Chapter I

CHAPTER – I

CONNECTEDNESS IN FUZZY TOPOLOGICAL SPACES

In this chapter we discuss the concept of semi-connectedness in fuzzy topological spaces introduced by Ghosh [16]. He [16] has introduced the concept of fuzzy semi-separated sets by using the concepts of fuzzy semi-open sets and quasi-coincidence. Some properties of fuzzy semi-separated sets are discussed. Using this notion he [16] has defined the concept of fuzzy semi-connected sets as a generalisation of fuzzy connected sets introduced by Ganguly and Saha [15]. It has been observed that every fuzzy semi-connected set is a fuzzy connected set but the converse is not necessarily true. An example is given to illustrate this. Some interesting characterizations of fuzzy semi-connected sets are obtained. Finally, it is shown that fuzzy semi-connectedness is preserved under a fuzzy irresolute mapping. First let us give the preliminary definitions and results needed for our discussion.

Section 1.1

Preliminary definitions and results

Definition : 1.1.1

Let X be a non-empty set and I be the unit interval [0, 1], $\mathrm{I}_0 = (0, 1]$ and $\mathrm{I}_1 = [0, 1)$. A **fuzzy set** in X is a function with domain X and values in I, that is, an element of I^X. For $\alpha \in \mathrm{I}$, $\overline{\alpha}$ denotes the constant function α. That is, $\overline{\alpha}(x) = \alpha \ \forall \ x \in X$.

Notation : 1.1.2

By 0_X and 1_X we will mean the constant fuzzy sets $\overline{0}$ and $\overline{1}$ taking on, respectively, the values 0 and 1 on X.

Let A, B $\in \mathrm{I}^X$. We define the following fuzzy sets :

i. A includes B (i.e., $B \leq A$) by $B(x) \leq A(x)$ for every $x \in X$.

ii. $A \wedge B \in \mathrm{I}^X$ by $(A \wedge B)(x) = \min\{A(x), B(x)\}$ for each $x \in X$.

iii. $A \vee B \in I^X$ by $(A \vee B)(x) = \max\{A(x), B(x)\}$ for each $x \in X$.

iv. The complement A' of A by $A'(x) = (1_X - A)(x)$

$$= 1 - A(x).$$

Let Λ be an indexing set and $\{A_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy sets in X. Then their union (or join) and intersection (or meet) are defined as follows :

$$(\vee A_\alpha)(x) = \sup\{A_\alpha(x) : \alpha \in \Lambda\}$$

$$(\wedge A_\alpha)(x) = \inf\{A_\alpha(x) : \alpha \in \Lambda\}.$$

Definition : 1.1.3

Let A be a fuzzy set in X. The set $\{x \in X : A(x) > 0\}$, denoted by **supp A** or $\mathbf{A_0}$, is called the **support of A**.

Definition : 1.1.4

Let f be a function from X to Y. Let A be a fuzzy set in Y. Then the **inverse image** of A or **preimage** of A, written as $\mathbf{f^{-1}(A)}$, is a fuzzy set in X defined by

$$f^{-1}(A(x)) = A(f(x)) \text{ for all } x \in X$$

Conversely, let B be a fuzzy set in X. The image of B, written as **f(B)** is a fuzzy set in Y defined by

$$f(B)(y) = \sup_{z \in f^{-1}(y)} B(z), \text{ if } f^{-1}(y) \text{ is non empty}$$

$$= 0, \qquad \text{otherwise}$$

for all $y \in Y$, where $f^{-1}(y) = \{x : f(x) = y\}$.

Definition : 1.1.5

A **fuzzy point** x_α, where $\alpha \in (0, 1]$ is a fuzzy set in X defined by $x_\alpha(x) = \alpha$ and $x_\alpha(x) = 0$ if $y \neq x$. x is called the **support** of x_α and α its **value**.

Definition : 1.1.6

A **fuzzy point x_α is said to be contained in a fuzzy set A** or belong to A denoted by $\mathbf{x_\alpha \in A}$, iff $\alpha \leq A(x)$.

Definition : 1.1.7 (Chang [11])

A **fuzzy topology** on a set X is a collection δ of fuzzy sets in X satisfying the following axioms :

i. $0_X, 1_X \in \delta$

ii. $A, B \in \delta \Rightarrow A \wedge B \in \delta$

iii. $A_\alpha \in \delta$ for $\alpha \in \Lambda \Rightarrow \bigvee_{\alpha \in \Lambda} A_\alpha \in \delta$.

