.1,0.3],0.5), ([0.2,0.3],0.6), ([0.6,0.7],0.8), \\ ([-0.8,-0.6],-0.2), ([-0.7,-0.3],-0.2), ([-0.9,-0.6],-0.4) \end{Bmatrix} \end{array} \right\rangle$$

$$Q = \left\langle \begin{array}{l} \left\{ \begin{array}{l} ab, ([0.3,0.1],0.7), ([0.6,0.8],0.5), ([0.4,0.7],0.1), \\ ([-0.4,-0.3],-0.5), ([-0.5,-0.2],-0.8), ([-0.9,-0.8],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} ac, ([0.3,0.5],0.8), ([0.4,0.6],0.7), ([0.5,0.6],0.4), \\ ([-0.4,-0.3],-0.2), ([-0.6,-0.3],-0.8), ([-0.9,-0.8],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} ad, ([0.1,0.3],0.5), ([0.2,0.3],0.6), ([0.6,0.7],0.5), \\ ([-0.4,-0.3],-0.2), ([-0.6,-0.3],-0.8), ([-0.9,-0.8],-0.4) \end{array} \right\} \\ \left\{ \begin{array}{l} bc, ([0.3,0.1],0.8), ([0.4,0.6],0.7), ([0.5,0.7],0.1), \\ ([-0.5,-0.4],-0.5), ([-0.5,-0.2],-0.1), ([-0.7,-0.6],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} bd, ([0.1,0.1],0.7), ([0.2,0.3],0.6), ([0.6,0.7],0.1), \\ ([-0.8,-0.6],-0.5), ([-0.5,-0.2],-0.2), ([-0.9,-0.6],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} cd, ([0.1,0.3],0.8), ([0.2,0.3],0.7), ([0.6,0.7],0.4), \\ ([-0.5,-0.4],-0.2), ([-0.6,-0.3],-0.2), ([-0.9,-0.6],-0.3) \end{array} \right\} \end{array} \right\rangle$$

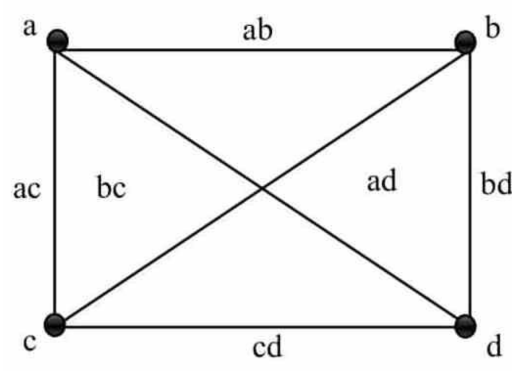

Fig. 4.1. The vertex set in *P* and the edge set in *Q* are represented for the graph *G*=(V,E)*

Remarks:

1. If $n \geq 3$ in the vertex set and $n \geq 3$ in the set of edges then the graphs is an bipolar neutrosophic cubic polygon only when we join each vertex to the corresponding vertex through an edge.
2. If we have infinite elements in the vertex set and by joining the edge and every edge with each other we get an bipolar neutrosophic cubic curve.

Definition 4.2.4: Let $G = (P, Q)$ be an bipolar spherical neutrosophic cubic graph. The order of bipolar spherical neutrosophic cubic graph is defined by

$$O(G)=\sum_{u\in V}\begin{Bmatrix}(T_A^{P+},T_\lambda^{P+})(u),(I_A^{P+},I_\lambda^{P+})(u),(F_A^{P+},F_\lambda^{P+})(u),\\(T_A^{P-},T_\lambda^{P-})(u),(I_A^{P-},I_\lambda^{P-})(u),(F_A^{P-},F_\lambda^{P-})(u)\end{Bmatrix}$$

and the degree of a vertex u of G is defined by

$$\deg(u)=\sum_{uv\in E}\begin{Bmatrix}(T_B^{P+},T_\mu^{P+})(uv),(I_B^{P+},I_\mu^{P+})(uv),(F_B^{P+},F_\mu^{P+})(uv),\\(T_B^{P-},T_\mu^{P-})(uv),(I_B^{P-},I_\mu^{P-})(uv),(F_B^{P-},F_\mu^{P-})(uv)\end{Bmatrix}$$

Example 4.2.5: In the above example, the order of a bipolar spherical neutrosophic cubic graph is

$$\deg(a)=\begin{Bmatrix}([0.7,0.9],2),([1.2,1.7],1.8),([1.5,2],1),\\([-1.2,-0.9],-0.9),([-1.7,-0.8],-2.4),([-2.7,-2.4],-0.8)\end{Bmatrix}$$

$$\deg(b)=\begin{Bmatrix}([0.7,0.3],2.2),([1.2,1.7],1.8),([1.5,2.1],0.3),\\([-1.7,-1.3],-1.5),([-1.5,-0.6],-1.1),([-2.5,-2],-0.3)\end{Bmatrix}$$

$$\deg(c)=\begin{Bmatrix}([0.7,0.9],2.4),([1,1.5],2.1),([1.6,2],0.9),\\([-1.4,-1.1],-0.9),([-1.7,-0.8],-1.1),([-2.5,-2],-0.7)\end{Bmatrix}$$

$$\deg(d)=\begin{Bmatrix}([0.3,0.7],2),([0.6,0.9],1.9),([1.8,2.1],1),\\([-1.7,-1.3],-0.9),([-1.7,-0.8],-1.2),([-2.7,-2],-0.8)\end{Bmatrix}$$

Definition 4.2.6: Let $G_1=(P_1,Q_1)$ be an bipolar spherical neutrosophic cubic graph of $G_1^*=(V_1,E_1)$ and $G_2=(P_2,Q_2)$ be an bipolar spherical neutrosophic cubic graph of $G_2^*=(V_2,E_2)$. Then Cartesian product of G_1 and G_2 is denoted by

$$G_1\times G_2=(P_1\times P_2,Q_1\times Q_2)$$

$$=\begin{pmatrix}(A_1^{P+},\lambda_1^{P+})\times(A_2^{P+},\lambda_2^{P+}),(A_1^{P-},\lambda_1^{P-})\times(A_2^{P-},\lambda_2^{P-}),\\(B_1^{P+},\mu_1^{P+})\times(B_2^{P+},\mu_2^{P+}),(B_1^{P-},\mu_1^{P-})\times(B_2^{P-},\mu_2^{P-})\end{pmatrix}$$

$$=\begin{pmatrix}(A_1^{P+}\times A_2^{P+},\lambda_1^{P+}\times\lambda_2^{P+}),(A_1^{P-}\times A_2^{P-},\lambda_1^{P-}\times\lambda_2^{P-}),\\(B_1^{P+}\times B_2^{P+},\mu_1^{P+}\times\mu_2^{P+}),(B_1^{P-}\times B_2^{P-},\mu_1^{P-}\times\mu_2^{P-})\end{pmatrix}$$

$$G_1 \times G_2 = \left\langle \begin{array}{l} (T^{P+}_{A_1\times A_2}, T^{P+}_{\lambda_1\times\lambda_2}), (I^{P+}_{A_1\times A_2}, I^{P+}_{\lambda_1\times\lambda_2}), (F^{P+}_{A_1\times A_2}, F^{P+}_{\lambda_1\times\lambda_2}) \\ (T^{P-}_{A_1\times A_2}, T^{P-}_{\lambda_1\times\lambda_2}), (I^{P-}_{A_1\times A_2}, I^{P-}_{\lambda_1\times\lambda_2}), (F^{P-}_{A_1\times A_2}, F^{P-}_{\lambda_1\times\lambda_2}) \\ (T^{P+}_{B_1\times B_2}, T^{P+}_{\mu_1\times\mu_2}), (I^{P+}_{B_1\times B_2}, I^{P+}_{\mu_1\times\mu_2}), (F^{P+}_{B_1\times B_2}, F^{P+}_{\mu_1\times\mu_2}) \\ (T^{P-}_{B_1\times B_2}, T^{P-}_{\mu_1\times\mu_2}), (I^{P-}_{B_1\times B_2}, I^{P-}_{\mu_1\times\mu_2}), (F^{P-}_{B_1\times B_2}, F^{P-}_{\mu_1\times\mu_2}) \end{array} \right\rangle$$

and is defined as follows:

1. $$\begin{pmatrix} T^{P+}_{A_1\times A_2}(u,v) = r\min\{T^{P+}_{A_1}(u), T^{P+}_{A_2}(v)\},\ T^{P+}_{\lambda_1\times\lambda_2}(u,v) = r\max\{T^{P+}_{\lambda_1}(u), T^{P+}_{\lambda_2}(v)\}, \\ T^{P-}_{A_1\times A_2}(u,v) = r\max\{T^{P-}_{A_1}(u), T^{P-}_{A_2}(v)\},\ T^{P-}_{\lambda_1\times\lambda_2}(u,v) = r\min\{T^{P-}_{\lambda_1}(u), T^{P-}_{\lambda_2}(v)\} \end{pmatrix}$$

2. $$\begin{pmatrix} I^{P+}_{A_1\times A_2}(u,v) = r\min\{I^{P+}_{A_1}(u), I^{P+}_{A_2}(v)\},\ I^{P+}_{\lambda_1\times\lambda_2}(u,v) = r\max\{I^{P+}_{\lambda_1}(u), I^{P+}_{\lambda_2}(v)\}, \\ I^{P-}_{A_1\times A_2}(u,v) = r\max\{I^{P-}_{A_1}(u), I^{P-}_{A_2}(v)\},\ I^{P-}_{\lambda_1\times\lambda_2}(u,v) = r\min\{I^{P-}_{\lambda_1}(u), I^{P-}_{\lambda_2}(v)\} \end{pmatrix}$$

3. $$\begin{pmatrix} F^{P+}_{A_1\times A_2}(u,v) = r\max\{F^{P+}_{A_1}(u), F^{P+}_{A_2}(v)\},\ F^{P+}_{\lambda_1\times\lambda_2}(u,v) = r\min\{F^{P+}_{\lambda_1}(u), F^{P+}_{\lambda_2}(v)\}, \\ F^{P-}_{A_1\times A_2}(u,v) = r\min\{F^{P-}_{A_1}(u), F^{P-}_{A_2}(v)\},\ F^{P-}_{\lambda_1\times\lambda_2}(u,v) = r\max\{F^{P-}_{\lambda_1}(u), F^{P-}_{\lambda_2}(v)\} \end{pmatrix}$$

4. $$\begin{pmatrix} T^{P+}_{B_1\times B_2}((u,v_1)(u,v_2)) = r\min\{T^{P+}_{A_1}(u), T^{P+}_{B_2}(v_1v_2)\}, \\ T^{P-}_{B_1\times B_2}((u,v_1)(u,v_2)) = r\max\{T^{P-}_{A_1}(u), T^{P+}_{B_2}(v_1v_2)\}, \\ T^{P+}_{\mu_1\times\mu_2}((u,v_1)(u,v_2)) = r\max\{T^{P+}_{\lambda_1}(u), T^{P+}_{\mu_2}(v_1v_2)\}, \\ T^{P-}_{\mu_1\times\mu_2}((u,v_1)(u,v_2)) = r\min\{T^{P-}_{\lambda_1}(u), T^{P+}_{\mu_2}(v_1v_2)\} \end{pmatrix}$$

5. $$\begin{pmatrix} I^{P+}_{B_1\times B_2}((u,v_1)(u,v_2)) = r\min\{I^{P+}_{A_1}(u), I^{P+}_{B_2}(v_1v_2)\}, \\ I^{P-}_{B_1\times B_2}((u,v_1)(u,v_2)) = r\max\{I^{P-}_{A_1}(u), I^{P+}_{B_2}(v_1v_2)\}, \\ I^{P+}_{\mu_1\times\mu_2}((u,v_1)(u,v_2)) = r\max\{I^{P+}_{\lambda_1}(u), I^{P+}_{\mu_2}(v_1v_2)\}, \\ I^{P-}_{\mu_1\times\mu_2}((u,v_1)(u,v_2)) = r\min\{I^{P-}_{\lambda_1}(u), I^{P+}_{\mu_2}(v_1v_2)\} \end{pmatrix}$$

6. $$\begin{pmatrix} F^{P+}_{B_1\times B_2}((u,v_1)(u,v_2)) = r\max\{F^{P+}_{A_1}(u), F^{P+}_{B_2}(v_1v_2)\}, \\ F^{P-}_{B_1\times B_2}((u,v_1)(u,v_2)) = r\min\{F^{P-}_{A_1}(u), F^{P+}_{B_2}(v_1v_2)\}, \\ F^{P+}_{\mu_1\times\mu_2}((u,v_1)(u,v_2)) = r\min\{F^{P+}_{\lambda_1}(u), F^{P+}_{\mu_2}(v_1v_2)\}, \\ F^{P-}_{\mu_1\times\mu_2}((u,v_1)(u,v_2)) = r\max\{F^{P-}_{\lambda_1}(u), F^{P+}_{\mu_2}(v_1v_2)\} \end{pmatrix}$$

$$7.\begin{pmatrix} T^{P+}_{B_1\times B_2}((u_1,v)(u_2,v)) = r\min\{T^{P+}_{B_1}(u_1,u_2), T^{P+}_{\lambda_2}(v)\}, \\ T^{P-}_{B_1\times B_2}((u_1,v)(u_2,v)) = r\max\{T^{P-}_{B_1}(u_1,u_2), T^{P+}_{\lambda_2}(v)\}, \\ T^{P+}_{\mu_1\times \mu_2}((u_1,v)(u_2,v)) = r\max\{T^{P+}_{\mu_1}(u_1,u_2), T^{P+}_{\lambda_2}(v)\}, \\ T^{P-}_{\mu_1\times \mu_2}((u_1,v)(u_2,v)) = r\min\{T^{P-}_{\mu_1}(u_1,u_2), T^{P+}_{\lambda_2}(v)\} \end{pmatrix}$$

$$8.\begin{pmatrix} I^{P+}_{B_1\times B_2}((u_1,v)(u_2,v)) = r\min\{I^{P+}_{B_1}(u_1,u_2), I^{P+}_{\lambda_2}(v)\}, \\ I^{P-}_{B_1\times B_2}((u_1,v)(u_2,v)) = r\max\{I^{P-}_{B_1}(u_1,u_2), I^{P+}_{\lambda_2}(v)\}, \\ I^{P+}_{\mu_1\times \mu_2}((u_1,v)(u_2,v)) = r\max\{I^{P+}_{\mu_1}(u_1,u_2), I^{P+}_{\lambda_2}(v)\}, \\ I^{P-}_{\mu_1\times \mu_2}((u_1,v)(u_2,v)) = r\min\{I^{P-}_{\mu_1}(u_1,u_2), I^{P+}_{\lambda_2}(v)\} \end{pmatrix}$$

$$9.\begin{pmatrix} F^{P+}_{B_1\times B_2}((u_1,v)(u_2,v)) = r\max\{F^{P+}_{B_1}(u_1,u_2), F^{P+}_{\lambda_2}(v)\}, \\ F^{P-}_{B_1\times B_2}((u_1,v)(u_2,v)) = r\min\{F^{P-}_{B_1}(u_1,u_2), F^{P+}_{\lambda_2}(v)\}, \\ F^{P+}_{\mu_1\times \mu_2}((u_1,v)(u_2,v)) = r\min\{F^{P+}_{\mu_1}(u_1,u_2), F^{P+}_{\lambda_2}(v)\}, \\ F^{P-}_{\mu_1\times \mu_2}((u_1,v)(u_2,v)) = r\max\{F^{P-}_{\mu_1}(u_1,u_2), F^{P+}_{\lambda_2}(v)\} \end{pmatrix}$$

Example 4.2.7: Let $G_1 = (P_1, Q_1)$ be a bipolar spherical neutrosophic cubic graph of $G_1^* = (V_1, E_1)$ as shown in Fig. 4.2 , Where $V_1 = \{a,b,c\}$, $E_1 = \{ab, bc, ac\}$.

