

Chapter III

CHAPTER III

INTERVAL-VALUED FUZZY SOFT SETS

Definition: 3.1

An **Interval-Valued Fuzzy Set** \hat{X} on a universe U is a mapping such that $\hat{X} : U \rightarrow \text{Int}([0,1])$, where $\text{Int}([0,1])$ stands for the set of all closed subintervals of $[0, 1]$, the set of all interval-valued fuzzy sets on U is denoted by $\tilde{F}(U)$.

Suppose that $\hat{X} \in \tilde{F}(U)$, $\forall x \in U$, $\mu_{\hat{X}}(x) = [\mu_{\hat{X}}^-(x), \mu_{\hat{X}}^+(x)]$ is called the degree of membership an element x to \hat{X} . $\mu_{\hat{X}}^-(x)$ and $\mu_{\hat{X}}^+(x)$ are referred to as the lower and upper degrees of membership x to \hat{X} where $0 \leq \mu_{\hat{X}}^-(x) \leq \mu_{\hat{X}}^+(x) \leq 1$.

Definition: 3.2

The **Complement, Intersection and Union of the Interval-Valued Fuzzy Sets** are defined as follows: Let $\hat{X}, \hat{Y} \in \tilde{F}(U)$, then

1) the complement of \hat{X} is denoted by \hat{X}^c where

$$\mu_{\hat{X}^c}(x) = 1 - \mu_{\hat{X}}(x) = [1 - \mu_{\hat{X}}^+(x), 1 - \mu_{\hat{X}}^-(x)]$$

2) the intersection of \hat{X} and \hat{Y} is denoted by $\hat{X} \cap \hat{Y}$ where

$$\mu_{\hat{X} \cap \hat{Y}}(x) = \inf[\mu_{\hat{X}}(x), \mu_{\hat{Y}}(x)] = [\inf(\mu_{\hat{X}}^-(x), \mu_{\hat{Y}}^-(x)), \inf(\mu_{\hat{X}}^+(x), \mu_{\hat{Y}}^+(x))]$$

3) the union of \hat{X} and \hat{Y} is denoted by $\hat{X} \cup \hat{Y}$ where

$$\mu_{\hat{X} \cup \hat{Y}}(x) = \sup[\mu_{\hat{X}}(x), \mu_{\hat{Y}}(x)] = [\sup(\mu_{\hat{X}}^-(x), \mu_{\hat{Y}}^-(x)), \sup(\mu_{\hat{X}}^+(x), \mu_{\hat{Y}}^+(x))]$$

Definition: 3.3

Let U be an initial universe and E be a set of parameters, a pair (F, E) is called an **Interval-Valued Fuzzy Soft Set** over $\tilde{F}(U)$, where F is a mapping given by $F : E \rightarrow \tilde{F}(U)$.

Definition: 3.4

An **Interval Fuzzy Number** is defined as $\hat{A} = [a_L, a_R] = \{ a : a_L \leq a \leq a_R \}$ where, $a_L, a_R \in [0, 1]$. A matrix of order $n \times n$ is said to be an **Interval Fuzzy Number Matrix (IFNM)** if all its elements are the Interval Fuzzy Numbers.

Example: 3.5

Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be the set of houses under consideration and $A = \{e_1, e_2, e_3, e_4\} = \{ \text{beautiful, wooden, cheap, in the green surroundings} \}$.

U	e_1	e_2	e_3	e_4
h_1	[0.7, 0.9]	[0.6, 0.7]	[0.3, 0.5]	[0.5, 0.8]
h_2	[0.6, 0.8]	[0.8, 1.0]	[0.8, 0.9]	[0.9, 1.0]
h_3	[0.5, 0.6]	[0.2, 0.4]	[0.5, 0.7]	[0.7, 0.9]
h_4	[0.6, 0.8]	[0.0, 0.1]	[0.7, 1.0]	[0.6, 0.8]
h_5	[0.8, 0.9]	[0.1, 0.3]	[0.9, 1.0]	[0.2, 0.5]
h_6	[0.8, 1.0]	[0.7, 0.8]	[0.2, 0.5]	[0.7, 1.0]

Table.1

The tabular representation of an interval-valued fuzzy soft set (F, A) is shown in Table.1. In Table.1, we can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation are given. For example, we cannot present the precise degree of how beautiful house h_1 is at least beautiful on the degree of 0.7 and it is at most beautiful on the degree of 0.9.

