CHAPTER III

CHAPTER-III

ON BITOPOLOGICAL $(1, 2)^*$ -GENERALIZED HOMEOMORPHISMS

SECTION-3.1

In this chapter the notion of $(1, 2)^*$ g-homeomorphism , $(1, 2)^*$ gc-homeomorphism, $(1, 2)^*$ gs- homeomorphism, $(1, 2)^*$ gsc- homeomorphism Ravi , Pious Missier and Salai Parkunan[41] are discussed. Properties, characterizations and applications are analysed.

Preliminaries:

Definition :3.1.1

A map $f: X \to Y$ is called **pre-$(1,2)^*$ -semi-open** if f(U) is $(1, 2)^*$ -semi-open of Y for every $(1, 2)^*$ -semi-open set U in X.

Definition :3.1.2

A map $f: X \to Y$ is called **pre-$(1,2)^*$ -semi-closed** if f(U) is $(1, 2)^*$ -semi-closed of Y for every $(1, 2)^*$ - semi-closed set U in X.

Definition :3.1.3

A bijection $f: X \to Y$ is called **$(1, 2)^*$ -homeomorphism** if f is bijection, $(1, 2)^*$ -continuous and $(1, 2)^*$ -open;

Definition :3.1.4

A bijection $f: X \to Y$ is called **$(1,2)^*$ -generalized homeomorphism** (briefly $(1,2)^*$-ghomeomorphism) if f is both $(1, 2)^*$ - g-continuous and $(1, 2)^*$- gopen.

Definition :3.1.5

A bijection $f: X \to Y$ is called **$(1,2)^*$ -gc-homeomorphism** if f is $(1, 2)^*$ -gc-irresolute and its inverse f^{-1} is also $(1, 2)^*$ -gc-irresolute.

Definition :3.1.6

A bijection $f : X \to Y$ is called **$(1,2)^*$-generalized semi-homeomorphism** (briefly $(1, 2)^*$-gshomeomorphism) if f is both $(1, 2)^*$-gs-continuous and $(1, 2)^*$-gs open.

Definition :3.1.7

A bijection $f : X \to Y$ is called **$(1,2)^*$-gsc-homeomorphism** if f is $(1, 2)^*$-gs-irresolute and its inverse f^{-1} is $(1, 2)^*$-gs-irresolute.

Definition :3.1.8

A map $f: X \to Y$ is called **$(1, 2)^*$-semi-irresolute** if $f^{-1}(V)$ is $(1, 2)^*$-semi-open in X for every $(1, 2)^*$-semi-open set V in Y.

SECTION – 3.2
CHARACTERIZATIONS AND PROPERTIES

Proposition :3.2.1

For any bijection $f: X \to Y$ the following statements are equivalent.

(a) $f^{-1}: Y \to X$ is $(1, 2)^*$-gs-continuous.

(b) f is $(1, 2)^*$-gs-open.

(c) f is $(1, 2)^*$-gs-closed.

Proof:

(a)$\Rightarrow$(b).

Let F be any $\tau_{1,2}$-open set of X.

Then X–F is $\tau_{1,2}$-closed in X.

Since f^{-1} is $(1, 2)^*$-gs-continuous, $(f^{-1})^{-1}(X–F) = f(X–F) = Y–f(F)$ is $(1, 2)^*$-gs-closed in Y. Then f (F) is $(1, 2)^*$-gs-open in Y.

Hence f is $(1, 2)^*$-gs-open.

(b)$\Rightarrow$(c)

Let F be any $\tau_{1,2}$-closed set in X. Then X–F is $\tau_{1,2}$-open in X.

Since f is $(1, 2)^*$-gs-open, f (X–F) =Y–f (F) is $(1, 2)^*$-gs-open in Y.

Then f(F) is $(1, 2)^*$-gs-closed in Y.

Hence f is $(1, 2)^*$-gs-closed.

(c) $\Rightarrow$ (a).

Let V be any $\tau_{1,2}$-closed set in X.

Since f : X $\rightarrow$ Y is $(1, 2)^*$-gs-closed, f(V) is $(1, 2)^*$-gs-closed in Y.

(i.e) $(f^{-1})^{-1}(V)$ is $(1, 2)^*$-gs-closed in Y.