$$P_1 = \left\langle \begin{matrix} \begin{Bmatrix} a, ([0.3,0.6],0.8), ([0.4,0.6],0.7), ([0.5,0.6],0.4), \\ ([-0.8,-0.6],-0.2), ([-0.7,-0.3],-0.2), ([-0.9,-0.6],-0.4) \end{Bmatrix} \\ \begin{Bmatrix} b, ([0.1,0.3],0.5), ([0.2,0.3],0.6), ([0.6,0.7],0.8), \\ ([-0.4,-0.3],-0.2), ([-0.6,-0.5],-0.8), ([-0.9,-0.8],-0.6) \end{Bmatrix} \\ \begin{Bmatrix} c, ([0.3,0.5],0.2), ([0.8,0.9],0.5), ([0.2,0.4],0.5), \\ ([-0.8,-0.7],-0.5), ([-0.5,-0.2],-0.1), ([-0.3,-0.2],-0.1) \end{Bmatrix} \end{matrix} \right\rangle$$

$$Q_1 = \left\langle \begin{matrix} \begin{Bmatrix} ab, ([0.1,0.3],0.8), ([0.2,0.3],0.7), ([0.6,0.7],0.4), \\ ([-0.4,-0.3],-0.2), ([-0.6,-0.3],-0.8), ([-0.9,-0.8],-0.4) \end{Bmatrix} \\ \begin{Bmatrix} ac, ([0.3,0.5],0.8), ([0.4,0.6],0.7), ([0.5,0.6],0.4), \\ ([-0.8,-0.6],-0.5), ([-0.5,-0.2],-0.2), ([-0.9,-0.6],-0.1) \end{Bmatrix} \\ \begin{Bmatrix} bc, ([0.1,0.3],0.5), ([0.2,0.3],0.6), ([0.6,0.7],0.4), \\ ([-0.4,-0.3],-0.5), ([-0.5,-0.2],-0.8), ([-0.9,-0.8],-0.1) \end{Bmatrix} \end{matrix} \right\rangle$$

and $G_2 = (P_2, Q_2)$ be an bipolar spherical neutrosophic cubic graph if $G_2^* = (V_2, E_2)$ as shown in Fig. 4.3, where $V_2 = \{x, y, z\}$ and $E_2 = \{xy, yz, xz\}$

$$P_2 = \left\langle \begin{matrix} \begin{Bmatrix} x, ([0.5,0.6],0.3), ([0.4,0.7],0.1), ([0.2,0.3],0.5), \\ ([-0.7,-0.6],-0.1), ([-0.4,-0.2],-0.5), ([-0.5,-0.4],-0.3) \end{Bmatrix} \\ \begin{Bmatrix} y, ([0.1,0.2],0.4), ([0.7,0.3],0.9), ([0.2,0.4],0.1), \\ ([-0.3,-0.2],-0.1), ([-0.5,-0.3],-0.2), ([-0.7,-0.5],-0.3) \end{Bmatrix} \\ \begin{Bmatrix} z, ([0.3,0.5],0.2), ([0.5,0.6],0.7), ([0.2,0.6],0.8), \\ ([-0.8,-0.7],-0.5), ([-0.7,-0.4],-0.3), ([-0.9,-0.4],-0.2) \end{Bmatrix} \end{matrix} \right\rangle$$

$$Q_2 = \left\langle \begin{matrix} \begin{Bmatrix} xy, ([0.1,0.2],0.4), ([0.4,0.3],0.9), ([0.2,0.4],0.1), \\ ([-0.3,-0.2],-0.1), ([-0.4,-0.2],-0.5), ([-0.7,-0.5],-0.3) \end{Bmatrix} \\ \begin{Bmatrix} xz, ([0.3,0.5],0.3), ([0.4,0.6],0.7), ([0.2,0.6],0.5), \\ ([-0.7,-0.6],-0.5), ([-0.4,-0.2],-0.5), ([-0.9,-0.4],-0.2) \end{Bmatrix} \\ \begin{Bmatrix} yz, ([0.1,0.2],0.4), ([0.5,0.3],0.9), ([0.2,0.6],0.1), \\ ([-0.3,-0.2],-0.5), ([-0.5,-0.3],-0.3), ([-0.9,-0.5],-0.2) \end{Bmatrix} \end{matrix} \right\rangle$$

Then $G_1 \times G_2$ is an bipolar spherical neutrosophic cubic graph of $G_1^* \times G_2^*$ as shown in Fig. 4.4, where $V_1 \times V_2 = \{(a,x),(a,y),(a,z),(b,x),(b,y),(b,z),(c,x),(c,y),(c,z)\}$ and

$$P_1 \times P_2 = \left\langle \begin{array}{l} \left\{ \begin{array}{l} (a,x),([0.3,0.6],0.8),([0.4,0.6],0.7),([0.5,0.6],0.4), \\ ([-0.7,-0.6],-0.2),([-0.4,-0.2],-0.5),([-0.9,-0.6],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} (a,y),([0.1,0.2],0.8),([0.4,0.3],0.9),([0.5,0.6],0.1), \\ ([-0.3,-0.2],-0.2),([-0.5,-0.3],-0.2),([-0.9,-0.6],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} (a,z),([0.3,0.5],0.8),([0.4,0.6],0.7),([0.5,0.6],0.4), \\ ([-0.8,-0.6],-0.5),([-0.7,-0.3],-0.3),([-0.9,-0.6],-0.2) \end{array} \right\} \\ \left\{ \begin{array}{l} (b,x),([0.1,0.3],0.5),([0.2,0.3],0.6),([0.6,0.7],0.5), \\ ([-0.4,-0.3],-0.2),([-0.4,-0.2],-0.8),([-0.9,-0.8],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} (b,y),([0.1,0.2],0.5),([0.2,0.3],0.9),([0.6,0.7],0.1), \\ ([-0.3,-0.2],-0.2),([-0.5,-0.3],-0.8),([-0.9,-0.8],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} (b,z),([0.1,0.3],0.5),([0.2,0.3],0.7),([0.6,0.7],0.8), \\ ([-0.4,-0.3],-0.5),([-0.6,-0.4],-0.8),([-0.9,-0.8],-0.2) \end{array} \right\} \\ \left\{ \begin{array}{l} (c,x),([0.3,0.5],0.3),([0.4,0.7],0.5),([0.2,0.4],0.5), \\ ([-0.7,-0.6],-0.5),([-0.4,-0.2],-0.5),([-0.5,-0.4],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} (c,y),([0.1,0.2],0.4),([0.7,0.3],0.9),([0.2,0.4],0.1), \\ ([-0.3,-0.2],-0.5),([-0.5,-0.2],-0.2),([-0.7,-0.5],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} (c,z),([0.3,0.5],0.2),([0.5,0.6],0.7),([0.2,0.6],0.5), \\ ([-0.8,-0.7],-0.5),([-0.5,-0.2],-0.3),([-0.9,-0.4],-0.1) \end{array} \right\} \end{array} \right\rangle$$

$$Q_1 \times Q_2 = \left\langle \begin{array}{l} \left\{ \begin{array}{l} ((a,x)(a,y)),([0.1,0.2],0.8),([0.4,0.3],0.9),([0.5,0.6],0.1), \\ ([-0.3,-0.2],-0.2),([-0.4,-0.2],-0.5),([-0.9,-0.6],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} ((a,y)(a,z)),([0.1,0.2],0.8),([0.4,0.3],0.9),([0.5,0.6],0.1), \\ ([-0.3,-0.2],-0.5),([-0.5,-0.3],-0.3),([-0.9,-0.6],-0.2) \end{array} \right\} \\ \left\{ \begin{array}{l} ((a,z)(b,z)),([0.1,0.3],0.8),([0.2,0.3],0.7),([0.6,0.7],0.4), \\ ([-0.4,-0.3],-0.5),([-0.6,-0.3],-0.8),([-0.9,-0.8],-0.2) \end{array} \right\} \\ \left\{ \begin{array}{l} ((b,x)(b,y)),([0.1,0.2],0.5),([0.2,0.3],0.9),([0.6,0.7],0.1), \\ ([-0.3,-0.2],-0.2),([-0.4,-0.2],-0.8),([-0.9,-0.8],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} ((b,x)(b,z)),([0.1,0.3],0.5),([0.2,0.3],0.7),([0.6,0.7],0.5), \\ ([-0.4,-0.3],-0.5),([-0.4,-0.2],-0.8),([-0.9,-0.8],-0.2) \end{array} \right\} \\ \left\{ \begin{array}{l} ((c,y)(c,z)),([0.1,0.2],0.4),([0.5,0.3],0.9),([0.2,0.6],0.1), \\ ([-0.3,-0.2],-0.5),([-0.5,-0.2],-0.3),([-0.9,-0.5],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} ((c,x)(c,z)),([0.3,0.5],0.3),([0.4,0.6],0.7),([0.2,0.6],0.5), \\ ([-0.7,-0.6],-0.5),([-0.4,-0.2],-0.5),([-0.9,-0.4],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} ((a,x)(c,x)),([0.3,0.5],0.8),([0.4,0.6],0.7),([0.5,0.6],0.4), \\ ([-0.7,-0.6],-0.5),([-0.4,-0.2],-0.5),([-0.9,-0.6],-0.1) \end{array} \right\} \end{array} \right\rangle$$

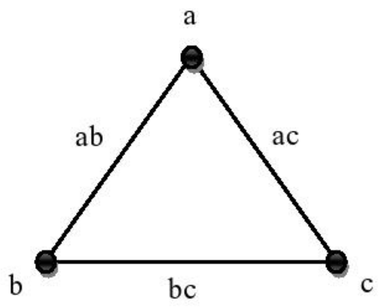

Fig. 4.2: The vertex set in P_1 and the edge set in Q_1 are represented for the graph $G_1=(P_1,Q_1)$

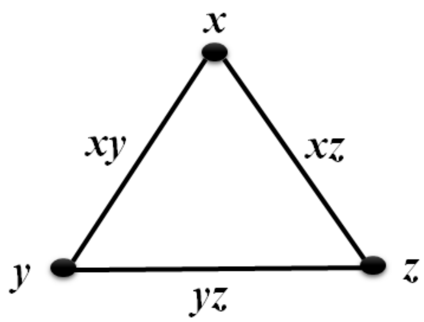

Fig. 4.3: The vertex set in P_2 and the edge set in Q_2 are represented for the graph $G_2=(P_2,Q_2)$

Theorem 4.2.8: The Cartesian product of two bipolar spherical neutrosophic cubic graphs is again an bipolar spherical neutrosophic cubic graph.

Proof: For $P_1 \times P_2$ the condition is obvious. Now we verify the conditions only for $Q_1 \times Q_2$,

where

$$Q_1 \times Q_2 = \begin{Bmatrix} (T^{P+}_{B_1 \times B_2}, T^{P+}_{\mu_1 \times \mu_2}), (I^{P+}_{B_1 \times B_2}, I^{P+}_{\mu_1 \times \mu_2}), (F^{P+}_{B_1 \times B_2}, F^{P+}_{\mu_1 \times \mu_2}), \\ (T^{P-}_{B_1 \times B_2}, T^{P-}_{\mu_1 \times \mu_2}), (I^{P-}_{B_1 \times B_2}, I^{P-}_{\mu_1 \times \mu_2}), (F^{P-}_{B_1 \times B_2}, F^{P-}_{\mu_1 \times \mu_2}) \end{Bmatrix}$$

Then

$$\begin{pmatrix} T^{P+}_{B_1\times B_2}((u,u_2)(u,v_2)) = r\min\{ T^{P+}_{A_1}(u), T^{P+}_{B_2}(u_2v_2)\}, \\ \leq r\min\{(T^{P+}_{A_1}(u)), (r\min(T^{P+}_{A_2}(u_2), T^{P+}_{A_2}(v_2)))\} \\ = r\min\{ r\min(T^{P+}_{A_1}(u), T^{P+}_{A_2}(u_2)), r\min(T^{P+}_{A_1}(u), T^{P+}_{A_2}(v_2))\} \\ = r\min\{((T^{P+}_{A_1}\times T^{P+}_{A_2})(uu_2)), ((T^{P+}_{A_1}\times T^{P+}_{A_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} T^{P-}_{B_1\times B_2}((u,u_2)(u,v_2)) = r\max\{ T^{P-}_{A_1}(u), T^{P-}_{B_2}(u_2v_2)\}, \\ \geq r\max\{(T^{P-}_{A_1}(u)), (r\max(T^{P-}_{A_2}(u_2), T^{P-}_{A_2}(v_2)))\} \\ = r\max\{ r\max(T^{P-}_{A_1}(u), T^{P-}_{A_2}(u_2)), r\max(T^{P-}_{A_1}(u), T^{P-}_{A_2}(v_2))\} \\ = r\max\{((T^{P-}_{A_1}\times T^{P-}_{A_2})(uu_2)), ((T^{P-}_{A_1}\times T^{P-}_{A_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} T^{P+}_{\mu_1\times \mu_2}((u,u_2)(u,v_2)) = r\max\{ T^{P+}_{\lambda_1}(u), T^{P+}_{\mu_2}(u_2v_2)\}, \\ \leq r\max\{(T^{P+}_{\lambda_1}(u)), (r\max(T^{P+}_{\lambda_2}(u_2), T^{P+}_{\lambda_2}(v_2)))\} \\ = r\max\{ r\max(T^{P+}_{\lambda_1}(u), T^{P+}_{\lambda_2}(u_2)), r\max(T^{P+}_{\lambda_1}(u), T^{P+}_{\lambda_2}(v_2))\} \\ = r\max\{((T^{P+}_{\lambda_1}\times T^{P+}_{\lambda_2})(uu_2)), ((T^{P+}_{\lambda_1}\times T^{P+}_{\lambda_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} T^{P-}_{\mu_1\times \mu_2}((u,u_2)(u,v_2)) = r\min\{ T^{P-}_{\lambda_1}(u), T^{P-}_{\mu_2}(u_2v_2)\}, \\ \geq r\min\{(T^{P-}_{\lambda_1}(u)), (r\min(T^{P-}_{\lambda_2}(u_2), T^{P-}_{\lambda_2}(v_2)))\} \\ = r\min\{ r\min(T^{P-}_{\lambda_1}(u), T^{P-}_{\lambda_2}(u_2)), r\min(T^{P-}_{\lambda_1}(u), T^{P-}_{\lambda_2}(v_2))\} \\ = r\min\{((T^{P-}_{\lambda_1}\times T^{P-}_{\lambda_2})(uu_2)), ((T^{P-}_{\lambda_1}\times T^{P-}_{\lambda_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} I^{P+}_{B_1\times B_2}((u,u_2)(u,v_2)) = r\min\{ I^{P+}_{A_1}(u), I^{P+}_{B_2}(u_2v_2)\}, \\ \leq r\min\{(I^{P+}_{A_1}(u)), (r\min(I^{P+}_{A_2}(u_2), I^{P+}_{A_2}(v_2)))\} \\ = r\min\{ r\min(I^{P+}_{A_1}(u), I^{P+}_{A_2}(u_2)), r\min(I^{P+}_{A_1}(u), I^{P+}_{A_2}(v_2))\} \\ = r\min\{((I^{P+}_{A_1}\times I^{P+}_{A_2})(uu_2)), ((I^{P+}_{A_1}\times I^{P+}_{A_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} I^{P-}_{B_1\times B_2}((u,u_2)(u,v_2)) = r\max\{ I^{P-}_{A_1}(u), I^{P-}_{B_2}(u_2v_2)\}, \\ \geq r\max\{(I^{P-}_{A_1}(u)), (r\max(I^{P-}_{A_2}(u_2), I^{P-}_{A_2}(v_2)))\} \\ = r\max\{ r\max(I^{P-}_{A_1}(u), I^{P-}_{A_2}(u_2)), r\max(I^{P-}_{A_1}(u), I^{P-}_{A_2}(v_2))\} \\ = r\max\{((I^{P-}_{A_1}\times I^{P-}_{A_2})(uu_2)), ((I^{P-}_{A_1}\times I^{P-}_{A_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} I^{P+}_{\mu_1\times\mu_2}((u,u_2)(u,v_2)) = r\max\{ I^{P+}_{\lambda_1}(u), I^{P+}_{\mu_2}(u_2v_2)\}, \\ \leq r\max\{(I^{P+}_{\lambda_1}(u)), (r\max(I^{P+}_{\lambda_2}(u_2), I^{P+}_{\lambda_2}(v_2)))\} \\ = r\max\{ r\max(I^{P+}_{\lambda_1}(u), I^{P+}_{\lambda_2}(u_2)), r\max(I^{P+}_{\lambda_1}(u), I^{P+}_{\lambda_2}(v_2))\} \\ = r\max\{((I^{P+}_{\lambda_1}\times I^{P+}_{\lambda_2})(uu_2)), ((I^{P+}_{\lambda_1}\times I^{P+}_{\lambda_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} I^{P-}_{\mu_1\times\mu_2}((u,u_2)(u,v_2)) = r\min\{ I^{P-}_{\lambda_1}(u), I^{P-}_{\mu_2}(u_2v_2)\}, \\ \geq r\min\{(I^{P-}_{\lambda_1}(u)), (r\min(I^{P-}_{\lambda_2}(u_2), I^{P-}_{\lambda_2}(v_2)))\} \\ = r\min\{ r\min(I^{P-}_{\lambda_1}(u), I^{P-}_{\lambda_2}(u_2)), r\min(I^{P-}_{\lambda_1}(u), I^{P-}_{\lambda_2}(v_2))\} \\ = r\min\{((I^{P-}_{\lambda_1}\times I^{P-}_{\lambda_2})(uu_2)), ((I^{P-}_{\lambda_1}\times I^{P-}_{\lambda_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} F^{P+}_{B_1\times B_2}((u,u_2)(u,v_2)) = r\max\{ F^{P+}_{A_1}(u), F^{P+}_{B_2}(u_2v_2)\}, \\ \leq r\max\{(F^{P+}_{A_1}(u)), (r\max(F^{P+}_{A_2}(u_2), F^{P+}_{A_2}(v_2)))\} \\ = r\max\{ r\max(F^{P+}_{A_1}(u), F^{P+}_{A_2}(u_2)), r\max(F^{P+}_{A_1}(u), F^{P+}_{A_2}(v_2))\} \\ = r\max\{((F^{P+}_{A_1}\times F^{P+}_{A_2})(uu_2)), ((F^{P+}_{A_1}\times F^{P+}_{A_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} F^{P-}_{B_1\times B_2}((u,u_2)(u,v_2)) = r\min\{ F^{P-}_{A_1}(u), F^{P-}_{B_2}(u_2v_2)\}, \\ \geq r\min\{(F^{P-}_{A_1}(u)), (r\min(F^{P-}_{A_2}(u_2), F^{P-}_{A_2}(v_2)))\} \\ = r\min\{ r\min(F^{P-}_{A_1}(u), F^{P-}_{A_2}(u_2)), r\min(F^{P-}_{A_1}(u), F^{P-}_{A_2}(v_2))\} \\ = r\min\{((F^{P-}_{A_1}\times F^{P-}_{A_2})(uu_2)), ((F^{P-}_{A_1}\times F^{P-}_{A_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} F^{P+}_{\mu_1\times\mu_2}((u,u_2)(u,v_2)) = r\min\{ F^{P+}_{\lambda_1}(u), F^{P+}_{\mu_2}(u_2v_2)\}, \\ \leq r\min\{(F^{P+}_{\lambda_1}(u)), (r\min(F^{P+}_{\lambda_2}(u_2), F^{P+}_{\lambda_2}(v_2)))\} \\ = r\min\{ r\min(F^{P+}_{\lambda_1}(u), F^{P+}_{\lambda_2}(u_2)), r\min(F^{P+}_{\lambda_1}(u), F^{P+}_{\lambda_2}(v_2))\} \\ = r\min\{((F^{P+}_{\lambda_1}\times F^{P+}_{\lambda_2})(uu_2)), ((F^{P+}_{\lambda_1}\times F^{P+}_{\lambda_2})(uv_2))\} \end{pmatrix}$$

