CHAPTER - VI

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πgb*-compact spaces and πgb*-neighborhoods

Introduction

In this chapter the idea of πgb*- compact spaces, πgb*-regular spaces, πgb*-normal spaces and πgb*-Hausdorff spaces were discussed its properties were studied. The chapter was concluded with the concept of πgb*-neighborhood at a point.

Section 1

Preliminaries

Definition 6,1

A topological space X is called **Hausdorff space** if for each pair x and y of distinct points of X, there exist disjoint open sets Uand V such that $x \in U$ and $y \in V$.

Definition 6.2

A topological space X is called **regular** if for each $x \in X$ and open set G not containing x there exist disjoint open sets Uand V such that $x \in U$ and $G \subset V$.

Definition 6.3

The topological space (X, τ) is said to be **normal** if for every pair of disjoint closed sets G and H in X there exist disjoint open sets U and V in X such that $G \subset U$ and $H \subset V$.

Section 2

πgb*-compact spaces

In this section the properties and characterizations of πgb*-compact spaces were discussed and its behavior under surjective πgb*-continuous function were analyzed. Further the concept of πgb*-regular spaces, πgb*-Hausdorff spaces and πgb*-normal spaces were discussed.

Definition 6.2.1

Let (X,τ) be a topological space. Then $\{G_i : i \in I\}$, where each G_i is πgb*-open in X is said to be **πgb*-open cover** of X if, $X \subset \bigcup_{i \in I} G_i$

Definition 6.2.2

A topological space (X, τ) is said to be **πgb*-compact** if every πgb*-open cover of X has a finite subcover.

Definition 6.2.3

A set A of a topological space (X, τ) is said to be **πgb*-compact relative to X** if every open cover of A by sets which are πgb*-open in X has a finite subcover.

Definition 6.2.4

A set A of a topological space (X, τ) is said to be **πgb*-compact** if A is πgb*-compact as a subspace of X.

Theorem 6.2.5

Every finite subset of a topological space (X, τ) is πgb*-compact.

Proof

Let A = $\{x_1, \ldots, x_n\}$ be a finite subset of (X, τ).

Let $\{U_{\alpha_j} : j\in I\}$ be a πgb*-open cover of A in X, then $A \subset \bigcup_{j \in I} U_{\alpha_j}$.

Thus for each I = 1, …, n, $x_i \in U_{\alpha_j}$ for some j.

Therefore $A \subset \bigcup_{j=1}^{n} U_{\alpha_j}$.

Hence A is compact.

Theorem 6.2.6

Every closed (pre-closed, semi-closed, b-closed) subset of a πgb*-compact space is πgb*-compact.

Proof

Let (X, τ) be a πgb*-compact space and let A be a closed (pre-closed, semi-closed, b-closed) subset of X.

Consider a πgb*-open cover $\{ U_i : i\in I \}$ of A.

Since A is closed (pre-closed, semi-closed, b-closed),

(X-A) is open (pre-open, semi-open, b-open).

As every open (pre-open, semi-open, b-open) set is πgb*-open,

(X-A) is πgb*-open.

Then(X-A) ∪ $(\bigcup_{i\in I} U_i)$ is a πgb*-open cover of (X, τ).

Since (X, τ) is πgb*-compact, every πgb*-open cover of X has a finite subcover.

Therefore X ⊂ (X-A) ∪ $(\bigcup_{i\in I_0,} U_i)$, where I_0 is a finite subset of I.

Hence A $\subset \bigcup_{i\in I_0} U_i \Rightarrow$ A is compact.

Theorem 6.2.7

Every g-closed (gp-closed, gs-closed, gb-closed) subset of a πgb*-compact space is πgb*-compact.

Proof

Let (X, τ) be a πgb*-compact space and let A be a g-closed (gp-closed, gs-closed, gb-closed) subset of X.

Consider a πgb*-open cover { U_i : i∈ I } of A.