Hence f^{-1} is $(1, 2)^*$-gs-continuous.

Proposition :3.2.2

Let f: X $\rightarrow$ Y be a bijective and $(1, 2)^*$-gs-continuous map.

Then the following statements are equivalent.

(a) f is $(1, 2)^*$-gs-open.

(b) f is $(1, 2)^*$-gs-homeomorphism.

(c) f is $(1, 2)^*$-gs-closed.

Proof:

(a)$\Rightarrow$(b).

Given f is bijective, $(1, 2)^*$-gs-continuous and $(1, 2)^*$-gs-open.

Hence f is $(1, 2)^*$-gs-homeomorphism.

(b)$\Rightarrow$(c)

Let f be $(1, 2)^*$-gs-homeomorphism. Hence f is $(1, 2)^*$-gs-open.

By Proposition 3.2.1, f is $(1, 2)^*$-gs-closed.

(c)$\Rightarrow$(a)

Follows from Proposition 3.2.1

Proposition :3.2.3

Every $(1, 2)^*$-sg-closed set is $(1, 2)^*$- gs-closed set.

Proof:

Let F be $\tau_{1,2}$-open set of X such that $S \subset F$.

Then F is $(1, 2)^*$-semi-open such that $S \subset F$.

Since S is $(1, 2)^*$-sg-closed, $(1, 2)^*$-scl(S)$\subset$F.

Thus S is $(1, 2)^*$-gs - closed.

Remark :3.2.4

A bijection f: $X \to Y$ is pre-$(1, 2)^*$-semi-open if and only if f is pre-$(1, 2)^*$-semi-closed.

Theorem :3.2.5

If a map f: $X \to Y$ is $(1, 2)^*$-semi-irresolute and pre-$(1, 2)^*$-semiclosed, Then

(a) For every $(1, 2)^*$-sg-closed set A of Y, $f^{-1}(A)$ is $(1, 2)^*$-sg-closed set in X and

(b) For every $(1, 2)^*$-sg-closed set B of X, f (B) is $(1, 2)^*$- sg-closed set in Y.

Proof:

(a) Let A be a $(1, 2)^*$-sg-closed set of Y.

Suppose that $f^{-1}(A) \subset O$ where O is $(1, 2)^*$-semi-open in X.

Since f is $(1, 2)^*$-semi-irresolute, $f((1, 2)^*\text{-scl}(f^{-1}(A)) \cap (X \setminus O)) \subset (1, 2)^*\text{-scl}(f(f^{-1}(A))) \cap f(f^{-1}(Y \setminus A)) \subset (1, 2)^*\text{-scl}(A) \setminus A$.

This means that $(1, 2)^*$-scl(A)$\setminus$A contains a $(1, 2)^*$-semi-closed subset $f((1, 2)^*\text{-scl}(f^{-1}(A) \cap (X \setminus O))$,

Since f is pre-$(1, 2)^*$-semi-closed , $f((1, 2)^*\text{-scl}(f^{-1}(A) \cap (X \setminus O)) = \varphi$ and hence $(1, 2)^*$-scl$(f^{-1}(A)) \subset O$.

This implies that $f^{-1}(A)$ is $(1, 2)^*$-sg-closed in X.

(b). Let B be a $(1, 2)^*$-sg-closed set in X.

Let $f(B) \subset O$ where O is any $(1, 2)^*$-semi-open set of Y.

Then, $B \subset f^{-1}(O)$ holds, and $f^{-1}(O)$ is $(1, 2)^*$-semi-open in X because f is $(1, 2)^*$-semi-irresolute.

Since B is $(1, 2)^*$-sg-closed, $(1, 2)^*$-scl(B) $\subset f^{-1}(O)$, and hence $f((1, 2)^*$-scl(B)) $\subset O$.
Since $(1, 2)^*$-scl(B) is $(1, 2)^*$-semi-closed set in X and f is pre-$(1, 2)^*$-semi-closed, $f((1, 2)^*$-scl(B)) is $(1, 2)^*$-semi-closed in Y.
Then $(1, 2)^*$-scl(f($(1, 2)^*$-scl(B))) = f($(1, 2)^*$-scl(B)).
Therefore, $(1, 2)^*$-scl(f(B)) $\subset$ $(1, 2)^*$-scl(f($(1, 2)^*$-scl(B))) = f($(1, 2)^*$-scl(B)) $\subset O$.

