Chapter 9

(i, j) - δP_S-Open sets in Bitopological Spaces

9.1 Introduction

In 1963, Kelly J.C. first introduced the concept of bitopological spaces, where X is a nonempty set τ_1, τ_2 are topologies on X. The classical theorems in general topological spaces become particular cases of the analogous theorems for bitopological spaces. In this chapter, we introduced the concept of a conditional preopen set in a bitopological spaces, and we find basic properties and relationships with other concepts of sets. A new class of sets, called (i, j)-δP_S-open sets in bitopological spaces was defined. By using this set, we introduced and defined a notion of (i, j)-δP_S-continuity and investigated some of its properties. In particular, (i, j)-δP_S-open sets and (i, j)-δP_S-continuity are used to extend some known results of continuity.

9.2 (i, j)-δP_S-Open Sets

In this section, we introduce and define a new type of sets in bitopological spaces and we find some of its properties.

Definition 9.2.1: A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j)**-**δP_S**-open**, if A is a j-δ-preopen set and for all x in A, there exists an i-semiclosed set F such that $x \in F \subseteq A$.

Remark 9.2.2: (a) The family of (i, j) δP_S-open subset of x is denoted by (i, j) $\delta P_S O(X)$.

(b) A subset B of X is called (i, j) δP_S-closed if and only if B^C is (i, j) δP_S-open.

Proposition 9.2.3: A subset A of a bitopological space X is (i, j) δP_S-open, if A is j δ-preopen set and it is a union of i-semi closed sets. This means that $A = \cup F_\gamma$, where A is a j δ-preopen and F_γ is an i-semi closed set for each γ.

Proof: The proof follows from Proposition 2.2.2.

Remark 9.2.4: By setting $\tau_i = \tau_j = \tau$ in the Definition 9.2.1, a (i, j) δP_S-open set is δP_S-open set in (X, τ).

Remark 9.2.5: In general, (i, j) $\delta P_S O(X) \neq (j, i)$ $\delta P_S O(X)$.

Example 9.2.6: Let $\tau_1 \subseteq \tau_2$, where $\tau_1 = \{X, \emptyset, \{a\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}\}$. Then $(1,2)$ $\delta P_S O(X) = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$ and $(2,1)$ $\delta P_S O(X) = \{X, \emptyset, \{a\}, \{b, c\}\}$. Then

(1,2) $\delta P_S O(X) \neq$ (2,1) $\delta P_S O(X)$. Since $\{c\} \in$ (1,2) $\delta P_S O(X)$ but not in (2,1) $\delta P_S O(X)$ and $\{a\} \in$ (2,1) $\delta P_S O(X)$ but not in (1,2) $\delta P_S O(X)$.

Remark 9.2.7: If $\tau_1 \subseteq \tau_2$, then (1,2) $\delta P_S O(X)$ and (2,1) $\delta P_S O(X)$ are independent. This hereditary property is not preserved for (1,2) δP_S-open sets in bitopology.

Remark 9.2.8: Every (i,j) $\delta P_S O(X)$ is a j $\delta PO(X)$ which follows from definition 9.2.2, but not conversely.

Example 9.2.9: Consider $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $\delta PO(X, \tau_j) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and (1,2) $\delta P_S O(X) = \{X, \emptyset, \{b\}\}$. Not all 2 δP-open set is a (1,2) δP_S-open set.

Proposition 9.2.10: For a bitopological space, (X, τ_i, τ_j), (i,j) $P_S O(X) \subseteq (i,j)$ $\delta P_S O(X)$.

Proof: Let $A \in (i,j)$ $P_S O(X)$. Then by Definition 1.4.4, A is a preopen set and hence A is a δ-preopen set. Since $A \in (i,j)$ $P_S O(X)$, for all $x \in A$, there exists a semi-closed set F such that $x \in F \subseteq A$. Hence by Definition 9.2.2 A is an (i,j) δP_S-open set.

Remark 9.2.11: An (i,j) δP_S-open set need not be an (i,j) δP_S-open set.

Example 9.2.12: Let $X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}, \{a, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a, c\}\}$.Then (1,2) $P_S O(X) = \{X, \emptyset, \{c\}, \{b, c\}\}$ and (i,j) $\delta P_S O(X) = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. Here $\{b\}$ is a (i,j) δP_S-open set but not a (i,j) P_S-open set.

Proposition 9.2.13: The union of any family of (i,j) δP_S-open sets in a space X is also (i,j) δP_S-open.

Remark 9.2.14: The intersection of two (i,j) δP_S-open sets is not (i,j) δP_S-open set in general, as shown in the following example:

Example 9.2.15: Let
$X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$,
$\tau_2 = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ then
(1,2) $\delta P_S O(X) = \{X, \emptyset, \{a\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}\}$ if $A = \{b, c, d\}$ and $B = \{a, c, d\}$ then $A \cap B = \{c, d\} \notin$ (1,2) $\delta P_S O(X)$.

Proposition 9.2.16: A subset A is an (i,j) δP_S-open set if it is both j-open and i-closed.

Proof: Let $A \subseteq X$. A is both j-open and i-closed. A is both j-open $\Rightarrow A$ is j δ-preopen. And A is i-closed $\Rightarrow A$ is i-semi-closed. By Proposition 9.2.2, A is (i,j) δP_S-closed.

Proposition 9.2.17: Let (X, τ_1, τ_2) be a bitopological space if (X, τ_i) is a semi-T_1-space, then $(i, j)\ \delta P_S O(X) = j\ \delta\ PO(X)$.

Proof: Let A be any subset of a space X and A is $j\ \delta$-preopen set, if $A = \emptyset$, then $A \in (i, j)\ \delta P_S O(X)$, if $A \neq \emptyset$, now let $x \in A$, since (X, τ_1) is semi-T_1 space, then by Theorem 1.1.10, every singleton is i-semi closed set, and hence $x \in \{x\} \subseteq A$, therefore $A \in (i, j)\ \delta P_S O(X)$, hence $j\ \delta\ PO(X) \subseteq (i, j)\ \delta P_S O(X)$ but $(i, j)\ \delta P_S O(X) \subseteq j\ \delta\ PO(X)$ generally, thus $(i, j)\ \delta P_S O(X) = j\ \delta\ PO(X)$.

Proposition 9.2.18: A subset A of a space (X, τ_1, τ_2) is $(i, j)\ \delta P_S$-open if and only if for each $x \in A$, there exists an $(i, j)\ \delta P_S$-open set B such that $x \in B \subseteq A$.

Proof. Assume that A is an $(i, j)\ \delta P_S$-open set in the (X, τ_1, τ_2)then for each $x \in A$, put $B = A$ is $(i, j)\ \delta P_S$-open set containing x such that $x \in B \subseteq A$.

Conversely, suppose that for each $x \in A$, there exists an $(i, j)\ \delta P_S$-open set B such that $x \in B_x \subseteq A$, thus $A = \cup B_x$, where $B_x \in (i, j)\ \delta P_S O(X)$ for each x, By Remark 9.2.4 A is $(i, j)\ \delta P_S$-open.

Corollary 9.2.19: Let (X, τ_1, τ_2) be a bitopological space if (X, τ_i) is a semi-T_1-space and (X, τ_j) is semi-regular space $\delta PO(X) = PO(X)$.

Proposition 9.2.20: In a bitopological space (X, τ_1, τ_2) if the space (X, τ_i) is hyperconnected then $(i, j)\ \delta P_S O(X) \neq \{X, \emptyset\}$.

Example 9.2.21: Let (X, τ_1, τ_2) be a bitopological space. $\tau_1 = \{X, \emptyset, \{\text{a}\}\}$ and $\tau_2 = \{X, \emptyset, \{\text{a}\}, \{\text{a, b}\}\}$. Then τ_1 is hyperconnected and $(1,2)\ \delta P_S O(X) = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\} \neq \{X, \emptyset\}$.

Remark 9.2.22: In the above conditions, from Theorem 1.4.6, it is proved that $(\boldsymbol{i}, \boldsymbol{j})\ \boldsymbol{P_S O(X)} = \{\boldsymbol{X}, \emptyset\}$. The result fails in the case of $(\boldsymbol{i}, \boldsymbol{j})\ \boldsymbol{\delta P_S O(X)}$.

Proposition 9.2.23: In a bitopological space (X, τ_1, τ_2) if a space (X, τ_i) is locally indiscrete then $(i, j)\ \delta P_S O(X) \subseteq \tau_i$.