The pair (X, δ) is referred to as a **fuzzy topological space (fts**, for short).

Definition : 1.1.8

If (X, δ) is a fuzzy topological space, members of δ are called **open fuzzy sets**.

A fuzzy set A is called a **closed fuzzy set** iff $A' \in \delta$.

Notation : 1.1.9

If δ is a fuzzy topology on X, then δ' stands for the family of closed fuzzy sets in X.

Definition : 1.1.10

The **closure** and **interior** of a fuzzy set $A \in I^X$ are defined, respectively, as

$$\text{cl } A = \wedge \{B : B \geq A, B' \in \delta\}$$

$$\text{int } A = \vee \{B : B \leq A, B \in \delta\}$$

It is easily seen that cl A is the smallest closed fuzzy set larger than A and int A is the largest open fuzzy set smaller than A.

Properties : 1.1.11

For any fuzzy set A

i. $1_X - \text{cl } A = \text{int}(1_X - A)$

ii. $1_X - \text{int } A = \text{cl}(1_X - A)$

Definition : 1.1.12

A function f from a fuzzy topological space (X, δ_1) to a fuzzy topological space (Y, δ_2) is said to be **fuzzy continuous** if for each fuzzy open set A in (Y, δ_2) the inverse image $f^{-1}(A)$ is a fuzzy open set in X.

Definition : 1.1.13

A function f from a fuzzy topological space (X, δ_1) to a fuzzy topological space (Y, δ_2) is said to be **fuzzy open** if the image of each fuzzy open set A in (X, δ_1) is a fuzzy open set in (Y, δ_2).

Definition : 1.1.14

A fuzzy point x_α is said to be **quasi-coincident** with A denoted by $x_\alpha \, q \, A$ iff $\alpha > A'(x)$ or $\alpha + A(x) > 1$.

Definition : 1.1.15

A fuzzy set A is said to be **quasi-coincident (q-coincident** in short) with B, denoted by **A q B**, iff there exists $x \in X$ such that $A(x) > 1 - B(x)$ or $A(x) + B(x) > 1$.

It is clear that if A and B are q-coincident at x, both A(x) and B(x) are not zero and hence A and B intersect at x.

A fuzzy set A is **not q-coincident** with a fuzzy set B denoted by $\boldsymbol{A \not{q} B}$ iff for every $x \in X$, $A(x) + B(x) \leq 1$.

Properties : 1.1.16

i. $A \not{q} B$ implies $A \leq 1_X - B$

ii. $A \not{q} B$ and $V \leq B$ implies $A \not{q} V$

Definition : 1.1.17

A fuzzy set A in X is called a **fuzzy semi-open set** if there exists a fuzzy open set B such that $B \leq A \leq \mathrm{cl}\,(B)$ or equivalently, $A \leq \mathrm{cl}(\mathrm{int}\,(A))$.

The complement of a fuzzy semi-open set is called a **fuzzy semi-closed set**.

Definition : 1.1.18

The intersection of all fuzzy semi-closed sets containing A is called the **fuzzy semi-closure of A** and is denoted by **scl (A)**.

Definition : 1.1.19

The union of all fuzzy semi-open sets contained in a fuzzy set A in X is called the **fuzzy semi-interior of A** and is denoted by **sint (A)**.

Remark : 1.1.20

A fuzzy set A is **fuzzy semi-closed** if and only if $A = \mathrm{scl}\,(A)$ and is **fuzzy semi-open** if and only if $A = \mathrm{sint}\,(A)$.

Section : 1.2

Fuzzy semi-connectedness

Definition : 1.2.1

Two non-zero fuzzy sets A and B (i.e. neither A nor B is 0_X) in an fts X are **fuzzy separated** iff $A \not{q}\ \mathrm{cl}\,B$ and $B \not{q}\ \mathrm{cl}\,A$.

Definition : 1.2.2

Two non-zero fuzzy sets A and B in an fts X (i.e., neither A nor B is 0_X) are said to be **fuzzy semi-separated** iff $A \not{q}\ \mathrm{scl}\,B$ and $B \not{q}\ \mathrm{scl}\,A$.

