# CHAPTER 4

# CHAPTER 4

# FUZZY n-INNER PRODUCT SPACE AND INTUITIONISTIC FUZZY n-INNER PRODUCT SPACE

In this chapter the notion of fuzzy n-inner product space and intuitionistic fuzzy n-inner product are discussed.

In section one of chapter 4, n-inner product space and fuzzy n-inner product space are studied.

In section two of chapter 4, fuzzy n-inner product space are generalized to intuitionistic fuzzy sets.

## SECTION: 4.1

## FUZZY n-INNER PRODUCT SPACE

**Definition: 4.1.1**

Let $n$ be a natural number greater than 1 and X be a real linear space of dimension greater than or equal to $n$ and let $(\bullet, \bullet|\bullet, \ldots, \bullet)$ be a real valued function on $\underbrace{X \times X \times \ldots \times X}_{n+1} = X^{n+1}$ satisfying the following conditions:

(1) $(i)(x, x|x_2, \ldots, x_n) \geq 0$,

$(ii)(x, x|x_2, \ldots, x_n) = 0$ if and only if $x, x_2, \ldots, x_n$ are linearly dependent,

$(2)(x, y|x_2, \ldots, x_n) = (y, x|x_2, \ldots, x_n)$,

$(3)(x, y|x_2, \ldots, x_n)$ is variant under any permutation of $x_2, \ldots, x_n$,

$(4)(x, x|x_2, \ldots, x_n) = (x_2, x_2|x, x_3 \ldots, x_n)$,

$(5)(ax, x|x_2, \ldots, x_n) = a(x, x|x_2, \ldots, x_n)$ for every $a \in \Re$(real),

$(6)(x + x', y|x_2, \ldots, x_n) = (x, y|x_2, \ldots, x_n) + (x', x|x_2, \ldots, x_n)$.

Then $(\bullet, \bullet|\bullet, \ldots, \bullet)$ is called an n-inner product on X and $\Big(X, (\bullet, \bullet|\bullet, \ldots, \bullet)\Big)$ is called an *n-inner product space*.

**Remark: 4.1.2**

If an n-inner product space $\left(X,(\bullet,\bullet|\bullet,\ldots,\bullet)\right)$ is given then $\|x_1,x_2,\ldots,x_n\| = \sqrt{(x_1,x_1|x_2,\ldots,x_n)}$ defines an n-norm on X. Further the following extension of Cauchy-Buniakowski inequality is also true.

$$|(x,y|x_2,\ldots,x_n)| \leq \sqrt{(x,x|x_2,\ldots,x_n)}\,\sqrt{(y,y|x_2,\ldots,x_n)}$$

**Definition: 4.1.3**

Let X be a linear space over a field F. A fuzzy subset $J : X^{n+1} \times \Re$ ($\Re$ set of real numbers) is called a fuzzy n-inner product on X if and only if:

(1) For all t$\in \Re$ with t$\leq$0, $J(x,x|x_2,\ldots,x_n,t)$=0,

(2) For all t$\in \Re$ with t>0, $J(x,x|x_2,\ldots,x_n,t)$= 1 if and only if $x,x_2,\ldots,x_n$ are linearly dependent,

(3) For all $t>0$, $J(x,y|x_2,\ldots,x_n,t) = J(y,x|x_2,\ldots,x_n,t)$,

(4) $J(x,y|x_2,\ldots,x_n,t)$ is invariant under any permutation of $x_2,\ldots,x_n$,

(5) For all $t>0$, $J(x,x|x_2,\ldots,x_n,t) = J(x_2,x_2|x,x_3\ldots,x_n,t)$,

(6) For all t>0,

$$J(ax,bx|x_2,\ldots,x_n,t) = J(x,x|x_2,\ldots,x_n,\frac{t}{|ab|}),\text{ a,b}\in \Re \text{ (real)},$$

(7) For all s,t$\in \Re$,

$$J(x+x',y|x_2,\ldots,x_n,t+s) \geq \min\{J(x,y|x_2,\ldots,x_n,t), J(x',y|x_2,\ldots,x_n,s)\},$$

(8) For all s,t$\in \Re$ with s> 0, t> 0

$$J(x,y|x_2,\ldots,x_n,\sqrt{ts}) \geq \min\{J(x,x|x_2,\ldots,x_n,t), J(y,y|x_2,\ldots,x_n,s)\},$$

(9) $J(x,y|x_2,\ldots,x_n,t)$ is a nondecreasing function of t$\in \Re$ and

$\lim_{t\to\infty} J(x,y|x_2,\ldots,x_n,t)$ =1.