Hence f(B) is $(1, 2)^*$-sg-closed in Y.

Theorem :3.2.6

(a) If f: X $\rightarrow$ Y is $(1, 2)^*$-semi-irresolute and pre-$(1, 2)^*$-semi-closed, then for every $(1, 2)^*$-sg-closed set A of Y, $f^{-1}(A)$ is $(1, 2)^*$-gs-closed.

(b) If f: X $\rightarrow$ Y is $(1, 2)^*$-continuous and pre-$(1, 2)^*$-semi-closed, then for every $(1, 2)^*$-gs-closed set A of X, f(A) is $(1, 2)^*$-gs-closed.

Proof :

(a) Let A be $(1, 2)^*$-sg-closed set in Y. By Theorem 4.2.7, $f^{-1}(A)$ is $(1, 2)^*$-sg-closed set in X. By Proposition 3.2.3, $f^{-1}(A)$ is $(1, 2)^*$-gs-closed set in X.

(b). Let O be a $\sigma_{1,2}$-open set of Y such that f(A) $\subset$ O. Then A $\subset f^{-1}(O)$ implies $(1, 2)^*$-scl(A) $\subset f^{-1}(O)$ since A is $(1, 2)^*$-gs-closed and $f^{-1}(O)$ is $\tau_{1,2}$-open in X.
Since f is pre-$(1, 2)^*$-semi-closed, $(1, 2)^*$-scl[f($(1, 2)^*$-scl(A))] = f[$(1, 2)^*$-scl(A)] $\subset$ O and hence $(1, 2)^*$-scl(f(A)) $\subset$ O.

Therefore f(A) is $(1, 2)^*$-gs-closed set.

Remark :3.2.7

The union of two disjoint $(1, 2)^*$-gs-open sets is not, in general, $(1, 2)^*$-gs-open

Theorem :3.2.8

Let S be a subset of X. Then $(1, 2)^*$-scl(S)=S $\cup$ $\tau_{1,2}$-int($\tau_{1,2}$-cl(S)).

Proposition :3.2.9

If A is $\tau_{1,2}$-open and $(1, 2)^*$-gs-closed set in X, then A is $(1, 2)^*$ - semi-closed.

Proof:

Since A is $\tau_{1,2}$- open and $(1, 2)^*$ -gs-closed, By definition of $(1, 2)^*$- gs closedness,$(1, 2)^*$ -scl(A)$\subset$A.

By Theorem 3.2.8, $(1, 2)^*$ -scl(A) = A$\subset$ $\tau_{1,2}$-int($\tau_{1,2}$-cl(A)).

Thus $\tau_{1,2}$-int($\tau_{1,2}$-cl(A))$\subset$A. Hence A is $(1, 2)^*$ - semi-closed.

Proposition :3.2.10

For any bijection f: X $\rightarrow$ Y the following statements are equivalent.

(a) f^{-1}: Y $\rightarrow$ X is $(1, 2)^*$ -g-continuous.

(b) f is $(1, 2)^*$ - g-open.

(c) f is $(1, 2)^*$ - g-closed.

Proof:

(a)$\Rightarrow$(b).

Let U be any $\tau_{1,2}$-open set of X. Then X–U is $\tau_{1,2}$-closed in X. Since f^{-1} is $(1, 2)^*$ - g-continuous, $(f^{-1})^{-1}$ (X–U) = f (X–U) =Y – f (U) is $(1, 2)^*$ - g-closed in Y. Then f(U) is $(1, 2)^*$ - g-open in Y. Hence f is $(1, 2)^*$ -g-open.

(b)$\Rightarrow$(c).

Let U be any $\tau_{1,2}$-closed set in X. Then X–U is $\tau_{1,2}$-open set in X. Since f is $(1, 2)^*$ -g-open, f (X-U) =Y – f(U) is $(1, 2)^*$ - g-open in Y. Then f(U) is $(1, 2)^*$ - g-closed in Y. Hence f is $(1, 2)^*$ -g-closed.

(c)$\Rightarrow$(a).