Note

Since scl $A \leq clA$, for any fuzzy set A, if follows that if A, B are fuzzy separated then they are fuzzy semi-separated.

Theorem : 1.2.3

Let A, B be non-empty fuzzy sets in an fts(X, δ).

a. If A, B are fuzzy semi-separated and A_1, B_1 are nonempty fuzzy sets such that $A_1 \leq A$ and $B_1 \leq B$, then A_1 and B_1 are also fuzzy semi-separated.

b. If $A \not{q} B$ and either both are fuzzy semi-open or both are fuzzy semi-closed then A, B are fuzzy semi-separated.

c. If A, B are either both fuzzy semi-open or both fuzzy semi-closed and if $C_A(B) = A \wedge (1_X - B)$, $C_B(A) = B \wedge (1_X - A)$, then $C_A(B)$ and $C_B(A)$ are fuzzy semi-separated.

Proof :

a. Assume A and B are fuzzy semi-separated then $A \not{q}$ scl B and

$B \not{q}$ scl A

Since $A_1 \leq A$ and $B_1 \leq B$, $A_1 \not{q}$ scl B and $B_1 \not{q}$ scl A (1)

Since $A_1 \leq A$ and $B_1 \leq B$,

scl $A_1 \leq$ scl A and scl $B_1 \leq B$

From (1), we get

$A_1 \not{q}$ scl B_1 and $B_1 \not{q}$ scl A_1

so A_1 and B_1 are fuzzy semi-separated.

b. Given $A \not{q} B$

Case (i)

Assume A and B are both fuzzy semi-open

$\therefore$ $1_X - A$ and $1_X - B$ are fuzzy semi-closed.

Hence $1_X - A = \text{scl}\,(1_X - A)$

and $1_X - B = \text{scl}\,(1_X - B)$

Since $A \not{q} B$, $A(x) + B(x) \leq 1 \quad \forall\, x \in X$

$\Rightarrow A(x) \leq 1 - B(x)$

$\Rightarrow A \leq 1_X - B$

$\Rightarrow$ scl $A \leq 1_X - B$ as $1_X - B$ is a fuzzy semi-closed set containing A.

$\Rightarrow B(x) + \text{scl}\, A(x) \leq 1 \quad \forall\, x \in X$

$\Rightarrow B \not{q}$ scl A

Similarly $A \not{q}$ scl B.

Case (ii)

Assume A and B are both fuzzy semi-closed

$\therefore A = \text{scl}\, A$ and $B = \text{scl}\, B$

Since $A \not{q} B$, $A(x) + B(x) \leq 1 \quad \forall\, x \in X$

$\Rightarrow A(x) + \text{scl}\, B(x) \leq 1 \quad \forall\, x \in X$

$\Rightarrow A \not{q}$ scl B

Similarly $B \not{q}$ scl A

$\therefore$ In both the cases we get A and B are fuzzy semi-separated.

C.

Case (i)

If both A and B are fuzzy semi-open, $1_X - A$ and $1_X - B$ are fuzzy semi-closed

$\therefore \text{scl}\,(1_X - A) = 1_X - A$ and $\text{scl}\,(1_X - B) = 1_X - B$.

Since $C_A(B) = A \wedge (1_X - B)$, $C_A(B) \leq 1_X - B$

$\Rightarrow \text{scl}\,(C_A(B)) \leq$ as $1_X - B$ is a fuzzy semi-closed set containing $C_A(B)$

$\Rightarrow B + \text{scl}\,(C_A(B)) \leq 1_X$

$\Rightarrow B \not{q} \text{ scl}\,(C_A(B))$

Similarly, $A \not{q} \text{ scl}\,(C_B(A))$.

Case (ii)

If both A and B are fuzzy semi-closed, then A = scl A and B = scl B.

$C_A(B) \leq 1_X - B$

$\Rightarrow \text{scl}\,(C_A(B)) \leq 1_X - B$

$\Rightarrow B + \text{scl}\,(C_A(B)) \leq 1_X$

$B \not{q}\ \text{scl}\,(C_A(B))$

Similarly, $A \not{q}\ \text{scl}\,(C_B(A))$

$\therefore$ $C_A(B)$ and $C_B(A)$ are fuzzy semi-separated.