$$\begin{pmatrix} F^{P-}_{\mu_1\times\mu_2}((u,u_2)(u,v_2)) = r\max\{ F^{P-}_{\lambda_1}(u), F^{P-}_{\mu_2}(u_2v_2)\}, \\ \geq r\max\{(F^{P-}_{\lambda_1}(u)), (r\max(F^{P-}_{\lambda_2}(u_2), F^{P-}_{\lambda_2}(v_2)))\} \\ = r\max\{ r\max(F^{P-}_{\lambda_1}(u), F^{P-}_{\lambda_2}(u_2)), r\max(F^{P-}_{\lambda_1}(u), F^{P-}_{\lambda_2}(v_2))\} \\ = r\max\{((F^{P-}_{\lambda_1}\times F^{P-}_{\lambda_2})(uu_2)), ((F^{P-}_{\lambda_1}\times F^{P-}_{\lambda_2})(uv_2))\} \end{pmatrix}$$

Similarly, we can prove it for $w \in V_2$ and $u_1, u_2 \in E_2$

Definition 4.2.9: Let $G_1=(P_1,Q_1)$ and $G_2=(P_2,Q_2)$ be two bipolar spherical neutrosophic cubic graphs. The degree of a vertex in $G_1\times G_2$ can be defined as follows for any $(u_1\times u_2)\in v_1\times v_2$

$$\begin{aligned}\deg(T_{A_1}^{P+}\times T_{A_2}^{P+})(u_1,u_2) &= \sum_{(u_1u_2)(v_1v_2)\in E_2} r\max(T_{B_1}^{P+}\times T_{B_2}^{P+})((u_1,u_2)(v_1,v_2))\\ &= \sum_{u_1=v_1=u,u_2v_2\in E_2} r\max(T_{A_1}^{P+}(u),T_{B_2}^{P+}(u_2,v_2))\\ &+ \sum_{u_2=v_2=w,u_1v_1\in E} r\max(T_{A_2}^{P+}(w),T_{B_1}^{P+}(u_1,v_1))\\ &+ \sum_{u_1=v_1\in E,u_2v_2\in E_2} r\max(T_{B_1}^{P+}(u_1,v_1),T_{B_2}^{P+}(u_2,v_2))\end{aligned}$$

$$\begin{aligned}\deg(T_{A_1}^{P-}\times T_{A_2}^{P-})(u_1,u_2) &= \sum_{(u_1u_2)(v_1v_2)\in E_2} r\min(T_{B_1}^{P-}\times T_{B_2}^{P-})((u_1,u_2)(v_1,v_2))\\ &= \sum_{u_1=v_1=u,u_2v_2\in E_2} r\max(T_{A_1}^{P-}(u),T_{B_2}^{P-}(u_2,v_2))\\ &+ \sum_{u_2=v_2=w,u_1v_1\in E} r\max(T_{A_2}^{P-}(w),T_{B_1}^{P-}(u_1,v_1))\\ &+ \sum_{u_1=v_1\in E,u_2v_2\in E_2} r\max(T_{B_1}^{P-}(u_1,v_1),T_{B_2}^{P-}(u_2,v_2))\end{aligned}$$

$$\begin{aligned}\deg(T_{\lambda_1}^{P+}\times T_{\lambda_2}^{P+})(u_1,u_2) &= \sum_{(u_1u_2)(v_1v_2)\in E_2} r\min(T_{\mu_1}^{P+}\times T_{\mu_2}^{P+})((u_1,u_2)(v_1,v_2))\\ &= \sum_{u_1=v_1=u,u_2v_2\in E_2} r\min(T_{\lambda_1}^{P+}(u),T_{\mu_2}^{P+}(u_2,v_2))\\ &+ \sum_{u_2=v_2=w,u_1v_1\in E} r\min(T_{\lambda_2}^{P+}(w),T_{\mu_1}^{P+}(u_1,v_1))\\ &+ \sum_{u_1=v_1\in E,u_2v_2\in E_2} r\min(T_{\mu_1}^{P+}(u_1,v_1),T_{\mu_2}^{P+}(u_2,v_2))\end{aligned}$$

$$\deg(T_{\lambda_1}^{P-}\times T_{\lambda_2}^{P-})(u_1,u_2)=\sum_{(u_1u_2)(v_1v_2)\in E_2} r\max(T_{\mu_1}^{P-}\times T_{\mu_2}^{P-})((u_1,u_2)(v_1,v_2))$$
$$=\sum_{u_1=v_1=u,u_2v_2\in E_2} r\max(T_{\lambda_1}^{P-}(u),T_{\mu_2}^{P-}(u_2,v_2))$$
$$+\sum_{u_2=v_2=w,u_1v_1\in E} r\max(T_{\lambda_2}^{P-}(w),T_{\mu_1}^{P-}(u_1,v_1))$$
$$+\sum_{u_1=v_1\in E,u_2v_2\in E_2} r\max(T_{\mu_1}^{P-}(u_1,v_1),T_{\mu_2}^{P-}(u_2,v_2))$$

$$\deg(I_{A_1}^{P+}\times I_{A_2}^{P+})(u_1,u_2)=\sum_{(u_1u_2)(v_1v_2)\in E_2} r\max(I_{B_1}^{P+}\times I_{B_2}^{P+})((u_1,u_2)(v_1,v_2))$$
$$=\sum_{u_1=v_1=u,u_2v_2\in E_2} r\max(I_{A_1}^{P+}(u),I_{B_2}^{P+}(u_2,v_2))$$
$$+\sum_{u_2=v_2=w,u_1v_1\in E} r\max(I_{A_2}^{P+}(w),I_{B_1}^{P+}(u_1,v_1))$$
$$+\sum_{u_1=v_1\in E,u_2v_2\in E_2} r\max(I_{B_1}^{P+}(u_1,v_1),I_{B_2}^{P+}(u_2,v_2))$$

$$\deg(I_{A_1}^{P-}\times I_{A_2}^{P-})(u_1,u_2)=\sum_{(u_1u_2)(v_1v_2)\in E_2} r\min(I_{B_1}^{P-}\times I_{B_2}^{P-})((u_1,u_2)(v_1,v_2))$$
$$=\sum_{u_1=v_1=u,u_2v_2\in E_2} r\max(I_{A_1}^{P-}(u),I_{B_2}^{P-}(u_2,v_2))$$
$$+\sum_{u_2=v_2=w,u_1v_1\in E} r\max(I_{A_2}^{P-}(w),I_{B_1}^{P-}(u_1,v_1))$$
$$+\sum_{u_1=v_1\in E,u_2v_2\in E_2} r\max(I_{B_1}^{P-}(u_1,v_1),I_{B_2}^{P-}(u_2,v_2))$$

$$\deg(I_{\lambda_1}^{P+}\times I_{\lambda_2}^{P+})(u_1,u_2)=\sum_{(u_1u_2)(v_1v_2)\in E_2} r\min(I_{\mu_1}^{P+}\times I_{\mu_2}^{P+})((u_1,u_2)(v_1,v_2))$$
$$=\sum_{u_1=v_1=u,u_2v_2\in E_2} r\min(I_{\lambda_1}^{P+}(u),I_{\mu_2}^{P+}(u_2,v_2))$$
$$+\sum_{u_2=v_2=w,u_1v_1\in E} r\min(I_{\lambda_2}^{P+}(w),I_{\mu_1}^{P+}(u_1,v_1))$$
$$+\sum_{u_1=v_1\in E,u_2v_2\in E_2} r\min(I_{\mu_1}^{P+}(u_1,v_1),I_{\mu_2}^{P+}(u_2,v_2))$$

$$\deg(I_{\lambda_1}^{P-} \times I_{\lambda_2}^{P-})(u_1,u_2) = \sum_{(u_1u_2)(v_1v_2)\in E_2} r\max(I_{\mu_1}^{P-} \times I_{\mu_2}^{P-})((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1=v_1=u, u_2v_2\in E_2} r\max(I_{\lambda_1}^{P-}(u), I_{\mu_2}^{P-}(u_2,v_2))$$

$$+ \sum_{u_2=v_2=w, u_1v_1\in E} r\max(I_{\lambda_2}^{P-}(w), I_{\mu_1}^{P-}(u_1,v_1))$$

$$+ \sum_{u_1=v_1\in E, u_2v_2\in E_2} r\max(I_{\mu_1}^{P-}(u_1,v_1), I_{\mu_2}^{P-}(u_2,v_2))$$

$$\deg(F_{A_1}^{P+} \times F_{A_2}^{P+})(u_1,u_2) = \sum_{(u_1u_2)(v_1v_2)\in E_2} r\min(F_{B_1}^{P+} \times F_{B_2}^{P+})((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1=v_1=u, u_2v_2\in E_2} r\min(F_{A_1}^{P+}(u), F_{B_2}^{P+}(u_2,v_2))$$

$$+ \sum_{u_2=v_2=w, u_1v_1\in E} r\min(F_{A_2}^{P+}(w), F_{B_1}^{P+}(u_1,v_1))$$

$$+ \sum_{u_1=v_1\in E, u_2v_2\in E_2} r\min(F_{B_1}^{P+}(u_1,v_1), F_{B_2}^{P+}(u_2,v_2))$$

$$\deg(F_{A_1}^{P-} \times F_{A_2}^{P-})(u_1,u_2) = \sum_{(u_1u_2)(v_1v_2)\in E_2} r\max(F_{B_1}^{P-} \times F_{B_2}^{P-})((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1=v_1=u, u_2v_2\in E_2} r\min(F_{A_1}^{P-}(u), F_{B_2}^{P-}(u_2,v_2))$$

$$+ \sum_{u_2=v_2=w, u_1v_1\in E} r\min(F_{A_2}^{P-}(w), F_{B_1}^{P-}(u_1,v_1))$$

$$+ \sum_{u_1=v_1\in E, u_2v_2\in E_2} r\min(F_{B_1}^{P-}(u_1,v_1), F_{B_2}^{P-}(u_2,v_2))$$

$$\deg(F_{\lambda_1}^{P+} \times F_{\lambda_2}^{P+})(u_1,u_2) = \sum_{(u_1u_2)(v_1v_2)\in E_2} r\max(F_{\mu_1}^{P+} \times F_{\mu_2}^{P+})((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_1=v_1=u, u_2v_2\in E_2} r\max(F_{\lambda_1}^{P+}(u), F_{\mu_2}^{P+}(u_2,v_2))$$

$$+ \sum_{u_2=v_2=w, u_1v_1\in E} r\max(F_{\lambda_2}^{P+}(w), F_{\mu_1}^{P+}(u_1,v_1))$$

$$+ \sum_{u_1=v_1\in E, u_2v_2\in E_2} r\max(F_{\mu_1}^{P+}(u_1,v_1), F_{\mu_2}^{P+}(u_2,v_2))$$

$$\deg(F_{\lambda_1}^{P-} \times F_{\lambda_2}^{P-})(u_1,u_2) = \sum_{(u_1u_2)(v_1v_2)\in E_2} r\min(F_{\mu_1}^{P-} \times F_{\mu_2}^{P-})((u_1,u_2)(v_1,v_2))$$
$$= \sum_{u_1=v_1=u, u_2v_2\in E_2} r\min(F_{\lambda_1}^{P-}(u), F_{\mu_2}^{P-}(u_2,v_2))$$
$$+ \sum_{u_2=v_2=w, u_1v_1\in E} r\min(F_{\lambda_2}^{P-}(w), F_{\mu_1}^{P-}(u_1,v_1))$$
$$+ \sum_{u_1=v_1\in E, u_2v_2\in E_2} r\min(F_{\mu_1}^{P-}(u_1,v_1), F_{\mu_2}^{P-}(u_2,v_2))$$

Definition 4.2.10: Let $G_1 = (P_1, Q_1)$ be a bipolar spherical neutrosophic cubic graph of $G_1^* = (V_1^*, E_1^*)$ and $G_2 = (P_2, Q_2)$ be a bipolar spherical neutrosophic cubic graph of $G_2^* = (V_2^*, E_2^*)$. Then the composition of G_1 and G_1 is denoted by $G_1[G_2]$ and defined as follows:

$$\begin{aligned} G_1[G_2] &= (P_1,Q_1)[P_2,Q_2] \\ &= \{(P_1[P_2], Q_1[Q_2]\} \\ &= \{(A_1,\lambda_1)[A_2,\lambda_2], (B_1,\mu_1)[B_2,\mu_2]\} \\ &= \{(A_1[A_2], \lambda_1[\lambda_2], (B_1[B_2], \mu_1[\mu_2])\} \end{aligned}$$

$$= \left\{ \begin{array}{l} \left\langle \begin{array}{l} ((T_{A_1}^{P+} \circ T_{A_2}^{P+}), (T_{\lambda_1}^{P+} \circ T_{\lambda_2}^{P+})), ((I_{A_1}^{P+} \circ I_{A_2}^{P+}), (I_{\lambda_1}^{P+} \circ I_{\lambda_2}^{P+})), ((F_{A_1}^{P+} \circ F_{A_2}^{P+}), (F_{\lambda_1}^{P+} \circ F_{\lambda_2}^{P+})), \\ ((T_{A_1}^{P-} \circ T_{A_2}^{P-}), (T_{\lambda_1}^{P-} \circ T_{\lambda_2}^{P-})), ((I_{A_1}^{P-} \circ I_{A_2}^{P-}), (I_{\lambda_1}^{P-} \circ I_{\lambda_2}^{P-})), ((F_{A_1}^{P-} \circ F_{A_2}^{P-}), (F_{\lambda_1}^{P-} \circ F_{\lambda_2}^{P-})) \end{array} \right\rangle \\ \left\langle \begin{array}{l} ((T_{B_1}^{P+} \circ T_{B_2}^{P+}), (T_{\mu_1}^{P+} \circ T_{\mu_2}^{P+})), ((I_{B_1}^{P+} \circ I_{B_2}^{P+}), (I_{\mu_1}^{P+} \circ I_{\mu_2}^{P+})), ((F_{B_1}^{P+} \circ F_{B_2}^{P+}), (F_{\mu_1}^{P+} \circ F_{\mu_2}^{P+})), \\ ((T_{B_1}^{P-} \circ T_{B_2}^{P-}), (T_{\mu_1}^{P-} \circ T_{\mu_2}^{P-})), ((I_{B_1}^{P-} \circ I_{B_2}^{P-}), (I_{\mu_1}^{P-} \circ I_{\mu_2}^{P-})), ((F_{B_1}^{P-} \circ F_{B_2}^{P-}), (F_{\mu_1}^{P-} \circ F_{\mu_2}^{P-})) \end{array} \right\rangle \end{array} \right\}$$