Let U be $\tau_{1,2}$-closed set of X. Since f is $(1, 2)^*$ -g-closed, f(U) is $(1, 2)^*$ - g-closed in Y. Then $(f^{-1})^{-1}$(U) is $(1, 2)^*$ -g-closed in Y. Hence f^{-1} is $(1, 2)^*$ - g-continuous.

Proposition :3.2.11

Let f: X $\rightarrow$ Y be a bijective and $(1, 2)^*$ -g-continuous map. Then the following statements are equivalent.

(a) f is $(1, 2)^*$ -g-open.

(b) f is $(1, 2)^*$ -g-homeomorphism.

(c) f is $(1, 2)^*$ -g-closed.

Proof.

(a)$\Rightarrow$(b).

Since f is bijective, $(1, 2)^*$ -g-continuous and $(1, 2)^*$ -g-open, f is $(1, 2)^*$ -g homeomorphism.

(b)$\Rightarrow$(c).

Since f is $(1, 2)^*$ -g-homeomorphism, f is $(1, 2)^*$ -g-open. By Proposition 3.2.8 f is $(1, 2)^*$ -g-closed.

(c)$\Rightarrow$(a).

Follows from Proposition 3.2.1

Proposition :3.2.12

If A is $(1, 2)^*$-gs-closed set in X and $A\subseteq B\subseteq(1, 2)^*$ -scl(A), then B is $(1, 2)^*$ -gs-closed in X.

Proof:

Let $B\subset U$ where U is $\tau_{1,2}$-open in X.

Since A is $(1, 2)^*$ -gs-closed set and $A\subseteq U$, $(1, 2)^*$ -scl(A)$\subseteq$U.

Since $B\subseteq$ 1, 2)* -scl(A), $(1, 2)^*$ -scl(B)$\subseteq(1, 2)^*$ -scl(A)$\subseteq$U.

Hence $(1, 2)^*$ -scl(B)$\subseteq$U and so B is $(1, 2)^*$ -gs-closed in X.

Theorem :3.2.13

If f: $X\rightarrow Y$ is $(1, 2)^*$-continuous $(1, 2)^*$ -gs-closed and A is $(1, 2)^*$ -g-closed set of X, then f(A) is $(1, 2)^*$ -gs-closed in Y.

Proof:

Let $f(A)\subset O$ where O is $\sigma_{1,2}$-open set in Y. Then $A\subset f^{-1}(O)$.

Since f is $(1, 2)^*$-continuous, $f^{-1}(O)$ is $\tau_{1,2}$-open set in X.

Hence $\tau_{1,2}$-cl(A)$\subset f^{-1}(O)$ as A is $(1, 2)^*$-g-closed set. Therefore f ($\tau_{1,2}$-cl(A))$\subset$O.

Since f is $(1, 2)^*$ -gs-closed and $\tau_{1,2}$-cl(A) is $\tau_{1,2}$-closed in X, f ($\tau_{1,2}$-cl(A)) is $(1, 2)^*$-gs-closed in Y.

Thus $(1, 2)^*$-scl[f($\tau_{1,2}$ -cl(A))] $\subset$ O.

Since f(A) $\subset$ f($\tau_{1,2}$ -cl(A)), $(1, 2)^*$-scl(f(A)) $\subset$ $(1, 2)^*$-scl[f($\tau_{1,2}$ -cl(A))] $\subset$ O.

Therefore f(A) is $(1, 2)^*$ -gs-closed in Y.

Proposition:3.2.14

Every $(1, 2)^*$ -g-closed set is $(1, 2)^*$-gs-closed.

Proof:

Let F be any $\tau_{1,2}$ -open set of X such that S $\subseteq$ F.

Since S is $(1, 2)^*$ -g-closed, $\tau_{1,2}$ -cl (S) $\subseteq$ F. But $(1, 2)^*$ -scl(S) $\subseteq$ $\tau_{1,2}$ -cl (S) .

Hence $(1, 2)^*$-scl(S) $\subseteq$ F.

Hence S is $(1, 2)^*$ -gs-closed.

Theorem :3.2.15

If f : X $\rightarrow$ Y is $(1, 2)^*$ -g-closed and g: Y $\rightarrow$ Z is $(1, 2)^*$ -continuous and $(1, 2)^*$ -gs-closed, then gof : X $\rightarrow$ Z is $(1, 2)^*$ -gs-closed.

Proof:

Let F be any $\tau_{1,2}$ -closed set of X.

