Separation Axioms for Bitopological Spaces

By

Mary Jam Shammi. T. A.

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE
AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE-641 043
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN MATHEMATICS

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CERTIFIED AS BONAFIDE RESEARCH WORK

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Introduction

INTRODUCTION

The concept of bitopological spaces was first introduced by Dr.J.C.Kelly in 1963. He has defined a bitopological space to be a set with two topologies defined on it. Here Dr.J.C.Kelly has established systematic generalization of Urysohn's lemma, Urysohn's metrization theorem and Tietze's extension theorem. Also he has established some elementary results about quasipseudo metrics. In the last decade a number of articles are published dealing with generalization of the concepts of separation axioms, connectedness, compactness and paracompactness etc to bitopological spaces.

In this thesis we shall present detailed survey of the following papers.

1. S.P.ARYA nd T.M.NOUR, "Separation Axioms for Bitopological spaces" [1]
2. MILA MRŠEVIČ (JUGOSLAVIA), "On Pairwise R_0 and pairwise R_1 Bitopological Spaces." [10]

This thesis has four chapters. In the first chapter, we give a brief review of literature. In the second chapter, we deal with preliminary definitions and results including the results of J.C.Kelly [6], I.L.Reilly [13] and M.J.Saegrove [14]. In chapter III, we discuss the generalizations of some new separation axioms introduced by Jain [3] for topological spaces to bitopological spaces and study its relationships among themselves as well as with other known separation axioms.

In chapter IV some new results concerning pairwise R_0 and pairwise R_1 bitopological spaces are studied.

Chapter I

CHAPTER I

REVIEW OF LITERATURE

A number of articles have been published in the past ten years. They deal with separation axioms, connected properties, compactness properties and quasi-matrizable spaces. In this chapter we shall give a brief review of seven interesting papers published on bitopological spaces.

1. "On Product of Minimal Pairwise Hausdorff Bitopological Spaces" by M.N.Mukherjee (1983) [22]

The purpose of this paper is to study whether the topological product of a family of minimal pairwise Hausdorff space is also minimal pairwise Hausdorff and also to study the converse problem.

2. "An Alternative Definition of Local Connectedness In Bitopological Spaces" by S.S.Lakshmi (1984) [15]

In this paper, an alternative definition of local connectedness in a bitopological space is given and some properties of locally connected bitopological spaces are derived.

3. "On Bitopological (P) Spaces" by P.T.Daniel Thanapalan (1984) [38]

In this paper the authors study properties of bitopological (P) spaces. A bitopological space (X, τ_1, τ_2) is said to be bitopological (P) space iff τ_1 and τ_2 are P-spaces. In this paper the authors prove that a pairwise-Lindelof, pairwise Hausdorff bitopological (P) space to be pairwise normal. Also it is proved that a locally pairwise - Lindelof, pairwise Hausdorff (P) space is pairwise regular.

4. "A Note On Bitopological Spaces Which Are Not Pairwise Hausdorff" by S.Ganguly and M.N.Mukherjee (1986) [10].

In this paper the concept of 'spiral' has been introduced in a bitopological space (X,τ_1,τ_2). $\tau_1\tau_2$ - spiral (x_0) of a point x_0 has been defined to be the set of all points x of X such that there is a net in X converging to x_0 with respect to τ_1 and to x with respect to τ_2. Expressions for $\tau_1\tau_2$ - spiral (x_0) use of the concept for characterising non-pairwise Hausdorff spaces and properties of the concept have been developed in this paper.

5. "Pairwise Set-Connected Mappings In Bitopological Spaces" by M.M.Mukherjee And G.K.Banerjee (September 1986) [25]

In this paper, attempts have been made to generalize the concept of set connected mappings in bitopological spaces. Such a mapping has been characterized and some of its properties have been studied. Finally its relationships with pairwise continous and pairwise weakly continuous and pairwise weakly continuous mapping have been investigated.

6. "On Pairwise Hausdorff **Bitopological Spaces**" by J.Ewert (1987) [8].

In this paper various characterizations of pairwise Hausdorff bitopological spaces are obtained.

7. "Characterizations of Quasi-Meteizable Bitopological Spaces" by T.G.Raghavan and I.L.Reilly (1988) [32].

In this paper it is proved that a pairwise Hausdorff bitopological space (X,τ_1,τ_2) is Quasi - metrizable if and only if for each point $x \in X$ and for $i,j=1,2, i \neq j$, one can assign τ_1- neighbourhood bases $\{S(n,i,x)/n=1,2,....\}$ such that

i) $y \notin S(n-1,i,x)$ implies

$$S(n,i,x) \cap S(n,i,y) = \emptyset$$

ii) $y \in S(n,i,x)$ implies

$$S(n,i,y) \subset S(n-1,i,y)$$

Chapter II

CHAPTER II

DEFINITION: 2.1 [6]

A space X on which are defined two (arbitrary) topologies τ_1 and τ_2 is called a bitopological space and denoted by (X,τ_1,τ_2).

DEFINITION: 2.2 [6]

In a space (X,τ_1,τ_2), τ_1 is said to be regular with respect to τ_2 if for each point x in X, there is a τ_1 - neighbourhood base of τ_2 - closed sets, or, equivalently, if for each point x in X and each τ_1 - closed set P such that $x \notin P$, there are a τ_1-open set U and a τ_2 open set V such that

$$x \in U,\ P \subset V \text{ and } U \cap V = \emptyset$$

The space (X,τ_1,τ_2) is called pairwise regular if τ_1 is regular with respect to τ_2 and vice versa.

DEFINITION : 2.3 [6]

A space (X,τ_1,τ_2) is said to be pairwise Hausdorff or pairwise T_2 if for each two distinct points x and y, there are a τ_1- neighbourhood U of x and a τ_2-neighbourhood V of y such that $U \cap V=\emptyset$

DEFINITION: 2.4 [6]

A space (X,τ_1,τ_2) is said to be pairwise normal if, given a τ_1-closed set A and a τ_2-closed set B with $A\cap B=\emptyset$, there exist a τ_2-open set U and a τ_1-open set V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Equivalently, (X, τ_1, τ_2) is pairwise normal if given a τ_2-closed set C and a τ_1-open set D such that $C \subset D$, there are a τ_1-open set G and a τ_2-closed set F such that

$$C \subseteq G \subseteq F \subseteq D$$

DEFINITION :2.5 [14]

Let (X, τ_1, τ_2) be a bitopological space, then the function $f: (X, \tau_1, \tau_2) \to (Y, \tau_3, \tau_4)$ is called pairwise continuous if the induced functions

$f: (X, \tau_1) \to (Y, \tau_3)$ and
$f: (X, \tau_2) \to (Y, \tau_4)$ are continuous

NOTATION [14]

In the space $(\mathbb{R}, R, L)$. $\mathbb{R}$ is the real line and R is the set of open right rays and L the set of open left rays.

According to M.J.Saegrove definition 2.4 is modified as follows:

"Let (X, τ_1, τ_2) be a bitopological space, $\mathcal{F}$ a family of τ_1-closed sets, and $\mathcal{G}$ a family of τ_2-closed sets. $(\mathcal{F}, \mathcal{G})$ is called a pairwise normal pair if and only if for each $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that $A \cap B = \emptyset$, there exists $C \in \mathcal{G}$ and $D \in \mathcal{F}$ such that $(X - C) \cap (X - D) = \emptyset$ and $A \subset X - C$ and $B \subset X - D$.

DEFINITION: 2.6 [14]

Let (X, τ_1, τ_2) be a bitopological space, $\mathcal{F}$ a family of τ_1-closed sets and $\mathcal{G}$ a family of τ_2-closed sets.

$(\mathcal{F}, \mathcal{G})$ is called a pairwise separating pair if and only if (i) & (ii) holds.

i) If F is τ_1-closed, $x \notin F$, then there exist $A \in \mathcal{G}$ and $B \in \mathcal{F}$ such that $x \in A$, $F \subset B$ and $A \cap B = \emptyset$.

ii) If F is τ_2-closed, $x \notin F$, then there exist $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that $x \in A$, $F \subset B$ and $A \cap B = \emptyset$.

DEFINITION :2.7 [12]

A set E is a τ_1-zero set iff there is a pairwise contiuous function $f: (X, \tau_1, \tau_2) \dashrightarrow (\mathcal{R}, R, L)$ such that $E = \{x / f(x) \leq 0\}$.

A set E is a τ_2-zero set iff there is a pairwise contiuous function $f: (X, \tau_1, \tau_2) \dashrightarrow (\mathcal{R}, R, L)$ such that $E = \{x / f(x) \geq 0\}$.

DEFINITION :2.8 [13]

Let (X, τ_1, τ_2) be a bitopological space, then τ_1 is completely regular with respect to τ_2 if for each τ_1-closed set C and each $x \notin C$ there is a real valued function f on X into [0,1] such that $f(x)=0$, $f(c)=1$ and f is τ_1-upper semicontinuous (u.s.c.) and τ_2-lower semi-continuous (l.s.c.). (X, τ_1, τ_2) is pairwise completely regular if τ_1 is completely regular with respect to τ_2 and τ_2 is completely regular with respect to τ_1.

In the following theorem **M.J.SAEGROVE** generalize a result of E.F.Steiner [16] to characterize pairwise complete regularity.

THEOREM 2.9 [14]

(X,τ_1,τ_2) is pairwise completely regular if and only if it possesses a pairwise normal pairwise separating pair.

PROOF

Assume that (X,τ_1,τ_2) is pairwise completely regular. It is obvious that the family $\mathcal{F}$ of all τ_1-zero sets and the family $\mathcal{G}$ of all τ_2-zero sets form a pairwise normal pairwise separating pair.

Conversely, assume that (X,τ_1,τ_2) possesses a pairwise normal pairwise separating pair. To prove the space (X,τ_1,τ_2) is completely regular, we use the generalization of Urysohn's procedure.

Assume that $(\mathcal{F},\mathcal{G})$ is a pairwise separating pair and F a τ_2- closed set with $x \notin F$. Then there are $F_0 \in \mathcal{F}$, $G_1 \in \mathcal{G}$ such that $x \in F_0$ and $F \subset G_1$ by the pairwise separating condition. By the pairwise normality condition there exists $G_{\frac{1}{2}} \in \mathcal{G}$ $F_{\frac{1}{2}} \in \mathcal{F}$ such that $(X-G_{\frac{1}{2}}) \cap (X - F_{\frac{1}{2}}) = \emptyset$ and $F_0 \subset X - G_{\frac{1}{2}}$ and $G_1 \subset X - F_{\frac{1}{2}}$. Thus $x \in F_0 \subset x - G_{\frac{1}{2}} \subset F_{\frac{1}{2}} \subset X - G_1$. Since $F_0 \cap G_{\frac{1}{2}} = \emptyset$, again by pairwise normality of $(\mathcal{F},\mathcal{G})$, there are $F_{\frac{1}{4}}$ and $G_{\frac{1}{4}}$ in $\mathcal{F}$ and $\mathcal{G}$ respectively such that

$$(X-F_{\frac{1}{4}}) \cap (X-G_{\frac{1}{4}}) = \emptyset \text{ and}$$

$$F_0 \subset X-G_{\frac{1}{4}} \text{ and } G_{\frac{1}{2}} \subset X - F_{\frac{1}{4}}$$

Similarly we get sets $F_{3/4}$, $G_{3/4}$ such that $F_{\frac{1}{4}} \subset X - G_{3/4}$. and $G_1 \subset X - F_{3/4}$

Thus we have

$x \in F_0 \subset X - G_{\frac{1}{4}} \subset F_{\frac{1}{4}} \subset X - G_{\frac{1}{2}} \subset F_{\frac{1}{2}} \subset X - G_{3/4} \subset F_{3/4} \subset X - G_1$

Proceeding like this we get collections $\{Fi\}_{i \in D} \subset \mathcal{F}$ and $\{G_i\}_{i \in D} \subset \mathcal{G}$ (where D is the set of diadic rationals between 0 and 1) such that for $i, j \in D$ and $i < j$,

$$F_0 \subset X - G_i \subset F_i \subset X - G_j \subset F_j \subset X - G_1$$

Now define a function $f : X \dashrightarrow [0,1]$ by

$$f(x) = \inf\{t \in D / x \in X - G_t\} \text{ and } f(x) = 1 \text{ for } x \in G_t$$

We show that f is a pairwise continuous function into $([1,0], R, L,)$ which is obviously o on $\{x\}$ and 1 on F.