Then (X,J) is called a *fuzzy n-inner product space or in short f-n-IPS.*

**Example: 4.1.4**

Let $\Big(X, (\bullet, \bullet|\bullet, \ldots, \bullet)\Big)$ be an n-inner product space. Let

$$J(x, y|x_2, \ldots, x_n, t) = \begin{cases} \dfrac{t}{t + |(x, y|x_2, \ldots, x_n)|} & \text{when } t > 0, t \in \Re, \\ & (x, y|x_2, \ldots, x_n) \in \underbrace{X \times \ldots, \times X}_{n+1} \\ 0 & \text{when } t \leq 0 \end{cases}$$

then (X,J)is an f-n-IPS.

**Proof:**

(1)For all t$\in \Re$ with t$\leq$ 0,

$$J(x, x|x_2, \ldots, x_n, t) = 0$$

(2)For all t $\in \Re$ with t>0,

$$\begin{aligned} J(x, x|x_2, \ldots, x_n, t) = 1 \\ & \Leftrightarrow \frac{t}{t + |(x, x|x_2, \ldots, x_n)|} = 1, \\ & \Leftrightarrow t = t + |(x, x|x_2, \ldots, x_n)|, \\ & \Leftrightarrow |(x, x|x_2, \ldots, x_n)| = 0, \\ & \Leftrightarrow (x, x|x_2, \ldots, x_n) = 0, \\ & \Leftrightarrow x, x_2, \ldots, x_n \text{are linearly dependent.} \end{aligned}$$

(3)For all t>0,

$$\begin{aligned} J(x, y|x_2, \ldots, x_n, t) &= \frac{t}{t + |(x, y|x_2, \ldots, x_n)|} \\ &= \frac{t}{t + |(y, x|x_2, \ldots, x_n)|} \\ &= J(y, x|x_2, \ldots, x_n, t) \end{aligned}$$

(4)As $(x, x|x_2, \ldots, x_n)$ is invariant under any permutation of $x_2, \ldots, x_n$, we have $J(x, y|x_2, \ldots, x_n, t)$ is invariant under any permutation.

(5)For all t>0,

$$\begin{aligned} J(x,x|x_2,\ldots,x_n,t) &= \frac{t}{t+|(x,x|x_2,\ldots,x_n)|} \\ &= \frac{t}{t+|(x_2,x_2|x,\ldots,x_n)|} \\ &= J(x_2,x_2|x,x_3\ldots,x_n,t) \end{aligned}$$

(6)For all t>0,

$$\begin{aligned} J(x,x|x_2,\ldots,x_n,\frac{t}{|ab|}) &= \frac{\frac{t}{|ab|}}{\frac{t}{|ab|}+|(x,x|x_2\ldots,x_n)|} \\ &= \frac{t}{t+|ab||(x,x|x_2,\ldots,x_n)|} \\ &= \frac{t}{t+|(ax,bx|x_2,\ldots,x_n)|} \\ &= J(ax,bx|x_2,\ldots,x_n,t) \end{aligned}$$

(7)we have to prove

$$J(x+x',y|x_2,\ldots,x_n,t+s) \geq \min\{J(x,y|x_2,\ldots,x_n,t), J(x',y|x_2,\ldots,x_n,s)\}.$$