Since f is $(1, 2)^*$-g-closed, f(F) is $(1, 2)^*$ -g-closed set in Y.

Since g is $(1, 2)^*$ -continuous and $(1, 2)^*$ -gs-closed and f(F) is $(1, 2)^*$ -g-closed set of Y,

By Theorem 3.2.11, g(f (F)) is $(1, 2)^*$ -gs-closed in Z.

Hence gof is $(1, 2)^*$ -gs-closed.

Theorem :3.2.16

If f: X $\rightarrow$ Y is $(1, 2)^*$-closed and g: Y $\rightarrow$ Z is $(1, 2)^*$ -gs-closed, then gof :X $\rightarrow$ Z is $(1, 2)^*$ - gs-closed.

Proof:

Let F be any $\tau_{1,2}$ -closed set of X.

Since f is $(1, 2)^*$ -closed, f(F) is $\sigma_{1,2}$ -closed in Y.

Since g is $(1, 2)^*$ -gs-closed, g(f(F))=(gof)(F) is $(1, 2)^*$ -gs-closed in Z.

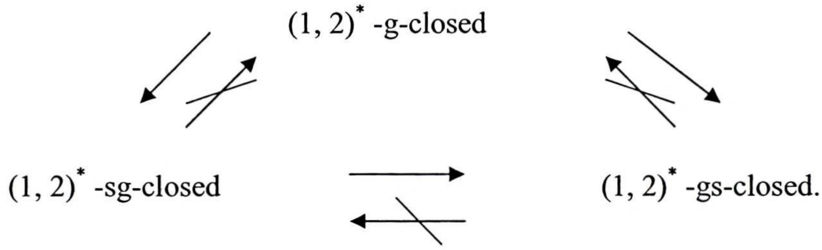
Hence gof is $(1, 2)^*$ -gs-close

SECTION -3.3

COMPARATIVE STUDY OF $(1, 2)^*$ -GENERALIZED HOMEOMORPHISMS

Remark: 3.3.1

For the sets we considered above we have the following diagram of implications where A $\nrightarrow$ B means A does not necessarily imply B.



Remark :3.3.2

$(1, 2)^*$ -g-closed set and $(1, 2)^*$ -sg-closed sets are in general,independent.

Remark :3.3.3

A $(1, 2)^*$ -gs-closed set need not be $(1, 2)^*$ -g-closed .

Example 3.3.4

Let Z={x,y,z), η_1={ φ, Z,{y}} and η_2={ φ,Z,{x,y}}. Then {x} is $(1, 2)^*$ -gs-closed set but it is not $(1, 2)^*$-g-closed.

Remark :3.3.5

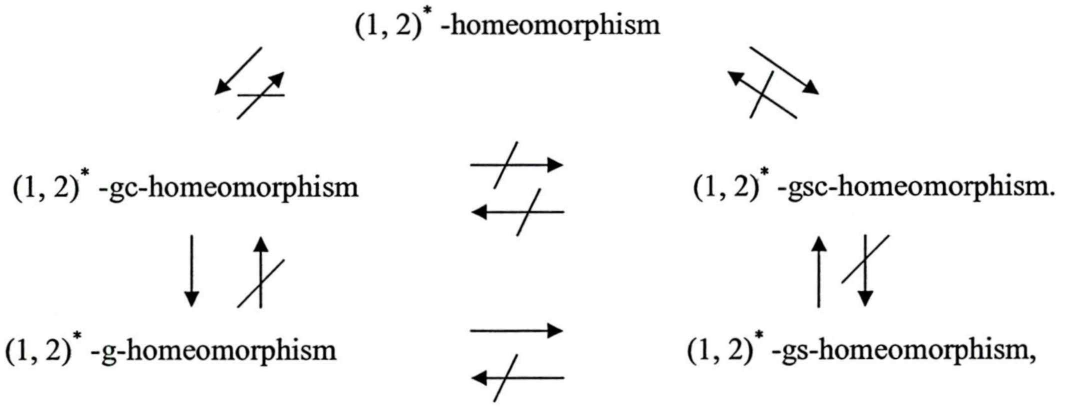
A $(1, 2)^*$ -gs-closed set need not be $(1, 2)^*$ -sg-closed.