To prove $f : (X, \tau_2) \dashrightarrow (\mathbb{R}, L)$ is continuous.

Let $x \in f^{-1}(-\infty, a)$; then $f(x) = t < a$, so there exists a diadic rational $t^1 \in D$ such that $f(x) = t < t^1 < a$

Since $x \in X - G_{t1} \subset \cup \{x - G_t / t < a,\ t \in D\}$,

$$f^{-1}(-\infty, a) \subset \cup \{x - G_t / t < a,\ t \in D\} \longrightarrow (1)$$

Let $x \in \cup \{X - G_t / t < a,\ t \in D\}$ then

$x \in X - G_s,\ s < a.$

Since

$f(x) = \inf \{t \in D / \ x \in X - G_t\} \leq s < a,$

$x \in f^{-1}(-\infty, a)$

Hence $\cup \{X - G_t / t < a,\ t \in D\} \subset f^{-1}(-\infty, a) \rightarrow (2)$

By (1) and (2)

$f^{-1}(-\infty, a) = \cup\{X - G_t / t < a,\ t \in D\}$,

which is τ_2-open.

Hence $f:(X,\tau_2) \dashrightarrow (\mathcal{R},L)$ is continuous.

To prove : $f : (X,\tau_1) \dashrightarrow (\mathcal{R}, R)$ is continuous . Since " $f(x) \leq a$ if and only if $x \in X - G_t$ for all $t \in D$ such that $t > a$" we have $f^{-1}(-\infty,a] = \cap \{X - G_t \mathbin{/} t > a, t \in D\}$.

Obviously, $\cap\{X - G_t \mathbin{/} t > a, t \in D\}$

$$\subset \cap\{\tau_1 - cl(X - G_t) \mathbin{/} t > a, t \in D\} \dashrightarrow (3)$$

Let $r \in D$ be such that $r > a$. Then there exists $s \in D$ such that $r>s>a$. Now, $\cap\{\tau_1 - cl\ (X - G_t)/ t > a, t \in D\}$

$$\subset \tau_1 - cl\ (X - G_s) \subset F_s \subset X - G_r$$

Thus, $\cap\{\tau_1 - cl\ (X - G_t)^{\circ} \mathbin{/} t > a, t \in D\} \subset X - G_r$ for all $r>a$;

so $\cap\{\tau_1 - cl\ (X - G_t)/ t>a, t \in D\}$

$$\subset \cap\{X - G_r / r > a, r \in D\} \dashrightarrow (4)$$

By (3) and (4)

$$\begin{aligned}\cap\{\tau_1 - cl\ (X - G_t) \mathbin{/} t>a, t \in D\} &= \cap\{X - G_t / t > a, t \in D\}\\ &= f^{-1}(-\infty,a]\end{aligned}$$

Hence $f : (X,\tau_1) \dashrightarrow (\mathcal{R},R)$ is continuous.

Similarly each τ_1-closed set is τ_1 completely separated with respect to τ_2 from points it excludes and hence (X,τ_1,τ_2) is pairwise completely regular.

J.C.Kelly has established the following important results.

THEOREM 2.10 (Generalization of Urysohn's lemma)

If (X,τ_1,τ_2) is pairwise normal, then given a τ_2-closed set F and a τ_1-closed set H with $F \cap H = \emptyset$ there exists a real valued function g on X such that

i) $g(x) = 0$ for every $x \in F$,

$g(x) = 1$ for every $x \in H$ and

$0 \leq g(x) \leq 1$ for every $x \in X$

ii) g is τ_1-upper semi-continuous and

τ_2-lower semi-continuous.

THEOREM: 2.11 (Generalization of Tietze's extension theorem)

Let (X, τ_1, τ_2) be pairwise normal bitopological spaces. Let $A \subseteq X$ be τ_1-closed and τ_2-closed. Let f be a real valued function defined on A which is τ_1-upper semi-continuous and τ_2-lower semi-continuous. Then there exists an extension F of f to the whole of X such that F is τ_1 - upper semi-continuous and τ_2 - lower semi-continuous. If f is bounded, then the extension F can be chosen so that

$\inf \{F(t) : t \in X\} = \inf \{f(t) : t \in A\}$ and

$\sup \{F(t) : t \in X\} = \sup \{f(t) : t \in A\}$

THEOREM : 2.12 (Generlization of Quasi-Metrization theorem)

Let (X, τ_1, τ_2) be a pairwise regular bitopological space satisfying the second axiom of countability. Then (X, τ_1, τ_2) is quasi-pseudo metrizable. If in addition (X, τ_1, τ_2) is pairwise Hausdorff it is quasi-metrizable.

The following elementary results state some basic facts about quasi-pseudo metrices according to J.C.Kelly.

THEOREM : 2.13

For a fixed point x in X, $p(x,y)$ is a τ_1-upper semi-continuous and τ_2-lower semi-continuous function of y; and for a fixed point y in X, $p(x,y)$ is a τ_1-lower semi-continuous and τ_2-upper semi-continuous function of x.

THEOREM 2.14

let τ_1 and τ_2 be the topologies on X determined by $p(,)$ and $q(,)$ respectively. Then (X,τ_1,τ_2) is pairwise regular and pairwise normal. (X,τ_1,τ_2) is pairwise Hausdorff if and only if $p(,)$ and $q(,)$ are quasie metrics.

PROOF

i) Since, for every x in X and each $k>0$ the set $\{y:p(x,y)\leq k\}$ is τ_2-closed, each point x has a τ_1-neighbourhood base of τ_2-closed sets.

Hence τ_1 is regular with respect to τ_2. Similarly, τ_2 is regular with respect to τ_1.

ii) Let A and B be disjoint subsets of X such that A is τ_1-closed and B is τ_2-closed. Define, for $x \in X$.

$$p(x, A) = \inf\{p(x, a) : a \in A\}$$

$$q(x, B) = \inf\{q(x, b) : b \in B\}$$

Then $A = \{y: p(y, A) = 0\}$ and

$B = \{y: q(y, B) = 0\}$

Let $U = \{x: p(x, A) < q(x, B)\}$ and

$V = \{y: q(x, B) < p(x, A)\}$

Then $U \cap V = \emptyset$, $A \subseteq U$ and $B \subseteq U$

To prove (X,τ_1,τ_2) is pairwise normal it is enough to prove that U and V are respectively τ_2-open and τ_1-open.

Suppose that $x0 \in V$ and that $p(x_0, A) - q(x_0, B) = k > 0$

Let $x \in p(x_0, k/4)$

Then

$$q(x, B) \le q(x, x_0) + q(x_0, B)$$

$$= p(x_0, x) + q(x_0, B) \longrightarrow (5)$$

and $p(x_0, A) \le p(x_0, x) + p(x, A)$

$\Longrightarrow p(x, A) \ge p(x_0, A) + p(x_0, x) \longrightarrow (6)$

By (5) and (6)

$$p(x, A) - q(x, B) \ge p(x_0, A) - p(x_0, x) - p(x_0, x) - q(x_0, B)$$

$$= p(x_0, A) - q(x_0, B) - 2p(x_0, x)$$

$$> k - 2k/4 = k/2$$

Thus $p(x, A) - q(x, B) > k/2$

$\Longrightarrow x \in V$

$\Longrightarrow p(x_0, k/4) \subsetneq V$

$\Longrightarrow$ V is τ_1-open. Similarly U is τ_2-open. Hence (X,τ_1,τ_2) is pairwise normal.

iii) Obviously (X,τ_1,τ_2) is pairwise Hausdorff if and only if $p(1)$ and $q(1)$ are quasi-metrics.

THEOREM : 2.15

i) If for each x in X, the function $p(x, y)$ is τ_1-continuous in y, then τ_1 is regular.

ii) If for each x in X, the function $p(x, y)$ is τ_2-continuous in y, then τ_2 is pseudo-metrizable.

DEFINITION: 2.16 [13]

The bitopological space (X,τ_1,τ_2) is pairwise T_0 if for each pair of distinct points of X there is a set which is either τ_1-open or τ_2-open set containing one of the points but not the other.

THEOREM :2.17 [13]

The bitopological space (X,τ_1,τ_2) is pairwise T_0 if and only if given two points of X either their τ_1-closures or τ_2-closures are distinct.

DEFINITION : 2.18 [13]

The bitopological space (X,τ_1,τ_2) is pairwise T_1 if for each pair of distinct points x and y of X there is a τ_1-open set U and a τ_2-open set V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

THEOREM :2.19 [13]

A space (X,τ_1,τ_2) is pairwise T_1 iff (X,τ_1) and (X,τ_2) are T_1.

Chapter III

CHAPTER III

SEPARATION AXIOMS FOR BITOPOLOGICAL SPACES

In this chapter we shall discuss the paper "Separation Axioms For Bitopological Spaces" by S.P.Arya. In this paper the author has introduced new separation axioms in bitopological spaces using the concepts of regularly open sets and regularly closed sets. The author has obtained the characterization of bitopological spaces possessing the separation axioms.

DEFINITION : 3.1 [9]

In a topological space X, a set A is said to be regularly open if it is the interior of its own closure, or, equivalently, if it is the interior of some closed set.

The complement of a regularly open set is said to be regularly closed.

DEFINITION : 3.2 [18]

A set is said to be δ-open if it is expressible as a union of regularly open sets.

The complement of a δ-open set is said to be δ-Closed .

DEFINITION : 3.3

A point in X is said to be a δ-adherent point of a set A of X if every regularly open set containing x has a non-empty intersection with A.

The set of all δ-adherent points of a set A is denoted by δ-cl A.

A is δ-closed if and only if A = δ-cl A.

PAIRWISE rT_0 - Spaces :

DEFINITION : 3.4 [3]

A topological space X is said to be rT_0 if for any two distinct points of X, there exists a regularly open set containing one of the points but not the other, or eqivalently, there exists a δ-open set containing one of the points but not the other.

Definition : 3.5

A space (X,τ_1,τ_2) is said to be pairwise rT_0 if for any two distinct points of X, there exists a set which is either τ_1-regularly open or τ_2-regularly open containing one of the points but not the other, or eqivalently, there exists either a τ_1-δ-closed or a τ_2-δ-open set containing one of the points but not the other.