Remark: 3.6

An interval-valued fuzzy soft set is a parameterized family of interval-valued fuzzy subsets of U , thus, its universe is the set of all interval-valued fuzzy sets of U , *i.e.* $\tilde{F}(U)$. An interval-valued fuzzy soft set is a special case of a soft set because it is still a mapping from parameters to $\tilde{F}(U)$. $\forall \varepsilon \in E$, $F(\varepsilon)$ is referred as the interval fuzzy value set of parameter ε , it is actually an interval-valued valued fuzzy set of U where $x \in U$ and $\varepsilon \in E$, it can be written as:

$$F(\varepsilon) = \left\{ \langle x, \mu_{F(\varepsilon)}(x) \rangle : x \in U \right\},$$

Here, $F(\varepsilon)$ is the interval-valued fuzzy membership degree that object x holds on parameter ε . If $\forall \varepsilon \in E, \forall x \in U, \mu_{F(\varepsilon)}^-(x) = \mu_{F(\varepsilon)}^+(x)$, then $F(\varepsilon)$ will degenerated to be a standard fuzzy set and then (F, E) will be degenerated to be a traditional fuzzy soft set.

Definition: 3.7

Suppose that (F, E) is an interval-valued fuzzy soft set over $\tilde{F}(U)$, $F(\varepsilon)$ is the interval fuzzy value set of parameter ε , then all interval fuzzy value sets in interval-valued fuzzy soft set (F, E) are referred to as the **Interval Fuzzy Value Class** of (F, E) and is denoted by $C_{(F,E)}$, then we have

$$C_{(F,E)} = \{F(\varepsilon) : \varepsilon \in E\}$$

Example: 3.8

From Example 3.5, we have $C_{(F,E)} = \{F(e_1), F(e_2), F(e_3), F(e_4)\}$ where:

$$F(e_1) = \left\{ \langle h_1, [0.7, 0.9] \rangle, \langle h_2, [0.6, 0.8] \rangle, \langle h_3, [0.5, 0.6] \rangle, \right. \\ \left. \langle h_4, [0.6, 0.8] \rangle, \langle h_5, [0.8, 0.9] \rangle, \langle h_6, [0.8, 1.0] \rangle \right\}$$

$$F(e_2) = \left\{ \begin{array}{l} \langle h_1, [0.6, 0.7] \rangle, \langle h_2, [0.8, 1.0] \rangle, \langle h_3, [0.2, 0.4] \rangle, \\ \langle h_4, [0.0, 0.1] \rangle, \langle h_5, [0.1, 0.3] \rangle, \langle h_6, [0.7, 0.8] \rangle \end{array} \right\}$$

$$F(e_3) = \left\{ \begin{array}{l} \langle h_1, [0.3, 0.5] \rangle, \langle h_2, [0.8, 0.9] \rangle, \langle h_3, [0.5, 0.7] \rangle, \\ \langle h_4, [0.7, 1.0] \rangle, \langle h_5, [0.9, 1.0] \rangle, \langle h_6, [0.2, 0.5] \rangle \end{array} \right\}$$

$$F(e_4) = \left\{ \begin{array}{l} \langle h_1, [0.5, 0.8] \rangle, \langle h_2, [0.9, 1.0] \rangle, \langle h_3, [0.7, 0.9] \rangle, \\ \langle h_4, [0.6, 0.8] \rangle, \langle h_5, [0.2, 0.5] \rangle, \langle h_6, [0.7, 1.0] \rangle \end{array} \right\}$$