Proof: Let $V \in (i, j)\ \delta P_S O(X)$ then $V \in j\ \delta PO(X)$ and for each $x \in V$, there exist i-semi-closed F in X such that $x \in F \subseteq V$, by Lemma 1.1.14(b), F is i-open, it follows that $V \in \tau_i$, and hence $(i, j)\ \delta P_S O(X) \subseteq \tau_i$.

The converse of Proposition 9.2.23 is not true in general, as shown in the following example:

Example 9.2.24: Let $X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}, \{b, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, then $(1,2)\ \delta P_S O(X) = \{X, \emptyset, \{a\}\}$ and it is clear that (X, τ_1) is locally indiscrete but τ_1 is not a subset of $(1,2)\ \delta P_S O(X)$.

Remark 9.2.25: In any bitopological space (X, τ_1, τ_2) we have:

a) If τ_i(resp., $j\ \delta PO(X)$) is indiscrete, then $(i,j)\ \delta P_S O(X)$ is also indiscrete.
b) If $(i,j)\ \delta P_S O(X)$ is discrete, then $j\ \delta\ PO(X)$ is also discrete.

Proposition 9.2.26: Let X_1, X_2 be two topological space and $X_1 \times X_2$ be the bitopological product, let $A_1 \in (i,j)\ \ \delta P_S O(X_1)$ and $A_2 \in (i,j)\ \ \delta P_S O(X_2)$ then $A_1 \times A_2 \in (i,j)\ \ \delta P_S O(X_1 \times X_2)$.

Proof: Let $(x_1, x_2) \in A_1 \times A_2$, then $x_1 \in A_1$ and $x_2 \in A_2$. Since $A_1 \in (i,j)\ \delta P_S O(X_1)$ and $A_2 \in (i,j)\ \ \delta P_S O(X_2)$, then $A_1 \in j\ \delta PO(X_1)$ and $A_2 \in j\ \delta PO(X_2)$ also, there exist $F_1 \in i\ SC(X_1)$ and $F_2 \in i\ SC(X_2)$ such that $x_1 \in F_1 \subseteq A_1$ and $x_2 \in F_2 \subseteq A_2$. Therefore $(x_1, x_2) \in F_1 \times F_2 \subseteq A_1 \times A_2$. Since $A_1 \in j\ \delta PO(X_1)$ and $A_2 \in j\ \delta PO(X_2)$, then by Lemma 1.4.7(a), $A_1 \times A_2 = j\ \delta pInt_{X_1}(A_1) \times j\ \delta pInt_{X_2}(A_2) = j\ \delta pInt_{X_1 \times X_2}(A_1 \times A_2)$, hence $A_1 \times A_2 \in j\ \delta PO(X_1 \times X_2)$. Since $F_1 \in i\ \ SC(X_1)$ and $F_2 \in i\ \ SC(X_2)$, then by Lemma 1.4.7(b), we get $F_1 \times F_2 = i\ \ sCl_{X_1}(F_1) \times i\ sCl_{X_2}(F_2) = i\ sCl_{X_1 \times X_2}(F_1 \times F_2)$. Hence $F_1 \times F_2 \in i\ SC(X_1 \times X_2)$, so $A_1 \times A_2 \in (i,j)\ \ \delta P_S O(X)$.

Proposition 9.2.27: For any bitopological space (X, τ_1, τ_2) if $A \in j\ \ \delta PO(X)$ and either $A \in i\ \eta O(X)$ or $A \in j\ \ S\theta O(X)$, if $A \neq \emptyset$, then $A \in (i,j)\ \delta P_S O(X)$.

Proof: Let $A \in i\ \eta O(X)$ and $A \in j\ \delta PO(X)$, if $A = \emptyset$, then $A \in (i,j)\ \delta P_S O(X)$. If $A \neq \emptyset$, since $A \in i\ \eta O(X)$, then $A = \cup F_\gamma$, where $F_\gamma \in i\ \delta C(X)$ for each γ. Since $i\ \delta C(X) \subseteq i\ SC(X)$, so $F_\gamma \in i\ \delta C(X)$ for each γ, and $A \in j\ \delta PO(X)$.

Hence by Corollary 9.2.2, $A \in (i,j)\ \delta P_S O(X)$

On the other hand, suppose that $A \in i\ S\theta O(X)$ and $A \in j\ \delta PO(X)$. If $A = \emptyset$, then $A \in (i,j)\ \delta P_S O(X)$. If $A \neq \emptyset$, since $A \in i\ S\theta O(X)$, then for each $x \in A$, there exists an i-semi-open set U such that $x \in U \subseteq i\ sCl(U) \subseteq A$. This implies that $x \in i\ sCl(U) \subseteq A$ and $A \in j\ \delta PO(X)$. Therefore, by Corollary 9.2.2, $A \in (i,j)\ \delta P_S O(X)$.

Theorem 9.2.28: If $\delta PO(X, \tau_j) = \mathcal{P}(X)$ then $SC(X, \tau_i) \subseteq (i,j)\ \delta P_S O(X)$.

Proof: Let $\delta PO(X, \tau_j) = \mathcal{P}(X)$ ⟶ (1)

Let $A \in SC(X, \tau_i)$ ⟶ (2)

From (1), A is a δ-preopen set. Hence, for all $x \in A$, choose the semi-open set A itself such that $x \in A \subseteq A$. Then A is an $(i,j)\ \delta P_S$-open set. Hence, $SC(X, \tau_i) \subseteq (i,j)\ \delta P_S O(X)$.

Proposition 9.2.29: If $i\ SC(X) = \mathcal{P}(X)$, then $(i,j)\ \delta P_S O(X) = j\ \delta PO(X)$.

Proof: In general, $(i, j)\ \delta P_S O(X) \subseteq j\ \delta PO(X)$, by definition. Let $A \in j\ \delta PO(X)$. Then for all $x \in A$, choose A itself the semi-open set satisfying $x \in A \subseteq A$, since $iSC(X) = \mathcal{P}(X)$. Hence A is in $(i, j)\ \delta P_S O(X)$. Thus, $(i, j)\ \delta P_S O(X) = j\ \delta PO(X)$.

Proposition 9.2.30: If $A, B \in (i, j)\ \delta P_S O(X)$ then $A \cup B \in (i, j)\ \delta P_S O(X)$.

Proof: Consider $A \cup B$, since A and B are $j\ \delta PO(X)$, then $A \cup B$ is a $j\ \delta PO(X)$. Let $x \in A \cup B$, then $x \in A$ or $x \in B$. Say $x \in A$, Now $A \in (i, j)\ \delta P_S O(X)$ then there exists $F \in iSC(X)$ such that $x \in F \subseteq A \subseteq A \cup B$

$$\therefore A \cup B \in (i, j)\ \delta P_S O(X)$$

The following result shows that any union of $(i, j)\ \delta P_S O(X)$ sets in bitopological space (X, τ_1, τ_2) is $(i, j)\ \delta P_S O(X)$.

Proposition 9.2.31: Let $\{A_\lambda : \lambda \in \Delta\}$be family of $(i, j)\ \delta P_S$-open sets in bitopological space (X, τ_1, τ_2), then $\cup \{A_\lambda : \lambda \in \Delta\}$ is an $(i, j)\ \delta P_S$-open set.

Proof. Let $\{A_\lambda : \lambda \in \Delta\}$ be family of $(i, j)\ \delta P_S$-open sets in bitopological space (X, τ_1, τ_2). Since A_λ is j-δ-preopen for each $\lambda \in \Delta$ then $\cup \{A_\lambda : \lambda \in \Delta\}$ is j-δ-preopen set in a space X. Suppose that $x \in \cup A_\lambda$, this implies that there exist $\lambda_0 \in \Delta$ such that $x \in A_{\lambda_0}$ and since A_{λ_0} is an $(i, j)\ \delta P_S$open set, so there exists i-semi-closed set F in X such that $x \in F \subseteq A_{\lambda_0} \subseteq \cup A_\lambda$ for all $\lambda \in \Delta$. Therefore, $\cup \{A_\lambda : \lambda \in \Delta\}$ is an $(i, j)\ \delta P_S$-open set.

Proposition 9.2.32: If A is i-closed and j-open then $A \in (i, j)\ \delta P_S O(X)$.

Proof. Let A be j-open then it is j-δ-preopen. Since A is also i-closed it is i-semi-closed. Hence for all $x \in A$, choose A itself as i-semi-closed such that $x \in A \subseteq A$. Thus, $A \in (i, j)\ \delta P_S O(X)$.