Since A is g-closed (gp-closed, gs-closed, gb-closed),

(X-A) is g-open (gp-open, gs-open, gb-open).

As every g-open (gp-open, gs-open, gb-open) set is πgb*-open,

(X-A) is πgb*-open.

Then(X-A) ∪ ($\bigcup_{i\in I} U_i$) is a πgb*-open cover of (X, τ).

Since (X, τ) is πgb*-compact,

every πgb*-open cover of X has a finite subcover.

Therefore X ⊂ (X-A) ∪ ($\bigcup_{i\in I_0,} U_i$), where I_0 is a finite subset of I.

Hence A $\subset \bigcup_{i\in I_0} U_i \Rightarrow$ A is compact.

Theorem 6.2.8

Every π-closed (πgp-closed, πgs-closed, πgb-closed) subset of a πgb*-compact space is πgb*-compact.

Proof

Let (X, τ) be a πgb*-compact space and let A be a π-closed (πgp-closed, πgs-closed, πgb-closed) subset of X.

Consider a πgb*-open cover {U_i : i∈ I } of A.

Since A is π-closed (πgp-closed, πgs-closed, πgb-closed),

(X-A) is π-open (πgp-open, πgs-open, πgb-open).

As every π-open (πgp-open, πgs-open, πgb-open) set is πgb*-open,

(X-A) is πgb*-open.

Then(X-A) ∪ ($\bigcup_{i\in I} U_i$) is a πgb*-open cover of (X, τ).

Since (X, τ) is πgb*-compact, every πgb*-open cover of X has a finite subcover.

Therefore $X \subset (X-A) \cup (\bigcup_{i\in I_0} U_i)$, where I_0 is a finite subset of I.

Hence $A \subset \bigcup_{i\in I_0} U_i \Rightarrow$ A is compact.

Theorem 6.2.9

Every πgb*-closed set of a πgb*-compact space is πgb*-compact.

Proof

Let (X, τ) be a πgb*-compact space and let A be a πgb*-closed subset of X.

Consider a πgb*-open cover $\{U_i : i\in I\}$ of A.

Since A is πgb*-closed, (X-A) is πgb*-open.

Then, $(X-A) \cup (\bigcup_{i\in I} U_i)$ is a πgb*-open cover of (X, τ).

Since (X, τ) is πgb*-compact, the πgb*-open cover $(X-A) \cup (\bigcup_{i\in I} U_i)$ of X has a finite subcover.

Therefore, $X \subset (X-A) \cup (\bigcup_{i\in I_0} U_i)$, where I_0 is a finite subset of I.

Hence $A \subset \bigcup_{i\in I_0} U_i$. Thus A is πgb*-compact.

Theorem 6.2.10

Let X be a πgb*-compact space and $f : X \longrightarrow Y$ be a surjective πgb*-continuous map then, Y is compact.

Proof

Let $f : X \longrightarrow Y$ be a surjective πgb*-continuous function and let $\{U_\alpha : \alpha \in I\}$ be any open cover of Y then $f^{-1}(U_\alpha)$ is πgb*-open in X.

Thus $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is a πgb*-open cover of X.

Since X is compact there exists a finite subset $I_0 \subset I$ such that $X = \bigcup_{\alpha\in I_0} f^{-1}(U_\alpha)$.

Since f is surjective, $Y = \bigcup_{\alpha\in I_0}(U_\alpha)$.

Hence Y is compact.

Theorem 6.2.11

If $f : X \longrightarrow Y$ is πgb*-irresolute and a subset $A \subset X$ is πgb*-compact relative to X then, the image f(A) is πgb*-compact relative to Y

Proof

Let $\{ U_i : i\in I \}$ be any πgb*-open cover of f(A) in Y, thus $f(A) \subset \bigcup_{i\in I} U_i$.

Since f is πgb*-irresolute, $f^{-1}(U_i)$ is πgb*-open for every $i \in I$.

Hence $\{f^{-1}(U_i) : i \in I\}$ is an πgb*-open cover of A.