NOTE : [11]

Every pairwise rT_0 space is pairwise T_0. But the converse need not be true can be seen in the following example :

EXAMPLE : 3.6

Let, $X = \{a,b,c\}$,

$\tau_1 = \{X,\emptyset,\{a\},\{b,c\}\}$

and $\tau_2 = \{X,\emptyset, \{b\}\}$

Consider the distinct points a,b then there exists a τ_1-open set {a} containing a but not b.

Consider the pair b,c where $b \neq c$, then there exists a τ_2 - open set {b} containing b but not c.

Consider the pair a,c with $a \neq c$, then there exists τ_1 -open set $\{a\}$ containing a but not c.

Therefore, the space (X,τ_1,τ_2) is pairwise τ_0.

$X - \tau_1 = \{\emptyset, X \ \{b,c\}, \{a\}\}$

$X - \tau_2 = \{\emptyset, X \ \{a,c\}\}$

$\tau_1\text{-int } (\tau_1\text{-cl } X) = \tau_1\text{-int } (X) = X$

$\tau_1\text{-int } (\tau_1\text{-cl } \emptyset) = \tau_1\text{-int } (\emptyset) = \emptyset$

$\tau_1 - \text{int } (\tau_1 - \text{cl } \{a\}) = \tau_1 - \text{int } \{a\} = \{a\}$

$\tau_1 - \text{int } (\tau_1 - \text{cl } \{b\}) = \tau_1 - \text{int } \{b,c\} = \{b,c\}$

$\tau_1 - \text{int } (\tau_1 - \text{cl } \{c\}) = \tau_1 - \text{int } \{b,c\} = \{b,c\}$

$\tau_1 - \text{int } (\tau_1 - \text{cl } \{a,b\}) = \tau_1 - \text{int } \{x\} = X$

$\tau_1 - \text{int } (\tau_1 - \text{cl } \{b,c\}) = \tau_1 - \text{int } \{b,c\} = \{b,c\}$

$\tau_1 - \text{int } (\tau_1 - \text{cl } \{a,c\}) = \tau_1 - \text{int } (X) = X$

The collection of τ_1 - regularly open sets $= \{X,\emptyset,\{a\},\{b,c\}\}$

$\tau_2 - \text{int } (\tau_2 - \text{cl } X) = \tau_2 - \text{int } (X) = X$

$\tau_2 - \text{int } (\tau_2 - \text{cl } \emptyset) = \tau_2 - \text{int } (\emptyset) = \emptyset$

$\tau_2 - \text{int } (\tau_2 - \text{cl } \{a\}) = \tau_2 - \text{int } \{a,c\} = X$

$\tau_2 - \text{int } (\tau_2 - \text{cl } \{b\}) = \tau_2 - \text{int } (X) = X$

$\tau_2 - \text{int } (\tau_2 - \text{cl } \{c\}) = \tau_2 - \text{int } \{a,c\} = X$

$\tau_2 - \text{int } (\tau_2 - \text{cl } \{a,b\}) = \tau_2 - \text{int } (X) = X$

$\tau_2 - \text{int } (\tau_2 - \text{cl } \{b,c\}) = \tau_2 - \text{int } (X) = X$

$\tau_2 - \text{int } (\tau_2 - \text{cl } \{c,a\}) = \tau_2 - \text{int } \{c,a\} = \emptyset$

The collection of τ_2 - regularly open sets $= \{X,\emptyset\}$

Consider the pair b,c with $b \neq c$, then there exists neither τ_1 - regularly open set nor τ_2 - regularly open set containing one but not the other.

Therefore the space (X,τ_1,τ_2) is not pairwise rT_0.

THEOREM : 3.7

A space (X,τ_1,τ_2) is pairwise rT_0 if and only if given two distinct points of X either their τ_1 - δ-closures are distinct or their τ_2 - δ-closures are distinct.

PROOF :

Let (X,τ_1,τ_2) is pairwise rT_0 space and let $x,y \in X$ be two distinct points. Suppose U is a τ_1-δ-open set containing x but not y.

Then,

$$y \in \tau_1 - \delta \text{ cl } \{y\} \subset X - U \text{ and}$$

So $$x \notin \tau_1 - \delta \text{ cl } \{y\}.$$

Hence $\tau_1 - \delta \text{ cl } \{x\} \neq \tau_1 - \delta \text{ cl } \{y\}$

Conversely, let x,y be two distinct points of X.

Then, either

$$\tau_1 - \delta\text{cl } \{x\} \neq \tau_1 - \delta \text{ cl } \{y\} \text{ or}$$

$$\tau_2 - \delta\text{cl } \{x\} \neq \tau_2 - \delta \text{ cl } \{y\}$$

Case (i)

Suppose p be a point of X such that,

$$p \in \tau_1 - \delta\text{cl } \{y\}$$

and $$p \notin \tau_1 - \delta\text{cl}\{x\}$$

We claim that,

$$y \notin \tau_1 - \delta \text{ cl } \{x\}$$

If, $y \in \tau_1 - \delta \text{ cl } \{x\}$, then

$$\tau_1 - \delta \text{ cl } \{y\} \subset \tau_1 - \delta \text{ cl } \{x\} \text{ , so that}$$

$p \in \tau_1 - \delta \text{ cl } \{y\} \subset \tau_1 - \delta \text{ cl } \{x\}$.

This contradicts the fact that $p \notin \tau_1 - \delta \text{ cl } \{x\}$

Hence $y \notin \tau_1 - \delta \text{ cl } \{x\}$.

Thus $U = X - \{ \tau_1 - \delta \text{ cl } \{x\} \}$ is a τ_1-δ-open set containing y but not x.

Case (ii)

If $\tau_2 - \delta \text{ cl } \{x\} \neq \tau_2 - \delta \text{ cl } \{y\}$, then we can prove that there exists τ_2-δ-open set containing by but not x as above. Hence (x,τ_1,τ_2) is a pairwise rT_0 space.

THEOREM : 3.8

A space (X,τ_1,τ_2) is pairwise rT_0 if either (X,τ_1) or (X,τ_2) is rT_0.

NOTE :

The converse of the previous theorem need not be true which can be seen in the following example.

EXAMPLE : 3.9

Let, $X = \{a,b,c\}$

$\tau_1 = \{ X, \emptyset, \{a\}, \{b,c\} \}$ and

$\tau_2 = \{ X, \emptyset, \{c\}, \{a,b\} \}$

$X - \tau_1 = \{\emptyset , X, \{b,c\}, \{a\} \}$

$X - \tau_2 = \{\emptyset , X, \{a,b\}, \{c\} \}$

$\tau_1 - \text{int}(\tau_1 - \text{cl } X) = \tau_1 - \text{int}(X) = X$

$\tau_1 - \text{int}(\tau_1 - \text{cl } \emptyset) = \tau_1 - \text{int}(\emptyset) = \emptyset$

$\tau_1 - \text{int}(\tau_1 - \text{cl } \{a\}) = \tau_1 - \text{int}\{a\} = \{a\}$

τ_1 - int(τ_1 - cl {b}) = τ_1 - int{b,c} = {b,c}

τ_1 - int(τ_1 - cl {c}) = τ_1 - int{b,c} = {b,c}

τ_1 - int(τ_1 - cl {a,b}) = τ_1 - int(X) = X

τ_1 - int(τ_1 - cl {b,c}) = τ_1 - int{b,c} = {b,c}

τ_1 - int(τ_1 - cl {a,c}) = τ_1 - int(X) = X

The collection of τ_1 - regularly open sets

= {X, ∅, {a}, {b,c}}

τ_2 - int(τ_2 - cl X) = τ_2 - int(X) = X

τ_2 - int(τ_2 - cl ∅) = τ_2 - int(∅) = ∅

τ_2 - int(τ_2 - cl {a}) = τ_2 - int{a,b} = {a,b}

τ_2 - int(τ_2 - cl {b}) = τ_2 - int{a,b} = {a,b}

τ_2 - int(τ_2 - cl {c}) = τ_2 - int{c} = {c}

τ_2 - int(τ_2 - cl {a,b}) = τ_2 - int{a,b} = {a,b}

τ_2 - int(τ_2 - cl {b,c}) = τ_2 - int{X} = X

τ_2 - int(τ_2 - cl {a,c}) = τ_2 - int{X} = X

The collection of τ_2 - regularly open sets

= {X, ∅, {a,b}, {c}}

Obviously for any two distinct points of X, there exists a set which is either τ_1 -regularly open set or τ_2 -regularly open set containing one of the points but not the other. Therefore the space (X, τ_1 , τ_2) is pairwise r T_0.

Consider the pair b, c with b ≠ c then there exists no τ_1-regularly open set containing one but not containing the other. Therefore the space (X, τ_1) is not rT_0.

Consider the pair a, b with a ≠ b. Then there exists no τ_2-regularly open set containing one but not containing the other. Therefore the space (X, τ_2) is not rT_0.

PAIRWISE rT_1 - SPACES :

DEFINITION : 3.10 [3]

A space X is said to be rT_1 if whenever x and y are distinct points in X, there exists a regularly open set containing x but not y, or equivalently, there exists a δ-open set containing x but not y.

DEFINITION : 3.11

A space (X, τ_1, τ_2) is said to be weakly pairwise rT_1 if for every pair of distinct points x, y of X, there exists either a τ_1-regularly open or a τ_2-regularly open set containing x but not y, or equivalently, there exists a τ_1-δ-open set or τ_2- δ-open set containing x but not y.

NOTE :

Every weakly pairwise rT_1 space is pairwise rT_0, but the converse need not be true as can be seen from the following example.

EXAMPLE : 3.12

Let, $X = \{a,b,c\}$

$\tau_1 = \{ X, \emptyset, \{a\}, \{b,c\} \}$ and

$\tau_2 = \{ X, \emptyset, \{a\}, \{b\}, \{a,b\} \}$

$X - \tau_1 = \{\emptyset , X, \{b,c\}, \{a\} \}$

$X - \tau_2 = \{\emptyset , X, \{b,c\}, \{a,c\}, \{c\} \}$

$\tau_1 - int(\tau_1 - cl\ X) = \tau_1 - int(X) = X$

$\tau_1 - int(\tau_1 - cl\ \{\emptyset\} = \tau_1 - int\{\emptyset\} = \emptyset$

$\tau_1 - int(\tau_1 - cl\ \{a\}) = \tau_1 - int\{a\} = \{a\}$

τ_1 - int(τ_1 - cl {b}) = τ_1 - int{b,c} = {b,c}

τ_1 - int(τ_1 - cl {c}) = τ_1 - int{b,c} = {b,c}

τ_1 - int(τ_1 - cl {a,b}) = τ_1 - int(X) = X

τ_1 - int(τ_1 - cl {b,c}) = τ_1 - int{b,c} = {b,c}

τ_1 - int(τ_1 - cl {c,a}) = τ_1 - int(X) = X

The collection of τ_1 - regularly open sets

= {X, ø, {a}, {b,c}}

τ_2 - int(τ_2 - cl X) = τ_2 - int(X) = X

τ_2 - int(τ_2 - cl ø) = τ_2 - int(ø) = ø

τ_2 - int(τ_2 - cl {a}) = τ_2 - int{a,c} = {a}

τ_2 - int(τ_2 - cl {b}) = τ_2 - int{b,c} = {b}

τ_2 - int(τ_2 - cl {c}) = τ_2 - int{c} = ø

τ_2 - int(τ_2 - cl {a,b}) = τ_2 - int(X) = X

τ_2 - int(τ_2 - cl {b,c}) = τ_2 - int{b,c} = {b}

τ_2 - int(τ_2 - cl {c,a}) = τ_2 - int{a,c} = {a}

The collection of τ_2 - regularly open sets

= {X, ø, {a}, {b}}

Then the space (X,τ_1,τ_2) is pairwise rT_0.