If

$(a)s+t<0$,

$(b)s=t=0$,

$(c)s+t>0; s>0, t<0; s<0, t>0$,then the above relation is obvious. If

(d)$s>0, t>0, s+t>0$, then without loss of generality assume that

$$\begin{aligned} J(x,y|x_2,\ldots,x_n,t) &\leq J(x',y|x_2,\ldots,x_n,s) \\ \Rightarrow \frac{t}{t+|(x,y|x_2,\ldots,x_n)|} &\leq \frac{s}{s+|(x',y|x_2,\ldots,x_n)|} \\ \frac{t+|(x,y|x_2,\ldots,x_n)|}{t} &\geq \frac{s+|(x',y|x_2,\ldots,x_n)|}{s} \\ 1+\frac{|(x,y|x_2,\ldots,x_n)|}{t} &\geq 1+\frac{|(x',y|x_2,\ldots,x_n)|}{s} \\ \frac{|(x,y|x_2,\ldots,x_n)|}{t} &\geq \frac{|(x',y|x_2,\ldots,x_n)|}{s} \end{aligned}$$

$$\begin{aligned}
\frac{s|(x,y|x_2,\ldots,x_n)|}{t} &\geq |(x',y|x_2,\ldots,x_n)| \\
|(x,y|x_2,\ldots,x_n)| + \frac{s|(x,y|x_2,\ldots,x_n)|}{t} &\geq |(x,y|x_2,\ldots,x_n)| + |(x',y|x_2,\ldots,x_n)| \\
\left(1+\frac{s}{t}\right)|(x,y|x_2,\ldots,x_n)| &\geq |(x+x',y|x_2,\ldots,x_n)| \\
\left(\frac{s+t}{t}\right)|(x,y|x_2,\ldots,x_n)| &\geq |(x+x',y|x_2,\ldots,x_n)| \\
\frac{|(x,y|x_2,\ldots,x_n)|}{t} &\geq \frac{|(x+x',y|x_2,\ldots,x_n)|}{s+t} \\
1+\frac{|(x,y|x_2,\ldots,x_n)|}{t} &\geq 1+\frac{|(x+x',y|x_2,\ldots,x_n)|}{s+t} \\
\frac{t+|(x,y|x_2,\ldots,x_n)|}{t} &\geq \frac{s+t+|(x+x',y|x_2,\ldots,x_n)|}{s+t} \\
\frac{t}{t+|(x,y|x_2,\ldots,x_n)|} &\leq \frac{s+t}{s+t+|(x+x',y|x_2,\ldots,x_n)|}
\end{aligned}$$

$$\Rightarrow \min\{J(x,y|x_2,\ldots,x_n,t), J(x',y|x_2,\ldots,x_n,s)\} \leq J(x+x',y|x_2,\ldots,x_n,s+t)$$

(8)without loss of generality assume that

$$\begin{aligned}
J(x,x|x_2,\ldots,x_n,t) &\leq J(y,y|x_2,\ldots,x_n,s) \text{ for all } s,t\in\Re \text{ with } s>0, t>0 \\
\frac{t}{t+|(x,x|x_2,\ldots,x_n)|} &\leq \frac{s}{s+|(y,y|x_2,\ldots,x_n)|} \\
\frac{t+|(x,x|x_2,\ldots,x_n)|}{t} &\geq \frac{s+|(y,y|x_2,\ldots,x_n)|}{s} \\
1+\frac{|(x,x|x_2,\ldots,x_n)|}{t} &\geq 1+\frac{|(y,y|x_2,\ldots,x_n)|}{s} \\
\frac{|(x,x|x_2,\ldots,x_n)|}{t} &\geq \frac{|(y,y|x_2,\ldots,x_n)|}{s} \\
\frac{s|(x,x|x_2,\ldots,x_n)|}{t} &\geq |(y,y|x_2,\ldots,x_n)| \\
\frac{|(x,x|x_2,\ldots,x_n)|s|(x,x|x_2,\ldots,x_n)|}{t} &\geq |(x,x|x_2,\ldots,x_n)||(y,y|x_2,\ldots,x_n)|
\end{aligned}$$

By Remark4.1.2,

$$\begin{aligned}
|(x,x|x_2,\ldots,x_n)|^2\frac{s}{t} &\geq |(x,y|x_2,\ldots,x_n)|^2 \\
|(x,x|x_2,\ldots,x_n)|^2\frac{s}{t^2} &\geq \frac{|(x,y|x_2,\ldots,x_n)|^2}{t} \\
\frac{|(x,x|x_2,\ldots,x_n)|^2}{t^2} &\geq \frac{|(x,y|x_2,\ldots,x_n)|^2}{st}
\end{aligned}$$