1. $\forall (u,v) \in (v_1, v_2) = V$

$$(T_{A_1}^{P+} \circ T_{A_2}^{P+})(u,v) = r\min(T_{A_1}^{P+}(u), T_{A_2}^{P+}(v)), (T_{\lambda_1}^{P+} \circ T_{\lambda_2}^{P+})(u,v) = \max(T_{\lambda_1}^{P+}(u), T_{\lambda_2}^{P+}(v))$$
$$(T_{A_1}^{P-} \circ T_{A_2}^{P-})(u,v) = r\max(T_{A_1}^{P-}(u), T_{A_2}^{P-}(v)), (T_{\lambda_1}^{P-} \circ T_{\lambda_2}^{P-})(u,v) = \min(T_{\lambda_1}^{P-}(u), T_{\lambda_2}^{P-}(v))$$

$$(I_{A_1}^{P+} \circ I_{A_2}^{P+})(u,v) = r\min(I_{A_1}^{P+}(u), I_{A_2}^{P+}(v)), (I_{\lambda_1}^{P+} \circ I_{\lambda_2}^{P+})(u,v) = \max(I_{\lambda_1}^{P+}(u), I_{\lambda_2}^{P+}(v))$$
$$(I_{A_1}^{P-} \circ I_{A_2}^{P-})(u,v) = r\max(I_{A_1}^{P-}(u), I_{A_2}^{P-}(v)), (I_{\lambda_1}^{P-} \circ I_{\lambda_2}^{P-})(u,v) = \min(I_{\lambda_1}^{P-}(u), I_{\lambda_2}^{P-}(v))$$

$$(F_{A_1}^{P+} \circ F_{A_2}^{P+})(u,v) = r\max(F_{A_1}^{P+}(u), F_{A_2}^{P+}(v)), (F_{\lambda_1}^{P+} \circ F_{\lambda_2}^{P+})(u,v) = \min(F_{\lambda_1}^{P+}(u), F_{\lambda_2}^{P+}(v))$$
$$(F_{A_1}^{P-} \circ F_{A_2}^{P-})(u,v) = r\min(F_{A_1}^{P-}(u), F_{A_2}^{P-}(v)), (F_{\lambda_1}^{P-} \circ F_{\lambda_2}^{P-})(u,v) = \max(F_{\lambda_1}^{P-}(u), F_{\lambda_2}^{P-}(v))$$

2. $\forall u \in V_1$ *and* $v_1 v_2 \in E$

$$(T_{B_1}^{P+} \circ T_{B_2}^{P+})((u,v_1)(u,v_2)) = r\min(T_{A_1}^{P+}(u), T_{B_2}^{P+}(v_1v_2)),$$
$$(T_{\mu_1}^{P+} \circ T_{\mu_2}^{P+})((u,v_1)(u,v_2)) = \max(T_{\lambda_1}^{P+}(u), T_{\mu_2}^{P+}(v_1v_2))$$

$$(T_{B_1}^{P-} \circ T_{B_2}^{P-})((u,v_1)(u,v_2)) = r\max(T_{A_1}^{P-}(u), T_{B_2}^{P-}(v_1v_2)),$$
$$(T_{\mu_1}^{P-} \circ T_{\mu_2}^{P-})((u,v_1)(u,v_2)) = \min(T_{\lambda_1}^{P-}(u), T_{\mu_2}^{P-}(v_1v_2))$$

$$(I_{B_1}^{P+} \circ I_{B_2}^{P+})((u,v_1)(u,v_2)) = r\min(I_{A_1}^{P+}(u), I_{B_2}^{P+}(v_1v_2)),$$
$$(I_{\mu_1}^{P+} \circ I_{\mu_2}^{P+})((u,v_1)(u,v_2)) = \max(I_{\lambda_1}^{P+}(u), I_{\mu_2}^{P+}(v_1v_2))$$

$$(I_{B_1}^{P-} \circ I_{B_2}^{P-})((u,v_1)(u,v_2)) = r\max(I_{A_1}^{P-}(u), I_{B_2}^{P-}(v_1v_2)),$$
$$(I_{\mu_1}^{P-} \circ I_{\mu_2}^{P-})((u,v_1)(u,v_2)) = \min(I_{\lambda_1}^{P-}(u), I_{\mu_2}^{P-}(v_1v_2))$$

$$(F_{B_1}^{P+} \circ F_{B_2}^{P+})((u,v_1)(u,v_2)) = r\max(F_{A_1}^{P+}(u), F_{B_2}^{P+}(v_1v_2)),$$
$$(F_{\mu_1}^{P+} \circ F_{\mu_2}^{P+})((u,v_1)(u,v_2)) = \min(F_{\lambda_1}^{P+}(u), F_{\mu_2}^{P+}(v_1v_2))$$

$$(F_{B_1}^{P-} \circ F_{B_2}^{P-})((u,v_1)(u,v_2)) = r\min(F_{A_1}^{P-}(u), F_{B_2}^{P-}(v_1v_2)),$$
$$(F_{\mu_1}^{P-} \circ F_{\mu_2}^{P-})((u,v_1)(u,v_2)) = \max(F_{\lambda_1}^{P-}(u), F_{\mu_2}^{P-}(v_1v_2))$$

3. $\forall v \in V_2$ *and* $u_1 u_2 \in E_1$

$$(T_{B_1}^{P+} \circ T_{B_2}^{P+})((u_1,v)(u_2,v)) = r\min(T_{B_1}^{P+}(u_1u_2), T_{A_2}^{P+}(v)),$$
$$(T_{\mu_1}^{P+} \circ T_{\mu_2}^{P+})((u_1,v)(u_2,v)) = \max(T_{\mu_1}^{P+}(u_1u_2), T_{\lambda_2}^{P+}(v))$$

$$(T_{B_1}^{P-} \circ T_{B_2}^{P-})((u_1,v)(u_2,v)) = r\max(T_{B_1}^{P-}(u_1u_2), T_{A_2}^{P-}(v)),$$
$$(T_{\mu_1}^{P-} \circ T_{\mu_2}^{P-})((u_1,v)(u_2,v)) = \min(T_{\mu_1}^{P-}(u_1u_2), T_{\lambda_2}^{P-}(v))$$

$$(I_{B_1}^{P+} \circ I_{B_2}^{P+})((u_1,v)(u_2,v)) = r\min(I_{B_1}^{P+}(u_1u_2), I_{A_2}^{P+}(v)),$$
$$(I_{\mu_1}^{P+} \circ I_{\mu_2}^{P+})((u_1,v)(u_2,v)) = \max(I_{\mu_1}^{P+}(u_1u_2), I_{\lambda_2}^{P+}(v))$$

$$(I_{B_1}^{P-} \text{o}\, I_{B_2}^{P-})((u_1,v)(u_2,v)) = r\max(I_{B_1}^{P-}(u_1u_2), I_{A_2}^{P-}(v)),$$
$$(I_{\mu_1}^{P-} \text{o}\, I_{\mu_2}^{P-})((u_1,v)(u_2,v)) = \min(I_{\mu_1}^{P-}(u_1u_2), I_{\lambda_2}^{P-}(v))$$

$$(F_{B_1}^{P+} \text{o}\, F_{B_2}^{P+})((u_1,v)(u_2,v)) = r\max(F_{B_1}^{P+}(u_1u_2), F_{A_2}^{P+}(v)),$$
$$(F_{\mu_1}^{P+} \text{o}\, F_{\mu_2}^{P+})((u_1,v)(u_2,v)) = \min(F_{\mu_1}^{P+}(u_1u_2), F_{\lambda_2}^{P+}(v))$$

$$(F_{B_1}^{P-} \text{o}\, F_{B_2}^{P-})((u_1,v)(u_2,v)) = r\min(F_{B_1}^{P-}(u_1u_2), F_{A_2}^{P-}(v)),$$
$$(F_{\mu_1}^{P-} \text{o}\, F_{\mu_2}^{P-})((u_1,v)(u_2,v)) = \max(F_{\mu_1}^{P-}(u_1u_2), F_{\lambda_2}^{P-}(v))$$

4. $\forall (u_1,v_1)(u_2,v_2) \in E^O - E$

$$(T_{B_1}^{P+} \text{o}\, T_{B_2}^{P+})((u_1,v_1)(u_2,v_2)) = r\min(T_{A_2}^{P+}(v_1), T_{A_2}^{P+}(v_2), T_{B_1}^{P+}(u_1,u_2)),$$
$$(T_{\mu_1}^{P+} \text{o}\, T_{\mu_2}^{P+})((u_1,v_1)(u_2,v_2)) = \max(T_{\lambda_2}^{P+}(v_1), T_{\lambda_2}^{P+}(v_2), T_{\mu_1}^{P+}(u_1,u_2))$$

$$(T_{B_1}^{P-} \text{o}\, T_{B_2}^{P-})((u_1,v_1)(u_2,v_2)) = r\max(T_{A_2}^{P-}(v_1), T_{A_2}^{P-}(v_2), T_{B_1}^{P-}(u_1,u_2)),$$
$$(T_{\mu_1}^{P-} \text{o}\, T_{\mu_2}^{P-})((u_1,v_1)(u_2,v_2)) = \min(T_{\lambda_2}^{P-}(v_1), T_{\lambda_2}^{P-}(v_2), T_{\mu_1}^{P-}(u_1,u_2))$$

$$(I_{B_1}^{P+} \text{o}\, I_{B_2}^{P+})((u_1,v_1)(u_2,v_2)) = r\min(I_{A_2}^{P+}(v_1), I_{A_2}^{P+}(v_2), I_{B_1}^{P+}(u_1,u_2)),$$
$$(I_{\mu_1}^{P+} \text{o}\, I_{\mu_2}^{P+})((u_1,v_1)(u_2,v_2)) = \max(I_{\lambda_2}^{P+}(v_1), I_{\lambda_2}^{P+}(v_2), I_{\mu_1}^{P+}(u_1,u_2))$$

$$(I_{B_1}^{P-} \text{o}\, I_{B_2}^{P-})((u_1,v_1)(u_2,v_2)) = r\max(I_{A_2}^{P-}(v_1), I_{A_2}^{P-}(v_2), I_{B_1}^{P-}(u_1,u_2)),$$
$$(I_{\mu_1}^{P-} \text{o}\, I_{\mu_2}^{P-})((u_1,v_1)(u_2,v_2)) = \min(I_{\lambda_2}^{P-}(v_1), I_{\lambda_2}^{P-}(v_2), I_{\mu_1}^{P-}(u_1,u_2))$$

$$(F_{B_1}^{P+} \text{o}\, F_{B_2}^{P+})((u_1,v_1)(u_2,v_2)) = r\max(F_{A_2}^{P+}(v_1), F_{A_2}^{P+}(v_2), F_{B_1}^{P+}(u_1,u_2)),$$
$$(F_{\mu_1}^{P+} \text{o}\, F_{\mu_2}^{P+})((u_1,v_1)(u_2,v_2)) = \min(F_{\lambda_2}^{P+}(v_1), F_{\lambda_2}^{P+}(v_2), F_{\mu_1}^{P+}(u_1,u_2))$$

$$(F_{B_1}^{P-} \text{o}\, F_{B_2}^{P-})((u_1,v_1)(u_2,v_2)) = r\min(F_{A_2}^{P-}(v_1), F_{A_2}^{P-}(v_2), F_{B_1}^{P-}(u_1,u_2)),$$
$$(F_{\mu_1}^{P-} \text{o}\, F_{\mu_2}^{P-})((u_1,v_1)(u_2,v_2)) = \max(F_{\lambda_2}^{P-}(v_1), F_{\lambda_2}^{P-}(v_2), F_{\mu_1}^{P-}(u_1,u_2))$$

Example 4.2.11: Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two bipolar spherical neutrosophic cubic graphs, where $V_1 = (a,b)$ and $V_2 = (c,d)$. Suppose P_1 and P_2 be the bipolar spherical neutrosophic cubic set representations of V_1 and

V_2 . Also Q_1 and Q_2 be the bipolar spherical neutrosophic cubic set representations of E_1 and E_2 defined as follows:

$$P_1 = \left\langle \begin{array}{l} \left\{ \begin{array}{l} a,([0.4,0.7],0.3),([0.3,0.5],0.6),([0.8,0.4],0.7), \\ ([-0.4,-0.3],-0.1),([-0.6,-0.5],-0.2),([-0.8,-0.6],-0.2) \end{array} \right\} \\ \left\{ \begin{array}{l} b,([0.8,0.5],0.1),([0.9,0.4],0.4),([0.6,0.8],0.9), \\ ([-0.8,-0.3],-0.2),([-0.9,-0.5],-0.5),([-0.4,-0.5],-0.2) \end{array} \right\} \end{array} \right\rangle$$

$$Q_1 = \left\langle \left\{ \begin{array}{l} ab,([0.4,0.5],0.3),([0.3,0.4],0.6),([0.8,0.8],0.7), \\ ([-0.4,-0.3],-0.2),([-0.6,-0.5],-0.5),([-0.8,-0.6],-0.2) \end{array} \right\} \right\rangle$$

$$P_2 = \left\langle \begin{array}{l} \left\{ \begin{array}{l} c,([0.3,0.6],0.9),([0.4,0.7],0.5),([1.0,0.2],0.3), \\ ([-0.5,-0.6],-0.3),([-0.4,-0.5],-0.2),([-0.8,-0.6],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} d,([0.5,0.1],0.2),([0.8,0.3],0.4),([0.9,0.4],0.1), \\ ([-0.8,-0.7],-0.3),([-0.7,-0.4],-0.2),([-0.6,-0.5],-0.4) \end{array} \right\} \end{array} \right\rangle$$

$$Q_2 = \left\langle \left\{ \begin{array}{l} cd,([0.3,0.1],0.9),([0.4,0.3],0.5),([0.9,0.4],0.1), \\ ([-0.5,-0.6],-0.3),([-0.4,-0.4],-0.2),([-0.8,-0.6],-0.1) \end{array} \right\} \right\rangle$$

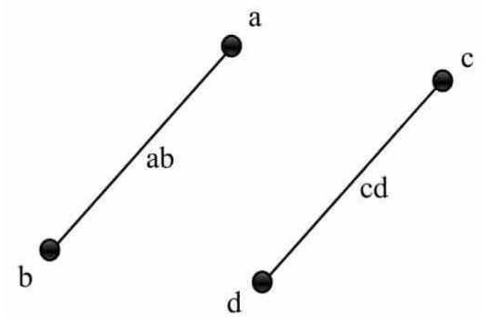

Fig. 4.4: For the two bipolar spherical neutrosophic cubic graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ the vertex sets $V_1 = (a,b)$ and $V_2 = (c,d)$ and the edge sets E_1 and E_2