Consider the pair of points b,c of X, we observe that any τ_1 regularly open set containing b also contains c and any τ_2 regularly open set containing c also contains b. Therefore the space (X,τ_1,τ_2) is not weakly pairwise rT_1.

DEFINITION 3.13 [11]

A space (X,τ_1,τ_2) is said to be pairwise T_1 if for every pair of distinct points of x and y of X, there exists a τ_1-open set or τ_2-open set containing x but not y.

NOTE : [11]

A pairwise T_1 space is called weakly pairwise T_1.

THEOREM : 3.14

Every weakly pairwise rT_1 bitopological space is weakly pairwise T_1.

NOTE :

The converse of the above theorem is not necessarily true.

THEOREM :3.15

The following statements are equivalent :

(i) (X, τ_1, τ_2) is a weakly pairwise rT_1 space.

(ii) $\tau_1 - \delta cl\{x\} \cap \tau_2 - \delta cl\{x\} = \{x\}$ for every $x \in X$.

(iii) For every $x \in X$, the intersection of all τ_1-δ-neighbourhoods and all τ_2-δ-neighbourhoods of x is $\{x\}$.

PROOF :

(i) ==> (ii)

Let $x \in X$ and $y \in \tau_1\text{-}\delta cl\{x\} \cap \tau_2\text{-}\delta cl\{x\}$ where $y \neq x$. Since x is weakly pairwise rT_1, therefore there is a τ_1-δ-open set U such that $y \in U$, $x \notin U$ or there is a τ_2-δ-open set V such that $y \in V$, $x \notin V$.

In either case, $y \notin \tau_1 - \delta\ cl\ \{x\} \cap \tau_2\text{-}\delta\ cl\{x\}$

For if, $y \in \tau_1 - \delta\ cl\ \{x\} \cap \tau_2\text{-}\delta\ cl\{x\}$

then, $y \in \tau_1 - \delta\ cl\ \{x\} \subset X - U$

and, $y \in \tau_2 - \delta\ cl\ \{x\} \subset X - V$

which is a contradiction.

Hence, $\{x\} = \tau_1\text{-}\delta\ cl\ \{x\} \cap \tau_2\text{-}\delta\ cl\{x\}$

(ii) ==> (iii)

If, $x, y \in X$, such that $x \neq y$, then $x \notin \tau_1 - \delta\ cl\ \{y\} \cap \tau_2 - \delta\ cl\ \{y\}$, so there is a τ_1-δ-open set or τ_2-δ-open set containing x but not y. Therefore y does not belong to the intersection of all τ_1 -δ-neighbourhoods and all τ_2- δ-neighbourhoods of x.

(iii) ==> (i)

Let x and y be two distinct points of X. By hypothesis, y does not belong to a τ_1-δ-neighborhoods or a τ_2-δ-neighborhoods of x. Therefore there exists a τ_1-δ-open set or a τ_2-δ-open set containing x but not y. Hence X is weakly pairwise rT_1 space.

DEFINITION : 3.16

A space (X, τ_1, τ_2) is said to be pairwise rT_1 if for each pair of distinct points x,y of X, there exists a τ_1 regularly open set containing x but not y and a τ_2- regularly open set containing y but not x, or equivalently, there exists a τ_1- δ-open set containing x but not y and a τ_2- δ-open set containing y but not x.

NOTE : [13]

Every pairwise rT_1 space is pairwise T_1, but the converse may be false as seen in the following example :

EXAMPLE : 3.17

Let R be the set of all real numbers and τ the co-countable topology. Then (X, τ_1, τ_2) is pairwise T_1 but it is not pairwise rT_1,because the only τ- regularly open sets or $\emptyset$

and R

THEOREM : 3.18

A space (X, τ_1, τ_2) is pairwise rT_1 if and only if (X,τ_1) and (X,τ_2) are rT_1.

PROOF :

Let (X, τ_1, τ_2) be pairwise rT_1 space. Let x,y be two distinct points of X, then there exists a τ_1 - regularly open set U such that $x \in U$, $y \notin U$. Thus (X,τ_1) is rT_1. Similarly, (X,τ_2) is rT_1. Converse is obvious.

NOTE :

Every pairwise rT_1 space is weakly pairwise rT_1 whereas a weakly pairwise rT_1 space need not be pairewise rT_1.

COROLLARY : 3.19

A space (X, τ_1, τ_2) is pairwise rT_1 if and only if each singleton is both τ_1-δ-closed and τ_2- δ-closed.

PAIRWISE rT_2 - SPACES :

DEFINITION : 3.20

A space (X, τ_1, τ_2) is said to be pairwise Semi - rT_2 if for every pair of distinct points x,y of X, there exists a τ_1 - regularly open set U and a disjoint τ_2-regularly open set V, such that $x \in U$, $y \in V$ or $x \in V$, $y \in U$.

NOTE : [8]

Every pairwise semi- rT_2 space is pairwise semi-τ_2. But the converse is false as can be seen in the following example.

EXAMPLE : 3.21

Let, $X = \{a,b\}$, τ_1 is the discrete topology on X and $\tau_2 = \{ X, \emptyset, \{a\} \}$.

Consider the pair a,b where $a \neq b$. Then there exists a τ_1 - open set $\{b\}$ containing b and there exists a τ_2 - open set $\{a\}$ containing a such that $\{a\} \cap \{b\} = \emptyset$

Therefore the space (X, τ_1, τ_2) is pairwise semi - T_2.

$X - \tau_1 = \{\emptyset, X, \{b\}, \{a\}\}$

$X - \tau_2 = \{\emptyset, X, \{b\}\}$.

$\tau_1 - \text{int}(\tau_1 - \text{cl } X) = \tau_1 - \text{int}(X) = X$

$\tau_1 - \text{int}(\tau_1 - \text{cl } \emptyset) = \tau_1 - \text{int}(\emptyset) = \emptyset$

$\tau_1 - \text{int}(\tau_1 - \text{cl } \{a\}) = \tau_1 - \text{int}\{a\} = \{a\}$

$\tau_1 - \text{int}(\tau_1 - \text{cl } \{b\}) = \tau_1 - \text{int}\{b\} = \{b\}$

The collection of τ_1 - regularly open sets = $\{X, \emptyset, \{a\},\{b\}\}$.

$\tau_2 - \text{int}(\tau_2 - \text{cl } X) = \tau_2 - \text{int}(X) = X$

$\tau_2 - \text{int}(\tau_2 - \text{cl } \emptyset) = \tau_2 - \text{int}(\emptyset) = \emptyset$

$\tau_2 - \text{int}(\tau_2 - \text{cl } \{a\}) = \tau_2 - \text{int}(X) = X$

$\tau_2 - \text{int}(\tau_2 - \text{cl } \{b\}) = \tau_2 - \text{int}\{b\} = \emptyset$

The collection of τ_2 - regularly open sets = $\{X,\emptyset\}$.

For the pair a,b with $a \neq b$, there exists τ_1 - regularly open set $\{a\}$ containing a but there exists no τ_2 - reglarly open set disjoint from $\{a\}$ containing b.

Therefore the space (X, τ_1, τ_2) is not pairwise semi - rT_2.

THEOREM : 3.22

If (X, τ_1, τ_2) is pairwise semi - rT_2 then (X,τ_1) and (X,τ_2) are both rT_0 spaces.

COROLLARY : 3.23

Every pairwise semi - rT_2 space is weakly pairwise rT_1.

DEFINITION : 3.24

A space (X, τ_1, τ_2) is said to be pairwise rT_2 if for every pair of distinct points x,y of X, there exists a τ_1 - regularly open set U and a τ_2 - regularly open set V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$

COROLLARY : 3.25

Every pairwise rT_2 space is pairwise rT_1. The converse need not be true.

DEFINITION : 3.26

A subset A of a space (X,τ) is said to be N-closed relative to τ if for every cover of A by regular open sets of X has a finite sub-cover.

THEOREM : 3.27

Let A be a subset of a pairwise rT_2 - space (X, τ_1, τ_2) which is N-closed relative to τ_1. Then A is τ_2- δ-closed.

PROOF :

If $A = X$, then A is τ_2- δ-closed. If $A \neq X$, then there is a point $x \in X-A$. Since X is pairwise rT_2, for each $y \in A$, there exists a τ_2 - regularly open set U_y and τ_1 - regularly open set V_y such that,

$$x \in U_y,\ y \in V_y \text{ and } U_y \cap V_y = \emptyset.$$

Then $\{ V_y : y \in A \}$ is a τ_1 - regularly open cover of the set A which is N-closed relative to τ_1.

So it has a finite sub cover say, V_{y_1} , V_{y_2} ,...., V_{y_n}.

Let, $$U = \bigcap_{i=1}^{n} U_{y_i}, \quad V = \bigcap_{i=1}^{n} V_{y_i}$$

Then U is τ_2-δ-open, V is τ_1-δ-open and $x \in U$, $A \subset V$ and $U \cap V = \emptyset$. Thus, $x \in U \subset X - A$, and also $X - A$ is τ_2-δ-open. Hence A is τ_2-δ-closed.

THE IMPLICATION BETWEEN PAIRWISE SEPARTION AXIOMS ARE INDICATED BY THE FOLLOWING DIAGRRAM :

$$\begin{tikzcd}
\text{Pairwise Semi - }T_2 \arrow[r, Rightarrow] & \text{Weakly pairwise }T_1 & \\
\text{Pairwise Semi - }rT_2 \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{Weakly pairwise }rT_1 \arrow[u, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Pairwise }rT_2 \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Pairwise }rT_1 \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Pairwise }rT_0 \arrow[d, Rightarrow] \\
\text{Pairwise }T_2 \arrow[r, Rightarrow] & \text{Pairwise }T_1 \arrow[r, Rightarrow] & \text{Pairwise }T_0
\end{tikzcd}$$

PAIRWISE rR_0 - SPACES :

DEFINITION : 3.28

In a topological space (X,τ) the δ-Kernal of a point x of X is the set.

$\delta\text{-Ker}\{X\} = \{ y : x \in \delta cl \{y\} \}$ and the δ-Kernal of a subset A of X is the set

$\delta\text{-Ker } A = \cap \{ U : U \text{ is } \delta\text{-open and } A \subset U \}.$

LEMMA : 3.29

Let (X,τ) be a topological space and let A be a subset of X, then $\delta\text{-Ker } A = \{x \in X : \delta cl \{x\} \cap A = \emptyset \}$

PROOF :

$x \notin \delta\text{-Ker } A$ implies $x \notin \cap \{U : U \text{ is } \delta\text{-open and } A \subset U\}$, so there is a δ-open set U such that $A \subset U$ and $x \notin U$. Therefore, $\delta\ cl \{x\} \cap U = \emptyset$ and

$\delta\ cl \{x\} \cap A = \emptyset$

Now, $\delta\ cl \{x\} \cap A = \emptyset$,

so $G = X - \delta\ cl \{x\}$ is a δ-open set such that $A \subset G$. Also x does not belong to the intersection of all δ-open neighbourhoods of A, so $x \notin \delta\text{-Ker } A$.