Taking square root on both sides,

$$\frac{|(x,x|x_2,\ldots,x_n)|}{t} \geq \frac{|(x,y|x_2,\ldots,x_n)|}{\sqrt{st}}$$
$$1+\frac{|(x,x|x_2,\ldots,x_n)|}{t} \geq 1+\frac{|(x,y|x_2,\ldots,x_n)|}{\sqrt{st}}$$
$$\frac{t+|(x,x|x_2,\ldots,x_n)|}{t} \geq \frac{\sqrt{st}+|(x,y|x_2,\ldots,x_n)|}{\sqrt{st}}$$
$$\frac{t}{t+|(x,x|x_2,\ldots,x_n)|} \leq \frac{\sqrt{st}}{\sqrt{st}+|(x,y|x_2,\ldots,x_n)|}$$

$\Rightarrow \min\{J(x,x|x_2,\ldots,x_n,t), J(y,y|x_2,\ldots,x_n,s)\} \leq J(x,y|x_2,\ldots,x_n,\sqrt{st})$

(9) For all $t_1, t_2 \in \Re$, if $t_1 < t_2 \leq 0$, then,

$J(x,y|x_2,\ldots,x_n,t_1) = J(x,y|x_2,\ldots,x_n,t_2) = 0$

Suppose $t_2 > t_1 > 0$, then

$$\frac{t_2}{t_2+|(x,y|x_2,\ldots,x_n)|} - \frac{t_1}{t_1+|(x,y|x_2,\ldots,x_n)|}$$

$$= \frac{|(x,y|x_2,\ldots,x_n)|(t_2-t_1)}{(t_2+|(x,y|x_2,\ldots,x_n)|)(t_1+|(x,y|x_2,\ldots,x_n)|)} \geq 0,$$

for all $(x,y|x_2,\ldots,x_n) \in X^{n+1}$

$$\frac{t_2}{t_2+|(x,y|x_2,\ldots,x_n)|} \geq \frac{t_1}{t_1+|(x,y|x_2,\ldots,x_n)|},$$

$$\Rightarrow J(x,y|x_2,\ldots,x_n,t_2) \geq J(x,y|x_2,\ldots,x_n,t_1)$$

Thus $J(x,y|x_2,\ldots,x_n,t)$ is a non decreasing function. Also,

$$\begin{aligned}\lim_{t\to\infty} J(x,y|x_2,\ldots,x_n,t) &= \lim_{t\to\infty} \frac{t}{t+|(x,y|x_2,\ldots,x_n)|} \\ &= \lim_{t\to\infty} \frac{t}{t(1+(1/t)|(x,y|x_2,\ldots,x_n)|)} \\ &= 1\end{aligned}$$

Thus (X,J) is an f-n-IPS.

**SECTION: 4.2**

**INTUITIONISTIC FUZZY n-INNER PRODUCT SPACE**

**Definition: 4.2.1**

An *intuitionistic fuzzy n-inner product space (or) in short i-f-n-IPS* is an object of the form

$A = \{(X, J(x, x|x_2, \ldots, x_n, t), K(x, x|x_2, \ldots, x_n, t)) : (x_1, x_2, \ldots, x_n) \in X^{n+1}\}$

where $X$ is a linear space over a field F, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and $J, K$ are fuzzy sets on $X^{n+1} \times \Re$, $J$ denotes the degree of membership and $K$ denotes the degree of non-membership of $(x, x|x_2, \ldots, x_n, t) \in X^{n+1} \times \Re$ satisfying the following conditions :

(1) $J(x, x|x_2, \ldots, x_n, t) + K(x, x|x_2, \ldots, x_n, t) \leq 1$,

(2) $J(x, x|x_2, \ldots, x_n, t) > 0$,

(3) $J(x, x|x_2, \ldots, x_n, t) = 1$ if and only if $x, x_2, \ldots, x_n$ are linearly dependent,