9.3. Bitopological Subspaces in $(i, j)\ \delta P_S$-Open Sets

Proposition 9.3.1: Let Y be a subspace of a bitopological space (X, τ_1, τ_2), if $A \in (i, j)\ \delta P_S O(X)$ and $A \subseteq Y$, then $A \in (i, j) \delta P_S O(Y)$.

Proof: Let $A \in (i, j) - \delta P_S O(X)$, then $A \in j\ \delta PO(X)$ and for each $x \in A$, there exists an i-semi-closed set F in X such that $x \in F \subseteq A$. Since $A \in j\ \delta PO(X)$ and $A \subseteq Y$, then by Lemma 1.1.28, $A \in j\ \delta PO(Y)$, and since $F \in i\ \ SC(X)$ and $F \subseteq Y$, then by Theorem 1.1.15(b), $F \in i\ \ SC(Y)$. Hence $A \in (i, j)\ \delta P_S O(Y)$.

Corollary 9.3.2: Let (X, τ_1, τ_2) be a bitopological space, A and Y be two subsets of X such that $A \subseteq Y \subseteq X$, $Y \in RO(X, \tau_j)$ and $Y \in RO(X, \tau_i)$. Then $A \in (i, j)\ \delta P_S O(Y)$ if and only if $A \in (i, j)\ \delta P_S O(X)$.

Proof: Let $A \in (i,j)\ \delta P_S O(X) \Rightarrow A \in (i,j)\ \delta P_S O(Y)$

Conversely, let $A \in (i,j)\ \delta P_S O(Y)$.

Given then $Y \in RO(X,\tau_j)$ and $Y \in RO(X,\tau_j)$

Now, $Y \in RO\left(X,\tau_j\right) \Rightarrow A \in j\delta O(X,\tau_j)$ $\longrightarrow$ (1)

By Lemma 1.1.30, $Y \in RO(X,\tau_i) \Rightarrow Y \in SC(X,\tau_i)$ $\longrightarrow$ (2)

Now $A \in (i,j)\ \delta P_S O(Y) \Rightarrow A \in j\ \delta PO(Y)$. By (1) and Theorem 1.1.16(c), $A \in j\ \delta PO(X)$. Then for all $x \in A$, there exists $F \in i\ SC(Y)$ then there exists $x \in F \subseteq A$. Again by (2) and Theorem 1.1.15(c), $F \in i\ SC(X)$.

Hence $A \in (i,j)\ \delta P_S O(X)$.

Proposition 9.3.3: Let Y be a subspace of a bitopological space (X,τ_1,τ_2). If $A \in (i,j)\ \delta P_S O(Y)$ and $Y \in i\ SC(X)$, then for each $x \in A$, there exists an i-semi-closed set F in X such that $x \in F \subseteq A$.

Proof: Let $A \in (i,j)\ \delta P_S O(Y)$ then $A \in j\ \delta PO(Y)$ and for each $x \in A$ there exists an i-semi-closed set F in Y such that $x \in F \subseteq A$. Since $Y \in i\ SC(X)$, so by Theorem 1.1.15(c), $F \in i\ SC(X)$, which completes the proof.

Proposition 9.3.4: Let A and Y be any subsets of a bitopological space X. If $A \in (i,j)\ \delta P_S O(X), Y \in RO(X,\tau_j)$ and $Y \in RO(X,\tau_i)$, then $A \cap Y \in (i,j)\ \delta P_S O(X)$.

Proof: Let $A \in (i,j)\ \delta P_S O(X)$, then $A \in j\ \delta PO(X)$ and $A = \cup F_\gamma$ where $F_\gamma \in i\ SC(X)$ for each γ, then $A \cap Y = \cup F_\gamma \cap Y = \cup (F_\gamma \cap Y)$. Since $Y \in RO\left(X,\tau_j\right)$, then Y is $j\ \delta$-open and by Theorem 1.1.16(a), $A \cap Y \in j\ \delta pO(X)$. Since $Y \in RO(X,\tau_i)$ $Y \in i\ SC(X)$ and hence $F_\gamma \cap Y \in i\ SC(X)$, for each γ. Therefore, by Corollary 9.2.2, $A \cap Y \in (i,j)\ \delta P_S O(X)$.

Proposition 9.3.5: Let A and Y be any subsets of a bitopological space X. If $A \in (i,j)\ \delta P_S O(X), Y \in RSO(X,\tau_i)$ and $Y \in \delta O(X,\tau_j)$, then $A \cap Y \in (i,j)\ \delta P_S O(Y)$.

Proof: Let $A \in (i,j)\ \delta P_S O(X)$, then $A \in j\ \delta PO(X)$ and $A = \cup F_\gamma$ where $F_\gamma \in i\ SC(X)$ for each γ, then $A \cap Y = \cup F_\gamma \cap Y = \cup (F_\gamma \cap Y)$. Since $Y \in \delta O\left(X,\tau_j\right)$, and $A \in j\ \delta PO(X)$. So, by Theorem 1.1.16(b), $A \cap Y \in j\ \delta PO(Y)$ Since $Y \in RSO(X,\tau_i)$ then $Y \in i\ SC(X)$ By Theorem 1.1.40. Hence $F_\gamma \cap Y \in i\ SC(X)$ for each γ. Since $F_\gamma \cap Y \subseteq Y$ and $F_\gamma \cap Y \in i\ SC(X)$ for each γ, then by Theorem 1.1.15(b), $F_\gamma \cap Y \in i\ SC(Y)$. Therefore, by Proposition 9.2.3, $A \cap Y \in (i,j)\ \delta P_S O(Y)$.

Proposition 9.3.6: If Y is an i-open and $j\ \delta$-open subspace of a bitopological space X and $A \in (i,j)\ \delta P_S O(X)$, then $A \cap Y \in (i,j)\ \delta P_S O(Y)$.

Proof: Let $A \in (i,j) - \delta P_S O(X)$, then $A \in j\ \ \delta PO(X)$ and $A = \cup F_\gamma$ where $F_\gamma \in i\ SC(X)$ for each γ, then $A \cap Y = \cup F_\gamma \cap Y = \cup (F_\gamma \cap Y)$. Since Y is $j\ \ \delta$-open subspace of X, then $Y \in j\ \ \delta SO(X)$ and hence by Lemma 1.1.29, $A \cap Y \in j\ \ \delta PO(Y)$. Since Y is an i-open subspace of X, then by Lemma 1.1.17, $F_\gamma \cap Y \in i\ \ SC(Y)$ for each γ. Therefore, by Corollary 9.2.2, $A \cap Y \in (i,j)\ \delta P_S O(Y)$.

Corollary 9.3.7: If either $Y \in RSO(X,\tau_j)$ and $Y \in \delta O(X,\tau_i)$ or Y is an i-open and $j\ \delta$-open subspace of a bitopological space X and $A \in (i,j)\ \delta P_S O(X)$ then $A \cap Y \in (i,j)\ \ \delta P_S O(Y)$.

Proposition 9.3.8: Let (X,τ_1,τ_2) be a bitopological space and A be a subset of X. If A is both i-regular semi-open and j-preregular set, then A is both $(i,j)\ \delta P_S$-open and $(i,j)\ \delta P_S$-closed set.

Proof: Suppose that A is both i-regular semi-open and j-preregular set, then A is both i-semi-closed Theorem 1.1.40 and j-preopen set, which is $j\ \delta$-preopen. Hence A is $(i,j)\ \delta P_S$-open. Again since A is both i-regular semi-open and j-preregular set, then A is both i-semi-open Theorem 1.1.40 and j-pre closed set, which is also $j\ \delta$-preclosed. Hence A is $(i,j)\ \delta P_S$-closed set.

9.4 $(i,j)\ \delta P_S$-Closed Sets in Bitopology

Definition 9.4.1: A subset B of a space X is called (i,j) **δP_S-closed** if $X \backslash B$ is $(i,j)\ \delta P_S$-open. The family of all $(i,j)\ \delta P_S$-closed subsets of bitopological space (X,τ_1,τ_2) is denoted by $(i,j)\ \delta P_S C(X,\tau_1,\tau_2)$ or $(i,j)\ \delta P_S C(X)$.

Corollary 9.4.2: A subset B of a bitopological space X is $(i,j)\ \delta P_S$-closed if and only if B is $j\ \delta$-preclosed and it is an intersection of i-semi-open sets.

Proof: Follows from Proposition 9.2.3 taking $A = B^C$.