Theorem : 1.2.4

Two non-empty fuzzy sets A and B are fuzzy semi-separated iff there exist two fuzzy semi-open sets U and V such that $A \leq U$, $B \leq V$, $A \not{q} V$ and $B \not{q} U$.

Proof :

Given A and B are fuzzy semi-separated. By definition,

$A \not{q}\ \text{scl}\,B$ and $B \not{q}\ \text{scl}\,A$.

Hence $A + \text{scl}\,B \leq 1_X$ and $B + \text{scl}\,A \leq 1_X$

$\Rightarrow A \leq 1_X - \text{scl}\,B$ and $B \leq 1_X - \text{scl}\,A$

Take $U = 1_X - \text{scl}\,B$ and $V = 1_X - \text{scl}\,A$

Then U and V are fuzzy semi-open such that $A \leq U$ and $B \leq V$.

$$\begin{aligned} \text{Consider } A(x) + V(x) &= A(x) + 1 - (\text{scl}\,A)(x) \\ &\leq (\text{scl}\,A)(x) + 1 - (\text{scl}\,A)(x) \\ &= 1 \end{aligned}$$

$\therefore A \not{q} V$

Similarly, $B \not{q} U$

Conversely, let U and V be fuzzy semi-open sets such that $A \leq U$, $B \leq V$, $A \not{q} V$ and $B \not{q} U$.

Since $A \not{q} V$ and $B \not{q} U$, we get

$A \leq 1_X - V$ and $B \leq 1_X - U$.

Since scl A is the smallest fuzzy semi-closed set containing A and $1_X - V$ is a fuzzy semi-closed set containing A we get $\text{scl } A \leq 1_X - V$.

$\Rightarrow V + \text{scl } A \leq 1_X$

$\Rightarrow V \not{q} \text{ scl } A$

Similarly, $U \not{q} \text{ scl } B$

Since $A \leq U$ and $B \leq V$ we get $A \not{q} \text{ scl } B$ and $B \not{q} \text{ scl } A$

∴ A and B are fuzzy semi-separated.

Definition : 1.2.5

A fuzzy set A in an fts X is said to be **fuzzy connected** iff A cannot be expressed as the join of two fuzzy separated sets.

Definition : 1.2.6

A fuzzy set which cannot be expressed as the join of two fuzzy semi-separated sets is said to be a **fuzzy semi-connected set**.

Theorem : 1.2.7

Every fuzzy semi-connected set is fuzzy connected.

Proof

Let A be a fuzzy semi-connected set.

Suppose A is not fuzzy connected then $A = B_1 \vee B_2$, where B_1 and B_2 are fuzzy semi-separated sets.

∴ $B_1 \not{q} \text{ cl } B_2$ and $B_2 \not{q} \text{ cl } B_1$

Since $\text{scl } B_1 \leq \text{cl } B_1$ and $\text{scl } B_2 \leq \text{cl } B_2$, we get

$B_1 \not{q} \text{ scl } B_2$ and $B_2 \not{q} \text{ scl } B_1$

∴ B_1 and B_2 are fuzzy semi-separated sets.

∴ A is the union of two fuzzy semi-separated sets.

Hence A is not fuzzy semi-connected which is a contradiction.

$\therefore$ A is fuzzy connected.

The converse of the above results is not necessarily true as is seen from the following example.

Example : 1.2.8

A fuzzy set which is fuzzy connected but not fuzzy semi-connected

Let $X = [0, 1]$ and $\delta = \{0_X, 1_X, A\}$, where $A(0) = 0.3$, and $A(x) = 0$, for $x \in (0, 1]$. Then (X, δ) is a fuzzy topological space.

Consider the fuzzy sets B and C defined by $B(0) = 0.5$, $C(0) = 0.4$ and $B(x) = C(x) = 0$, for all $x \in (0, 1]$.

$A'(0) = 1 - 0.3 = 0.7$, $A'(x) = 1$ for $x \in (0, 1]$

Consider

$$\begin{aligned} \text{int } A' &= \vee\{S \,/\, S \text{ is a fuzzy open set and } s \leq A\} \\ &= \vee\{0_X, A\} \\ \text{int } A' &= A \end{aligned}$$

Since A is fuzzy open, A′ is fuzzy closed.