(4) $J(x, y|x_2, \ldots, x_n, t) = J(y, x|x_2, \ldots, x_n, t)$,

(5) $J(x, y|x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_2, \ldots, x_n$,

(6) $J(x, x|x_2, \ldots, x_n, t) = J(x_2, x_2|x, x_3 \ldots, x_n, t)$,

(7) $J(ax, bx|x_2, \ldots, x_n, t) = J(x, x|x_2, \ldots, x_n, \frac{t}{|ab|})$, a,b$\in \Re$ (real),

(8) $J(x, y|x_2, \ldots, x_n, t) * J(x', y|x_2, \ldots, x_n, s) \leq J(x + x', y|x_2, \ldots, x_n, s + t)$,

(9) $J(x, x|x_2, \ldots, x_n, t) * J(y, y|x_2, \ldots, x_n, s) \leq J(x, y|x_2, \ldots, x_n, \sqrt{ts})$,

(10) $J(x, y|x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t,

(11) $K(x, x|x_2, \ldots, x_n, t) > 0$,

(12) $K(x, x|x_2, \ldots, x_n, t) = 0$ if and only if $x, x_2, \ldots, x_n$ are linearly dependent,

(13 $K(x, y|x_2, \ldots, x_n, t) = K(y, x|x_2, \ldots, x_n, t)$,

(14) $K(x, y|x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_2, \ldots, x_n$,

(15) $K(x, x|x_2, \ldots, x_n, t) = K(x_2, x_2|x, x_3 \ldots, x_n, t)$,

(16) $K(ax, bx|x_2, \ldots, x_n, t) = K(x, x|x_2, \ldots, x_n, \frac{t}{|ab|})$, a,b$\in \Re$ (real),

(17) $K(x, y|x_2, \ldots, x_n, t) \diamond K(x', y|x_2, \ldots, x_n, s) \geq K(x + x', y|x_2, \ldots, x_n, t + s)$,

(18) $K(x,x|x_2,\ldots,x_n,t)\diamond K(y,y|x_2,\ldots,x_n,s)\geq K(x,y|x_2,\ldots,x_n,\sqrt{ts})$,

(19) $K(x,x|x_2,\ldots,x_n,t):(0,\infty)\rightarrow[0,1]$ is continuous in t.

**Example: 4.2.2**

Let $\Big(X,(\bullet,\bullet|\bullet,\ldots,\bullet)\Big)$ be an n-inner product space. Let $a*b=\min\{a,b\}$ and $a\diamond b=\max\{a,b\}$ and

$$J(x,y|x_2,\ldots,x_n,t)=\frac{t}{t+|(x,y|x_2,\ldots,x_n)|},$$

$$K(x,y|x_2,\ldots,x_n,t)=\frac{|(x,y|x_2,\ldots,x_n)|}{t+|(x,y|x_2,\ldots,x_n)|}$$

then (X,J)is an i-f-n-IPS.

**Proof:**

(1)Clearly $J(x,x|x_2,\ldots,x_n,t)+K(x,x|x_2,\ldots,x_n,t)\ \leq\ 1$

(2)It is obvious that $J(x,y|x_2,\ldots,x_n,t)>0$

The results

(3)$J(x,x|x_2,\ldots,x_n,t)$= 1 if and only if $x,x_2,\ldots,x_n$ are linearly dependent,

(4) $J(x,y|x_2,\ldots,x_n,t)\ =\ J(y,x|x_2,\ldots,x_n,t)$,

(5) $J(x,y|x_2,\ldots,x_n,t)$ is invariant under any permutation of $x_2,\ldots,x_n$,

(6) $J(x,x|x_2,\ldots,x_n,t)\ =\ J(x_2,x_2|x,x_3\ldots,x_n,t)$,

(7) $J(ax,bx|x_2,\ldots,x_n,t)\ =\ J(x,x|x_2,\ldots,x_n,\frac{t}{|ab|})$, a,b$\in\Re$ (real),

(8) $J(x,y|x_2,\ldots,x_n,t)*J(x',y|x_2,\ldots,x_n,s)\leq J(x+x',y|x_2,\ldots,x_n,s+t)$,

(9) $J(x,x|x_2,\ldots,x_n,t)*J(y,y|x_2,\ldots,x_n,s)\leq J(x,y|x_2,\ldots,x_n,\sqrt{ts})$,

(10) $J(x,y|x_2,\ldots,x_n,t):(0,\infty)\rightarrow[0,1]$ is continuous in t.

are showed in Example 4.1.4.