Proposition 9.4.3: Let $\{B_\alpha, \alpha \in \Delta\}$ be a collection of $(i,j)\ \delta P_S$-closed sets in a bitopological space X. Then $\cap \{B_\alpha, \alpha \in \Delta\}$ is $(i,j)\ \delta P_S$-closed set.

Proof. Follows from Proposition 9.2.31

All of the following results are true by using complement.

Proposition 9.4.4: If (X,τ_j) is semi-T_1 space, then $(i,j)\ \delta P_S C(X) \equiv i\ C(X)$.

Proof. The proof follows from Proposition 9.2.17.

Remark 9.4.5: Let (X,τ_j) is T_1 space, then the family of $(j,i)\ SO(X)$ is discrete topology in X.

Corollary 9.4.6: Each (i,j) θ-closed set is (i,j) δP_S-closed.

Corollary 9.4.7: Every (i,j)-regular closed set is (i,j) δP_S-closed.

Corollary 9.4.8: For any subset B of a space (X,τ_1,τ_2)The following statements are equivalent:

a) B is i-closed and j-open.

b) B is (i,j) δP_S-closed and j-open.

c) B is (i,j) α -closed and j-open.

d) B is (i,j) δ-pre-closed and j-open.

Proof. Similar to Corollary 2.2.38 taking $A = X \backslash B$.

Corollary 9.4.9: For any subset A of a space (X,τ_1,τ_2)The following statements are equivalent:

a) A is (i,j) regular closed.

b) A is (i,j) δP_S-closed and (j,i)-semi open.

c) A is i-closed and (j,i)-semi open.

d) A is (i,j) α -closed and (j,i)-semi open.

e) A is (i,j) δ-pre-closed and (j,i)-semi open.

Proof. Similar to Corollary 2.2.39 taking $A = X \backslash B$.

Proposition 9.4.10: Let Y be a subset of a space (X,τ_1,τ_2).If $B \in (i,j)\ \delta P_S C(X)$ and $B \subseteq Y$, then $B \in (i,j)\ \delta P_S C(Y)$.

Proof. The proof is similar to Proposition 9.3.1.

Proposition 9.4.11: Let Y be a subset of a space (X,τ_1,τ_2). If $B \in (i,j)\ \delta P_S C(Y)$ and $Y \in (i,j)\ RC(X)$, then $B \in (i,j)\ \delta P_S C(X)$.

Proof. The proof is similar to Corollary 9.4.2.

From Proposition 9.4.10 and Proposition 9.4.11 we obtain the following result:

Corollary 9.4.12: Let (X,τ_1,τ_2) be a bitopological space and B, Y subsets of X such that $B \subseteq Y \subseteq X$ and $Y \in (i,j)\ RC(X)$. Then $B \in (i,j)\ \delta P_S C(Y)$ if and only if $B \in (i,j)\ \delta P_S C(X)$

Proposition 9.4.13: Let B and Y be any subsets of a space X. If $B \in (i,j)\ \delta P_S C(X)$ and $Y \in (i,j)\ RC(X)$, then $B \cup Y \in (i,j)\ \delta P_S C(X)$.

Proof. The proof is directly from Proposition 9.4.4, and using complements.

9.5. Properties of $(\boldsymbol{i},\boldsymbol{j})$ $\boldsymbol{\delta P_S}$-open sets in Bitopology

Definition 9.5.1: If A is a subset of a bitopological space (X,τ_1,τ_2), then **the $(\boldsymbol{i},\boldsymbol{j})$ $\boldsymbol{\delta P_S}$-interior ($(\boldsymbol{i},\boldsymbol{j})$ $\boldsymbol{\delta P_S Int(A)}$)**, the **$(\boldsymbol{i},\boldsymbol{j})$ $\boldsymbol{\delta P_S}$-closure ($(\boldsymbol{i},\boldsymbol{j})$ $\boldsymbol{\delta P_S Cl(A)}$)** and the **$(\boldsymbol{i},\boldsymbol{j})$ $\boldsymbol{\delta P_S}$-boundary ($(\boldsymbol{i},\boldsymbol{j})$ $\boldsymbol{\delta P_S Bd(A)}$)** of A are defined as follows:

a) $(i,j)\ \delta P_S Cl(A) = \cap\{F: A\subseteq F, X-F\in (i,j)\ \delta P_S O(X)\}$.
b) $(i,j)\ \delta P_S Int(A) = \cup\{U: U\subseteq A, U\in (i,j)\ \delta P_S O(X)\}$.
c) $(i,j)\ \delta P_S Bd(A) = (i,j)\ \delta P_S Cl(A) - (i,j)\ \delta P_S Int(A)$.

Theorem 9.5.2: For any two subsets A and B of a bitopological space X, the following statements are true:

a) $(i,j)\ \delta P_S Cl(X-A) = X\ ((i,j)\ \delta P_S Int(A))$ and $(i,j)\ \delta P_S Int(X-A) = X - ((i,j)\ \delta P_S Cl(A))$,
b) $(i,j)\ \delta P_S Cl(X) = X, (i,j)\ \delta P_S Cl(\emptyset) = \emptyset, (i,j)\ \delta P_S Int(X) = X$ and $(i,j)\ \delta P_S Int(\emptyset) = \emptyset$,
c) If $A\subseteq B$, then $(i,j)\ \delta P_S Cl(A) \subseteq (i,j)\delta P_S Cl(B)$ and $(i,j)\ \delta P_S Int(A) \subseteq (i,j)\delta P_S Int(B)$,
d) $A\subseteq (i,j)\ \delta P_S Cl(A)$ and $(i,j)\ \delta P_S Int(A)\subseteq A$.
e) $(i,j)\ \delta P_S Cl\big((i,j)\ \delta P_S Cl(A)\big) = (i,j)\ \delta P_S Cl(A)$ and
f) $(i,j)\ \delta P_S Int\big((i,j)\ \delta P_S Int(A)\big) = (i,j)\ \delta P_S Int(A)$.

Remark 9.5.3: (a) $(i,j)\ \delta P_S Int(A) \cup (i,j)\ \delta P_S Int(B) \neq (i,j)\ \delta P_S Int(A\cup B)$
(b), $(i,j)\ \delta P_S Int(A\cap B) \neq (i,j)\ \delta P_S Int(A) \cap (i,j)\ \delta P_S Int(B)$, it shown in the following example:

Example 9.5.4: Let $X=\{a,b,c\}, \tau_1 = \{X,\emptyset,\{a\},\{a,b\}\}$ and $\tau_2 = \{X,\emptyset,\{a,b\}\}$, then $(1,2)\ \delta P_S O(X) = \{X,\emptyset,\{b\},\{c\},\{b,c\}\}$. If we take $A=\{a\}$ and $B=\{b\}$, then $(1,2)\ \delta P_S Int(A) = \emptyset$ and $(1,2)\ \delta P_S Int(B) = \{b\}$. Therefore, $(1,2)\ \delta P_S Int(A) \cup (i,j)\ \delta P_S Int(B) = \{b\}$ but $(1,2)\ \delta P_S Int(A\cup B) = \{b,c\}$. So, $(i,j)\ \delta P_S Int(A\cup B) \neq (i,j)\ \delta P_S Int(A) \cap (i,j)\ \delta P_S Int(B)$.

Example 9.5.5: Let $X=\{a,b,c,d\}, \tau_1 = \{X,\emptyset,\{b\},\{c\},\{d\},\{a,b\},\{b,c\},\{b,d\},\{c,d\}, \{a,b,d\},\{b,c,d\}\}$ and $\tau_2 = \{X,\emptyset,\{a\}\}$ then $(1,2)\delta P_S O(X) = \{X,\emptyset,\{b\},\{c\},\{d\},\{a,b\},\{b,c\}, \{b,d\},\{c,d\},\{a,b,d\},\{b,c,d\}\}$. If we take $A=\{a,b,c\}$ and $B=\{a,b,d\}$. Then $(1,2)\delta P_S Int(A) = \{b,c\}$ and $(1,2)\delta P_S Int(B) = \{a,b,d\}$. Therefore, $(1,2)\delta P_S Int(A) \cap (1,2)\delta P_S Int(B) = \{b\}$ but $(1,2)\delta P_S Int(A\cap B) = (1,2)\delta P_S Int(\{a,b\}) = \{a,b\} \neq (1,2)\delta P_S Int(A) \cap (1,2)\delta P_S Int(B)$

The proof of the following result is obvious.