DEFINITION : 3.30 [11]

A space (X, τ_1, τ_2) is said to be pairwise R_o, if for every τ_i - open set G, $x \in G$ implies

$\tau_j - cl \{x\} \subset G,\ i,j = 1,2,\ i \neq j.$

DEFINITION: 3.31

A space (X, τ_1, τ_2) is said to be pairwise rR_o if for every τ_1 - δ-open set G, $x \in G$ implies $\tau_j = \delta\ cl \{x\} \subset G$, $i,j = 1,2,\ i \neq j$.

EXAMPLE: 3.32

Let $X = \{ a, b, c \}$

$\tau_1 = \{ X, \emptyset, \{a,b\}, \{a,c\}, \{a\} \}$ and

$\tau_2 = \{ X, \emptyset, \{b,c\}, \{a,b\}, \{b\} \}$

Then, τ_1 - regularly open sets $= \{X,\emptyset\}$

and τ_2 - regularly open sets = $\{X, \emptyset\}$

Then the space (X, τ_1, τ_2) is pairwise rR_o but not pairwise R_o.

EXAMPLE : 3.33

Let, $X = \{ a, b, c \}$

$\tau_1 = \{ X, \emptyset, \{a\}, \{b\}, \{a,b\} \}$ and

$\tau_2 = \{ X, \emptyset, \{a,c\}, \{b,c\}, \{c\} \}$

Then τ_1 - regularly open sets = $\{ X, \emptyset, \{a\}, \{b\}\}$

and, τ_2 - regularly open sets = $\{ X, \emptyset, \}$

Then the space (X, τ_1, τ_2) is pairwise R_o, but not pairwise rR_o.

From the above two examples, we note that the pairwise rR_o and pairwise R_o are independent concepts.

THEOREM : 3.34

A space (X, τ_1, τ_2) is pairwise rR_o if and only if for each pair of distinct points x,y of X.

τ_1-δ-cl $\{x\} \cap \tau_2$-δ cl $\{y\} = \emptyset$ or

$\{x,y\} \subset \tau_1$ - δ cl $\{x\} \cap \tau_2$-δ cl $\{y\}$

PROOF :

Let, τ_1-δ-cl $\{x\} \cap \tau_2$ - δ cl $\{y\} = \emptyset$ and

$\{x,y\} \not\subset \tau_1$ - δ cl $\{x\} \cap \tau_2$-δ cl $\{y\}$

Let, $z \in \tau_1$ - δ cl $\cap \tau_2$ - δ cl $\{y\}$ and

$x \notin \tau_1$ - δ cl $\{x\} \cap \tau_2$ - δ cl $\{y\}$.

Then, $x \notin \tau_2$ - δ cl $\{y\}$ which implies that

$x \in X$ - $\{ \tau_2$ - δ cl $\{y\} \}$ which is τ_2 - δ-open set,

but, $\tau_1 - \delta \text{ cl } \{x\} \not\subset X - \{ \tau_2 - \delta \text{ cl } \{y\} \}$ because
$z \in \tau_2 - \delta \text{ cl } \{y\}$,

so the space (X, τ_1, τ_2) is not pairwise rR_0. Conversly, let G be a τ_1 - δ-open set containing a point x of X. Suppose, $\tau_2 - \delta \text{ cl } \{x\} \not\subset G$, then there is a point $y \in \tau_2 - \delta \text{ cl } \{x\}$ such that $y \notin G$ and $\tau_1 - \delta \text{ cl } \{y\} \cap G = \emptyset$ Since $X - G$ is τ_1- δ-closed and $y \in X - G$,

hence $\{x,y\} \subset \tau_1 - \delta \text{ cl } \{y\} \cap \tau_2 - \delta \text{ cl } \{x\}$

and so, $\tau_1 - \delta \text{ cl } \{y\} \cap \tau_2 - \delta \text{ cl } \{x\} \neq \emptyset$

THEOREM : 3.35

The following statements are equivalent :

a. (X, τ_1, τ_2) is a pairwise rR_0 space.

b. For each $x \in X, \tau_i - \delta \text{ cl } \{x\} \subset \tau_j - \delta \text{ Ker } \{x\}$, $i, j = 1,2$, $i \neq j$.

c. For any $x,y \in X$, $Y \in \tau_1 - \delta \text{ Ker } \{x\}$ if and only if $x \in \tau_j - \delta\text{-Ker } \{y\}$, $i,j = 1,2$, $i \neq j$.

d. For any $x,y \in X$, $y \in \tau_i - \delta\text{-cl } [x\}$ if and only if $x \in \tau_j - \delta\text{-cl } \{y\}$, $i,j = 1,2$, $i \neq j$.

e. For any τ_i - δ-closed set F and a point $x \notin F$, there exists a τ_j- δ-open set U such that $x \notin U$ and $F \subset U$, $i,j = 1,2$, $i \neq j$.

f. Each τ_i- δ-closed set F can be expressed as
$F = \{ G : \cap G$ is τ_j-δ-open and $F \subset G \}$, $i,j = 1,2$, $i \neq j$.

g. Each τ_i -δ-open set G can be expressed as the union of τ_j -δ-closed set contained in G, $i,j = 1,2$, $i \neq j$.

h. For each τ_i- δ-closed set F, $x \notin F$ implies,

$$\tau_j - \delta\ cl\ \{x\} \cap F = \emptyset,\ i,\ j = 1,2,\ i \neq j.$$

PROOF :

To prove (a) ==> (b),

Assume that (X, τ_1, τ_2) is pairwise rR_0 space.

To prove : For each $x \in X$, $\tau_i - \delta\ cl\ \{x\} \subset \tau_j - \delta\ Ker\ \{x\}$, $i, j = 1,2$, $i \neq j$.

By definition 3.28, for any $x \in X$, we have $\tau_j - \delta\ Ker\ \{x\} = \cap \{G/G$ is τ_j-δ-open and $x \in G\}$ and by definition 3.31 each τ_j -δ-open set G containing x contains τ_i- $\delta\ cl\ \{x\}$.

Hence τ_i- $\delta\ cl\ \{x\} \subset \tau_j$-$\delta$ Ker $\{x\}$, $i,j = 1,2$, $i \neq j$.

To prove (b) ==> (c)

Assume that for each $x \in X$, τ_i-δ cl $\{x\} \subset \tau_j$ -δ Ker$\{x\}$, $i,j = 1,2$, $i \neq j$.

To prove : For any $x, y \in X$, $y \in \tau_i$ -δ Ker $\{x\}$ if and only if $x \in \tau_j$-δ Ker $\{y\}$, $i,j = 1,2$, $i \neq j$.

For any $x,y \in X$, if $y \in \tau_i$-δ Ker$\{x\}$ then $x \in \tau_i$ -δ cl $\{y\}$.

Then by our assumption,

$$x \in \tau_j -\delta\ Ker\ \{y\}.$$

To Prove (c) ==> (d)

Assume that for any $x,y \in X$, $y \in \tau_i$-δ Ker $\{x\}$ if and only if $x \in \tau_j$-δ Ker $\{y\}$, $i,j = 1,2$, $i \neq j$.

To Prove: For any $x, y \in X$, $y \in \tau_i$ -δ cl $\{x\}$ if and only if $x \in \tau_j$-δ-cl $\{y\}$, $i,j = 1,2$, $i \neq j$.

For any $x,y \in X$, if $y \in \tau_i$-δ cl$\{x\}$ then $x \in \tau_j$ -δ Ker$\{y\}$ by the definition and hence by our assumption,

$y \in \tau_j - \delta$ Ker$\{x\}$ implies

$x \in \tau_j - \delta$ cl$\{y\}$.

To Prove (d) ==> (e)

Assume that for any $x,y \in X$, $y \in \tau_i$-δ cl$\{x\}$ if and only if $x \in \tau_j$-δ cl$\{y\}$, $i,j = 1,2$, $i \neq j$.

To Prove : For any τ_j- δ-closed set F and a point $x \notin F$, there exists a τ_j- δ-open set U such that $x \notin U$ and $F \subset U$, $i,j = 1,2$, $i \neq j$. Let F be a τ_i-δ-closed set and a point $x \notin F$. Then for any $y \in F$, $\tau_i - \delta$ cl$\{y\} \subset F$ and so $x \notin \tau_i$- δ cl$\{y\}$ Now, by our assumption $x \notin \tau_i$ -δ cl$\{y\}$, implies $y \notin \tau_j$ -δ cl $\{x\}$, that is there exists a τ_j-δ-open set Gy such that $y \in$ Gy and $x \notin G_y$.

Let, $G = \bigcup_{y \in F} \{$ Gy : Gy is τ_j-δ open, $y \in$ Gy and $x \notin$ Gy $\}$

Then G is a τ_j-δ-open set such that

$x \notin G$ and $F \subset G$.

To Prove (e) ==> (f)

Assume that for any τ_i- δ- closed set F and a point $x \notin F$, there exists a τ_j-δ-open set U such that $x \notin U$ and $F \subset U$, $i,j = 1,2$, $i \neq j$.

To Prove :

Each τ_i- δ- closed set F can be expressed as $F = \{G : \cap G$ is τ_j-δ- open and $F \subset G\}$, $i,j = 1,2$, $i \neq j$.

Let F be a τ_i-δ-closed set and suppose that $H = \cap \{ G : G$ is τ_j-δ- open, and $F \subset G \}$.

Then $F \subset H$ and to show that $H \subset F$.

Let, $x \notin F$. Then by our assumption there is a τ_j- δ- open set G such that $x \notin G$ and $F \subset G$ and hence $x \notin H$.

Therefore, each τ_i-δ-closed set F can be written as $F = \cap \{ G : G$ is τ_j-δ-open and $H \subset G \}$

To Prove (f) ==> (g)

Assume that for every τ_i-δ closed set F can be expressed as,

$F = \{ G : \cap G$ is τ_j-δ-open and $F \subset G \}$, $i,j = 1,2$, $i \neq j$.

Then clearly each τ_i-δ-open set G can be expressed as the union of τ_j- δ-closed sets contained in G, $i,j = 1,2$, $i \neq j$.

To Prove (g) ==> (h) .

Assume that each τ_i-δ-open set G can be expressed as the union of τ_j-δ-closed sets contained in G, $i,j = 1,2$, $i \neq j$.

To Prove : For each τ_i-δ-closed set F, $x \notin F$ implies τ_j-δ cl $\{x\} \cap F = \emptyset$, $i, j=1,2$, $i \neq j$.

let F be a τ_i-δ-closed set and $x \notin F$. Then $X - F = G$ (say) is a τ_i-δ-open set containing x. Then by our assumption, G can be written as the union of τ_j-δ closed sets and so there is a τ_j-δ-closed set H such that $x \in H \subset G$ and hence,

τ_j - δ cl $\{x\} \subset G$

Thus τ_j - δ cl $\{x\} \cap F = \emptyset$.

To prove (h) ==> (a)

Assume that for each τ_i-δ-closed set F, $x \notin F$ impiles,

τ_j - δ cl $\{x\} \cap F = \emptyset$, $i, j=1,2$, $i \neq j$.

To Prove : (X, τ_1, τ_2) is a pairwise rR_0 space.

That is to prove for every τ_i-δ-open set G, $x \in G$ implies τ_j - δ cl $\{x\} \subset G$, $i, j=1,2$, $i \neq j$.