Example: 3.3.6

Let X={a, b, c), τ_1={ φ,X} and τ_2={ φ,X,{a}}. Then {a, b} is $(1, 2)^*$ -gs-closed set but it is not $(1, 2)^*$ -sg-closed.

Remark :3.3.7

For the maps we considered above, we have the following diagram where $A \not\rightarrow B$ means A does not necessarily imply B.



Remark:3.3.8

If $f: X \rightarrow Y$ is $(1, 2)^*$ -semi-continuous and $(1, 2)^*$-open map,then f is $(1, 2)^*$-semi-irresolute.

Proposition :3.3.9

If $f: X \rightarrow Y$ is $(1, 2)^*$ -homeomorphism, then f and its inverse f^{-1} are pre-$(1, 2)^*$ -semi-closed and also $(1, 2)^*$ -semi-irresolute.

Proof:

Since f is $(1, 2)^*$ -open and $(1, 2)^*$ -continuous, by Remark 3.3.8 f is $(1, 2)^*$ -semi-irresolute.

The bijectivity of f implies that its inverse f^{-1} is pre-$(1, 2)^*$ -semi-closed.

Similarly since f^{-1} is $(1, 2)^*$ -open and $(1, 2)^*$ -continuous we have that f^{-1} is $(1, 2)^*$ -semi-irresolute and f is pre-$(1, 2)^*$ -semi-closed.

Remark:3.3.10

a) Every $(1, 2)^*$-continuous map is $(1, 2)^*$-g-continuous but not conversely.
b) If $f: X \rightarrow Y$ is bijective, $(1, 2)^*$-open and $(1, 2)^*$-g-continuous map,then f is $(1, 2)^*$-gc-irresolute.

Theorem:3.3.11

Every $(1, 2)^*$ -homeomorphism is $(1, 2)^*$ -gc-homeomorphism.

Proof:

Let f be $(1, 2)^*$ -homeomorphism. Then f and f^{-1} are continuous.

By Result (a) and (b) of 3.3.10, f and f^{-1} are $(1, 2)^*$ -g-continuous and f and f^{-1} are $(1, 2)^*$ -gcirresolute,

Hence f is $(1, 2)^*$ -gc-homeomorphism.

Remark:3.3.12

The converse of the above theorem need not be true.

Example :3.3.13

Let X={a, b, c}, τ_1 ={ φ , X, {a, b}} and τ_2 ={ φ ,X}. Let Y ={p, q,r}, σ_1 ={ φ , Y, {q}} and σ_2 ={ φ , Y, {p, q}}. Define f: X $\rightarrow$ Y by f(a) = p, f(b)= q, f(c) = r. Then f is bijective, both f and f^{-1} are $(1, 2)^*$ -gc-irresolute. Hence f is $(1, 2)^*$ -gc-homeomorphism. But f is not $(1, 2)^*$ -homeomorphism, since f is not $(1, 2)^*$ -continuous.

Proposition 3.3.14

Every $(1, 2)^*$ -gc-homeomorphism is $(1, 2)^*$ -g-homeomorphism.

Proof:

Let f be $(1, 2)^*$ -gc-homeomorphism.

Then f is $(1, 2)^*$ -gc-irresolute and its inverse f^{-1} is also $(1, 2)^*$ -gc-irresolute.

Every $(1, 2)^*$ -gc-irresolute map is $(1, 2)^*$ -g-continuous.

Hence f and f^{-1} are $(1, 2)^*$ -g-continuous. By Proposition 3.2.11, f is $(1, 2)^*$ -g-open.

By Proposition 3.2.12, f is $(1, 2)^*$ - g-homeomorphism.

Remark :3.3.15

The converse of the above Proposition need not be true.

Proposition :3.3.16

Every $(1, 2)^*$ -g-continuous map is $(1, 2)^*$ -gs-continuous.

Proof.

Let V be $\sigma_{1,2}$-closed set of Y.

Let f: X → Y be $(1, 2)^*$ -g-continuous.

Then f^{-1} (V) is $(1, 2)^*$ -g-closed in X.

By Proposition 3.2.15, f^{-1}(V) is $(1, 2)^*$ -gs-closed in X.

Hence f is $(1, 2)^*$ -gs-continuous.

Proposition 3.3.17

Every $(1, 2)^*$- g-open map is $(1, 2)^*$ -gs-open.