Definition: 3.9

Let U be an initial universe and E be a set of parameters, suppose that $A, B \subset E$, (F, A) and (G, B) are two interval-valued fuzzy soft sets, we say that (F, A) is an **Interval-Valued Fuzzy Soft Subset** of (G, B) if and only if

- 1) $A \subset B$
- 2) $\forall \varepsilon \in A$, $F(\varepsilon)$ is an interval-valued fuzzy subset of $G(\varepsilon)$.

Which can be denoted by $(F, A) \tilde{\subset} (G, B)$.

Remark: 3.10

By the Definition 3.9, we can see that (F, A) is an interval-valued fuzzy soft subset of (G, B) if the following two conditions hold:

- 1) the set of parameters of interval-valued fuzzy soft set (F, A) is a subset of the set of parameters of interval-valued fuzzy soft set (G, B)
- 2) interval-valued fuzzy set $F(\varepsilon)$ is a subset of interval-valued fuzzy set $G(\varepsilon)$ where ε is an arbitrary parameter held by (F, A) and (G, B) .

Definition: 3.11

(F, A) is said to be an **Interval-Valued Fuzzy Soft Super Set** of (G, B) , if (G, B) is an interval-valued fuzzy soft subset of (F, A) . We denote it by $(F, A) \tilde{\supset} (G, B)$.

Example: 3.12

Given two interval-valued fuzzy soft sets (F, A) and (G, B) , let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be the set of houses,

$$A = \{e_1, e_2\} = \{\text{beautiful, wooden}\},$$

$$B = \{e_1, e_2, e_3\} = \{\text{beautiful, wooden, cheap}\} \text{ and}$$

$$F(e_1) = \left\{ \langle h_1, [0.7, 0.9] \rangle, \langle h_2, [0.6, 0.8] \rangle, \langle h_3, [0.5, 0.6] \rangle, \right. \\ \left. \langle h_4, [0.6, 0.8] \rangle, \langle h_5, [0.8, 0.9] \rangle, \langle h_6, [0.8, 1.0] \rangle \right\}$$

$$F(e_2) = \left\{ \langle h_1, [0.6, 0.7] \rangle, \langle h_2, [0.8, 1.0] \rangle, \langle h_3, [0.2, 0.4] \rangle, \right. \\ \left. \langle h_4, [0.0, 0.1] \rangle, \langle h_5, [0.1, 0.3] \rangle, \langle h_6, [0.7, 0.8] \rangle \right\}$$

$$G(e_1) = \left\{ \langle h_1, [0.8, 1.0] \rangle, \langle h_2, [0.6, 0.8] \rangle, \langle h_3, [0.7, 0.9] \rangle, \right. \\ \left. \langle h_4, [0.6, 0.8] \rangle, \langle h_5, [0.8, 1.0] \rangle, \langle h_6, [0.8, 1.0] \rangle \right\}$$

$$G(e_2) = \left\{ \langle h_1, [0.6, 0.7] \rangle, \langle h_2, [0.9, 1.0] \rangle, \langle h_3, [0.4, 0.5] \rangle, \right. \\ \left. \langle h_4, [0.1, 0.3] \rangle, \langle h_5, [0.1, 0.3] \rangle, \langle h_6, [0.9, 1.0] \rangle \right\}$$

$$G(e_3) = \left\{ \langle h_1, [0.3, 0.5] \rangle, \langle h_2, [0.8, 0.9] \rangle, \langle h_3, [0.5, 0.7] \rangle, \right. \\ \left. \langle h_4, [0.7, 0.1] \rangle, \langle h_5, [0.9, 0.1] \rangle, \langle h_6, [0.2, 0.5] \rangle \right\}$$

Clearly, we have $(F, A) \tilde{=} (G, B)$.