Proposition 9.5.6: If A is any subset of a bitopological space X, then $(i,j)\ \delta P_S Int(A) \subseteq j\ \ \delta\ pInt(A) \subseteq A \subseteq j\ \delta\ pCl(A) \subseteq (i,j)\ \delta P_S Cl(A)$.

In general, $(i,j)\ \delta P_S Int(A) \neq j\ \delta\ pInt(A)$ and $(i,j)\ \delta P_S Cl(A) \neq j\ \delta\ pCl(A)$ as it is shown in the following example:

Example 9.5.7: Let $X = \{a,b,c\}, \tau_1 = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$ and $\tau_2 = \{X, \emptyset, \{a,b\}\}$ then $(i,j)\ \delta P_S O(X) = \{X, \emptyset, \{b\}, \{c\}, \{b,c\}\}$. If we take $A = \{a\}$,then $j\ \delta\ pInt(A) = A$ and $(i,j)\ \delta P_S Int(A) = \emptyset$. This shows that $(i,j)\ \delta P_S Int(A) \neq j\ \delta\ pInt(A)$.

Example 9.5.8: Let $X = \{a,b,c\}, \tau_1 = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$ and $\tau_2 = \{X, \emptyset, \{a,b\}\}$ then $(i,j)\ \delta P_S O(X) = \{X, \emptyset, \{b\}, \{c\}, \{b,c\}\}$. If we take $F = \{c\}$, then $j\ \delta\ pCl(F) = \{c\}$ and $(i,j)\ \delta P_S Cl(F) = \{a,c\}$. Hence $(i,j)\ \delta P_S Cl(F) \neq j\ \delta PCl(F)$.

The following results can be proved by straightforward statements:

Proposition 9.5.9: Let A be a subset of a bitopological space X, then $(i,j)\ \delta P_S Bd(A) = \emptyset$ if and only if A is both $(i,j)\ \delta P_S$-open and $(i,j)\ \delta P_S$-closed set.

Proof: Let A be $(i,j)\ \delta P_S$- open and $(i,j)\ \delta P_S$-closed, then $A = (i,j)\ \delta P_S\ Int(A) = (i,j)\ \delta P_S Cl(A)$, hence by Definition 9.5.1(c) $A = (i,j)\ \delta P_S\ Cl(A) - (i,j)\ \delta P_S Int(A)^c = \emptyset$.

Theorem 9.5.10: For any subset A of a bitopological space X, the following statements are true:

a) $(i,j)\ \delta P_S Bd(A) = (i,j)\ \delta P_S Bd(X - A)$
b) $A \in (i,j)\ \delta P_S O(X)$ if and only if $(i,j)\ \delta P_S Bd(A) \subseteq X - A$, i.e., $A \cap (i,j)\ \delta P_S Bd(A) = \emptyset$.
c) $A \in (i,j)\ \delta P_S C(X)$ if and only if $(i,j)\ \ \delta P_S Bd(A) \subseteq A$.
d) $(i,j)\ \delta P_S Bd\big((i,j)\ \ \delta P_S Bd(A)\big) \subseteq (i,j)\ \delta P_S Bd(A)$
e) $(i,j)\ \delta P_S Bd\big((i,j)\ \delta P_S Int(A)\big) \subseteq (i,j)\ \delta P_S Bd(A)$
f) $(i,j)\ \delta P_S Bd\big((i,j)\ \ \delta P_S Cl(A)\big) \subseteq (i,j)\ \delta P_S Bd(A)$
g) $(i,j)\ \delta P_S Int(A) = A - ((i,j)\ \delta P_S Bd(A))$

Proof: a) From Definition 9.5.1(c),

$(i,j)\ \ \delta P_S Bd(A) = (i,j)\ \delta P_S Cl(A) - (i,j)\ \delta P_S Int(A) \longrightarrow$ (1)

$\therefore (i,j)\ \ \delta P_S Bd(X - A) = (i,j)\ \delta P_S Cl(X - A) - (i,j)\ \delta P_S\ Int(X - A)$

$= [X - (i,j)\ \delta P_S Int(X - (X - A))] - [X - (i,j)\ \delta P_S\ Cl(X - (X - A))]$

$= (i,j)\ \delta P_S Cl(A) - (i,j)\ \delta P_S Int(A) \longrightarrow$ (2)

From (1) & (2), (a) is proved.

(b) Necessity: Let $A \in (i,j)\ \delta P_S O(X)$. Then $A = (i,j)\ Int\ \delta P_S(A)$

$$\therefore\ (i,j)\ \delta P_S BC(A) = (i,j)\ \delta P_S\ Cl(A) - (i,j)\ Int\ \delta P_S(A)$$
$$= (i,j)\ \delta P_S Cl(A) - A$$
$$\subseteq X - A$$

Sufficiency: Let $(i,j)\ \delta P_S Bd(A) \subseteq X - A$ ⟶ (3)

$x \in (i,j)\ \delta P_S Bd(A) \Rightarrow x \in (i,j)\ \delta P_S\ Cl(A)$ and

$x \notin (i,j)\ \delta P_S\ Int(A) \Rightarrow x \in X - A \Rightarrow x \notin A$

Now (3) $\Rightarrow x \in X - A \Rightarrow x \notin A$

$\Rightarrow A \subseteq (i,j)\ \delta P_S Int(A)$

$\therefore A \in (i,j)\ \delta P_S O(X)$

c) obvious from (b)

d) $(i,j)\ \delta P_S Bd\big(Bd(A)\big) \subseteq (i,j)\ \delta P_S Bd(A)$

Let $x \in (i,j)\ \delta P_S Bd((i,j)Bd(A)) \Rightarrow x \in ((i,j)\ \delta P_S Bd\big((i,j)\ \delta P_S Cl(A) - (i,j)\ \delta P_S Int(A)\big)$

$\Rightarrow x \in \Big[(i,j)\ \delta P_S\ Cl\big((i,j)\ \delta P_S Cl(A\big) - (i,j)\ \delta P_S Int(A)\big) - (i,j)\delta P_S\ Int((i,j)\ \delta P_S Cl(A) - Int(A))]$

$\Rightarrow x \in [(i,j)\ \delta P_S\ Cl\ \big((i,j)\ \delta P_S\ Cl(A)\big) - (i,j)\ \delta P_S\ Int\ \big((i,j)\ \delta P_S Int(A)\big)]$

$\Rightarrow x \in (i,j)\ \delta P_S Bd(A)$.

e)$(i,j)\ \delta P_S Bd((i,j)\ \delta P_S Cl(A)) =$

$(i,j)\ \ \delta P_S\ Cl\big((i,j)\ \delta P_S Cl(A)\big) - (i,j)\ \delta P_S Int\big((i,j)\ \delta P_S Cl(A)\big) \subseteq (i,j)\ \delta P_S Cl(A) -$

$(i,j)\ \delta P_S\ Int(A) = (i,j)\ \delta P_S Bd(A)$

f)$(i,j)\ \delta P_S Bd\big((i,j)\ \delta P_S Int(A)\big) =$

$(i,j)\ \delta P_S\ Cl((i,j)\ \delta P_S Int(A)) - (i,j)\ \delta P_S\ Int((i,j)\ \delta P_S Int(A)) \subseteq (i,j)\delta P_S\ Cl(A) -$

$(i,j)\ \delta P_S\ Int(A) = (i,j)\ \delta P_S Bd(A)$

g) To Prove: $(i,j)\delta P_S Int(A) \subseteq A - (i,j)\ \delta P_S Bd(A)$

$$A - (i,j)\ \delta P_S Bd(A) = A - \big((i,j)\ \delta P_S\ Cl(A) - (i,j)\ \delta P_S Int(A)\big) \supseteq (i,j)\delta P_S Int(A)$$

$$\therefore (i,j)\ \delta P_S Int(A) \subseteq A - ((i,j)\ \delta P_S Bd(A))$$

Definition 9.5.11: A subset N of a bitopological space (X, τ_1, τ_2) is called $(\boldsymbol{i,j})$ $\boldsymbol{\delta P_S}$**-neighbourhood** of a subset A of X if there exists an $(i,j)\ \delta P_S$-open set U such that $A \subseteq U \subseteq N$. When $A = \{x\}$, we say that N is $(i,j)\ \delta P_S$- neighbourhood of x.

Proposition 9.5.12: Let X be a bitopological space and A ⊆ X, $x \in X$, then x is (i,j) δP_S-interior of A if and only if A is an (i,j) δP_S-neighbourhood of x.

Proposition 9.5.13: A subset G of a bitopological space X is (i,j) δP_S-open if and only if it is an (i,j) δP_S-neighbourhood of each of its points.