Also $\text{int } A' = A \leq B \leq A'$

$\therefore$ There exist a fuzzy closed set A′ such that $\text{int } A' \leq B \leq A'$

$\therefore$ By definition, B is fuzzy semi-closed

$\therefore$ $B = \text{scl } B$

Also $\text{int } A' = A \leq C \leq A'$

$\therefore$ C is fuzzy semi-closed

$\therefore$ $C = \text{scl } C$

$$\begin{aligned} \text{Consider } B(0) + (\text{scl } C)(0) &= 0.5 + 0.4 \\ &= 0.9 \\ &< 1 \end{aligned}$$

Take $x \in (0, 1]$

Then $B(x) + (\text{scl } c)(x) = 0 + 0$

$= 0$

< 1

$\therefore B \not q \text{ scl } C$

Similarly $C(0) + (\text{scl } B)(0) = 0.4 + 0.5$

$= 0.9$

< 1

Take $x \in (0, 1]$

Then $C(x) + (\text{scl } B)(x) = 0 + 0$

$= 0$

< 1

$\therefore C \not q \text{ scl } B$

$\therefore$ B and C are fuzzy semi-separated.

Consider $\text{cl } B = \wedge \{S / S \text{ is a fuzzy closed set and } S \geq B\}$.

$\delta' = \{0_X', 1_X', A'\}$

$= \{1_X, 0_X, A'\}$

$\text{cl } B = \wedge \{1_X, A'\}$

$= A'$

$\text{cl } C = \wedge \{S / S \text{ is a fuzzy closed set and } S \geq C\}$

$\text{cl } C = \wedge \{1_X, A'\}$

$= A'$

$\therefore \quad \text{cl } B = \text{cl } C = A'$

To verify $B \, q \text{ cl } C$ or not.

Consider $B(0) + (\text{cl } C)(0) = 0.5 + 0.7$

$= 1.2$

> 1

$\therefore B \, q \text{ cl } C$

$\therefore$ B and C are not fuzzy-separated.

Consider

$(B \vee C)(0) = \max\{B(0), C(0)\}$

$= \max\{0.5, 0.4\}$

$= 0.5$

Take $x \in (0, 1]$.

$(B \vee C)(x) = \max\{B(x), C(x)\}$

$= \max\{0, 0\}$

$= 0.$

$C(0) = 0.5 = (B \vee C)(0)$

$B(x) = 0 = (B \vee C)(x)$ for $x \in (0, 1]$

$\therefore B = B \vee C.$

Since B and C are fuzzy semi-separated, B is not fuzzy semi-connected.

Claim :

B is fuzzy connected.

Let $B = B_1 \vee B_2$ where B_1 and B_2 are fuzzy sets in X.

Since $B(0) = 0.5$,

$B_1 \vee B_2(0) = 0.5$

$\Rightarrow \max\{B_1(0), B_2(0)\} = 0.5$

$\therefore$ Either $B_1(0) = 0.5$ or $B_2(0) = 0.5$

suppose $B_1(0) = 0.5$

For $x \in (0, 1]$, $B(x) = 0$

$\therefore B_1(x) = 0$ and $B_2(x) = 0 \qquad \forall x \in (0, 1]$

$\mathrm{cl}\, B_1 = \wedge\{S / S \text{ is fuzzy closed and } S \geq B_1\}$

$= \wedge\{1_X, A'\}$

$= A'.$

$\mathrm{cl}\, B_2 = \wedge\{S / S \text{ is fuzzy closed and } S \geq B_2\}$

$= \wedge\{1_X, A'\}$

$= A'.$

To verify B q cl B_2 or not

Consider

$$\begin{aligned} B_1(0) + (\mathrm{cl}\, B_2)(0) &= 0.5 + 0.7 \\ &= 1.2 \\ &> 1. \end{aligned}$$

$\therefore$ B_1 q cl B_2.

$\therefore$ B_1 and B_2 are not fuzzy separated. Hence B is fuzzy connected.

Theorem : 1.2.9

Let A be a non-empty fuzzy semi-connected set in X. Then whenever A is contained in the join of two fuzzy semi-separated sets P and Q, exactly one of the following possibilities (a) and (b) holds : (a) $A \leq P$ and $A \wedge Q = 0_X$; (b) $A \leq Q$ and $A \wedge P = 0_X$.

Proof

Given $A \leq P \vee Q$ where P and Q are fuzzy semi-separated.