The composition of two bipolar spherical neutrosophic cubic graphs G_1 and G_2 is again an bipolar spherical neutrosophic cubic graph, where

$$P_1[P_2] = \left\langle \begin{array}{l} \left\{ \begin{array}{l} (a,c),([0.3,0.6],0.9),([0.3,0.5],0.6),([1.0,0.4],0.3), \\ ([-0.4,-0.3],-0.3),([-0.4,-0.5],-0.2),([-0.8,-0.6],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} (a,d),([0.4,0.1],0.3),([0.3,0.3],0.6),([0.9,0.4],0.1), \\ ([-0.4,-0.3],-0.3),([-0.6,-0.4],-0.2),([-0.8,-0.6],-0.2) \end{array} \right\} \\ \left\{ \begin{array}{l} (b,c),([0.3,0.5],0.9),([0.4,0.4],0.5),([1.0,0.8],0.3), \\ ([-0.5,-0.3],-0.3),([-0.4,-0.5],-0.5),([-0.8,-0.6],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} (b,d),([0.5,0.1],0.2),([0.8,0.3],0.4),([0.9,0.8],0.1), \\ ([-0.8,-0.3],-0.3),([-0.7,-0.4],-0.5),([-0.6,-0.5],-0.2) \end{array} \right\} \end{array} \right\rangle$$

$$Q_1[Q_2] = \left\langle \begin{array}{l} \left\{ \begin{array}{l} ((a,c)(a,d)),([0.3,0.1],0.9),([0.3,0.3],0.6),([1.0,0.4],0.1), \\ ([-0.4,-0.3],-0.3),([-0.4,-0.4],-0.2),([-0.8,-0.6],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} ((a,d)(b,d)),([0.4,0.1],0.3),([0.3,0.3],0.6),([0.9,0.8],0.1), \\ ([-0.4,-0.3],-0.3),([-0.6,-0.4],-0.5),([-0.8,-0.6],-0.2) \end{array} \right\} \\ \left\{ \begin{array}{l} ((b,d)(b,c)),([0.3,0.1],0.9),([0.4,0.3],0.5),([1.0,0.8],0.1), \\ ([-0.5,-0.3],-0.3),([-0.4,-0.4],-0.5),([-0.8,-0.6],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} ((b,c)(a,c)),([0.3,0.5],0.9),([0.3,0.4],0.6),([1.0,0.8],0.3), \\ ([-0.4,-0.3],-0.3),([-0.4,-0.5],-0.5),([-0.8,-0.6],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} ((a,c)(b,d)),([0.3,0.1],0.9),([0.3,0.3],0.6),([1.0,0.8],0.1), \\ ([-0.4,-0.3],-0.3),([-0.4,-0.4],-0.5),([-0.8,-0.6],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} ((a,d)(b,c)),([0.3,0.1],0.9),([0.3,0.3],0.6),([1.0,0.8],0.1), \\ ([-0.4,-0.3],-0.3),([-0.4,-0.4],-0.5),([-0.8,-0.6],-0.1) \end{array} \right\} \end{array} \right\rangle$$

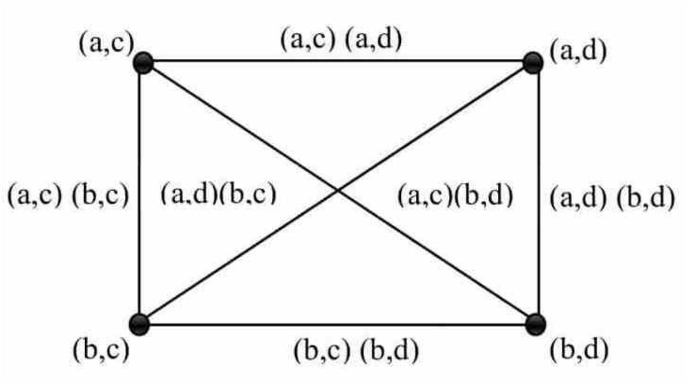

Fig. 4.5. The composition of two bipolar spherical neutrosophic cubic graphs G_1 and G_2.

Definition 4.2.12: Let $G_1=(P_1,Q_1)$ and $G_2=(P_2,Q_2)$ be two bipolar spherical neutrosophic cubic graphs of the graph G_1^* and G_2^* respectively. Then M-union is denoted by $G_1 \cup_M G_2$ and is defined as

$$G_1 \cup_M G_2 = \{(P_1,Q_1) \cup_M (P_2,Q_2)\} = \{P_1 \cup_M P_2, Q_1 \cup_M Q_2\}$$

$$=\left\{\begin{array}{l}\left\langle\begin{array}{l}((T_{A_1}^{P+} \cup_M T_{A_2}^{P+}),(T_{\lambda_1}^{P+} \cup_M T_{\lambda_2}^{P+})),((I_{A_1}^{P+} \cup_M I_{A_2}^{P+}),(I_{\lambda_1}^{P+} \cup_M I_{\lambda_2}^{P+})),((F_{A_1}^{P+} \cup_M F_{A_2}^{P+}),(F_{\lambda_1}^{P+} \cup_M F_{\lambda_2}^{P+}))\\((T_{A_1}^{P-} \cup_M T_{A_2}^{P-}),(T_{\lambda_1}^{P-} \cup_M T_{\lambda_2}^{P-})),((I_{A_1}^{P-} \cup_M I_{A_2}^{P-}),(I_{\lambda_1}^{P-} \cup_M I_{\lambda_2}^{P-})),((F_{A_1}^{P-} \cup_M F_{A_2}^{P-}),(F_{\lambda_1}^{P-} \cup_M F_{\lambda_2}^{P-}))\end{array}\right\rangle\\ \left\langle\begin{array}{l}((T_{B_1}^{P+} \cup_M T_{B_2}^{P+}),(T_{\mu_1}^{P+} \cup_M T_{\mu_2}^{P+})),((I_{B_1}^{P+} \cup_M I_{B_2}^{P+}),(I_{\mu_1}^{P+} \cup_M I_{\mu_2}^{P+})),((F_{B_1}^{P+} \cup_M F_{B_2}^{P+}),(F_{\mu_1}^{P+} \cup_M F_{\mu_2}^{P+}))\\((T_{B_1}^{P-} \cup_M T_{B_2}^{P-}),(T_{\mu_1}^{P-} \cup_M T_{\mu_2}^{P-})),((I_{B_1}^{P-} \cup_M I_{B_2}^{P-}),(I_{\mu_1}^{P-} \cup_M I_{\mu_2}^{P-})),((F_{B_1}^{P-} \cup_M F_{B_2}^{P-}),(F_{\mu_1}^{P-} \cup_M F_{\mu_2}^{P-}))\end{array}\right\rangle\end{array}\right\}$$

where

$$(T_{A_1}^{P+} \cup_M T_{A_2}^{P+})(u)=\begin{cases}T_{A_1}^{P+}(u), & ,if\ u \in v_1-v_2\\ T_{A_2}^{P+}(u), & ,if\ u \in v_2-v_1\\ r\max\{T_{A_1}^{P+}(u),T_{A_2}^{P+}(u)\} & ,if\ u \in v_1 \cap v_2\end{cases}$$

$$(T_{A_1}^{P-} \cup_M T_{A_2}^{P-})(u)=\begin{cases}T_{A_1}^{P-}(u), & ,if\ u \in v_1-v_2\\ T_{A_2}^{P-}(u), & ,if\ u \in v_2-v_1\\ r\min\{T_{A_1}^{P-}(u),T_{A_2}^{P-}(u)\} & ,if\ u \in v_1 \cap v_2\end{cases}$$

$$(T_{\lambda_1}^{P+} \cup_M T_{\lambda_2}^{P+})(u)=\begin{cases}T_{\lambda_1}^{P+}(u), & ,if\ u \in v_1-v_2\\ T_{\lambda_2}^{P+}(u), & ,if\ u \in v_2-v_1\\ \max\{T_{\lambda_1}^{P+}(u),T_{\lambda_2}^{P+}(u)\} & ,if\ u \in v_1 \cap v_2\end{cases}$$

$$(T_{\lambda_1}^{P-} \cup_M T_{\lambda_2}^{P-})(u)=\begin{cases}T_{\lambda_1}^{P-}(u), & ,if\ u \in v_1-v_2\\ T_{\lambda_2}^{P-}(u), & ,if\ u \in v_2-v_1\\ \min\{T_{\lambda_1}^{P-}(u),T_{\lambda_2}^{P-}(u)\} & ,if\ u \in v_1 \cap v_2\end{cases}$$

$$(I_{A_1}^{P+} \cup_M I_{A_2}^{P+})(u) = \begin{cases} I_{A_1}^{P+}(u), & ,if \ u \in v_1 - v_2 \\ I_{A_2}^{P+}(u), & ,if \ u \in v_2 - v_1 \\ r\max\{ I_{A_1}^{P+}(u), I_{A_2}^{P+}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(I_{A_1}^{P-} \cup_M I_{A_2}^{P-})(u) = \begin{cases} I_{A_1}^{P-}(u), & ,if \ u \in v_1 - v_2 \\ I_{A_2}^{P-}(u), & ,if \ u \in v_2 - v_1 \\ r\min\{ I_{A_1}^{P-}(u), I_{A_2}^{P-}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(I_{\lambda_1}^{P+} \cup_M I_{\lambda_2}^{P+})(u) = \begin{cases} I_{\lambda_1}^{P+}(u), & ,if \ u \in v_1 - v_2 \\ I_{\lambda_2}^{P+}(u), & ,if \ u \in v_2 - v_1 \\ \max\{ I_{\lambda_1}^{P+}(u), I_{\lambda_2}^{P+}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(I_{\lambda_1}^{P-} \cup_M I_{\lambda_2}^{P-})(u) = \begin{cases} I_{\lambda_1}^{P-}(u), & ,if \ u \in v_1 - v_2 \\ I_{\lambda_2}^{P-}(u), & ,if \ u \in v_2 - v_1 \\ \min\{ I_{\lambda_1}^{P-}(u), I_{\lambda_2}^{P-}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(F_{A_1}^{P+} \cup_M F_{A_2}^{P+})(u) = \begin{cases} F_{A_1}^{P+}(u), & ,if \ u \in v_1 - v_2 \\ F_{A_2}^{P+}(u), & ,if \ u \in v_2 - v_1 \\ r\min\{ F_{A_1}^{P+}(u), F_{A_2}^{P+}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(F_{A_1}^{P-} \cup_M F_{A_2}^{P-})(u) = \begin{cases} F_{A_1}^{P-}(u), & ,if \ u \in v_1 - v_2 \\ F_{A_2}^{P-}(u), & ,if \ u \in v_2 - v_1 \\ r\max\{ F_{A_1}^{P-}(u), F_{A_2}^{P-}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(F_{\lambda_1}^{P+} \cup_M F_{\lambda_2}^{P+})(u) = \begin{cases} F_{\lambda_1}^{P+}(u), & ,if \ u \in v_1 - v_2 \\ F_{\lambda_2}^{P+}(u), & ,if \ u \in v_2 - v_1 \\ \min\{ F_{\lambda_1}^{P+}(u), F_{\lambda_2}^{P+}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(F_{\lambda_1}^{P-} \cup_M F_{\lambda_2}^{P-})(u) = \begin{cases} F_{\lambda_1}^{P-}(u), & ,if \ u \in v_1 - v_2 \\ F_{\lambda_2}^{P-}(u), & ,if \ u \in v_2 - v_1 \\ \max\{ F_{\lambda_1}^{P-}(u), F_{\lambda_2}^{P-}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(T_{B_1}^{P+}\cup_M T_{B_2}^{P+})(u_2v_2)=\begin{cases} T_{B_1}^{P+}(u_2v_2), & ,if\ u_2v_2\in v_1-v_2 \\ T_{B_2}^{P+}(u_2v_2), & ,if\ u_2v_2\in v_2-v_1 \\ r\max\{T_{B_1}^{P+}(u_2v_2),T_{B_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2\in E_1\cap E_2 \end{cases}$$

$$(T_{B_1}^{P-}\cup_M T_{B_2}^{P-})(u_2v_2)=\begin{cases} T_{B_1}^{P-}(u_2v_2), & ,if\ u_2v_2\in v_1-v_2 \\ T_{B_2}^{P-}(u_2v_2), & ,if\ u_2v_2\in v_2-v_1 \\ r\min\{T_{B_1}^{P-}(u_2v_2),T_{B_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2\in E_1\cap E_2 \end{cases}$$

$$(T_{\mu_1}^{P+}\cup_M T_{\mu_2}^{P+})(u_2v_2)=\begin{cases} T_{\mu_1}^{P+}(u_2v_2), & ,if\ u_2v_2\in v_1-v_2 \\ T_{\mu_2}^{P+}(u_2v_2), & ,if\ u_2v_2\in v_2-v_1 \\ \max\{T_{\mu_1}^{P+}(u_2v_2),T_{\mu_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2\in E_1\cap E_2 \end{cases}$$

$$(T_{\mu_1}^{P-}\cup_M T_{\mu_2}^{P-})(u_2v_2)=\begin{cases} T_{\mu_1}^{P-}(u_2v_2), & ,if\ u_2v_2\in v_1-v_2 \\ T_{\mu_2}^{P-}(u_2v_2), & ,if\ u_2v_2\in v_2-v_1 \\ \min\{T_{\mu_1}^{P-}(u_2v_2),T_{\mu_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2\in E_1\cap E_2 \end{cases}$$

$$(I_{B_1}^{P+}\cup_M I_{B_2}^{P+})(u_2v_2)=\begin{cases} I_{B_1}^{P+}(u_2v_2), & ,if\ u_2v_2\in v_1-v_2 \\ I_{B_2}^{P+}(u_2v_2), & ,if\ u_2v_2\in v_2-v_1 \\ r\max\{I_{B_1}^{P+}(u_2v_2),I_{B_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2\in E_1\cap E_2 \end{cases}$$

$$(I_{B_1}^{P-}\cup_M I_{B_2}^{P-})(u_2v_2)=\begin{cases} I_{B_1}^{P-}(u_2v_2), & ,if\ u_2v_2\in v_1-v_2 \\ I_{B_2}^{P-}(u_2v_2), & ,if\ u_2v_2\in v_2-v_1 \\ r\min\{I_{B_1}^{P-}(u_2v_2),I_{B_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2\in E_1\cap E_2 \end{cases}$$

$$(I_{\mu_1}^{P+}\cup_M I_{\mu_2}^{P+})(u_2v_2)=\begin{cases} I_{\mu_1}^{P+}(u_2v_2), & ,if\ u_2v_2\in v_1-v_2 \\ I_{\mu_2}^{P+}(u_2v_2), & ,if\ u_2v_2\in v_2-v_1 \\ \max\{I_{\mu_1}^{P+}(u_2v_2),I_{\mu_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2\in E_1\cap E_2 \end{cases}$$

$$(I_{\mu_1}^{P-}\cup_M I_{\mu_2}^{P-})(u_2v_2)=\begin{cases} I_{\mu_1}^{P-}(u_2v_2), & ,if\ u_2v_2\in v_1-v_2 \\ I_{\mu_2}^{P-}(u_2v_2), & ,if\ u_2v_2\in v_2-v_1 \\ \min\{I_{\mu_1}^{P-}(u_2v_2),I_{\mu_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2\in E_1\cap E_2 \end{cases}$$

$$(F_{B_1}^{P+} \cup_M F_{B_2}^{P+})(u_2v_2) = \begin{cases} F_{B_1}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ F_{B_2}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ r\min\{ F_{B_1}^{P+}(u_2v_2), F_{B_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(F_{B_1}^{P-} \cup_M F_{B_2}^{P-})(u_2v_2) = \begin{cases} F_{B_1}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ F_{B_2}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ r\max\{ F_{B_1}^{P-}(u_2v_2), F_{B_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(F_{\mu_1}^{P+} \cup_M F_{\mu_2}^{P+})(u_2v_2) = \begin{cases} F_{\mu_1}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ F_{\mu_2}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ \min\{ F_{\mu_1}^{P+}(u_2v_2), F_{\mu_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(F_{\mu_1}^{P-} \cup_M F_{\mu_2}^{P-})(u_2v_2) = \begin{cases} F_{\mu_1}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ F_{\mu_2}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ \max\{ F_{\mu_1}^{P-}(u_2v_2), F_{\mu_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

and the N-union is denoted by $G_1 \cup_N G_2$ and is defined as follows:

$$G_1 \cup_N G_2 = \{(P_1, Q_1) \cup_N (P_2, Q_2)\} = \{P_1 \cup_N P_2, Q_1 \cup_N Q_2\}$$