Let G be a τ_i-δ-open set and $x \in G$. Then $x \notin X-G$ which is

τ_i-δ-closed set. Then by our assumption,

$$\tau_j - \delta\ \mathrm{cl}\ \{x\} \cap (X - G) = \emptyset,$$

which implies that,

$$\tau_j - \delta\ \mathrm{cl}\ \{x\} \subset G,\ i,\ j=1,2,\ i \neq j.$$

Thus, (X, τ_1, τ_2) is pairwise rR_0.

DEFINITION : 3.36

In a bitopological space, (X, τ_1, τ_2) for any $x \in X$, we introduce the following notations :

(i) $b_i - \delta\ \mathrm{cl}\ \{x\} = \tau_1 - \delta\ \mathrm{cl}\ \{x\} \cap \tau_2 - \delta\ \mathrm{cl}\ \{x\}$.

(ii) $b_i - \delta\ \mathrm{Ker}\ \{x\} = \tau_1 - \delta\ \mathrm{Ker}\ \{x\} \cap \tau_2 - \delta\ \mathrm{Ker}\ \{x\}$.

THEOREM : 3.37

In a pairwise rR_0 space (X, τ_1, τ_2) for any x and y in X, we have either,

$$b_i - \delta\ \mathrm{cl}\ \{x\} = b_i - \delta\ \mathrm{cl}\ \{y\} \text{ or}$$

$$b_i - \delta\ \mathrm{cl}\ \{x\} \cap b_i - \delta\ \mathrm{cl}\ \{y\} = \emptyset$$

PROOF :

Suppose that,

$$b_i - \delta\ \mathrm{cl}\ \{x\} \cap b_i - \delta\ \mathrm{cl}\ \{y\} \neq \emptyset$$

To Prove : $b_i - \delta\ \mathrm{cl}\ \{x\} = b_i - \delta\ \mathrm{cl}\ \{y\}$

Let, $z \in b_i - \delta\ \mathrm{cl}\ \{x\} \cap b_i - \delta\ \mathrm{cl}\ \{y\}$

That is,

$$z \in [\tau_1-\delta\ \mathrm{cl}\{x\} \cap \tau_2 - \delta\ \mathrm{cl}\ \{x\}\ ] \cap [\tau_1 - \delta\ \mathrm{cl}\ \{y\} \cap \tau_2 - \delta\ \mathrm{cl}\ \{y\}\ ]$$

Then, $[\tau_1-\delta\ \mathrm{cl}\{z\} \subset \tau_1 - \delta\ \mathrm{cl}\ \{x\}\ ] \cap [\tau_1 - \delta\ \mathrm{cl}\ \{y\}]$ and

$[\tau_2-\delta\ \mathrm{cl}\{z\} \subset \tau_2 - \delta\ \mathrm{cl}\ \{x\}\ ] \cap [\tau_2 - \delta\ \mathrm{cl}\ \{y\}]$.

Also, $z \in [\tau_1-\delta\ \mathrm{cl}\{x\}]$ implies that

$$\tau_2\text{-}\delta\ cl\{x\} \subset \tau_2 - \delta\ cl\ \{y\} \dashrightarrow (1)$$

this is so because by (d) of theorem 3.35, $z \in \tau_1\text{-}\delta\ cl\ \{x\}$

then, $x \in \tau_2\text{-}\delta\ cl\ \{z\}$

implies $[\tau_2\text{-}\delta\ cl\{x\}] \subset [\tau_2 - \delta\ cl\ \{z\}\ ] \subset [\tau_2 - \delta\ cl\ \{y\}]$.

Similarly, $z \in \tau_2\text{-}\delta\ cl\{x\}$ implies

$$\tau_1\text{-}\delta\ cl\{x\} \subset \tau_1 - \delta\ cl\ \{y\} \dashrightarrow (2)$$

and $z \in \tau_1\text{-}\delta\ cl\{y\}$ implies

$\tau_2\text{-}\delta\ cl\{y\} \subset \tau_2 - \delta\ cl\ \{x\} \dashrightarrow (3)$ and

$z \in \tau_2\text{-}\delta\ cl\{y\}$ implies

$$\tau_1\text{-}\delta\ cl\{y\} \subset \tau_1 - \delta\ cl\ \{x\} \dashrightarrow (4)$$

From (1) and (2),

$$\tau_1\text{-}\delta\ cl\{x\} \cap \tau_2\text{-}\delta\ cl\{x\} \subset \tau_1\text{-}\ \delta\ cl\{y\} \cap \tau_2\text{-}\ \delta\ cl\ \{y\} \dashrightarrow (5)$$

From (3) and (4),

$$\tau_1\text{-}\delta\ cl\{y\} \cap \tau_2\text{-}\delta\ cl\{y\} \subset \tau_1\text{-}\delta\ cl\{x\} \cap \tau_2 - \delta\ cl\ \{x\} \dashrightarrow (6)$$

From (5) and (6),

$$\tau_1\text{-}\delta\ cl\{x\} \cap \tau_2\text{-}\delta\ cl\{x\} = \tau_1 - \delta\ cl\ \{y\} \cap \tau_2 - \delta\ cl\ \{y\}$$

That is,

$$b_i\text{-}\delta\ cl\{x\} = b_i\text{-}\delta\ cl\{y\}$$

This proves the theorem.

THEOREM : 3.38

In a pairwise rR_0 space, (X, τ_1, τ_2) for any x and y in X, we have either

$$b_i\text{-}\delta\ Ker\ \{x\} = b_i\text{-}\delta\ Ker\ \{y\} \text{ or}$$

$$b_i\text{-}\delta\ Ker\ \{x\} \cap b_i\text{-}\delta\ Ker\ \{y\} = \emptyset.$$

PROOF :

By the statement (c), of the therorem 3.35 the proof is simialar to that of therorem 3.37 which is obvious.

THEOREM : 3.39

If a space (X, τ_1, τ_2) is pairwise $r\tau_1$, then it is pairwise rR_0.

PROOF :

If (X, τ_1, τ_2) is pairwise rT_1, then by theorem 3.18, (X,τ_1) and (X,τ_2) are both rT_1 spaces which implies that

$$\tau_1\text{-}\delta\ cl\{x\} = \{x\} = \tau_2 - \delta\ cl\ \{x\}.$$

Thus (X, τ_1, τ_2) is pairwise rR_0.

THEOREM : 3.40

Every pairwise rT_0, pairwise rR_0 space is weakly pairwise rT_1

PROOF :

Let (X, τ_1, τ_2) be pairwise rT_0 and pairwise rR_0 space. Let x,y be any pair of distinct points of X. Therefore, there exists a τ_i- δ-open set G, i = 1 or 2, containing one of the points but not the other, since (X, τ_1, τ_2) is pairwise rT_0.

Suppose G contains x (say). Since X is pairwise rR_0 , $x \in G$ implies

$$\tau_j\text{-}\ \delta\ cl\ \{x\} \subset G,\ i,j = 1,2\ ,\ i \neq j.$$

Now, $x \notin X - \tau_j - \delta\ cl\ \{x\} = M$ (say)

Thus G is a τ_i- δ-open set containing x but not y and M is τ_j-δ-open set containing y but not x.

Thus (X, τ_1, τ_2) is weakly pairwise $r\tau_1$.

PAIRWISE rR_1 SPACES :

DEFINITION : 3.41

A space (X, τ_1, τ_2) is said to be pairwise rR_1 if for every pair of distinct points x and y of X such that τ_i-δ cl$\{x\} \neq \tau_j$-δ cl$\{y\}$, there exists a τ_j-δ-open set U and a τ_i-δ-open set V such that,

$x \in U$, $y \in V$ and $U \cap V = \emptyset$, $i,j = 1,2$, $i \neq j$.

THEOREM : 3.42

Every pairwise rR_1 spaces is pairwise rR_0.

PROOF :

Let (X, τ_1, τ_2) be pairwise rR_1. Let G be any τ_i-δ-open set and $x \in G$. For every point $y \in X - G$,

τ_j - δ cl $\{x\} \neq \tau_j$ - δ cl$\{y\}$,

so there exists a τ_i- δ- open set Uy and a τ_j-δ- open set Vy such that,

$x \in Uy$, $y \in Vy$ and $Uy \cap Vy = \emptyset$.

If, $A = \cup \{ Vy : y \in X - G \}$,

then, $X - G \subset A$ and $x \notin A$

τ_j-δ-open sets of A implies

τ_j -δ cl $\{x\} \subset X - A \subset G$

Hence, X is pairwise rR_0.

THEOREM : 3.43

A space (X, τ_1, τ_2) is pairwise rR_1 if and only if for every pair of points x and y of x such that, τ_i-δ cl $\{x\} \neq \tau_j$ - δ cl$\{y\}$, there exists a τ_i- δ-open set U and a τ_j- δ-open set V such that τ_i-δ cl$\{x\} \subset V$, τ_j-δ cl $\{y\} \subset U$ and,

$U \cap V = \emptyset$, $i, j = 1,2, i \neq j$.

PROOF :

Let, (X, τ_1, τ_2) be pairwise rR_1 space.

τ_i-δ cl $\{x\} \neq \tau_j$-δ cl $\{y\}$, $i,j = 1,2$, $i \neq j$.

Then there exists a τ_i- δ- open set U and a τ_j-δ-open set V such that $x \in V$, $y \in U$ and $U \cap V = \emptyset$. Since a pairwise rR_1 space is pairwise rR_0, therefore, $x \in V$ which implies,

τ_i-δ cl$\{x\} \subset V$ and $y \in U$ implies,

τ_j-δ cl$\{y\} \subset U$, $i,j = 1,2$, $i \neq j$.

Hence, the theorem. The converse is obvious.

THEOREM : 3.44

Every pairwise rT_2 space is pairwise rR_1.

COROLLARY : 3.45

A space (X, τ_1, τ_2) is pairwise rT_2 if and only if it is pairwise rT_1 and pairwise rR_1.

THEOREM : 3.46

Let A be a subset of a pairwise rR_1 space (X, τ_1, τ_2) which is N-closed relative to τ_2 such that $A \cap \tau_2$- δ cl$\{x\}$ = $\emptyset$ for some $x \in X$. Then there exists a τ_1-δ-open set U and a τ_2-δ-open set V such that τ_2-δ cl$\{x\} \subset U$, $A \subset V$ and $U \cap V = \emptyset$.

PROOF :

For each $y \in A$,

τ_1-δ cl$\{y\} \neq \tau_2$-δ cl$\{x\}$.

Since X is pairwise rR_1, there exists a τ_1-δ-open set U_y and a τ_2- δ-open set V_y such that,

τ_1 - δ-cl $\{y\} \subset V_y$, τ_2 -δ cl$\{x\} \subset U_y$

and $U_y \cap V_y = \emptyset$

Since A is N-closed relative to τ_2, the collection $\{V_y : y \in A\}$ is a τ_2-δ open cover of A, which has a finite sub-cover, say $\{V_{y_1} : i = 1,2,\ldots n]$

Let, $U = \bigcap_{i=1}^{n} U_{y_i}$, $V = \bigcap_{i=1}^{n} V_{y_i}$.

Then U is τ_1 -δ-open, V is τ_2 -δ-open and $A \subset V$.

Also τ_2 -δ $cl\{x\} \subset U$ and $U \cap V = \emptyset$. Hence the theorem.