Since A is πgb*-compact, there exist a finite subset $I_0 \subset I$ such that $A \subset \bigcup_{i \in I_0} f^{-1}(U_i)$.

Therefore $f(A) \subset \bigcup_{i \in I_0} U_i \Rightarrow f(A)$ is πgb*-compact in Y.

Theorem 6.2.12

If $f : X \rightarrow Y$ is a πgb*-irresolute bijection and X is a πgb*-compact space then Y is πgb*-compact.

Proof

Let $\{ U_i : i \in I \}$ be any πgb*-open cover of Y.

Then $Y = \bigcup_{i \in I} U_i$ and $X = f^{-1}(Y) = f^{-1}(\bigcup_{i \in I} U_i)) = \bigcup_{i \in I} f^{-1}(U_i)$.

Since f is πgb*-irresolute, $f^{-1}(U_i)$ is πgb*-open for every $i \in I$.

Hence $\{f^{-1}(U_i): i \in I\}$ is an πgb*-open cover of X.

Since X is πgb*-compact, there exist a finite subset $I_0 \subset I$ such that $X= \bigcup_{i \in I_0} f^{-1}(U_i)$.

Therefore, $Y = f(X) = f(\bigcup_{i \in I_0} f^{-1}(U_i)) = \bigcup_{i \in I_0} f(f^{-1}(U_i)) = \bigcup_{i \in I_0} U_i$.

Hence Y is πgb*-compact.

Theorem 6.2.13

If $f : X \rightarrow Y$ is πgb*-open bijection and Y is πgb*-compact then X is πgb*-compact.

Proof

Let $\{ U_i : i \in I \}$ be any πgb*-open cover of X,

thus $Y = f(X) = f(\bigcup_{i \in I} U_i)) = \bigcup_{i \in I} f(U_i)$.

Since f is πgb*-open, $f(U_i)$ is πgb*-open for every $i \in I$.

Hence $\{f(U_i): i \in I\}$ is an πgb*-open cover of Y.

Since Y is πgb*-compact, there exists a finite subset $I_0 \subset I$ such that $Y = \bigcup_{i \in I_0} f(U_i)$.

Therefore $X = f^{-1}(Y) = f^{-1}(\bigcup_{i \in I_0} f(U_i)) = \bigcup_{i \in I_0} f^{-1}(f(U_i)) = \bigcup_{i \in I_0} U_i$.

Hence X is πgb*-compact.

Theorem 6.2.14

If $f : X \rightarrow Y$ is πgb*-irresolute then f^{-1} is πgb*-open.

Proof

Let f : X $\rightarrow$Y be a πgb*-irresolute and U be πgb*-open in Y. Since f is πgb*-irresolute, $f^{-1}(U)$ is πgb*-open in X. Hence f^{-1} is πgb*-open.

Definition 6.2.15

A topological space (X, τ) is said to be **πgb*-Hausdorff** if for every pair of distinct points x and y in X there exist disjoint πgb*- open sets U and V in X such that x$\in$ U and y$\in$Y.

Definition 6.2.16

The topological space (X, τ) is said to be **πgb*-regular** if for every x $\in$ X and π-closed set G not containing x there exist disjoint πgb*-open sets U and V in X such that x $\in$ X and G $\subset$V.

Definition 6.2.17

The topological space (X, τ) is said to be **πgb*-normal** if for every pair of disjoint π-closed sets G and H in X there exist disjoint πgb*-open sets U and V in X such that G $\subset$ U and H $\subset$ V.

Theorem 6.2.18

Every finite point set in a πgb*-Hausdorff space is πgb*-closed.

Proof

It is enough to show that every one point set $\{x_0\}$ is πgb*-closed.

Let x $\in$ X such that x $\neq$ x_0.

Since X is πgb*-Hausdorff there exists disjoint πgb*-open sets U and V in X such that x_0 $\in$ U and x $\in$ V.