Definition: 3.13

Let (F, A) and (G, B) be two interval-valued fuzzy soft sets, are said to be **Interval-Valued Fuzzy Soft Equal** if and only if

- 1) (F, A) is an interval-valued fuzzy soft subset of (G, B)
- 2) (G, B) is an interval-valued fuzzy soft subset of (F, A)

which can be denoted by $(F, A) = (G, B)$.

Definition: 3.14

The **Complement of an Interval-Valued Fuzzy Soft Set** (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \neg A)$, where $\forall \alpha \in A, \neg \alpha = \text{not } \alpha$, is the not set of the parameter α , which holds the opposite meanings of parameter α ; $F^c : \neg A \rightarrow \tilde{F}(U)$ is a mapping given by $F^c(\beta) = (F(\neg \beta))^c, \forall \beta \in \neg A$.

Example: 3.15

Consider another interval-valued fuzzy soft set (G, B) shown in Table.2. Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ is the set of the houses under consideration, $B = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{\text{convenient traffic, modern style, in good repair}\}$ is the set of parameters.

U	ε_1	ε_2	ε_3
h_1	[0.3, 0.5]	[0.9, 1.0]	[0.6, 0.7]
h_2	[0.7, 0.9]	[0.8, 1.0]	[0.1, 0.3]
h_3	[0.5, 0.7]	[0.2, 0.5]	[0.9, 1.0]
h_4	[0.2, 0.3]	[0.1, 0.3]	[0.2, 0.4]
h_5	[0.8, 0.9]	[0.9, 1.0]	[0.5, 0.7]
h_6	[0.9, 1.0]	[0.5, 0.6]	[0.8, 1.0]

Table.2

Thus by Definition 3.14, we have

$$G^c(\neg \varepsilon_1) = \left\{ \langle h_1, [0.5, 0.7] \rangle, \langle h_2, [0.1, 0.3] \rangle, \langle h_3, [0.3, 0.4] \rangle, \right. \\ \left. \langle h_4, [0.7, 0.8] \rangle, \langle h_5, [0.1, 0.2] \rangle, \langle h_6, [0.0, 0.1] \rangle \right\}$$

$$G^C(\neg \varepsilon_2) = \left\{ \langle h_1, [0.0, 0.1] \rangle, \langle h_2, [0.0, 0.2] \rangle, \langle h_3, [0.5, 0.8] \rangle, \right. \\ \left. \langle h_4, [0.7, 0.9] \rangle, \langle h_5, [0.0, 0.1] \rangle, \langle h_6, [0.4, 0.5] \rangle \right\}$$

$$G^C(\neg \varepsilon_3) = \left\{ \langle h_1, [0.3, 0.4] \rangle, \langle h_2, [0.7, 0.9] \rangle, \langle h_3, [0.0, 0.1] \rangle, \right. \\ \left. \langle h_4, [0.6, 0.8] \rangle, \langle h_5, [0.3, 0.5] \rangle, \langle h_6, [0.0, 0.2] \rangle \right\}$$

Definition: 3.16

The “AND” operation on the two Interval-Valued Fuzzy Soft Sets (F, A) , (G, B) is defined by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$, $\forall (\alpha, \beta) \in A \times B$, where ‘ \cap ’ is the intersection of two interval-valued fuzzy sets.

Definition: 3.17

The “OR” operation on the two Interval-Valued Fuzzy Soft Sets (F, A) , (G, B) is defined by $(F, A) \vee (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cup G(\beta)$, $\forall (\alpha, \beta) \in A \times B$, where ‘ \cup ’ is the union of two interval-valued fuzzy sets.