$$= \left\{ \begin{array}{l} \left\langle \begin{array}{l} ((T_{A_1}^{P+} \cup_N T_{A_2}^{P+}), (T_{\lambda_1}^{P+} \cup_N T_{\lambda_2}^{P+})), ((I_{A_1}^{P+} \cup_N I_{A_2}^{P+}), (I_{\lambda_1}^{P+} \cup_N I_{\lambda_2}^{P+})), ((F_{A_1}^{P+} \cup_N F_{A_2}^{P+}), (F_{\lambda_1}^{P+} \cup_N F_{\lambda_2}^{P+})) \\ ((T_{A_1}^{P-} \cup_N T_{A_2}^{P-}), (T_{\lambda_1}^{P-} \cup_N T_{\lambda_2}^{P-})), ((I_{A_1}^{P-} \cup_N I_{A_2}^{P-}), (I_{\lambda_1}^{P-} \cup_N I_{\lambda_2}^{P-})), ((F_{A_1}^{P-} \cup_N F_{A_2}^{P-}), (F_{\lambda_1}^{P-} \cup_N F_{\lambda_2}^{P-})) \end{array} \right\rangle \\ \left\langle \begin{array}{l} ((T_{B_1}^{P+} \cup_N T_{B_2}^{P+}), (T_{\mu_1}^{P+} \cup_N T_{\mu_2}^{P+})), ((I_{B_1}^{P+} \cup_N I_{B_2}^{P+}), (I_{\mu_1}^{P+} \cup_N I_{\mu_2}^{P+})), ((F_{B_1}^{P+} \cup_N F_{B_2}^{P+}), (F_{\mu_1}^{P+} \cup_N F_{\mu_2}^{P+})) \\ ((T_{B_1}^{P-} \cup_N T_{B_2}^{P-}), (T_{\mu_1}^{P-} \cup_N T_{\mu_2}^{P-})), ((I_{B_1}^{P-} \cup_N I_{B_2}^{P-}), (I_{\mu_1}^{P-} \cup_N I_{\mu_2}^{P-})), ((F_{B_1}^{P-} \cup_N F_{B_2}^{P-}), (F_{\mu_1}^{P-} \cup_N F_{\mu_2}^{P-})) \end{array} \right\rangle \end{array} \right\}$$

where

$$(T_{A_1}^{P+} \cup_N T_{A_2}^{P+})(u) = \begin{cases} T_{A_1}^{P+}(u), & ,if\ u \in v_1 - v_2 \\ T_{A_2}^{P+}(u), & ,if\ u \in v_2 - v_1 \\ r\max\{T_{A_1}^{P+}(u), T_{A_2}^{P+}(u)\} & ,if\ u \in v_1 \cap v_2 \end{cases}$$

$$(T_{A_1}^{P-} \cup_N T_{A_2}^{P-})(u) = \begin{cases} T_{A_1}^{P-}(u), & ,if \ u \in v_1 - v_2 \\ T_{A_2}^{P-}(u), & ,if \ u \in v_2 - v_1 \\ r\min\{ T_{A_1}^{P-}(u), T_{A_2}^{P-}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(T_{\lambda_1}^{P+} \cup_N T_{\lambda_2}^{P+})(u) = \begin{cases} T_{\lambda_1}^{P+}(u), & ,if \ u \in v_1 - v_2 \\ T_{\lambda_2}^{P+}(u), & ,if \ u \in v_2 - v_1 \\ \max\{ T_{\lambda_1}^{P+}(u), T_{\lambda_2}^{P+}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(T_{\lambda_1}^{P-} \cup_N T_{\lambda_2}^{P-})(u) = \begin{cases} T_{\lambda_1}^{P-}(u), & ,if \ u \in v_1 - v_2 \\ T_{\lambda_2}^{P-}(u), & ,if \ u \in v_2 - v_1 \\ \min\{ T_{\lambda_1}^{P-}(u), T_{\lambda_2}^{P-}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(I_{A_1}^{P+} \cup_N I_{A_2}^{P+})(u) = \begin{cases} I_{A_1}^{P+}(u), & ,if \ u \in v_1 - v_2 \\ I_{A_2}^{P+}(u), & ,if \ u \in v_2 - v_1 \\ r\max\{ I_{A_1}^{P+}(u), I_{A_2}^{P+}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(I_{A_1}^{P-} \cup_N I_{A_2}^{P-})(u) = \begin{cases} I_{A_1}^{P-}(u), & ,if \ u \in v_1 - v_2 \\ I_{A_2}^{P-}(u), & ,if \ u \in v_2 - v_1 \\ r\min\{ I_{A_1}^{P-}(u), I_{A_2}^{P-}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(I_{\lambda_1}^{P+} \cup_N I_{\lambda_2}^{P+})(u) = \begin{cases} I_{\lambda_1}^{P+}(u), & ,if \ u \in v_1 - v_2 \\ I_{\lambda_2}^{P+}(u), & ,if \ u \in v_2 - v_1 \\ \max\{ I_{\lambda_1}^{P+}(u), I_{\lambda_2}^{P+}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(I_{\lambda_1}^{P-} \cup_N I_{\lambda_2}^{P-})(u) = \begin{cases} I_{\lambda_1}^{P-}(u), & ,if \ u \in v_1 - v_2 \\ I_{\lambda_2}^{P-}(u), & ,if \ u \in v_2 - v_1 \\ \min\{ I_{\lambda_1}^{P-}(u), I_{\lambda_2}^{P-}(u)\} & ,if \ u \in v_1 \cap v_2 \end{cases}$$

$$(F_{A_1}^{P+} \cup_N F_{A_2}^{P+})(u) = \begin{cases} F_{A_1}^{P+}(u), & ,if\ u \in v_1 - v_2 \\ F_{A_2}^{P+}(u), & ,if\ u \in v_2 - v_1 \\ r\min\{ F_{A_1}^{P+}(u), F_{A_2}^{P+}(u)\} & ,if\ u \in v_1 \cap v_2 \end{cases}$$

$$(F_{A_1}^{P-} \cup_N F_{A_2}^{P-})(u) = \begin{cases} F_{A_1}^{P-}(u), & ,if\ u \in v_1 - v_2 \\ F_{A_2}^{P-}(u), & ,if\ u \in v_2 - v_1 \\ r\max\{ F_{A_1}^{P-}(u), F_{A_2}^{P-}(u)\} & ,if\ u \in v_1 \cap v_2 \end{cases}$$

$$(F_{\lambda_1}^{P+} \cup_N F_{\lambda_2}^{P+})(u) = \begin{cases} F_{\lambda_1}^{P+}(u), & ,if\ u \in v_1 - v_2 \\ F_{\lambda_2}^{P+}(u), & ,if\ u \in v_2 - v_1 \\ \min\{ F_{\lambda_1}^{P+}(u), F_{\lambda_2}^{P+}(u)\} & ,if\ u \in v_1 \cap v_2 \end{cases}$$

$$(F_{\lambda_1}^{P-} \cup_N F_{\lambda_2}^{P-})(u) = \begin{cases} F_{\lambda_1}^{P-}(u), & ,if\ u \in v_1 - v_2 \\ F_{\lambda_2}^{P-}(u), & ,if\ u \in v_2 - v_1 \\ \max\{ F_{\lambda_1}^{P-}(u), F_{\lambda_2}^{P-}(u)\} & ,if\ u \in v_1 \cap v_2 \end{cases}$$

$$(T_{B_1}^{P+} \cup_N T_{B_2}^{P+})(u_2v_2) = \begin{cases} T_{B_1}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ T_{B_2}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ r\max\{T_{B_1}^{P+}(u_2v_2), T_{B_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(T_{B_1}^{P-} \cup_N T_{B_2}^{P-})(u_2v_2) = \begin{cases} T_{B_1}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ T_{B_2}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ r\min\{ T_{B_1}^{P-}(u_2v_2), T_{B_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(T_{\mu_1}^{P+} \cup_N T_{\mu_2}^{P+})(u_2v_2) = \begin{cases} T_{\mu_1}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ T_{\mu_2}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ \max\{T_{\mu_1}^{P+}(u_2v_2), T_{\mu_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(T_{\mu_1}^{P-} \cup_N T_{\mu_2}^{P-})(u_2v_2) = \begin{cases} T_{\mu_1}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ T_{\mu_2}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ \min\{ T_{\mu_1}^{P-}(u_2v_2), T_{\mu_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(I_{B_1}^{P+} \cup_N I_{B_2}^{P+})(u_2v_2) = \begin{cases} I_{B_1}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ I_{B_2}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ r\max\{ I_{B_1}^{P+}(u_2v_2), I_{B_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(I_{B_1}^{P-} \cup_N I_{B_2}^{P-})(u_2v_2) = \begin{cases} I_{B_1}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ I_{B_2}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ r\min\{ I_{B_1}^{P-}(u_2v_2), I_{B_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(I_{\mu_1}^{P+} \cup_N I_{\mu_2}^{P+})(u_2v_2) = \begin{cases} I_{\mu_1}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ I_{\mu_2}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ \max\{ I_{\mu_1}^{P+}(u_2v_2), I_{\mu_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(I_{\mu_1}^{P-} \cup_N I_{\mu_2}^{P-})(u_2v_2) = \begin{cases} I_{\mu_1}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ I_{\mu_2}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ \min\{ I_{\mu_1}^{P-}(u_2v_2), I_{\mu_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(F_{B_1}^{P+} \cup_N F_{B_2}^{P+})(u_2v_2) = \begin{cases} F_{B_1}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ F_{B_2}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ r\min\{ F_{B_1}^{P+}(u_2v_2), F_{B_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(F_{B_1}^{P-} \cup_N F_{B_2}^{P-})(u_2v_2) = \begin{cases} F_{B_1}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ F_{B_2}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ r\max\{ F_{B_1}^{P-}(u_2v_2), F_{B_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(F_{\mu_1}^{P+} \cup_N F_{\mu_2}^{P+})(u_2v_2) = \begin{cases} F_{\mu_1}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ F_{\mu_2}^{P+}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ \min\{ F_{\mu_1}^{P+}(u_2v_2), F_{\mu_2}^{P+}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

$$(F_{\mu_1}^{P-} \cup_N F_{\mu_2}^{P-})(u_2v_2) = \begin{cases} F_{\mu_1}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_1 - v_2 \\ F_{\mu_2}^{P-}(u_2v_2), & ,if\ u_2v_2 \in v_2 - v_1 \\ \max\{ F_{\mu_1}^{P-}(u_2v_2), F_{\mu_2}^{P-}(u_2v_2)\} & ,if\ u_2v_2 \in E_1 \cap E_2 \end{cases}$$

Example 4.2.13: Let us consider the two bipolar spherical neutrosophic cubic graphs as $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$

$$P_1 = \left\langle \begin{matrix} \left\{ \begin{matrix} a, ([0.3,0.4],0.7), ([0.7,0.5],0.1), ([0.9,0.1],0.3), \\ ([-0.6,-0.2],-0.3), ([-0.7,-0.8],-0.1), ([-0.4,-0.7],-0.5) \end{matrix} \right\} \\ \left\{ \begin{matrix} b, ([0.2,0.6],0.8), ([0.4,0.6],0.3), ([0.7,0.8],0.4), \\ ([-0.5,-0.8],-0.6), ([-0.3,-0.5],-0.2), ([-0.4,-0.9],-0.3) \end{matrix} \right\} \\ \left\{ \begin{matrix} c, ([0.7,0.5],0.1), ([0.8,0.3],0.7), ([0.6,0.7],0.4), \\ ([-0.4,-0.2],-0.9), ([-0.1,-0.2],-0.9), ([-0.8,-0.5],-0.1) \end{matrix} \right\} \end{matrix} \right\rangle$$

$$Q_1 = \left\langle \begin{matrix} \left\{ \begin{matrix} ab, ([0.2,0.4],0.8), ([0.4,0.5],0.3), ([0.9,0.8],0.3), \\ ([-0.5,-0.2],-0.6), ([-0.3,-0.5],-0.2), ([-0.4,-0.9],-0.3) \end{matrix} \right\} \\ \left\{ \begin{matrix} ac, ([0.3,0.4],0.7), ([0.7,0.3],0.7), ([0.9,0.7],0.3), \\ ([-0.4,-0.2],-0.9), ([-0.1,-0.2],-0.9), ([-0.8,-0.7],-0.1) \end{matrix} \right\} \\ \left\{ \begin{matrix} bc, ([0.2,0.5],0.8), ([0.4,0.3],0.7), ([0.7,0.8],0.4), \\ ([-0.4,-0.2],-0.9), ([-0.1,-0.2],-0.9), ([-0.8,-0.9],-0.1) \end{matrix} \right\} \end{matrix} \right\rangle$$

$$P_2 = \left\langle \begin{matrix} \left\{ \begin{matrix} a, ([0.5,0.7],0.3), ([0.2,0.7],0.8), ([0.2,0.4],0.3), \\ ([-0.9,-0.8],-0.1), ([-0.6,-0.3],-0.2), ([-0.3,-0.7],-0.8) \end{matrix} \right\} \\ \left\{ \begin{matrix} b, ([0.5,0.4],0.3), ([0.3,0.5],0.6), ([0.7,0.5],0.3), \\ ([-0.9,-0.4],-0.2), ([-0.7,-0.4],-0.3), ([-0.1,-0.4],-0.6) \end{matrix} \right\} \\ \left\{ \begin{matrix} c, ([0.2,0.4],0.8), ([0.2,0.4],0.3), ([0.3,0.5],0.3), \\ ([-0.2,-0.5],-0.4), ([-0.7,-0.4],-0.5), ([-0.5,-0.4],-0.3) \end{matrix} \right\} \end{matrix} \right\rangle$$

$$Q_2 = \left\langle \begin{matrix} \left\{ \begin{matrix} ab, ([0.5,0.4],0.3), ([0.2,0.5],0.8), ([0.7,0.5],0.3), \\ ([-0.9,-0.4],-0.2), ([-0.6,-0.3],-0.3), ([-0.3,-0.7],-0.6) \end{matrix} \right\} \\ \left\{ \begin{matrix} ac, ([0.2,0.4],0.8), ([0.2,0.4],0.8), ([0.3,0.5],0.3), \\ ([-0.2,-0.5],-0.4), ([-0.6,-0.3],-0.5), ([-0.5,-0.7],-0.3) \end{matrix} \right\} \\ \left\{ \begin{matrix} bc, ([0.2,0.4],0.8), ([0.2,0.4],0.6), ([0.7,0.5],0.3), \\ ([-0.2,-0.4],-0.4), ([-0.7,-0.4],-0.5), ([-0.5,-0.4],-0.3) \end{matrix} \right\} \end{matrix} \right\rangle$$

Here M-union of the bipolar spherical neutrosophic cubic graph $G_1 \cup_M G_2$ as follows:

$$P_1 U_M P_2 = \left\langle \begin{array}{l} \left\{ \begin{array}{l} a,([0.5,0.7],0.7),([0.7,0.7],0.8),([0.9,0.4],0.3), \\ ([-0.9,-0.8],-0.3),([-0.7,-0.8],-0.2),([-0.4,-0.7],-0.8) \end{array} \right\} \\ \left\{ \begin{array}{l} b,([0.5,0.6],0.8),([0.4,0.6],0.6),([0.7,0.8],0.4), \\ ([-0.9,-0.8],-0.6),([-0.7,-0.5],-0.3),([-0.4,-0.9],-0.6) \end{array} \right\} \\ \left\{ \begin{array}{l} a,([0.7,0.5],0.8),([0.8,0.4],0.7),([0.6,0.7],0.4), \\ ([-0.4,-0.5],-0.9),([-0.7,-0.4],-0.9),([-0.8,-0.5],-0.3) \end{array} \right\} \end{array} \right\rangle$$

$$Q_1 U_M Q_2 = \left\langle \begin{array}{l} \left\{ \begin{array}{l} ab,([0.5,0.4],0.8),([0.4,0.5],0.8),([0.9,0.8],0.3), \\ ([-0.9,-0.4],-0.6),([-0.6,-0.5],-0.3),([-0.4,-0.9],-0.6) \end{array} \right\} \\ \left\{ \begin{array}{l} ac,([0.3,0.4],0.8),([0.7,0.4],0.8),([0.9,0.7],0.3), \\ ([-0.4,-0.5],-0.9),([-0.6,-0.3],-0.9),([-0.8,-0.7],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} bc,([0.2,0.5],0.8),([0.4,0.4],0.7),([0.7,0.8],0.4), \\ ([-0.4,-0.4],-0.9),([-0.7,-0.4],-0.9),([-0.8,-0.9],-0.3) \end{array} \right\} \end{array} \right\rangle$$

Here N-union of the bipolar spherical neutrosophic cubic graph $G_1 \cup_N G_2$ as follows:

$$P_1 U_N P_2 = \left\langle \begin{array}{l} \left\{ \begin{array}{l} a,([0.5,0.7],0.3),([0.7,0.7],0.1),([0.9,0.4],0.3), \\ ([-0.9,-0.1],-0.3),([-0.7,-0.8],-0.1),([-0.4,-0.7],-0.5) \end{array} \right\} \\ \left\{ \begin{array}{l} b,([0.5,0.6],0.3),([0.4,0.6],0.3),([0.7,0.8],0.3), \\ ([-0.9,-0.8],-0.2),([-0.7,-0.5],-0.2),([-0.4,-0.9],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} a,([0.7,0.5],0.1),([0.8,0.4],0.3),([0.6,0.7],0.3), \\ ([-0.4,-0.5],-0.4),([-0.7,-0.4],-0.5),([-0.8,-0.5],-0.1) \end{array} \right\} \end{array} \right\rangle$$

$$Q_1 U_N Q_2 = \left\langle \begin{array}{l} \left\{ \begin{array}{l} ab,([0.5,0.4],0.3),([0.4,0.5],0.3),([0.9,0.8],0.3), \\ ([-0.9,-0.4],-0.2),([-0.6,-0.5],-0.2),([-0.4,-0.9],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} ac,([0.3,0.4],0.7),([0.7,0.4],0.7),([0.9,0.7],0.3), \\ ([-0.4,-0.5],-0.4),([-0.6,-0.3],-0.5),([-0.8,-0.7],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} bc,([0.2,0.5],0.8),([0.4,0.4],0.6),([0.7,0.8],0.3), \\ ([-0.4,-0.4],-0.4),([-0.7,-0.4],-0.5),([-0.8,-0.9],-0.1) \end{array} \right\} \end{array} \right\rangle$$

Theorem 4.2.14: The M-union and N-union of the two bipolar spherical neutrosophic cubic graphs is again an bipolar spherical neutrosophic cubic graph.