Assume $A \wedge Q = 0_X$

$\therefore$ Then $(A \wedge Q)(x) = 0, \quad \forall x \in X$

$\therefore$ Either $A(x) = 0$ or $Q(x) = 0$

If $A(x) = 0$ then as $P(x) \geq 0$ we get $A(x) \leq P(x)$

If $Q(x) = 0$ then as $A \leq P \vee Q$ we get

$$\begin{aligned} A(x) &\leq \max\{P(x), Q(x)\} \\ &= \max\{P(x), 0\} \\ &= P(x) \end{aligned}$$

$\therefore$ $A(x) \leq P(x)$.

$\therefore$ In both the cases $A(x) \leq P(x)$

Hence $A \leq P$.

Similarly, when $A \wedge P = 0_X$, we get $A \leq Q$.

Claim 1 :

Both (a) and (b) cannot hold simultaneously suppose $A \wedge P = 0_X$ and $A \wedge Q = 0_X$.

Since $A \neq 0_X$, there exists $x \in X$ such that $A(x) \neq 0$.

For this x, $(A \wedge P)(x) = 0_X(x)$

$\Rightarrow \min\{A(x), P(x)\} = 0$

$\Rightarrow P(x) = 0$, since $A(x) \neq 0$.

Similarly, $A \wedge Q = 0_X$ implies $Q(x) = 0$

$$\begin{aligned} \therefore (P \vee Q)(x) &= \max\{P(x), Q(x)\} \\ &= \max\{0, 0\} \\ &= 0. \end{aligned}$$

But $A(x) \neq 0$.

$\therefore A(x) > (P \vee Q)(x)$.

This is a contradiction since $A \leq P \vee Q$

$\therefore$ (a) and (b) cannot hold simultaneously.

Claim 2 :

Both (a) and (b) cannot be false simultaneously.

Suppose $A \wedge P \neq 0_X$ and $A \wedge Q \neq 0_X$. Since P and Q are fuzzy semi-separated and $A \wedge P \leq P$ and $A \wedge Q \leq Q$.

By Theorem 1.2.3 (a) we get $A \wedge P$ and $A \wedge Q$ are fuzzy semi-separated.

$$\begin{aligned} \text{Consider } (A \wedge P) \vee (A \wedge Q) &= A \wedge (P \vee Q) \\ &= A \text{ (since } A \leq P \vee Q). \end{aligned}$$

$\therefore$ A is not fuzzy semi connected which is a contradiction.

Hence (a) and (b) cannot be false simultaneously.

$\therefore$ By claims 1 and 2 we get that exactly one of the cases (a) and (b) holds.

Theorem : 1.2.10

A non-empty fuzzy set A in X is fuzzy semi-connected provided whenever A is contained in the join of two fuzzy semi-separated sets P and Q then either $A \wedge P = 0_X$ or $A \wedge Q = 0_X$.

Proof

Suppose A is not fuzzy semi-connected. Then by definition $A = B \vee C$ where B and C are fuzzy semi-separated.

By hypothesis either $A \wedge B = 0_X$ or $A \wedge C = 0_X$.

Suppose $A \wedge B = 0_X$.

Now $A(x) = 0 \Rightarrow B(x) = 0$ and $C(x) = 0$ since $A = B \vee C$

$A(x) \neq 0 \Rightarrow B(x) = 0$ since $A \wedge B = 0_X$.

$\therefore B(x) = 0 \quad \forall x$

$\therefore B = 0_X$ which is a contradiction.

Similarly, if we assume $A \wedge C = 0_X$ we get a contradiction.

$\therefore$ A is fuzzy semi-connected.

Theorem : 1.2.11

Let $\{A_\alpha : \alpha \in \lambda\}$ be a collection of fuzzy semi-connected sets in the fts X. Then $A = \vee \{A_\alpha : \alpha \in \lambda\}$ is fuzzy semi-connected provided there exists an $\alpha_0 \in \lambda$ such that either i) A_α and A_{α_0} are not fuzzy semi separated, for each $\alpha \in \lambda$ or ii) $A_\alpha \wedge A_{\alpha_0} \neq 0_X$ for each $\alpha \in \lambda$.