(11)It is obvious that $K(x,y|x_2,\ldots,x_n,t)>0$

(12)For all t $\in\Re$ with t>0,

$$K(x,x|x_2,\ldots,x_n,t)=0$$

$$\Leftrightarrow\frac{|(x,x|x_2,\ldots,x_n)|}{t+|(x,x|x_2,\ldots,x_n)|}=0,$$

$$\Leftrightarrow |(x,x|x_2,\ldots,x_n)| = 0,$$

$$\Leftrightarrow (x,x|x_2,\ldots,x_n) = 0,$$

$$\Leftrightarrow x,x_2,\ldots,x_n \text{are linearly dependent.}$$

(13)For all t>0,

$$\begin{aligned} K(x,y|x_2,\ldots,x_n,t) &= \frac{|(x,y|x_2,\ldots,x_n)|}{t+|(x,y|x_2,\ldots,x_n)|} \\ &= \frac{|(y,x|x_2,\ldots,x_n)|}{t+|(y,x|x_2,\ldots,x_n)|} \\ &= K(y,x|x_2,\ldots,x_n,t) \end{aligned}$$

(14)As $(x,x|x_2,\ldots,x_n)$ is invariant under any permutation of $x_2,\ldots,x_n$, we have $K(x,y|x_2,\ldots,x_n,t)$ is invariant under any permutation.

(15)For all t>0,

$$\begin{aligned} K(x,x|x_2,\ldots,x_n,t) &= \frac{|(x,x|x_2,\ldots,x_n)|}{t+|(x,x|x_2,\ldots,x_n)|} \\ &= \frac{|(x_2,x_2|x,x_3,\ldots,x_n)|}{t+|(x_2,x_2|x,x_3,\ldots,x_n)|} \\ &= K(x_2,x_2|x,x_3,\ldots,x_n,t) \end{aligned}$$

(16)For all t>0,

$$\begin{aligned} K(ax,bx|x_2,\ldots,x_n,t) &= \frac{|(ax,bx|x_2\ldots,x_n)|}{t+|(ax,bx|x_2\ldots,x_n)|} \\ &= \frac{|ab||(x,x|x_2,\ldots,x_n)|}{t+|ab||(x,x|x_2,\ldots,x_n)|} \\ &= \frac{|(x,x|x_2,\ldots,x_n)|}{\frac{t}{|ab|}+|(x,x|x_2,\ldots,x_n)|} \\ &= K(x,x|x_2,\ldots,x_n,\frac{t}{|ab|}) \end{aligned}$$

(17)Without loss of generality assume that

$$\begin{aligned} K(x',y|x_2,\ldots,x_n,s) &\leq K(x,y|x_2,\ldots,x_n,t) \\ \frac{|(x',y|x_2,\ldots,x_n)|}{s+|(x',y|x_2,\ldots,x_n)|} &\leq \frac{|(x,y|x_2,\ldots,x_n)|}{t+|(x,y|x_2,\ldots,x_n)|} \\ t|(x',y|x_2,\ldots,x_n)| &\leq s|(x,y|x_2,\ldots,x_n)| \end{aligned}$$

$$t|(x', y|x_2, \ldots, x_n)| - s|(x, y|x_2, \ldots, x_n)| \leq 0 \tag{4.1}$$

Now,

$$\frac{|(x+x', y|x_2, \ldots, x_n)|}{s+t+|(x+x', y|x_2, \ldots, x_n)|} - \frac{|(x, y|x_2, \ldots, x_n)|}{t+|(x, y|x_2, \ldots, x_n)|}$$

$$\leq \frac{|(x, y|x_2, \ldots, x_n)| + |(x', y|x_2, \ldots, x_n)|}{s+t+|(x, y|x_2, \ldots, x_n)| + |(x', y|x_2, \ldots, x_n)|} - \frac{|(x, y|x_2, \ldots, x_n)|}{t+|(x, y|x_2, \ldots, x_n)|}$$

$$= \frac{t|(x', y|x_2, \ldots, x_n)| - s|(x, y|x_2, \ldots, x_n)|}{(s+t+|(x, y|x_2, \ldots, x_n)| + |(x', y|x_2, \ldots, x_n)|)(t+|(x, y|x_2, \ldots, x_n)|)}$$