Theorem: 3.18 DeMorgan’s Laws of Interval-Valued Fuzzy Soft Sets:

Let (F, A) and (G, B) are two interval-valued fuzzy soft sets, we have

- 1) $((F, A) \wedge (G, B))^C = (F, A)^C \vee (G, B)^C$
- 2) $((F, A) \vee (G, B))^C = (F, A)^C \wedge (G, B)^C$

Proof:

$$1) (F, A)^C \vee (G, B)^C = (F^C, \neg A) \vee (G^C, \neg B) \\ = (I, \neg A \times \neg B) \text{ where } I(\neg \alpha, \neg \beta) = F^C(\neg \alpha) \cup G^C(\neg \beta) \\ = (I, \neg(A \times B))$$

Suppose that $(F, A) \wedge (G, B) = (H, A \times B)$,

then we have $((F, A) \wedge (G, B))^c = (H, A \times B)^c$

$$= (H^c, \neg(A \times B))$$

$\forall (\alpha, \beta) \in A \times B$, we have

$$\begin{aligned} H^c(\neg\alpha, \neg\beta) &= (H(\alpha, \beta))^c \\ &= (F(\alpha) \cap G(\beta))^c \\ &= (F(\alpha))^c \cup (G(\beta))^c \\ &= F^c(\neg\alpha) \cup G^c(\neg\beta) \end{aligned}$$

From the discussion above, we have $((F, A) \wedge (G, B))^c = (F, A)^c \vee (G, B)^c$.

2) Similar to (1), we can prove $((F, A) \vee (G, B))^c = (F, A)^c \wedge (G, B)^c$.

Theorem: 3.19 Distribution Law of Interval-Valued Fuzzy Soft Sets:

Let (F, A) , (G, B) and (H, C) be three interval-valued fuzzy soft sets, then we have

- 1) $(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$
- 2) $(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$

Proof:

- 1) $\forall \alpha \in A, \forall \beta \in B$ and $\forall \gamma \in C$, by the properties of interval-valued fuzzy sets, we have $F(\alpha) \cap (G(\beta) \cup H(\gamma)) = (F(\alpha) \cap G(\beta)) \cup (F(\alpha) \cap H(\gamma))$, from which we can conclude that

$$(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C)) \text{ holds.}$$

- 2) Similarity, we also have

$$(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C)).$$

Example: 3.20

U	e_1, ε_1	e_1, ε_2	e_1, ε_3	e_2, ε_1	e_2, ε_2	e_2, ε_3
h_1	[0.3, 0.5]	[0.7, 0.9]	[0.6, 0.7]	[0.3, 0.5]	[0.6, 0.7]	[0.6, 0.7]
h_2	[0.6, 0.8]	[0.6, 0.8]	[0.1, 0.3]	[0.7, 0.9]	[0.8, 1.0]	[0.1, 0.3]
h_3	[0.5, 0.6]	[0.2, 0.5]	[0.5, 0.6]	[0.2, 0.4]	[0.2, 0.4]	[0.2, 0.4]
h_4	[0.2, 0.3]	[0.1, 0.3]	[0.2, 0.4]	[0.0, 0.1]	[0.0, 0.1]	[0.0, 0.1]
h_5	[0.8, 0.9]	[0.8, 0.9]	[0.5, 0.7]	[0.1, 0.3]	[0.1, 0.3]	[0.1, 0.3]
h_6	[0.8, 1.0]	[0.5, 0.6]	[0.8, 1.0]	[0.7, 0.8]	[0.5, 0.6]	[0.7, 0.8]
U	e_3, ε_1	e_3, ε_2	e_3, ε_3	e_4, ε_1	e_4, ε_2	e_4, ε_3
h_1	[0.3, 0.5]	[0.3, 0.5]	[0.3, 0.5]	[0.3, 0.5]	[0.5, 0.8]	[0.5, 0.7]
h_2	[0.7, 0.9]	[0.8, 0.9]	[0.1, 0.3]	[0.7, 0.9]	[0.8, 1.0]	[0.1, 0.3]
h_3	[0.5, 0.7]	[0.2, 0.5]	[0.5, 0.7]	[0.5, 0.7]	[0.2, 0.5]	[0.7, 0.9]
h_4	[0.2, 0.3]	[0.1, 0.3]	[0.2, 0.4]	[0.2, 0.3]	[0.1, 0.3]	[0.2, 0.4]
h_5	[0.8, 0.9]	[0.9, 1.0]	[0.5, 0.7]	[0.2, 0.5]	[0.2, 0.5]	[0.2, 0.5]
h_6	[0.2, 0.5]	[0.2, 0.5]	[0.2, 0.5]	[0.7, 1.0]	[0.5, 0.6]	[0.7, 1.0]