Proof

Given $\{A_\alpha : \alpha \in \lambda\}$ is a collection of fuzzy semi-connected sets. Also there exists $\alpha_0 \in \lambda$ such that that either

i) A_α and A_{α_0} are not fuzzy semi-separated for each $\alpha \in \lambda$

(or) ii) $A_\alpha \wedge A_{\alpha_0} \neq 0_X$ for each $\alpha \in \lambda$.

To prove :

$A = \bigvee \{A_\alpha : \alpha \in \lambda\}$ is fuzzy semi-connected. Suppose not, then $A = P \vee Q$ where P and Q fuzzy semi-separated.

Since $A_\alpha \leq A$ for each $\alpha \in \lambda$ we get by Theorem 1.2.9 either

(a) $A_\alpha \leq P$ with $A_\alpha \wedge Q = 0_X$ (or)

(b) $A_\alpha \leq Q$ with $A_\alpha \wedge P = 0_X$

Similarly,

either (c) $A_{\alpha_0} \leq P$ with $A_{\alpha_0} \wedge Q = 0_X$

(or) (d) $A_{\alpha_0} \leq Q$ with $A_{\alpha_0} \wedge P = 0_X$

Without loss of generality, we can assume each A_α ($\alpha \in \lambda$) to be non-zero and hence exactly one of the possibilities (a) and (b), and exactly one of (c) and (d) will hold.

By condition (ii) $A_\alpha \wedge A_{\alpha_0} \neq 0_X$ for each $\alpha \in \lambda$. Hence there exists atleast one $x_0 \in X$ such that $A_\alpha(x_0) \neq 0$ and $A_{\alpha_0}(x_0) \neq 0$.

Claim : 1

(a) and (d) cannot happen simultaneously.

Suppose (a) and (d) are true.

By (a) $A_\alpha \wedge Q = 0_X$

$\Rightarrow Q(x_0) = 0$ since $A_\alpha(x_0) \neq 0$

By (d) $A_{\alpha_0} \leq Q$

$\Rightarrow A_{\alpha_0}(x_0) \leq Q(x_0)$

$\Rightarrow A_{\alpha_0}(x_0) \leq 0$

$\Rightarrow A_{\alpha_0}(x_0) = 0.$

This is a contradiction

Hence claim (1).

Claim : 2

(b) and (c) cannot happen simultaneously.

Proof similar to claim 1.

Hence by claims (1) and (2) we get that for condition (ii) the possibilities (a) and (d) cannot happen and (b) and (c) cannot hold simultaneously.

By condition (i), A_α and A_{α_0} are not fuzzy semi-separated for each $\alpha \in \lambda$.

Claim : 3

(a) and (d) cannot hold simultaneously

Suppose not, by (a) $A_\alpha \leq P$

By (d) $A_{\alpha_0} \leq Q$

Since P and Q are fuzzy semi-separated. We get A_α and A_{α_0} are fuzzy semi-separated.

This is a contradiction.

Hence claim (3).

Claim : 4

(b) and (c) cannot hold.

Proof similar to claim 3.

Hence by claims (3) and (4) we get for condition (i) also the possibilities (a) and (d) cannot happen and (b) and (c) cannot hold simultaneously.

Thus in any case either $A_\alpha \leq P$ with $A_\alpha \wedge Q = 0_X$ or $A_\alpha \leq Q$ with $A_\alpha \wedge P = 0_X$ but not both simultaneously.

Suppose $A_\alpha \leq P$ with $A_\alpha \wedge Q = 0_X$ for each $\alpha \in \Lambda$.

Then $\vee A_\alpha \leq P$ and $\vee A_\alpha \wedge Q = 0_X$

Hence $A \leq P$ and $A \wedge Q = 0_X$

$\therefore Q = 0_X$

This is a contradiction.

Similarly, $A_\alpha \leq Q$ and $A_\alpha \wedge P = 0_X \quad \forall \alpha \in \Lambda$

Implies $P = 0_X$ which is again a contradiction.

Theorem : 1.2.12

Let A be a fuzzy set in an fts X such that there exists at least one point $x \in X$ with $A(x) > 1/2$. Then A is fuzzy semi-connected iff any two fuzzy points of A are contained in a fuzzy semi-connected set contained in A.

Proof

Assume A is fuzzy semi-connected. Then the statement holds since A itself is a fuzzy semi-connected set.

Conversely, assume that the condition in the statement holds.