Proposition 9.5.14: Let A be any subset of a bitopological space X. If a point x in the (i,j) δP_S-Int(A), then there exists a i-semi-closed set F of X containing x and $F \subseteq A$.

Proof. Suppose that x ∈ (i,j) δP_S-Int(A), then there exists an (i,j) δP_S-open set U of X containing x such that $x \in U \subseteq A$. Since U is an (i,j) δP_S-open set, so there exists an i-semi-closed set F such that x ∈ F ⊆ U ⊆ A. Hence, x ∈ F ⊆ A.

Definition 9.5.15: Let A be a subset of a bitopological space X, A point x ∈ X is said to be $\boldsymbol{(i,j)}$ $\boldsymbol{\delta P_S}$**-limit point** of A if for each (i,j) δP_S-open set U containing $x, U \cap A \neq \emptyset$. The set of all (i,j) δP_S-limit point of A is called (i,j) δP_S-derived set of A and is denoted by (i,j) $\delta P_S D(A)$.

In general, it is clear that (i,j) $\delta P_S D(A) \subseteq j\ D(A)$, but the converse may not be true as shown in the following example:

Example 9.5.16: Considering the space X as defined in Example 9.2.15 if we take A = {a, c}, So (i,j) $\delta P_S D(A) = \{a, b\}$ and j-D(A) = {b}, hence (i,j) $\delta P_S D(A)$ is not a subset of j-D(A)

Theorem 9.5.17: Let X be a bitopological space and A be a subset of X, then A ∪ (i,j) $\delta P_S D(A)$ is (i,j) δP_S- closed.

Proof. Let $x \notin A \cup (i,j)$ δP_SD(A). This implies that $x \notin A$ and $x \notin (i,j)$ $\delta P_S D(A)$. Since $x \notin (i,j)$ $\delta P_S D(A)$, then there exists an (i,j) δP_S-open U of X which contains no point of A other than x, but $x \notin A$, so U contains no point of A, which implies that U ⊆ X \A. Again, U is an (i,j) δP_S-open set for each of its points. But as U does not contain any point of A, no point of U can be (i,j) δP_S-limit point of A. Therefore, no point of U can belong to(i,j) δP_SD(A). This implies that U ⊆ X \ (i,j) δP_SD(A). Hence, it follows that x ∈ X \ A ∩ (X \ (i,j) δP_SD(A)) = X \ (A∪(i,j) δP_SD(A)), Therefore A∪(i,j) $\delta P_S D(A)$ is an (i,j) δP_S-closed. Hence (i,j) $\delta P_S Cl(A) \subseteq A \cup (i,j)$ $\delta P_S D(A)$.

Corollary 9.5.18: If a subset A of a bitopological space X is $(i,j)\ \delta P_S$-closed, then A contains the set of all of its $(i,j)\ \delta P_S$-limit points.

Theorem 9.5.19: Let A be any subset of a bitopological space X, then the following statements are true:

a) $((i,j)\ \delta P_S D(((i,j)\ \delta P_S D(A)))\backslash A \subseteq (i,j)\ \delta P_S D(A)$
b) $(i,j)\ \delta P_S D(A) \cup (i,j)\ \delta P_S D(A) \subseteq A \cup (i,j)\ \delta P_S D(A)$

Proof. Obvious

Theorem 9.5.20: Let X be a bitopological space and A be a subset of X, then:
$(i,j)\ \delta P_S\ Int(A) = A\backslash(i,j)\ \delta P_S D(X\backslash A)$.

Proof. Obvious.

9.6 (i,j) δP_S-Continuous Functions

Definition 9.6.1: A function $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called **(i,j) δP_S-continuous** at a point $x \in X$, if for each i-open set V of Y containing $f(x)$, there exists $(i,j)\ \delta P_S$-open U of X containing x such that $f(U) \subseteq V$. If f is $(i,j)\ \delta P_S$-continuous at every point x of X, then it is called $(i,j)\ \delta P_S$-continuous.

Definition 9.6.2: A function $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called **almost (i,j) δP_S-continuous** at a point $x \in X$, if for each i-open set V of Y containing $f(x)$, there exists $(i,j)\ \delta P_S$-open U of X containing x such that $f(U) \subseteq i\ Int(i\ \ Cl(V))$. If f is almost $(i,j)\ \delta P_S$-continuous at every point x of X, then it is called almost $(i,j)\ \delta P_S$-continuous.

It is obvious from definition that $(i,j)\ \delta P_S$-continuity implies almost $(i,j)\ \delta P_S$-continuity however the converse is not true in general, as shown in the following example:

Example 9.6.3: Let $X = \{a,b,c\}, \tau_1 = \{X, \emptyset, \{a\}, \{a,c\}\}$ and $\tau_2 = \{X, \emptyset, \{a,c\}\}$ then the family of (1,2) δP_S-open subsets of X is $(1,2)\ \delta P_S O(X) = \{X, \emptyset, \{b\}, \{c\}, \{b,c\}\}$. Let $f:(X,\tau_1,\tau_2) \to (X,\tau_1,\tau_2)$ be the identity function, then it is almost $(i,j)\ \delta P_S$-continuous. Since $\{a\}$ is an i-open set in X containing $f(a) = a$ but there is no $(i,j)\ \delta P_S$-open set U of X containing a such that $f(U) \subseteq \{a\}$, so it is not $(i,j)\ \ \delta P_S$-continuous.

It is clear that every $(i,j)\ \delta P_S$-continuous function is j-pre-continuous but the converse is not true in general, as shown in the following example:

Example 9.6.4: In Example 9.6.3, f is j-pre continuous, but it is not $(i,j)\ \delta P_S$-continuous.

The following is the characterization theorem of $(i,j)\ \delta P_S$-continuous:

Theorem 9.6.5: For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following statements are equivalent:

a) f is $(i,j)\ \delta P_S$-continuous,
b) $f^{-1}(V)$ is $(i,j)\ \delta P_S$-open set in X, for each i-open set V in Y,
c) $f^{-1}(F)$ is $(i,j)\ \delta P_S$-closed set in X, for each i-closed set F in Y,
d) $f\big((i,j)\ \delta P_S Cl(A)\big)\subseteq i\ Cl(f(A))$, for each subset A of X,
e) $(i,j)\ \delta P_S Cl(f^{-1}(B))\subseteq f^{-1}(i\ Cl(B))$, for each subset B of Y,
f) $f^{-1}(i\ Int(B))\subseteq(i,j)\ \delta P_S Int(f^{-1}(B))$, for each subset B of Y,
g) $i\ Int(f(A))\subseteq f((i,j)\ \delta P_S Int(A))$, for each subset A of X.

Proof: $(a)\Rightarrow(b)$: Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be $(i,j)\ \delta P_S$-continuous function.

To Prove. $f^{-1}(V)$ is $(i,j)\ \delta P_S$-open in X $\forall$ open set V in Y

Let V be i-open in Y. Let $x\in f^{-1}(V)$. Then $f(x)\in V$

By Definition 9.6.1, there exists a $(i,j)\ \delta P_S$-open set U of X containing x $\ni f(U)\subseteq V\ \Rightarrow x\in U\subseteq f^{-1}(V)$.

By Definition 9.5.1(b) x is an $(i,j)\ \delta P_S\ Int$ point of $f^{-1}(V)$. Since x is arbitrary, $f^{-1}(V)$ is $(i,j)\ \delta P_S$-open.

$(b)\Rightarrow(c)$: Let F be any i-closed set of Y, then $Y-F$ is an i-open set of Y. By (b), we have $f^{-1}(Y-F)=X-f^{-1}(F)$ is $(i,j)\ \delta P_S$-open set in X. Hence $f^{-1}(F)$ is $(i,j)\ \delta P_S$-closed set in X.

$(c)\Rightarrow(d)$: Let A be any subset of X, then $f(A)\subseteq i\ Cl(f(A))$ and $i\ Cl(f(A))$ is a i-closed set in Y. Hence $A\subseteq f^{-1}(i\ Cl\big(f(A)\big))$, by (c), $f^{-1}(i\ Cl(f(A))$ is an $(i,j)\ \delta P_S$-closed set in X. Therefore, $(i,j)\ \delta P_S Cl(A)\subseteq f^{-1}(i\ Cl\big(f(A)\big))$, so $f\big((i,j)\ \delta P_S Cl(A)\big)\subseteq i\ Cl(f(A))$.