Chapter IV

CHAPTER IV

ON PAIRWISE R_0 AND PAIRWISE R_1 BITOPOLOGICAL SPACES

In this chapter we shall discuss the paper "On Pairwise R_0 and Pairwise R_1 Bitopological spaces" by MILA MRŠEVIČ. The author introduces the concept of quasiopen and quasiclosed sets. He has obtained the charterization of pairwise R_0, pairwise R_1 spaces.

DEFINITION : 4.1

In a bitopological space (X,τ_1,τ_2) a set is said to be quasiopen if it is a union of τ_1-open sets and τ_2-open sets.

A set is quasiclosed if its complement is quasiopen.

DEFINITION : 4.2

In a bitopological space (X,τ_1,τ_2) a set is semiopen of τ open (semiclosed or τ closed) if it is open (closed) in the topological space (x,τ), where $\tau=\tau_1 V\tau_2$, is the least upper bound topology of τ_1 and τ_2.

DEFINITION 4.3

A space (X,τ_1,τ_2) is quasicompact if the topological space (x,τ) is compact.

DEFINITION 4.4

In a topological space (x,τ_i) the kernal of a subset A is,

$$\mathrm{Ker}_i A = \cap\{U \in \tau_i / A \subset U\}$$

NOTE:

The equivalence relation $\sim_i$ on a topological space (x,τ_i) is defined by

$x \sim_i y \iff cl_i\{x\} = cl_i\{y\}$

NOTE:

Let $[x]_i$ be the equivalence class of the element x in the space (x, τ_i) then,

$[x]_i = cl_i\{x\} \cap Ker_i\{x\}$

DEFINITION : 4.5

Let (X,τ_1,τ_2) be a bitopological space and $A \subset X$. The quasikernal of A, denoted by qKer A is the intersection of all quasiopen subsets of X containing A.

DEFINITION : 4.6

In a bitopological space (X,τ_1,τ_2) the kernal of A, denoted by Ker A, we mean the intersection of all semiopen subsets of X containing A. That is the kernal of A in the space (X,τ) wgere $\tau=\tau_1 V \tau_2$.

THEOREM : 4.7

For each $A \subset X$ we have

$q\,Ker\,A = Ker\,A$

PROOF :

For each $A \subset X$,

it is obvious that

$Ker\,A \subset q\,Ker\,A \quad \rightarrow (1)$

Since each τ open subset W,

$$W = \bigcup_{\lambda\in\Lambda} (U_\lambda \cap V_\lambda) = \bigcap_{\lambda\in\Lambda_1} [(\bigcup_{\lambda\in\Lambda_i} U) \cup (\bigcup_{\lambda\in\Lambda-\Lambda_i} V)]$$

where $U_\lambda \in \tau_1$ and $V_\lambda \in \tau_2$, is an intersection of quasiopen sets, it follows that

$$q \text{ Ker } A \subset \text{Ker } A \quad \rightarrow (2)$$

By (1) and (2) we have

$$q \text{ Ker } A = \text{Ker } A$$

THEOREM : 4.8

For each $x \in X$ the following holds:

i) q Ker $\{x\} = \text{Ker}_1 \{x\} \cap \text{Ker}_2\{x\}$ and

ii) $q\text{cl}\{x\} = \text{cl}\{x\}$

PROOF:

It is clear that for each $A \subset X$,

$$q \text{ Ker } A \subset \text{Ker}_1 A \cap \text{Ker}_2 A \quad \rightarrow (1)$$

The converse inclusion holds for singletons since for each $U \in \tau_1$ and each $V \in \tau_2$,

$x \in U \cap V$ if and only if $x \in U$ or $x \in V$

implies that

$$\text{Ker}_1 A \cap \text{Ker}_2 A \subset q \text{ Ker } A \quad \rightarrow (2)$$

By (1) and (2) we have,

$$q \text{ Ker } A = \text{Ker}_1 A \cap \text{ker}_2 A.$$

ii) It is clear that for each $A \subset X$

$$\text{cl } A \subset q\text{cl } A \quad \rightarrow (1)$$

Let $y \notin \text{cl } \{x\}$

There is a τ closed subset F containing x but not y.

Since $F = \bigcap_{\lambda \in \Lambda} (F_{1_\lambda} \cup F_{2_\lambda})$,

where F_{i_λ} is τ_i closed for $i = 1.2$,

$y \notin F$ implies $y \notin cli\{x\}$ or $y \notin cl_2\{x\}$,

so $y \notin q\,cl\,\{x\}$

implies that for each $x \in X$

$$qcl\,\{x\} \subset cl\,\{x\} \quad \rightarrow (2)$$

By (1) and (2) we have

for each $x \in X$,

$$qcl\,\{x\} = cl\,\{x\}$$

NOTE:

In general, for all $A \subset X$, the equalities are not holds good which can be shown in the following example.

EXAMPLE : 4.9

Let $X = R$, τ_1 be the left hand topology and τ_2 be the right hand topology.

For $A = \{0,1\}$

$Ker_1 A = (-\infty,1] = cl_2 A$ and

$Ker_2 A = [0,+\infty) = cl_1 A$

Hence $Ker_1\, A \cap Ker_2\, A = [0,1] = qcl\, A$,

while $qKerA = KerA = A = clA$, since the real line is a T_1 -space.

An equivalence relation on a bitopological spaces.

DEFINITION : 4.10

Let (X,τ_1,τ_2) be a bitopological space and $\sim$ be the binary relation on x defined $x \sim y \Leftrightarrow q\,cl\,\{x\} = qcl\{y\}$.

NOTE :

$\sim$ is an equivalence relation. Reflexivity and symmetric properties are trivial.

Transitivity:

$x \sim y \Rightarrow qcl\ \{x\} = qcl\ \{y\}$ and

$y \sim z \Rightarrow qcl\ \{y\} = qcl\ \{z\}$

$\Rightarrow qcl\ \{x\} = qcl\ \{z\} \quad \Rightarrow x \sim z$

NOTATION :

Let $[x]$ denote the equivalence class of the element x and let $\tilde{X} = X/\sim$ be the quotient set. $\tilde{\tau}_1, \tilde{\tau}_2$ and $\tilde{\tau}$ be the quotient topologies on $\tilde{X}$ of τ_1 τ_2 and τ respectively and $p: X \dashrightarrow \tilde{X}$ be the quotient mapping.

THEOREM : 4.11

For each $x \in X$,

i) $[x] = cl\ \{x\} \cap Ker\ \{x\}$

ii) $[x] = qcl\ \{x\} \cap qKer\ \{x\}$

iii) $[x] = [x]_1 \cap [x]_2$

This theorem can be proved by using the results of the paper "separation Axioms Between T_0 and T_1" by C.E.AULL [2].

COROLLARY : 4.12

For each semiopen (semiclosed) subsets A of X $p^{-1}(p(A)) = A$ holds.

THEOREM : 4.13

The topology $\tilde{\tau}$ is the least upper bound topology of $\tilde{\tau}_1$ and $\tilde{\tau}_2$,that is, $\tilde{\tau} = \tilde{\tau}_1 \vee \tilde{\tau}_2$

PROOF:

By theorem 2.12 the quotient mapping p induces a lattice isomorphism.

$$\hat{p} : \tau \to \tilde{\tau} \text{ defined by}$$

$$\hat{p}\,(U) = P(U)$$

Moreover,

$\tilde{\tau}_i = \{P(U)/U\in\tau_i\}$, for i = 1, 2.

$\{P(U)/U\in\tau_i\} \subset \tilde{\tau}_i$, since for each $U\in \tau_i$, $U = P^{-1}(P(U))$ holds.

Conversely,

$\tilde{U}\in\tilde{\tau}_i$ implies $p^{-1}(\tilde{U}) \in \tilde{\tau}$

and hence

$$U = P(P^{-1}(\tilde{U})) \in \{P(U)/U\in\tau_i\}$$

To prove $\tilde{\tau}=\tilde{\tau}_1 \vee \tilde{\tau}_2$

Since $\tilde{\tau}_i \subset \tilde{\tau}$ for i = 1,2,

we have,

$$\tilde{\tau}_1 \vee \tilde{\tau}_2 \subset \tilde{\tau} \longrightarrow (1)$$

Now for each $\tilde{\tau}$ open set $\tilde{W}$ there are τ_1-open sets U_λ and τ_2 open sets V_λ such that

$$p^{-1}(\tilde{W}) = \cup(U_\lambda \cap V_\lambda)$$

Hence $\tilde{W} = \bigcup_{\lambda\in\Lambda} P\,[P^{-1}(P(U_\lambda)) \cap P^{-1}\,(P(V_\lambda))]$

$$= \bigcup_{\lambda\in\Lambda} [P(U_\lambda) \cap P\,(V_\lambda)] \in \tilde{\tau}_1 \vee \tilde{\tau}_2$$

$$\Rightarrow \tilde{\tau} \subset \tau_1 \vee \tilde{\tau}_2 \longrightarrow (2)$$

By (1) and (2)

$$\tilde{\tau} = \tilde{\tau}_1 \vee \tilde{\tau}_2$$

COROLLARY : 4.14

The quotient mapping p is pairwise open and pairwise closed.

COROLLARY : 4.15

For each $x \in X$ the set $[x]$ is quasicompact in X, hence τ_1 and τ_2 compact.

COROLLARY 4.16

If one of the subsets A or B is semiopen or semiclosed in X, then

$$P(A \cap B) = p(A) \cap P(B)$$

THEOREM : 4.17

For each $A \subset X$ we have

$$cl_i p(A) \subset p\,(cl_i\ A) \text{ for } i=1,2$$

and

$$qcl\{p(A)\} = p(qcl\ A)$$

PROOF :

$\tilde{\tau}_i clp(A) = p\ (\tau_i cl\ A)$, sincep is pairwise closed and pairwise continuous mapping.

Also using corollary 4.16

$$q\ cl\ p(A) = cl_1(p(A)) \cap cl_2 p(A)$$

$$= p(cl_1 A) \cap p(cl_2 A) \text{ by proposition 4.17}$$

$$= p(cl_1 A \cap cl_2 A) \text{ by corollary 4.16}$$

$$= p(qclA)$$

THEOREM : 4.18

Let M be the lattice of all τ_i closed $(i=1,2)$ quasiclosed or semiclosed subsets of a bitopological space (X,τ_1,τ_2). It is the corresponding lattice in the quotient space $(X,\tilde{\tau}_1,\tilde{\tau}_2)$. M and $\tilde{M}$ are isomorphic.

PROOF:

We know that the quotient mapping p induces a lattice isomorphism between τ and $\tilde{\tau}$ [2]. Hence it induces a lattice isomorphism q^* between $S^*(x)$ and $S^*(\tilde{x})$, the corresponding classes of all semiclosed subsets as well as between the classes of all τ_i and $\tilde{\tau}_i$ closed subsets respectively for $i=1,2$.

Let M be the lattice of all quasiclosed subsets of X.