Definition 4.2.15: Let $G_1=(P_1,Q_1)$ and $G_2=(P_2,Q_2)$ be two bipolar spherical neutrosophic cubic graphs of the graphs G_1^* and G_2^* respectively, then M-join is denoted by $G_1+_M G_2$ and is defined as follows:

$$G_1+_M G_2=(P_1,Q_1)+_M (P_2,Q_2)=(P_1+_M P_2,Q_1+_M Q_2)$$

$$=\left\{\begin{array}{l}\left\langle\begin{array}{l}((T_{A_1}^{P+}+_M T_{A_2}^{P+}),(T_{\lambda_1}^{P+}+_M T_{\lambda_2}^{P+})),((I_{A_1}^{P+}+_M I_{A_2}^{P+}),(I_{\lambda_1}^{P+}+_M I_{\lambda_2}^{P+})),((F_{A_1}^{P+}+_M F_{A_2}^{P+}),(F_{\lambda_1}^{P+}+_M F_{\lambda_2}^{P+}))\\((T_{A_1}^{P-}+_M T_{A_2}^{P-}),(T_{\lambda_1}^{P-}+_M T_{\lambda_2}^{P-})),((I_{A_1}^{P-}+_M I_{A_2}^{P-}),(I_{\lambda_1}^{P-}+_M I_{\lambda_2}^{P-})),((F_{A_1}^{P-}+_M F_{A_2}^{P-}),(F_{\lambda_1}^{P-}+_M F_{\lambda_2}^{P-}))\end{array}\right\rangle\\\left\langle\begin{array}{l}((T_{B_1}^{P+}+_M T_{B_2}^{P+}),(T_{\mu_1}^{P+}+_M T_{\mu_2}^{P+})),((I_{B_1}^{P+}+_M I_{B_2}^{P+}),(I_{\mu_1}^{P+}+_M I_{\mu_2}^{P+})),((F_{B_1}^{P+}+_M F_{B_2}^{P+}),(F_{\mu_1}^{P+}+_M F_{\mu_2}^{P+}))\\((T_{B_1}^{P-}+_M T_{B_2}^{P-}),(T_{\mu_1}^{P-}+_M T_{\mu_2}^{P-})),((I_{B_1}^{P-}+_M I_{B_2}^{P-}),(I_{\mu_1}^{P-}+_M I_{\mu_2}^{P-})),((F_{B_1}^{P-}+_M F_{B_2}^{P-}),(F_{\mu_1}^{P-}+_M F_{\mu_2}^{P-}))\end{array}\right\rangle\end{array}\right\}$$

where

$(i)\ if\ u\in v_1\cup v_2$

$$(T_{A_1}^{P+}+_M T_{A_2}^{P+})(u)=(T_{A_1}^{P+}\cup_M T_{A_2}^{P+})(u),(T_{\lambda_1}^{P+}+_M T_{\lambda_2}^{P+})(u)=(T_{\lambda_1}^{P+}\cup_M T_{\lambda_2}^{P+})(u)$$

$$(T_{A_1}^{P-}+_M T_{A_2}^{P-})(u)=(T_{A_1}^{P-}\cup_M T_{A_2}^{P-})(u),(T_{\lambda_1}^{P-}+_M T_{\lambda_2}^{P-})(u)=(T_{\lambda_1}^{P-}\cup_M T_{\lambda_2}^{P-})(u)$$

$$(I_{A_1}^{P+}+_M I_{A_2}^{P+})(u)=(I_{A_1}^{P+}\cup_M I_{A_2}^{P+})(u),(I_{\lambda_1}^{P+}+_M I_{\lambda_2}^{P+})(u)=(I_{\lambda_1}^{P+}\cup_M I_{\lambda_2}^{P+})(u)$$

$$(I_{A_1}^{P-}+_M I_{A_2}^{P-})(u)=(I_{A_1}^{P-}\cup_M I_{A_2}^{P-})(u),(I_{\lambda_1}^{P-}+_M I_{\lambda_2}^{P-})(u)=(I_{\lambda_1}^{P-}\cup_M I_{\lambda_2}^{P-})(u)$$

$$(F_{A_1}^{P+}+_M F_{A_2}^{P+})(u)=(F_{A_1}^{P+}\cup_M F_{A_2}^{P+})(u),(F_{\lambda_1}^{P+}+_M F_{\lambda_2}^{P+})(u)=(F_{\lambda_1}^{P+}\cup_M F_{\lambda_2}^{P+})(u)$$

$$(F_{A_1}^{P-}+_M F_{A_2}^{P-})(u)=(F_{A_1}^{P-}\cup_M F_{A_2}^{P-})(u),(F_{\lambda_1}^{P-}+_M F_{\lambda_2}^{P-})(u)=(F_{\lambda_1}^{P-}\cup_M F_{\lambda_2}^{P-})(u)$$

$(ii)\ if\ uv\in E_1\cup E_2$

$$(T^{P+}_{B_1} +_M T^{P+}_{B_2})(uv) = (T^{P+}_{B_1} \cup_M T^{P+}_{B_2})(uv), (T^{P+}_{\mu_1} +_M T^{P+}_{\mu_2})(uv) = (T^{P+}_{\mu_1} \cup_M T^{P+}_{\mu_2})(uv)$$
$$(T^{P-}_{B_1} +_M T^{P-}_{B_2})(uv) = (T^{P-}_{B_1} \cup_M T^{P-}_{B_2})(uv), (T^{P-}_{\mu_1} +_M T^{P-}_{\mu_2})(uv) = (T^{P-}_{\mu_1} \cup_M T^{P-}_{\mu_2})(uv)$$
$$(I^{P+}_{B_1} +_M I^{P+}_{B_2})(uv) = (I^{P+}_{B_1} \cup_M I^{P+}_{B_2})(uv), (I^{P+}_{\mu_1} +_M I^{P+}_{\mu_2})(uv) = (I^{P+}_{\mu_1} \cup_M I^{P+}_{\mu_2})(uv)$$
$$(I^{P-}_{B_1} +_M I^{P-}_{B_2})(uv) = (I^{P-}_{B_1} \cup_M I^{P-}_{B_2})(uv), (I^{P-}_{\mu_1} +_M I^{P-}_{\mu_2})(uv) = (I^{P-}_{\mu_1} \cup_M I^{P-}_{\mu_2})(uv)$$
$$(F^{P+}_{B_1} +_M F^{P+}_{B_2})(uv) = (F^{P+}_{B_1} \cup_M F^{P+}_{B_2})(uv), (F^{P+}_{\mu_1} +_M F^{P+}_{\mu_2})(uv) = (F^{P+}_{\mu_1} \cup_M F^{P+}_{\mu_2})(uv)$$
$$(F^{P-}_{B_1} +_M F^{P-}_{B_2})(uv) = (F^{P-}_{B_1} \cup_M F^{P-}_{B_2})(uv), (F^{P-}_{\mu_1} +_M F^{P-}_{\mu_2})(uv) = (F^{P-}_{\mu_1} \cup_M F^{P-}_{\mu_2})(uv)$$

(ii) *if* $uv \in E^*$, where E^* is the set of all edges joining the vertices of $v_1 \& v_2$.

$$(T^{P+}_{B_1} +_M T^{P+}_{B_2})(uv) = r\min\{ T^{P+}_{A_1}(u), T^{P+}_{A_2}(v)\}, (T^{P+}_{\mu_1} +_M T^{P+}_{\mu_2})(uv) = \min\{ T^{P+}_{\lambda_1}(u), T^{P+}_{\lambda_2}(v)\}$$
$$(T^{P-}_{B_1} +_M T^{P-}_{B_2})(uv) = r\max\{ T^{P-}_{A_1}(u), T^{P-}_{A_2}(v)\}, (T^{P-}_{\mu_1} +_M T^{P-}_{\mu_2})(uv) = \max\{ T^{P-}_{\lambda_1}(u), T^{P-}_{\lambda_2}(v)\}$$
$$(I^{P+}_{B_1} +_M I^{P+}_{B_2})(uv) = r\min\{ I^{P+}_{A_1}(u), I^{P+}_{A_2}(v)\}, (I^{P+}_{\mu_1} +_M I^{P+}_{\mu_2})(uv) = \min\{ I^{P+}_{\lambda_1}(u), I^{P+}_{\lambda_2}(v)\}$$
$$(I^{P-}_{B_1} +_M I^{P-}_{B_2})(uv) = r\max\{ I^{P-}_{A_1}(u), I^{P-}_{A_2}(v)\}, (I^{P-}_{\mu_1} +_M I^{P-}_{\mu_2})(uv) = \max\{ I^{P-}_{\lambda_1}(u), I^{P-}_{\lambda_2}(v)\}$$
$$(F^{P+}_{B_1} +_M F^{P+}_{B_2})(uv) = r\min\{ F^{P+}_{A_1}(u), F^{P+}_{A_2}(v)\}, (F^{P+}_{\mu_1} +_M F^{P+}_{\mu_2})(uv) = \min\{ F^{P+}_{\lambda_1}(u), F^{P+}_{\lambda_2}(v)\}$$
$$(F^{P-}_{B_1} +_M F^{P-}_{B_2})(uv) = r\max\{ F^{P-}_{A_1}(u), F^{P-}_{A_2}(v)\}, (F^{P-}_{\mu_1} +_M F^{P-}_{\mu_2})(uv) = \max\{ F^{P-}_{\lambda_1}(u), F^{P-}_{\lambda_2}(v)\}$$

Definition 4.2.16: Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two bipolar spherical neutrosophic cubic graphs of the graphs G_1^* and G_2^* respectively, then N-join is denoted by $G_1 +_N G_2$ and is defined as follows:

$$G_1 +_N G_2 = (P_1, Q_1) +_N (P_2, Q_2) = (P_1 +_N P_2, Q_1 +_N Q_2)$$

$$= \left\{ \begin{array}{l} \left\langle \begin{array}{l} ((T^{P+}_{A_1} +_N T^{P+}_{A_2}), (T^{P+}_{\lambda_1} +_N T^{P+}_{\lambda_2})), ((I^{P+}_{A_1} +_N I^{P+}_{A_2}), (I^{P+}_{\lambda_1} +_N I^{P+}_{\lambda_2})), ((F^{P+}_{A_1} +_N F^{P+}_{A_2}), (F^{P+}_{\lambda_1} +_N F^{P+}_{\lambda_2})) \\ ((T^{P-}_{A_1} +_N T^{P-}_{A_2}), (T^{P-}_{\lambda_1} +_N T^{P-}_{\lambda_2})), ((I^{P-}_{A_1} +_N I^{P-}_{A_2}), (I^{P-}_{\lambda_1} +_N I^{P-}_{\lambda_2})), ((F^{P-}_{A_1} +_N F^{P-}_{A_2}), (F^{P-}_{\lambda_1} +_N F^{P-}_{\lambda_2})) \end{array} \right\rangle \\ \left\langle \begin{array}{l} ((T^{P+}_{B_1} +_N T^{P+}_{B_2}), (T^{P+}_{\mu_1} +_N T^{P+}_{\mu_2})), ((I^{P+}_{B_1} +_N I^{P+}_{B_2}), (I^{P+}_{\mu_1} +_N I^{P+}_{\mu_2})), ((F^{P+}_{B_1} +_N F^{P+}_{B_2}), (F^{P+}_{\mu_1} +_N F^{P+}_{\mu_2})) \\ ((T^{P-}_{B_1} +_N T^{P-}_{B_2}), (T^{P-}_{\mu_1} +_N T^{P-}_{\mu_2})), ((I^{P-}_{B_1} +_N I^{P-}_{B_2}), (I^{P-}_{\mu_1} +_N I^{P-}_{\mu_2})), ((F^{P-}_{B_1} +_N F^{P-}_{B_2}), (F^{P-}_{\mu_1} +_N F^{P-}_{\mu_2})) \end{array} \right\rangle \end{array} \right\}$$

where

$(i)\ if\ u \in V_1 \cup V_2$

$$(T_{A_1}^{P+} +_N T_{A_2}^{P+})(u) = (T_{A_1}^{P+} \cup_N T_{A_2}^{P+})(u), (T_{\lambda_1}^{P+} +_N T_{\lambda_2}^{P+})(u) = (T_{\lambda_1}^{P+} \cup_N T_{\lambda_2}^{P+})(u)$$

$$(T_{A_1}^{P-} +_N T_{A_2}^{P-})(u) = (T_{A_1}^{P-} \cup_N T_{A_2}^{P-})(u), (T_{\lambda_1}^{P-} +_N T_{\lambda_2}^{P-})(u) = (T_{\lambda_1}^{P-} \cup_N T_{\lambda_2}^{P-})(u)$$

$$(I_{A_1}^{P+} +_N I_{A_2}^{P+})(u) = (I_{A_1}^{P+} \cup_N I_{A_2}^{P+})(u), (I_{\lambda_1}^{P+} +_N I_{\lambda_2}^{P+})(u) = (I_{\lambda_1}^{P+} \cup_N I_{\lambda_2}^{P+})(u)$$

$$(I_{A_1}^{P-} +_N I_{A_2}^{P-})(u) = (I_{A_1}^{P-} \cup_N I_{A_2}^{P-})(u), (I_{\lambda_1}^{P-} +_N I_{\lambda_2}^{P-})(u) = (I_{\lambda_1}^{P-} \cup_N I_{\lambda_2}^{P-})(u)$$

$$(F_{A_1}^{P+} +_N F_{A_2}^{P+})(u) = (F_{A_1}^{P+} \cup_N F_{A_2}^{P+})(u), (F_{\lambda_1}^{P+} +_N F_{\lambda_2}^{P+})(u) = (F_{\lambda_1}^{P+} \cup_N F_{\lambda_2}^{P+})(u)$$

$$(F_{A_1}^{P-} +_N F_{A_2}^{P-})(u) = (F_{A_1}^{P-} \cup_N F_{A_2}^{P-})(u), (F_{\lambda_1}^{P-} +_N F_{\lambda_2}^{P-})(u) = (F_{\lambda_1}^{P-} \cup_N F_{\lambda_2}^{P-})(u)$$