Clearly V does not intersect $\{x_0\}$.

Therefore x $\notin$ πgb*-cl($\{x_0\}$), hence πgb*-cl($\{x_0\}$) = $\{x_0\}$.

Thus $\{x_0\}$ is πgb*-closed.

Theorem 6.2.19

Every b-normal space is πgb*-normal.

Proof

Let X be b-normal and let A and B be two π-closed sets in X.

Since every π-closed set iss closed and X is b-normal there exist disjoint b-open sets U and V in X such that A $\subset$ U and B $\subset$ V.

Since every b-open sets is πgb^*-open we get, $A \subset U$ and $B \subset V$ where U and V are πgb^*-open.

Hence X is πgb^*-normal.

Section 3

πgb*-NEIGHBORHOODS

In this section πgb^*-neighborhood at a point were discussed and its properties were analyzed.

Definition 6.3.1

A subset N of a space X is called an **πgb*-neighborhood** (briefly, πgb^*-nbd) of $x \in X$ if there exists a πgb^*-open set G containing x such that $x \in G \subset N$.

Theorem 6.3.2

Every neighborhood N of $x \in X$ is a πgb^*-nbd of x.

Proof

Let N be a nbd of $x \in X$, then there exists an open set G such that $x \in G \subset N$.

Since every open set is πgb^*-open, G is πgb^*-open.

Hence N is a πgb^*-nbd of x.

Remark 6.3.3

The converse of above theorem need not be true as seen from the following example.

Example 6.3.4

Let X = {a, b, c} with the topology τ = {X, ϕ , {a}, {a, b}, {a, c}} then{b} is πgb^*-nbd of b but not a neighborhood of b.

Theorem 6.3.5

Let X be a topological space. If F is a πgb^*-closed subset of X and $x \in F^c$. Then there exists a πgb^*-nbd N of x such that $N \cap F = \phi$.

Proof

Let F be a πgb^*-closed subset of X, then F^c is πgb^*-open.

Therefore F^c is πgb^*-nbd of each of its points.

Hence there exists a πgb^*-nbd N of x such that $x \in N \subset F^c$.

Thus $N \cap F = \phi$.

Definition 6.3.6

Let x be a point in a topological space X. The set of all πgb*-nbd of x is called the **πgb*-nbd system at x** and is denoted by **πgb*-N(x)**.

Theorem 6.3.7

Let X be a topological space and N be a πgb*-nbd of x $\in$ X. Let πgb*-N(x) denote the neighborhood system at x. Then we have the following results.

1) $\forall$ x $\in$ X, πgb*-N(x) $\neq \phi$.
2) N $\in$ πgb*-N(x) $\Rightarrow$ x $\in$ N.
3) N $\in$ πgb*-N(x), M $\supset$ N $\Rightarrow$ M $\in$ πgb*-N(x).
4) L $\in$ πgb*-N(x) $\Rightarrow$ there exists M $\in$ πgb*-N(x) such that M $\subset$ L and M $\in$ πgb*-N(y) $\forall$ y $\in$ M.

Proof

1) In every topological space the set X is πgb*-open and every πgb*-open set is a πgb*-nbd of each of its points, πgb*-N(x) $\neq \phi$.
2) Let N $\in$ πgb*-N(x).

   Then N is a πgb*-nbd of x.

   Thus, clearly x $\in$ N.
3) Let N $\in$ πgb*-N(x) and M $\supset$ N.

   Since N is πgb*-nbd of x there exists a πgb*-open set L such that L $\subset$ N and since N $\subset$ M we have L $\subset$ N $\subset$ M.

   Thus M is a πgb*-nbd of x.

   Hence M $\in$ πgb*-N(x).
4) Let N $\in$ πgb*-N(x).

   Then there exists a πgb*-open set M such that M $\subset$ N.

   Since M is πgb*-open, it is a πgb*-nbd of each of its points.

   Therefore M $\in$ πgb*-N(y) $\forall$ y $\in$ M.