$(d)\Rightarrow(e)$: Let B be any subset of Y, then $f^{-1}(B)$ is a subset of X, by (d) we have $f\left((i,j)\ \delta P_S Cl\big(f^{-1}(B)\big)\right)\subseteq i\ Cl(f(f^{-1}(B)))=i\ Cl(B)$, hence $(i,j)\ \delta P_S Cl(f^{-1}(B))\subseteq f^{-1}(i\ Cl(B))$.

$(e) \Rightarrow (f)$: Let B be any subset of Y, then by applying condition (e) on $Y - B$ we obtain,

$(i,j)\ \delta P_S Cl(f^{-1}(Y - B)) \subseteq f^{-1}(iCl(Y - B))$

$\Leftrightarrow (i,j)\ \delta P_S Cl(X - f^{-1}(B)) \subseteq f^{-1}(Y - iInt(B))$

$\Leftrightarrow X - (i,j)\ \delta P_S Int\big(f^{-1}(B)\big) \subseteq Y - f^{-1}(iInt(B))$

$\Leftrightarrow f^{-1}\big(iInt(B)\big) \subseteq (i,j)\delta P_S Int(f^{-1}(B))$

$(f) \Rightarrow (g)$: Let A be any subset of X, then $f(A)$ is a subset of Y. By (f), we have $f^{-1}\left(i\ Int\big(f(A)\big)\right) \subseteq (i,j)\ \delta P_S Int\left(f^{-1}\big(f(A)\big)\right) = (i,j)\ \delta P_S Int(A)$. Therefore, $i\ \ Int(f(A)) \subseteq ((i,j)\ \ \delta P_S Int(A))$.

$(g) \Rightarrow (a)$: Let $x \in X$ and let V be any i-open set of Y containing $f(x)$, then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. Hence, by (g), we have $i - Int(f(f^{-1}(V)) \subseteq f((i,j)\ \delta P_S Int(f^{-1}(V)))$. So, $i\ \ Int(V) \subseteq f((i,j)\ \ \delta P_S Int(f^{-1}(V)))$. Since V is an i-open set in Y, then $V \subseteq f((i,j)\ \ \delta P_S Int(f^{-1}(V)))$ which implies that $f^{-1}(V)) \subseteq (i,j)\ \delta P_S Int(f^{-1}(V)))$. Therefore, $f^{-1}(V)$ is $(i,j)\ \ \delta P_S$-open set in X containing x and clearly $f(f^{-1}(V)) \subseteq V$. Hence f is $(i,j)\ \ \delta P_S$-continuous.

Theorem 9.6.6: For a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

a) f is almost $(i,j)\ \delta P_S$-continuous,

b) For each $x \in X$ and each i-open set V of Y containing $f(x)$, there exists an $(i,j)\ \ \delta P_S$-open set U of X containing x such that $f(U) \subseteq i\ \ sCl(V)$

c) For each $x \in X$ and each i-regular open set V of Y containing $f(x)$, there exists an $(i,j)\ \delta P_S$-open set U of X containing x such that $f(U) \subseteq V$.

d) For each $x \in X$ and each i- open set V of Y containing $f(x)$, there exists an $(i,j)\ \delta P_S$-open set U of X containing x such that $f(U) \subseteq V$.

Proof: $(a) \Rightarrow (b)$: Let $x \in X$ and let V be any i-open set of Y containing $f(x)$, by (a) there exists an $(i,j)\ \delta P_S$-open set U of X containing x such that $f(U) \subseteq\ Int(i\ Cl(V))$. Since V is $i\ \delta$-open and hence V is an $i\ \delta$-preopen set, so $Int\big(i\ Cl(V)\big) = i\ sCl(V)$. Therefore, $f(U) \subseteq i\ \delta\ sCl(V)$.

$(b) \Rightarrow (c)$: Let $x \in X$ and V be any i-regular open set of Y containing $f(x)$, then V is a i δ-open set of Y containing $f(x)$. So by (b), there exists an (i,j) δP_S-open set U of X containing x such that $f(U) \subseteq i\ sCl(V)$. Since V is i-regular open and hence V is i-open, so $i\ sCl(V) = Int\big(i\ Cl(V)\big)$. Therefore, $f(U) \subseteq iInt\big(i\ Cl(V)\big)$ and since V is i-regular open and hence V is i- regular open, then $f(U) \subseteq V$.

$(c) \Rightarrow (d)$: Let $x \in X$, and V be any i- open set of Y containing $f(x)$, then for each $f(x) \in V$ there exists an i-open set G containing $f(x)$ such that $G \subseteq i\ \ Int(i\ Cl(G)) \subseteq V$. Since $i\ Int(i\ Cl(G))$ is i-regular open set of Y containing $f(x)$, so by (c) there exists an (i,j) δP_S-open set U of X containing x such that $f(U) \subseteq i\ Int(i\ Cl(G)) \subseteq V$. This implies that $f(U) \subseteq V$.

$(d) \Rightarrow (a)$: Let $x \in X$ and let V be any i-open set of Y containing $f(x)$, then $i\ Int(i\ Cl(V))$ is i -open set of Y containing $f(x)$. Hence, by (d), there exists an (i,j) δP_S-open set U of X containing x such that $f(U) \subseteq i\ Int(i\ Cl(V))$. Therefore, f is almost (i,j) δP_S-continuous.

Theorem 9.6.7: For a function $f\colon (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following statements are equivalent:

a) f is almost (i,j) δP_S-continuous,

b) $f^{-1}(i\ Int(i\ Cl(V)))$ is (i,j) δP_S-open set in X, for each i-open set V in Y.

c) $f^{-1}(i\ Cl(i\ Int(F)))$ is (i,j) δP_S-closed set in X, for each i-closed set F in Y.

d) $f^{-1}(F)$ is (i,j) δP_S-closed set in X, for each i-regular closed set F in Y.

e) $f^{-1}(V)$ is (i,j) δP_S-open set in X, for each i-regular open set V in Y.

Proof. $(a) \Rightarrow (b)$. Let V be any i-open set in Y. We have to show that $f^{-1}(Int(Cl(V))$ is a (i,j) δP_S-open set in X. Let $x \in f^{-1}(i\ Int(i\ Cl(V))$ and $i\ Int\big(i\ Cl(V)\big)$ is a i-regular open set in Y. Since f is almost (i,j) δP_S-continuous, by Theorem 9.6.6(b), there exists a (i,j) δP_S-open set U of X containing x such that $f(U) \subseteq Int(i\ Cl(V))$, which implies that $x \in U \subseteq f^{-1}(Int(i\ Cl(V))$. Therefore, $f^{-1}(i\ Int(i\ Cl(V))$ is (i,j) δP_S-open set in X.

$(b) \Rightarrow (c)$. Let F be any i-closed set of Y. Then $Y - F$ is an i-open set of Y. By (b), $f^{-1}(i\ Int(i\ Cl(Y \backslash F))$ is (i,j) δP_S-open set in X and $f^{-1}(i\ Int\big(i\ Cl(Y\backslash F)\big) = f^{-1}\big(Int(Y\backslash F)\big) = f^{-1}(Y \backslash i\ Cl\big(i\ Int(F)\big) = X \backslash f^{-1}(i\ Cl(i\ Int(F))$ is (i,j) δP_S-open set in X and hence $f^{-1}(i\ Cl(i\ Int(F))$ is an (i,j) δP_S-closed set in X .

$(c) \Rightarrow (d)$. Let F be any i-regular closed set of Y. Then F is a i-closed set of Y. By (c), $f^{-1}(i\,Cl(i\,Int(F))$ is $(i,j)\ \delta P_S$-closed set in X. Since F is i-regular closed set, then $f^{-1}(i\,Cl(i\,Int(F)) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is an $(i,j)\ \delta P_S$-closed set in X.

$(d) \Rightarrow (e)$. Let V be any i-regular open set of Y. Then $Y \backslash V$ is i-regular closed set of Y and by (d), we have $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is an $(i,j)\ \delta P_S$-closed set in X and hence $f^{-1}(V)$ is $(i,j)\ \delta P_S$-open in X.

$(e) \Rightarrow (a)$. Let $x \in X$ and let V be any i-regular open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$. By (e), we have $f^{-1}(V)$ is an $(i,j)\ \delta P_S$-open set in X. Therefore, we obtain $f(f^{-1}(V)) \subseteq V$. Hence by Theorem 9.6.6(c), f is almost $(i,j)\ \delta P_S$-continuous.