$(ii)\ if\ uv \in E_1 \cup E_2$

$$(T_{B_1}^{P+} +_N T_{B_2}^{P+})(uv) = (T_{B_1}^{P+} \cup_N T_{B_2}^{P+})(uv), (T_{\mu_1}^{P+} +_N T_{\mu_2}^{P+})(uv) = (T_{\mu_1}^{P+} \cup_N T_{\mu_2}^{P+})(uv)$$

$$(T_{B_1}^{P-} +_N T_{B_2}^{P-})(uv) = (T_{B_1}^{P-} \cup_N T_{B_2}^{P-})(uv), (T_{\mu_1}^{P-} +_N T_{\mu_2}^{P-})(uv) = (T_{\mu_1}^{P-} \cup_N T_{\mu_2}^{P-})(uv)$$

$$(I_{B_1}^{P+} +_N I_{B_2}^{P+})(uv) = (I_{B_1}^{P+} \cup_N I_{B_2}^{P+})(uv), (I_{\mu_1}^{P+} +_N I_{\mu_2}^{P+})(uv) = (I_{\mu_1}^{P+} \cup_N I_{\mu_2}^{P+})(uv)$$

$$(I_{B_1}^{P-} +_N I_{B_2}^{P-})(uv) = (I_{B_1}^{P-} \cup_N I_{B_2}^{P-})(uv), (I_{\mu_1}^{P-} +_N I_{\mu_2}^{P-})(uv) = (I_{\mu_1}^{P-} \cup_N I_{\mu_2}^{P-})(uv)$$

$$(F_{B_1}^{P+} +_N F_{B_2}^{P+})(uv) = (F_{B_1}^{P+} \cup_N F_{B_2}^{P+})(uv), (F_{\mu_1}^{P+} +_N F_{\mu_2}^{P+})(uv) = (F_{\mu_1}^{P+} \cup_N F_{\mu_2}^{P+})(uv)$$

$$(F_{B_1}^{P-} +_N F_{B_2}^{P-})(uv) = (F_{B_1}^{P-} \cup_N F_{B_2}^{P-})(uv), (F_{\mu_1}^{P-} +_N F_{\mu_2}^{P-})(uv) = (F_{\mu_1}^{P-} \cup_N F_{\mu_2}^{P-})(uv)$$

$(iii)\ if\ uv \in E^*$, where E^* is the set of all edges joining the vertices of $v_1 \& v_2$.

$$(T_{B_1}^{P+} +_N T_{B_2}^{P+})(uv) = r\min\{\ T_{A_1}^{P+}(u), T_{A_2}^{P+}(v)\}, (T_{\mu_1}^{P+} +_N T_{\mu_2}^{P+})(uv) = \max\{\ T_{\lambda_1}^{P+}(u), T_{\lambda_2}^{P+}(v)\}$$

$$(T_{B_1}^{P-} +_N T_{B_2}^{P-})(uv) = r\max\{\ T_{A_1}^{P-}(u), T_{A_2}^{P-}(v)\}, (T_{\mu_1}^{P-} +_N T_{\mu_2}^{P-})(uv) = \min\{\ T_{\lambda_1}^{P-}(u), T_{\lambda_2}^{P-}(v)\}$$

$$(I_{B_1}^{P+} +_N I_{B_2}^{P+})(uv) = r\min\{\ I_{A_1}^{P+}(u), I_{A_2}^{P+}(v)\}, (I_{\mu_1}^{P+} +_N I_{\mu_2}^{P+})(uv) = \max\{\ I_{\lambda_1}^{P+}(u), I_{\lambda_2}^{P+}(v)\}$$

$$(I_{B_1}^{P-} +_N I_{B_2}^{P-})(uv) = r\max\{ I_{A_1}^{P-}(u), I_{A_2}^{P-}(v)\}, (I_{\mu_1}^{P-} +_N I_{\mu_2}^{P-})(uv) = \min\{ I_{\lambda_1}^{P-}(u), I_{\lambda_2}^{P-}(v)\}$$
$$(F_{B_1}^{P+} +_N F_{B_2}^{P+})(uv) = r\min\{ F_{A_1}^{P+}(u), F_{A_2}^{P+}(v)\}, (F_{\mu_1}^{P+} +_N F_{\mu_2}^{P+})(uv) = \max\{ F_{\lambda_1}^{P+}(u), F_{\lambda_2}^{P+}(v)\}$$
$$(F_{B_1}^{P-} +_N F_{B_2}^{P-})(uv) = r\max\{ F_{A_1}^{P-}(u), F_{A_2}^{P-}(v)\}, (F_{\mu_1}^{P-} +_N F_{\mu_2}^{P-})(uv) = \min\{ F_{\lambda_1}^{P-}(u), F_{\lambda_2}^{P-}(v)\}$$

Theorem 4.2.17: The M-join and N-join of two bipolar spherical neutrosophic cubic graphs is again an bipolar spherical neutrosophic cubic graph.

4.3 Applications of Bipolar spherical Neutrosophic Cubic Graphs

In this section, the real life applications of bipolar spherical neutrosophic cubic graph are presented.

Example 4.3.1: Let us consider three factors that influence the e-learning effectiveness represented by the vertex set $V = \{X, Y, Z\}$. And let the truth-value denotes the e-learning material, the indeterminacy-value denotes the quality of web learning platform, the false-value denotes the e-learning course flexibility.

Let the vertex is given as follows:

$$P = \left\langle \begin{matrix} \left\{ \begin{matrix} X, ([0.8, 0.2], 0.3), ([0.5, 0.3], 0.4), ([0.7, 0.5], 0.6), \\ ([-0.4, -0.9], -0.2), ([-0.8, -0.4], -0.1), ([-0.5, -0.1], -0.7) \end{matrix} \right\} \\ \left\{ \begin{matrix} Y, ([0.7, 0.1], 0.6), ([0.9, 0.1], 0.8), ([0.5, 0.6], 0.7), \\ ([-0.2, -0.7], -0.4), ([-0.4, -0.6], -0.2), ([-0.6, -0.5], -0.3) \end{matrix} \right\} \\ \left\{ \begin{matrix} Z, ([0.1, 0.8], 0.4), ([0.5, 0.7], 0.2), ([0.9, 0.3], 0.4), \\ ([-0.8, -0.4], -0.2), ([-0.4, -0.3], -0.7), ([-0.7, -0.2], -0.5) \end{matrix} \right\} \end{matrix} \right\rangle$$

where the interval-valued membership indicates the effectiveness of an e-learning system at present and the fixed single-valued membership indicates the possibility of effectiveness of an e-learning system. So on the basis of the vertex set *P* we get the edge set *Q* defined as follows:

$$Q = \left\langle \begin{array}{l} \left\{ \begin{array}{l} XY, ([0.7,0.1],0.6), ([0.5,0.1],0.8), ([0.7,0.6],0.6), \\ ([-0.2,-0.7],-0.4), ([-0.4,-0.4],-0.2), ([-0.6,-0.5],-0.3) \end{array} \right\} \\ \left\{ \begin{array}{l} XZ, ([0.1,0.2],0.4), ([0.5,0.3],0.4), ([0.9,0.5],0.4), \\ ([-0.4,-0.4],-0.2), ([-0.4,-0.3],-0.7), ([-0.7,-0.2],-0.5) \end{array} \right\} \\ \left\{ \begin{array}{l} YZ, ([0.1,0.1],0.6), ([0.5,0.1],0.8), ([0.9,0.6],0.4), \\ ([-0.2,-0.4],-0.4), ([-0.4,-0.3],-0.7), ([-0.7,-0.5],-0.3) \end{array} \right\} \end{array} \right\rangle$$

Finally, we see that the effectiveness of an e-learning with other factors.

$$order(G) = \left\{ \begin{array}{l} ([1.6,1.1],1.3), ([1.9,1.1],1.4), ([2.1,1.4],1.7), \\ ([-1.4,-2.0],-0.8), ([-1.6,-1.3],-1.0), ([-1.8,-0.8],-1.5) \end{array} \right\}$$

$$\deg(X) = \left\{ \begin{array}{l} ([0.7,0.3], 2.2), ([1.2,1.7],1.8), ([1.5, 2.1], 0.3), \\ ([-1.7,-1.3],-1.5), ([-1.5,-0.6],-1.1), ([-2.5,-2],-0.3) \end{array} \right\}$$

$$\deg(Y) = \left\{ \begin{array}{l} ([0.7,0.9], 2.4), ([1,1.5],2.1), ([1.6, 2], 0.9), \\ ([-1.4,-1.1],-0.9), ([-1.7,-0.8],-1.1), ([-2.5,-2],-0.7) \end{array} \right\}$$

$$\deg(Z) = \left\{ \begin{array}{l} ([0.3,0.7], 2), ([0.6,0.9],1.9), ([1.8, 2.1],1), \\ ([-1.7,-1.3],-0.9), ([-1.7,-0.8],-1.2), ([-2.7,-2],-0.8) \end{array} \right\}$$

The order of *G* represents the overall effectiveness of an e-learning. Degree of *X* represents the combination of the e-learning material and quality of web learning platform, degree of *Y* represents the e-learning material and e-learning course flexibility and degree of *Z* represents the quality of web learning platform and e-learning course flexibility.

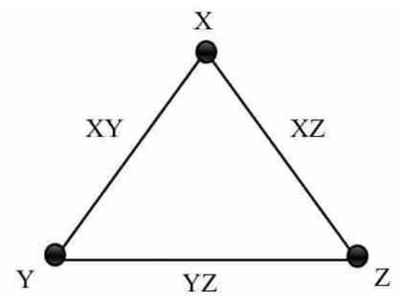

Fig. 4.6: The vertex set in *P* and the edge set in *Q* are represented for the graph *G=(P,Q)*

Example 4.3.2: Let us consider the construction company and we evaluate the overall performance of the company. The important criteria considered are strong structure, having own skilled crew, innovative designs, high-quality materials and competitive pricing. The above said criteria are taken in the form of single-valued to represent present type and in the form of interval-valued on future.

$$P = \left\langle \begin{array}{l} \left\{ \begin{array}{l} A,([0.7,0.6],0.4),([0.9,0.5],0.8),([0.7,0.8],0.6), \\ ([-0.4,-0.9],-0.2),([-0.8,-0.7],-0.4),([-0.5,-0.9],-0.1) \end{array} \right\} \\ \left\{ \begin{array}{l} B,([0.6,0.5],0.7),([0.7,0.5],0.3),([0.9,0.3],0.7), \\ ([-0.8,-0.6],-0.1),([-0.5,-0.6],-0.7),([-0.2,-0.6],-0.8) \end{array} \right\} \\ \left\{ \begin{array}{l} C,([0.1,0.7],0.6),([0.7,0.8],0.1),([0.2,0.5],0.9), \\ ([-0.5,-0.7],-0.9),([-0.7,-0.4],-0.9),([-0.5,-0.8],-0.9) \end{array} \right\} \\ \left\{ \begin{array}{l} D,([0.6,0.7],0.3),([0.5,0.9],0.4),([0.3,0.8],0.9), \\ ([-0.9,-0.5],-0.2),([-0.7,-0.8],-0.7),([-0.4,-0.5],-0.5) \end{array} \right\} \\ \left\{ \begin{array}{l} E,([0.8,0.4],0.5),([0.6,0.7],0.6),([0.6,0.7],0.4), \\ ([-0.6,-0.7],-0.8),([-0.8,-0.6],-0.4),([-0.9,-0.5],-0.3) \end{array} \right\} \end{array} \right\rangle$$

$$Q = \left\langle \begin{matrix} \left\{ \begin{matrix} AB,([0.6,0.5],0.7),([0.7,0.5],0.8),([0.9,0.8],0.6), \\ ([-0.4,-0.6],-0.2),([-0.5,-0.6],-0.7),([-0.5,-0.9],-0.1) \end{matrix} \right\} \\ \left\{ \begin{matrix} AC,([0.1,0.6],0.6),([0.7,0.5],0.8),([0.7,0.8],0.6), \\ ([-0.4,-0.7],-0.9),([-0.7,-0.4],-0.9),([-0.5,-0.9],-0.1) \end{matrix} \right\} \\ \left\{ \begin{matrix} AD,([0.6,0.6],0.4),([0.5,0.5],0.8),([0.7,0.8],0.6), \\ ([-0.4,-0.5],-0.2),([-0.7,-0.7],-0.7),([-0.5,-0.9],-0.1) \end{matrix} \right\} \\ \left\{ \begin{matrix} AE,([0.7,0.4],0.5),([0.6,0.5],0.8),([0.7,0.8],0.4), \\ ([-0.4,-0.7],-0.8),([-0.8,-0.6],-0.4),([-0.9,-0.9],-0.1) \end{matrix} \right\} \\ \left\{ \begin{matrix} BC,([0.1,0.5],0.7),([0.7,0.5],0.3),([0.9,0.5],0.7), \\ ([-0.5,-0.6],-0.9),([-0.5,-0.4],-0.9),([-0.5,-0.8],-0.8) \end{matrix} \right\} \\ \left\{ \begin{matrix} BD,([0.6,0.5],0.7),([0.5,0.5],0.4),([0.9,0.8],0.7), \\ ([-0.8,-0.5],-0.2),([-0.5,-0.6],-0.7),([-0.4,-0.6],-0.5) \end{matrix} \right\} \\ \left\{ \begin{matrix} BE,([0.6,0.4],0.7),([0.6,0.5],0.6),([0.9,0.7],0.4), \\ ([-0.6,-0.6],-0.8),([-0.5,-0.6],-0.7),([-0.9,-0.6],-0.3) \end{matrix} \right\} \\ \left\{ \begin{matrix} CD,([0.1,0.7],0.6),([0.5,0.8],0.4),([0.3,0.8],0.9), \\ ([-0.5,-0.5],-0.9),([-0.7,-0.4],-0.9),([-0.5,-0.8],-0.5) \end{matrix} \right\} \\ \left\{ \begin{matrix} CE,([0.1,0.4],0.6),([0.6,0.7],0.6),([0.6,0.7],0.4), \\ ([-0.5,-0.7],-0.9),([-0.7,-0.4],-0.9),([-0.9,-0.8],-0.3) \end{matrix} \right\} \\ \left\{ \begin{matrix} DE,([0.6,0.4],0.5),([0.5,0.7],0.6),([0.6,0.8],0.4), \\ ([-0.6,-0.5],-0.8),([-0.7,-0.6],-0.7),([-0.9,-0.5],-0.3) \end{matrix} \right\} \end{matrix} \right\rangle$$

where the edge $\left\{ \begin{matrix} AB,([0.6,0.5],0.7),([0.7,0.5],0.8),([0.9,0.8],0.6), \\ ([-0.4,-0.6],-0.2),([-0.5,-0.6],-0.7),([-0.5,-0.9],-0.1) \end{matrix} \right\}$ denotes the combined effect of strong structure and company having own skilled crew.

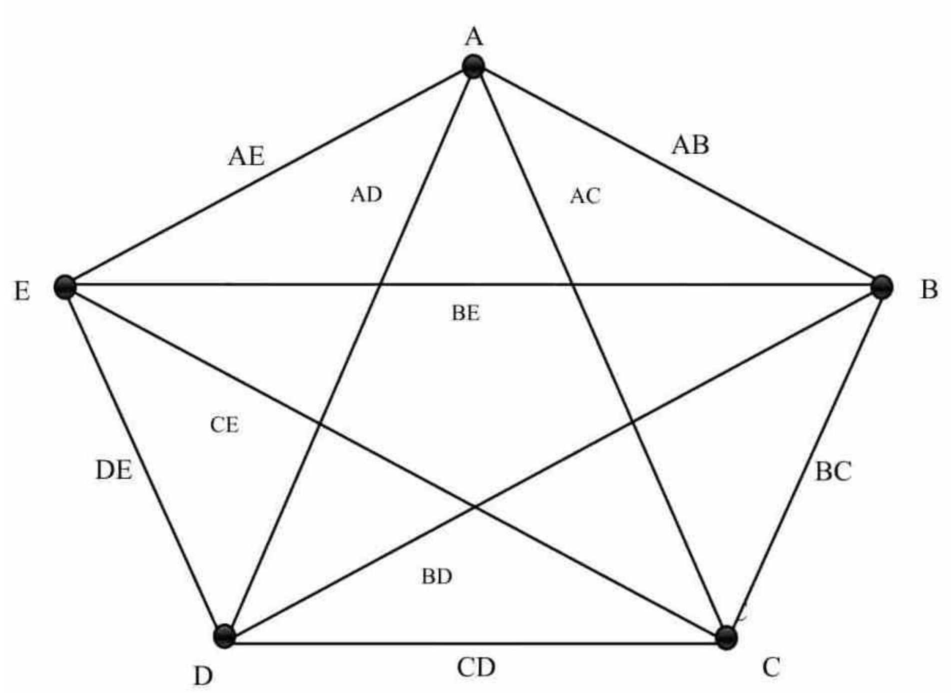

Fig. 4.7: The vertex set in *P* and the edge set in *Q* are represented for the graph *G=(P,Q)*