Proof.

Let f: X → Y be $(1, 2)^*$ -g-open and F be any $\tau_{1,2}$-open set of X.

Then f(F) is $(1, 2)^*$ -g-open in Y and Y–f(F) is $(1, 2)^*$ -g-closed in Y.

By Proposition 3.2.15, Y–f(F) is $(1, 2)^*$ -gs-closed in Y.

Therefore f(F) is $(1, 2)^*$ -gs-open in Y.

Hence f is $(1, 2)^*$ -gs-open map.

Remark:3.3.18

The converse of the above Proposition need not be true .

Example: 3.3.19

Let X ={a, b, c}, τ_1= { φ , X, {a}, {a, b}} and τ_2 = { φ , X, {a, c}}.

Let Y={p, q, r}, σ_1= { φ , Y, {q}} and σ_2 ={ φ , Y, {p, q}}. Define f: X → Y by f(a) = q, f(b) = p, f(c) = r. Then f is $(1, 2)^*$ -gs-homeomorphism. But f is not $(1, 2)^*$-g open because f ({a, c}) ={q, r} is not $(1, 2)^*$-g-open in Y where {a, c} is $\tau_{1,2}$-open in X. Hence f is not $(1, 2)^*$ -g-homeomorphism.

Remark:3.3.20

If f is $(1, 2)^*$-open and $(1, 2)^*$-gc-irresolute,then f is $(1, 2)^*$-gs-irresolute.

Proposition :3.3.21

Every $(1, 2)^*$ - homeomorphism is $(1, 2)^*$ -gsc-homeomorphism.

Proof:

Let f: $X \rightarrow Y$ be $(1, 2)^*$ - homeomorphism. By Theorem 4.3.12, f is $(1, 2)^*$ -gc-homeomorphism.

Hence f and f^{-1} are both $(1, 2)^*$ -gc-irresolute.

By Proposition 3.3.18, f and f^{-1} are both $(1, 2)^*$ -gs-irresolute.

Hence f is $(1, 2)^*$ -gsc-homeomorphism.

Proposition: 3.3.22

Every $(1, 2)^*$ - gsc-homeomorphism is $(1, 2)^*$ -gs-homeomorphism.

Proof:

Let f: $X \rightarrow Y$ be $(1, 2)^*$ -gsc- homeomorphism.

Hence f and f^{-1} are $(1, 2)^*$ -gs-irresolute and hence $(1, 2)^*$ -gs-continuous.

Since f^{-1} is $(1, 2)^*$ -gs-continuous, by Proposition 3.2.1 f is $(1, 2)^*$ -gs-open.

Hence f is $(1, 2)^*$ -gs-homeomorphism.

Remark: 3.3.23

The converse of the above Proposition need not be true.

Example: 3.3.24

Let X={a, b, c}, τ_1 ={ φ , X, {b}} and τ_2 ={ φ , X, {a, b}}. Let Y={p,q, r}, σ_1 ={ φ , Y} and σ_2 ={ φ , Y, {p}}. Define f: $X \rightarrow Y$ by f(a) = q, f(b) = p, f(c) = r. Then f is $(1, 2)^*$ -gs-homeomorphism. But f is not $(1, 2)^*$-gsc- homeomorphism because f^{-1}({p, q})={a, b} is not $(1, 2)^*$ -gs-closed in X where {p, q} is $(1, 2)^*$ -gs-closed in Y.

SECTION –3.4
(1,2)*-SGO-COMPACT SPACES AND (1,2)*-GSO-COMPACT SPACES

Definition :3.4.1

A space (X, τ_1, τ_2) is called a **(1,2)*-T_d space** if every $(1, 2)^*$ -gs-closed subset of X is $(1, 2)^*$-g-closed in X.

Definition :3.4.2

A space (X, τ_1, τ_2) is called a **$(1, 2)^*$ -T_b space** if every $(1, 2)^*$ -gs-closed subset of X is $\tau_{1,2}$ -closed in X.

Definition :3.4.3

A space (X, τ_1, τ_2) is called a **$(1, 2)^*$-$T_{1/2}$ space** if every $(1, 2)^*$ -g-closed subset of X is $\tau_{1,2}$ -closed in X.