Take $x \in X$ such that $A(x) > 1/2$

Take $y \in A_0 - \{x\}$

$\therefore$ $y \neq x$ and $A(x) > 0$

Let $A(x) = \alpha$ and $A(y) = \beta$

Consider the two fuzzy points x_α and y_β. Then $x_\alpha \in A$ and $y_\beta \in A$.

Hence by hypothesis there exists a fuzzy semi-connected set.

$B_y \leq A$ such that $x_\alpha, y_\beta \in B_y$

$x_\alpha \in B_y$ implies $\alpha \leq B_y(x) \leq (\vee B_y)(x)$

i.e., $A(x) \leq (\vee B_y)(x)$

$y_\beta \in B_y$ implies $\beta \leq B_y(y) \leq (\vee B_y)(y)$

i.e., $A(y) \leq (\vee B_y)(y)$

$\therefore$ $\forall z \in X$, $A(z) \leq (\vee B_y)(z)$ (1)

$A \leq \vee B_y$

Since $B_y \leq A$, $\vee B_y \leq A$

Hence $A = \vee \{B_y / y \in A_0 - \{x\}$ (2)

Since $A(x) > 1/2$, $(\vee B_y)(x) > 1/2$

By sup property there exists $y_0 \in A_0 - \{x\}$ such that $1/2 < B_{y_0}(x)$

Since $x_\alpha \in B_y$, $\alpha \leq B_y(x)$

Since $A(x) = \alpha$ and $A(x) > 1/2$

We get $\alpha > ½$

$\therefore B_y(x) \geq \alpha > ½$

$\min\{B_y(x), B_{y_0}(x)\} > 1/2$

Hence $(B_y \wedge B_{y_0})(x) > ½$

$\therefore B_y \wedge B_{y_0} \neq 0_x \qquad \forall\, y \in A_0 - \{x\}$ (3)

$\therefore$ From (2) and (3) and by Theorem 1.2.11 (ii) we get A is fuzzy semi-connected.

Corollary : 1.2.13

An fts X is fuzzy semi-connected iff every pair of fuzzy points is contained in a fuzzy semi-connected set.

Theorem : 1.2.14

Let f be a fuzzy irresolute mapping from an fts X onto an fts Y. If A is a fuzzy semi-connected set in X, then so is f(A) in Y.

Proof

Given $f : X \rightarrow Y$ is a fuzzy irresolute mapping.

Given A is a fuzzy semi-connected set in X.

To prove : f(A) is fuzzy semi-connected in Y.

Suppose not, $f(A) = B \vee C$ where B and C are fuzzy semi-separated sets in Y. By Theorem 1.2.4 there exist two fuzzy semi-open sets U and V such that $B \leq U$, $C \leq V$, $B \not{q}\, V$, $C \not{q}\, U$

Since f is fuzzy irresolute $f^{-1}(B)$ and $f^{-1}(C)$ are fuzzy semi-open sets in x

Since f is onto, $A = f^{-1}(f(A))$

$= f^{-1}(B \vee C)$

$= f^{-1}(B) \vee f^{-1}(C)$

Claim : 1

$f^{-1}(B)$ and $f^{-1}(C)$ are fuzzy semi-separate in X.

Since $B \leq U$ and $C \leq U$ we get

$f^{-1}(B) \leq f^{-1}(U)$ and $f^{-1}(C) \leq f^{-1}(V)$ (1)

Take $x \in X$

$$\text{Consider } f^{-1}(B)(x) + f^{-1}(V)(x) = B(f(x)) + V(f(x))$$

$$\leq 1 \; [\because B \not q V]$$

$\therefore f^{-1}(B) \not q f^{-1}(V)$ (2)

Similarly,

$f^{-1}(C) \not q f^{-1}(U)$ (3)

Since f is fuzzy irresolute and U and V are fuzzy semi-open sets we get $f^{-1}(U)$ and $f^{-1}(V)$ are fuzzy semi-open sets (4)

From (1), (2), (3) and (4) by Theorem 1.2.4 we get that $f^{-1}(B)$ and $f^{-1}(C)$ are fuzzy semi-separated.

Also $A = f^{-1}(B) \vee f^{-1}(C)$

$\therefore$ A is not fuzzy semi-connected.

This is a contradiction.

Hence f(A) is fuzzy semi-connected.