For $A,B \in M$,

$\inf\{A,B\} = A \cap B$

and $\sup\{A,B\} = qcl\,(A \cup B)$

The isomorphism

$q^*: S^*(x) \dashrightarrow S^*(\tilde{X})$

induces an isomorphism

$\hat{q} = q^* / M : M \dashrightarrow \tilde{M}$

since for each $F \in M$ the set

$p(F) \in \tilde{M}$ and

$\hat{q}\,(A \cap B) = \hat{q}(A) \cap \hat{q}(B)$ by corollary 4.16 and

$\hat{q}\,(qcl(A \cup B)) = p(qcl(A \cup B))$

$$= qcl(\hat{q}(A) \cup \hat{q}(B))$$

$$= \sup \{\hat{q}(A), \hat{q}(B)\}$$ by theorem 4.17

Here onwards we denote the subset $[0,1]$ of R by I and the left hand topology and the right hand topology on I by $\mathcal{L}$ and $\mathcal{D}$ respectively.

THEOREM : 4.19

Let $f:(X,\tau_1,\tau_2) \dashrightarrow (I,\mathcal{L},\mathcal{D})$ be a pairwise continuous function.

Then f induces a pairwise continuous function

$\tilde{f}: (\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2) \dashrightarrow (I,\mathcal{L},\mathcal{D})$ such that

$f = \tilde{f} \circ p$ holds

PROOF:

Let $x \in X$. Since f is pairwise continuous, the set $\{f(x)\}$ is quasiclosed in $(I,\mathcal{L},\mathcal{D})$ implies $f^{-1}(\{f(x)\})$ is quasiclosed in X

Hence $[x] = f^{-1}(\{f(x)\})$ and $f([x]) = \{f(x)\}$

It follows that f is constant on equivalence classes.

It induces a mapping

$\tilde{f} :(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2) \dashrightarrow (I,\mathcal{L},\mathcal{D})$

such that

$f= \tilde{f} \circ p$. Hence the theorem.

DEFINITION : 4.20

A bitopological space (X,τ_1,τ_2) is weak pairwise stone if for each two distinct points in X, there is a pairwise continuous function.

$f:(X,\tau_1,\tau_2) \dashrightarrow (I,\mathcal{L},\mathfrak{D})$

carrying one point into 0 and the other into 1.

The space is pairwise stone if for each pair (x,y) of distinct points in x, there is a pairwise continuous function.

$$f:(X,\tau_1,\tau_2) \dashrightarrow (I,\mathcal{L},\mathfrak{D})$$

such that

$$f(x) = 0 \text{ and } f(y) = 1$$

DEFINITION : 4.21

A bitopological space (X,τ_1,τ_2) is weak pairwise Urysohn if for each two distinct points in X, there is a τ_1 open set U and a τ_2 open set V such that

$$\tau_2 \mathrm{cl} U \cap \tau_1 \mathrm{cl}\, V = \emptyset,$$

one of the points belongs to U and the other to V.

The space is pairwise Urysohn if for each pair (x,y) of distinct points in X there is a τ_1-open set U and a τ_2 open set V such that $x \in U$, $y \in V$ and

$$\tau_2 \mathrm{cl} U \cap \tau_1 \mathrm{cl} V = \emptyset$$

THEOREM : 4.22

Let (X,τ_1,τ_2)be a bitopological space and

$(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2)$ be its quotient space. The following properties are shared by X and $\tilde{X}$: quasicompactness, (1,2) - local quasicompactness and pairwise, local quasicompactness, pairwise, R_0 and pairwise R_1 properties, τ_1 regularity with respect to τ_2 and pairwise regularity, τ_1 complete regularity with respect to τ_2 and pairwise complete regularity, pairwise normality, pairwise connectedness (1,2) - local connectedness and pairwise local connectedness, (1,2) - zero dimensionality and pairwise. zero dimensionality.

The following properties are carried over from X to $\tilde{X}$ weak pairwise total disconnectedness and pairwise total disconnectedness, weak pairwise stone and pairwise stone properties, weak pairwise Urysohn and pairwise Urysohn, weak pairwise T_k and pairwise T_k for k=0,1,2.

THEOREM : 4.23

For every bitopological space (X, τ_1, τ_2) its quotient space $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2)$ is weak pairwise T_0.

PROOF:

If $[x] \neq [y]$ then

$x \notin qcl\,\{y\}$ or

$y \notin qcl\,\{x\}$

It follows that

$x \notin cl_1\,\{y\}$ or $x \notin cl_2\,\{y\}$ or

$y \notin cl_1\,\{x\}$ or $y \notin cl_2\,\{x\}$ or

By corollaries 4.12 and 4.16 and theorem 4.17 atleast one of the following relations hold:

$[x] \in cl_1 \{[y]\}^c$,

$[x] \in cl_2 \{[y]\}^c$,

$[y] \in cl_1 \{[x]\}^c$ or

$[y] \in cl_2 \{[x]\}^c$

Hence the proof.

NOTE:

The quotient space need not be pairwise T_0 as shown in the example 4.9

NOTE:

For each $x \in R$, $qcl\{x\} = \{x\}$ and $\tilde{X}$ is pairwise homeomorphic to X

THEOREM :4.24

If a bitopological space (X,τ_1,τ_2) is pairwise R_0 the topological space (X,τ) is an R_0 space.

PROOF:

Let (X,τ_1,τ_2) be pairwise R_0 and $x \in X$. Let B be a basic element of the topology τ such that $x \in B$.

There is a τ_1-open set U and τ_2-open set V such that

$$B = U \cap V$$

It follows that

$cl_1\{x\} \cap cl_2\{x\} \subset U \cap V$

Hence

$qcl\{x\} = \tau\ cl\{x\} \subset B$

Hence (X,τ) is R_0.

THEOREM : 4.25

If a bitopological space (X,τ_1,τ_2) is pairwise R_1, the topological space (X,τ) is an R_1 space.

PROOF:

Let (X,τ_1,τ_2) be pairwise R_1 and

$cl\{x\} \neq cl\{y\}$

By theorem 4.8 this coniditon is equivalent to

$x \in qcl\{y\}$ or $y \in qcl\{x\}$

So there is an $i \in \{1,2\}$ such that

$x \in cl_i\{y\}$ or $y \in cl_i\{x\}$. By definition of pairwise R_1 spaces,

there is a $U \in \tau_i \subset \tau$ and a $V \in \tau_j \subset \tau$

such that

$U \cap V = \emptyset$ and

$x \in U$, $y \in V$ or $y \in U$, $x \in V$

Hence (X,τ) is R_1.

THEOREM : 4.26

Every weak pairwise T_1 pairwise R_1 bitopological space is weak pairwise T_2.

PROOF:

let (X,τ_1,τ_2) be weak pairwise T_0 pairwise R_1 bitopological space. Let $x,y \in X$, $x \neq y$. There is a τ_i-open set W,
$i \in \{1,2\}$ containing one of these points but not the other.
Let $x \in W$ and $y \notin W$.

Then $cl_i\{y\} \subset W^c$, so $x \notin cl_i \{y\}$
and there is a
$U \in \tau_i$ and a $V \in \tau_j$
such that

$x \in U$, $y \in V$,

$U \cap V = \emptyset$, for $i \in \{1,2\}$

and $i \neq j$. So X is weak pairwise T_2.

COROLLARY : 4.27

A pairwise R_1 bitopological space is weakly pairwise T_2 if and only if it is weak pairwise T_0.

The converse of theorem 4.24, 4.25 and 4.26 do not hold in general.

The space (X,τ_1,τ_2) is weak pairwise T_2, but not pairwise R_0 hence not pairwise R_1, while (X,τ) is R_1, hence R_0, since τ is the discrete topology.

THEOREM : 4.28

If (X,τ_1,τ_2) is pairwise R_0, then

$[x] = qcl\ \{x\}$ for each $x \in X$

PROOF:

If (X,τ_1,τ_2) is pairwise R_0, the space (X,τ) is R_0 and $[x] = \text{qcl}\,\{x\}$

COROLLARY : 4.29

If a bitopological space is pairwise R_0, its quotient space is weak pairwise T_1.

COROLLARY : 4.30.

If a bitopological space is pairwise R_1, its quotient space is weak pairwise T_2.

COROLLARY : 4.31

If a bitopological space is pairwise regular, its quotient space is weak pairwise T_3.

Let (X,τ_1,τ_2) be a non-quasicompact bitopological space and $(X_\infty,\tau_{1_\infty},\tau_{2_\infty})$ its one point quasicompactification.
If (X,τ_1,τ_2) is the quotient space of (X,τ_1,τ_2) with respect to the relation ~, let $(X_\infty,\tau_{1_\infty},\tau_{2_\infty})$ be the quotient space of $(X_\infty,\tau_{1_\infty},\tau_{2_\infty})$ with respect to the corresponding equivalence relation $\tilde{}_\infty$ defined on X_∞.

THEOREM : 4.32

The quotient space $(\tilde{X}_\infty,\tilde{\tau}_{1_\infty},\tilde{\tau}_{2_\infty})$ is the one point compactification of the space $(\tilde{X},\tilde{\tau}_{1_\infty},\tilde{\tau}_{2_\infty})$.

COROLLARY : 4.33

If (X,τ_1,τ_2) is pairwise locally quasicompact and pairwise R_1, its quotient space is pairwise locally quasicompact and weak pairwise and $(\tilde{X}_{\infty}, \tilde{\tau}_{1\infty}, \tilde{\tau}_{2\infty})$ is weak pairwise Hausdorff

NOTE :

The one-point compactification of an R_0 topological space is R_0.

NOTE :

The bitopological analogue does not hold. Namely the one point quasicompactification of a pairsie R_0 space need not be pairwise R_0 as shown in the example 4.9.

The space $(R, \mathcal{L}, \mathcal{D})$ is pairwise R_0 but the space $(R_{\infty}, \mathcal{L}_{\infty}, \mathcal{D}_{\infty})$ where

$\mathcal{L}_{\infty} = \mathcal{L} \cup \{R_{\infty}\}$ and

$\mathcal{D}_{\infty} = \mathcal{D} \cup \{R_{\infty}\}$ is not pairwise R_0

THEOREM : 4.34

The one-point quasicompactification of a pairwise R_0 pairwise locally quasicompact space is a pairwise R_0 space.

PROOF:

In a pairwise locally quasicompact space X

$\tau_{i\infty}$- cl $\{x\} = \tau_i$- cl $\{x\}$

for each $x \in X$ and $i = 1,2$.

Also $\tau_{i\infty}\{\infty\} = \{\infty\}$

NOTE:

The converse of the above theorem does not hold in general.

If $(X_{\infty}, \tau_{1\infty}, \tau_{2\infty})$ is pairwise R_0, its subspace (X, τ_1, τ_2) is also pairwise R_0. But it need not be pairwise locally quasicompact.

COROLLARY : 4.35

If (X,τ_1,τ_2) is pairwise locally quasicompact and pairwise R_0, its quotient space is weak pairwise T_1 and pairwise locally quasicompact and $(\tilde{X}_\infty,\tilde{\tau}_{1\infty},\tilde{\tau}_{2\infty})$ is weak pairwise T_1

Summary and Conclusion

SUMMARY AND CONCLUSION

Kelly [6] initiated the study of separation properties for bitopological spaces and introduced the terms pairwise Hausdorff, Pairwise regular and pairwise normal.

In this thesis we have made an attempt to give a brief account of results dealing with pairwise rT_0, pairwise rT_1, pairwise rT_2, pairwise rR_0, pairwise rR_1, pairwise R_0 and pairwise R_1 spaces in bitopological spaces.

From the literature available in bitopological spaces, we find that bitopoligical spaces can be studied by investigating many topological properties like compactness, connectedness and countability etc. Such study will lead to many interesting research problems.

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