By (4.1) we have

$$\frac{|(x+x', y|x_2, \ldots, x_n)|}{s+t+|(x+x', y|x_2, \ldots, x_n)|} \leq \frac{|(x, y|x_2, \ldots, x_n)|}{t+|(x, y|x_2, \ldots, x_n)|}$$

Similarrly,

$$\frac{|(x+x', y|x_2, \ldots, x_n)|}{s+t+|(x+x', y|x_2, \ldots, x_n)|} \leq \frac{|(x', y|x_2, \ldots, x_n)|}{s+|(x', y|x_2, \ldots, x_n)|}$$

$\Rightarrow K(x+x', y|x_2, \ldots, x_n, s+t) \leq \max\{K(x, y|x_2, \ldots, x_n, t), K(x', y|x_2, \ldots, x_n, s)\}$

(18)Without loss of generality assume that

$$\begin{aligned}
K(y, y|x_2, \ldots, x_n, s) &\leq K(x, x|x_2, \ldots, x_n, t) \text{ for all } s, t \in \Re \text{with} s > 0, t > 0. \\
\frac{|(y, y|x_2, \ldots, x_n)|}{s+|(y, y|x_2, \ldots, x_n)|} &\leq \frac{|(x, x|x_2, \ldots, x_n)|}{t+|(x, x|x_2, \ldots, x_n)|} \\
t|(y, y|x_2, \ldots, x_n)| &\leq s|(x, x|x_2, \ldots, x_n)| \\
t|(y, y|x_2, \ldots, x_n)||(x, x|x_2, \ldots, x_n)| &\leq s|(x, x|x_2, \ldots, x_n)|^2 \\
\frac{|(x, y|x_2, \ldots, x_n)|^2}{st} &\leq \frac{|(x, x|x_2, \ldots, x_n)|^2}{t^2}
\end{aligned}$$

Taking square root,

$$\begin{aligned}
\frac{|(x, y|x_2, \ldots, x_n)|}{\sqrt{st}} &\leq \frac{|(x, x|x_2, \ldots, x_n)|}{t} \\
t|(x, y|x_2, \ldots, x_n)| - \sqrt{st}|(x, x|x_2, \ldots, x_n)| &\leq 0
\end{aligned}$$

Now,

$$\frac{|(x,y|x_2,\dots,x_n)|}{\sqrt{st}+|(x,y|x_2,\dots,x_n)|}-\frac{|(x,x|x_2,\dots,x_n)|}{t+|(x,x|x_2,\dots,x_n)|}$$

$$=\frac{t|(x,y|x_2,\dots,x_n)|-\sqrt{st}|(x,x|x_2,\dots,x_n)|}{(\sqrt{st}+|(x,y|x_2,\dots,x_n)|)(t+|(x,x|x_2,\dots,x_n)|)}\leq 0$$

$$\frac{|(x,y|x_2,\dots,x_n)|}{\sqrt{st}+|(x,y|x_2,\dots,x_n)|}\leq\frac{|(x,x|x_2,\dots,x_n)|}{t+|(x,x|x_2,\dots,x_n)|}$$

Similarrly,

$$\frac{|(x,y|x_2,\dots,x_n)|}{\sqrt{st}+|(x,y|x_2,\dots,x_n)|}\leq\frac{|(y,y|x_2,\dots,x_n)|}{s+|(y,y|x_2,\dots,x_n)|}$$

$K(x,y|x_2,\dots,x_n,\sqrt{st})\leq \max\{K(x,x|x_2,\dots,x_n,t),K(y,y|x_2,\dots,x_n,s)\}$

(19)Clearly $K(x,y|x_2,\dots,x_n,t):(0,\infty)\to[0,1]$ is continuous in t.