Table.3

The result of the “AND” operation on the two interval-valued fuzzy soft sets (F, A) , (G, B) in Table.1 and Table.2 is shown in Table.3

Example: 3.21

U	e_1, ε_1	e_1, ε_2	e_1, ε_3	e_2, ε_1	e_2, ε_2	e_2, ε_3
h_1	[0.7, 0.9]	[0.9, 1.0]	[0.7, 0.9]	[0.6, 0.7]	[0.9, 1.0]	[0.6, 0.7]
h_2	[0.7, 0.9]	[0.8, 1.0]	[0.6, 0.8]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]
h_3	[0.5, 0.7]	[0.5, 0.6]	[0.9, 1.0]	[0.5, 0.7]	[0.2, 0.5]	[0.9, 1.0]
h_4	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.2, 0.3]	[0.1, 0.3]	[0.2, 0.4]
h_5	[0.8, 0.9]	[0.9, 1.0]	[0.8, 0.9]	[0.8, 0.9]	[0.9, 1.0]	[0.5, 0.7]
h_6	[0.9, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.9, 1.0]	[0.7, 0.8]	[0.8, 1.0]
U	e_3, ε_1	e_3, ε_2	e_3, ε_3	e_4, ε_1	e_4, ε_2	e_4, ε_3
h_1	[0.3, 0.5]	[0.9, 1.0]	[0.6, 0.7]	[0.5, 0.8]	[0.9, 1.0]	[0.6, 0.8]
h_2	[0.8, 0.9]	[0.8, 1.0]	[0.8, 0.9]	[0.9, 1.0]	[0.9, 1.0]	[0.9, 1.0]
h_3	[0.5, 0.7]	[0.5, 0.7]	[0.9, 1.0]	[0.7, 0.9]	[0.7, 0.9]	[0.9, 1.0]
h_4	[0.7, 1.0]	[0.7, 1.0]	[0.7, 1.0]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]
h_5	[0.9, 1.0]	[0.9, 1.0]	[0.9, 1.0]	[0.8, 0.9]	[0.9, 1.0]	[0.5, 0.7]
h_6	[0.9, 1.0]	[0.5, 0.6]	[0.8, 1.0]	[0.9, 1.0]	[0.7, 1.0]	[0.8, 1.0]

Table.4

The result of the “OR” operation on the two interval-valued fuzzy soft sets (F, A), (G, B) in Table.1 and Table.2 is shown in Table.4

Theorem: 3.22 Associative Law of Interval-Valued Fuzzy Soft Sets:

Let (F, A) , (G, B) and (H, C) be three interval-valued fuzzy soft sets, then we have

1) $(F, A) \wedge ((G, B) \wedge (H, C)) = ((F, A) \wedge (G, B)) \wedge (H, C)$

2) $(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee (G, B)) \vee (H, C)$

Proof:

1) $\forall \alpha \in A, \forall \beta \in B$ and $\forall \gamma \in C$, we have

$F(\alpha) \cap (G(\beta) \cap H(\gamma)) = (F(\alpha) \cap G(\beta)) \cap H(\gamma)$, from which we can conclude that $(F, A) \wedge ((G, B) \wedge (H, C)) = ((F, A) \wedge (G, B)) \wedge (H, C)$ holds.

2) Similarity, we also have $(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee (G, B)) \vee (H, C)$.