Proposition :3.4.4

(a) Every $(1, 2)^*$ -gsc- homeomorphism from a $(1, 2)^*$ - T_d space onto itself is a $(1, 2)^*$-gc- homeomorphism.

(b) Every $(1, 2)^*$-gc-homeomorphism from a $(1, 2)^*$ - $T_{1/2}$ space onto itself is a $(1, 2)^*$ -homeomorphism.

Proposition :3.4.5

Every $(1, 2)^*$ -gs-homeomorphism from a $(1, 2)^*$ -T_b space onto itself is a $(1, 2)^*$-homeomorphism.

Definition:3.4.6

A subset B of a bitopological (X, τ_1, τ_2) is said to be **$(1, 2)^*$ -SGO-compact** (resp. $(1, 2)^*$ -GSO-compact) **relative to X** if for every cover $\{A_i : i \in \Lambda)$ of B by

...

$(1, 2)^*$ -sg-open (resp. $(1, 2)^*$ -gs-open) subsets of (X, τ_1, τ_2) , i.e. $B \subset \cup \{A_i : i \in \Lambda)$ where A_i ($i \in \Lambda$) are $(1, 2)^*$ -sg-open (resp. $(1, 2)^*$ -gs-open) sets of (X, τ_1, τ_2) , there exists a finite subset Λ_0 of Λ such that $B \subset \cup \{A_i : i \in \Lambda_0)$.

Definition:3.4.7

If X is $(1, 2)^*$ -SGO-compact (resp. $(1, 2)^*$ -GSO-compact) relative to X, (X, τ_1, τ_2) is said to be a **$(1, 2)^*$ -SGO-compact space** (resp. $(1, 2)^*$ -GSO-compact space), shortly. It is evident that the $(1, 2)^*$ -GSO-compactness implies the $(1, 2)^*$ -SGO-compactness and the $(1, 2)^*$ - SGO-compactness implies the $(1, 2)^*$ -compactness .

Proposition :3.4.8

(i) A $(1, 2)^*$ -sg-closed subset of a $(1, 2)^*$ -SGO-compact space (X, τ_1, τ_2) is $(1, 2)^*$ -SGO-compact relative to X.

(ii) A $(1, 2)^*$ -gs-closed subset of a $(1, 2)^*$ -GSO-compact space (X, τ_1, τ_2) is $(1, 2)^*$ -GSO-compact relative to X.

Since the proof is similar to the case of $(1, 2)^*$ -compactness, it is omitted.

Proposition:3.4.9

(i) If $f : X \to Y$ is $(1, 2)^*$ -sg-continuous (resp. $(1, 2)^*$ -sg-irresolute) and a subset B of X is $(1, 2)^*$-SGO-compact relative to X, then f(B) is $(1, 2)^*$-compact in Y (resp. $(1, 2)^*$-SGO-compact relative to Y).

(ii) If $f : X \to Y$ is $(1, 2)^*$-gs-continuous (resp. $(1, 2)^*$-gs-irresolute) and a subset B of X is $(1, 2)^*$-GSO-compact relative to X, then f(B) is $(1, 2)^*$-compact in Y (resp. $(1, 2)^*$-GSO-compact relative to Y).

Proof:

(i) Let $\{U_i : i \in \Lambda\}$ be any collection of $\sigma_{1,2}$-open (resp. $(1, 2)^*$ -sg-open) subsets of Y such that $f(B) \subset \cup \{U_i : i \in \Lambda)$.

Then $B \subset \cup \{f^{-1}(U_i): i \in \Lambda\}$ holds and there exists a finite subset Λ_0 of Λ such that $B \subset \cup \{ f^{-1}(U_i): i \in \Lambda_0\}$.

. . .

Therefore, we have $f(B) \subset \cup \{U_i : i \in \Lambda_0)$ which shows that f(B) is $(1, 2)^*$ -compact in Y (resp. $(1, 2)^*$ -SGO-compact relative to Y).

(ii) The proof is similar to that of (i) by using $(1, 2)^*$-GSO-compactness and $(1, 2)^*$ -gs-continuity (resp. $(1, 2)^c$ -gs-irresolute) in the place of $(1, 2)^*$ - SGO compactness and $(1, 2)^*$-sg-continuity (resp. $(1, 2)^*$-sg-irresolute) respectively.