Theorem 9.6.8: For a function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

a) f is almost $(i,j)\ \delta P_S$-continuous.
b) $(i,j)\ \delta P_S Cl(f^{-1}(V) \subseteq f^{-1}(iCl(V))$, for each $i\ \beta$-open set V of Y.
c) $f^{-1}(i\,Int(F)) \subseteq (i,j)\ \delta P_S Int(f^{-1}(F))$, for each $i\ \beta$-closed set F of Y
d) $f^{-1}(i\,Int(F)) \subseteq (i,j)\ \delta P_S\ Int(f^{-1}(F))$, for each i-semi closed set F of Y.
e) $(i,j)\ \delta P_S Cl(f^{-1}(V)) \subseteq f^{-1}(iCl(V))$ for each i-semi open set V of Y.

Proof. $(a) \Rightarrow (b)$ Let V be any $i\ \beta$-open set of Y. By Lemma 1.3.11, $i\,Cl(V)$ is an i-regular closed set in Y and f is almost $(i,j)\ \delta P_S$-continuous. Then by Theorem 9.6.7(d), $f^{-1}(i\,Cl(V))$ is an $(i,j)\ \delta P_S$-closed set in X. Therefore, we obtain $(i,j)\ \delta P_S Cl(f^{-1}(V)) = f^{-1}(i\,Cl(V))$. Now $V \subseteq iCl(V) \Rightarrow f^{-1}(V) \subseteq f^{-1}(iCl(V)) \Rightarrow (i,j)\delta P_S Cl(f^{-1}(V)) \subseteq (i,j)\delta P_S Cl(f^{-1}(i\,Cl(V)) = f^{-1}(i\,Cl(V))$
Hence $(i,j)\delta P_S Cl(f^{-1}(V) \subseteq f^{-1}(i\,Cl(V))$.

$(b) \Rightarrow (c)$ Let F be any $i\ \beta$-closed set of Y. Then $Y \backslash F$ is $i\ \beta$-open set of Y and by (b), we have $(i,j)\ \delta P_S Cl(f^{-1}(Y \backslash F)) \subseteq f^{-1}(\ i\,Cl(Y \backslash F)) \Leftrightarrow (i,j)\ \delta P_S Cl(X \backslash f^{-1}(F)) \subseteq f^{-1}(Y \backslash i\,Int(F)) \Leftrightarrow X \backslash (i,j)\ \delta P_S(f^{-1}(F)) \subseteq X \backslash f^{-1}(\ i\,Int(F))$. Therefore, $f^{-1}(i\,Int(F)) \subseteq (i,j)\ \delta P_S Int(f^{-1}(F))$.

$(c) \Rightarrow (d)$. This is obvious since every semi closed set is β-closed set.

$(d) \Rightarrow (e)$. Let V be any i-semi open set of Y. Then $Y \backslash V$ is i-semi closed set and by (d), we have $f^{-1}\big(i\ Int(Y \backslash V)\big) \subseteq (i,j)\ \delta P_S\ Int\big(f^{-1}(Y \backslash V)\big) \Leftrightarrow f^{-1}\big(Y\ i\ Cl(V)\big) \subseteq (i,j)\ \delta P_S Int\big(X \backslash f^{-1}\ (V)\big) \Leftrightarrow X \backslash f^{-1}\big(i\ Cl(V)\big) \subseteq X \backslash (i,j)\ \delta P_S(f^{-1}(V))$. Therefore, $(i,j)\ \delta P_S\ Cl(f^{-1}(V)) \subseteq f^{-1}(\ i\ Cl(V))$.

$(e) \Rightarrow (a)$. Let F be any i-regular closed set of Y. Then F is i-semi open set of Y. By (e), we have $(i,j)\ \delta P_S\ Cl\big(f^{-1}(F)\big) \subseteq f^{-1}\big(\ i\ Cl(F)\big) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is $(i,j)\ \delta P_S$-closed set in X. Therefore, by Theorem 9.6.6(d), f is almost $(i,j)\ \delta P_S$-continuous.

Proposition 9.6.9: A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is almost $(i,j)\ \delta P_S$-continuous if and only if $f^{-1}(V) \subseteq (i,j)\ \delta P_S Int(f^{-1}(Int\big(j\ Cl(V)\big))$ for each ı-open set V of Y.

Proof. Necessity. Let V be any i-open set of Y. Then $V \subseteq i\ Int(\ i\ Cl(V))$ and $i\ Int\ (i\ Cl(V))$ is an i-regular open set in Y. Since f is almost $(i,j)\ \delta P_S$-continuous, by Theorem 9.6.7(e), $f^{-1}(\ i\ Int\big(\ i\ Cl(V)\big))$ is $(i,j)\ \delta P_S$-open set in X and hence we obtain that $f^{-1}(V) \subseteq f^{-1}\left(iInt\big(iCl(V)\big)\right) = (i,j)\ \delta P_S\ Int(f^{-1}(\ i\ Int\big(\ i\ Cl(V)\big))$.

Sufficiency. Let V be any i-regular open set of Y. Then V is i-open set of Y. By hypothesis, we have $f^{-1}(V) \subseteq (i,j)\ \delta P_S(f^{-1}\left(\ i\ Int\big(i\ Cl(V)\big)\right) = (i,j)\ \delta P_S Int(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an $(i,j)\ \delta P_S$-open set in X and hence by Theorem 9.6.7(e), f is almost $(i,j)\ \delta P_S$-continuous.

Corollary 9.6.10: A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is almost $(i,j)\ \delta P_S$-continuous if and only if $(i,j)\ \delta P_S Cl(f^{-1}\left(\ i\ Cl\big(\ i\ Int(F)\big)\right) \subseteq f^{-1}(F)$ for each i-closed set F of Y.

Proposition 9.6.11: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an almost $(i,j)\ \delta P_S$-continuous function and let V be any i-open subset of Y. If $x \in (i,j)\ \delta P_S\ Cl(f^{-1}(V)) \backslash f^{-1}(V)$, then $f(x) \in (i,j)\ \delta P_S Cl(V)$.

Proof. Let $x \in X$ be such that $x \in (i,j)\ \delta P_S\ Cl(f^{-1}(V)) \backslash f^{-1}(V)$ and suppose $f(x) \notin (i,j)\ \delta P_S Cl(V)$. Then there exists an $(i,j)\ \delta P_S$-open set H containing $f(x)$ such that $H \cap V = \emptyset$. Then $i\ Cl(H) \cap V = \emptyset$ implies $i\ Int(i\ Cl(H) \cap V = \emptyset$ and $i\ Int(i\ Cl(H))$ is i-regular open set. Since f is almost $(i,j)\ \delta P_S$-continuous, by Theorem 9.6.6(c), there exists an $(i,j)\ \delta P_S$-open set U in X containing x such that $f(U) \subseteq iInt(\ i\ Cl(H))$. Therefore, $f(U) \cap V = \emptyset$. However, since $x \in (i,j)\ \delta P_S\ Cl\big(f^{-1}(V)\big), U \cap f^{-1}(V) \neq \emptyset$ for every $(i,j)\ \delta P_S$-open set U in X containing x, so that $f(U) \cap V \neq \emptyset$. We have a contradiction. It follows that $f(x) \in (i,j)\ \delta P_S Cl(V)$.

Proposition 9.6.12: A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(i,j)\ \ \delta P_S$-continuous if and only if f is j- δ-pre continuous and for each $x \in X$ and each i-open set V of Y containing $f(x)$ there exists an i-semi-closed set F of X containing x such that $f(F) \subseteq V$.

Proof: Necessity. Let f be an $(i,j)\ \delta P_S$-continuous function. Let $x \in X$ and each i-open set of Y containing $f(x)$, then by hypothesis there exists an $(i,j)\ \ \delta P_S$-open set U of X containing x such that $f(U) \subseteq V$. Since U is $(i,j)\ \ \delta P_S$-open, then for each $x \in U$ there exists an i-semi-closed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$ and $(i,j)\ \delta P_S$-continuity always implies j- δ-pre continuity.

Sufficiency. Let V be any i-open set of Y. To show that $f^{-1}(V)$ is an $(i,j)\ \delta P_S$-open set in X. Since $f$is j- δ-precontinuous, then $f^{-1}(V)$ is an j- δ-preopen set in X. Let $x \in f^{-1}(V)$, then $f(x) \in V$. So by hypothesis, there exists an i-semi-closed set F of X containing x such that $f(F) \in V$. This implies that $x \in F \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is an $(i,j)\ \delta P_S$-open set in X.Therefore, so by Thereom 9.6.5(b), f is $(i,j)\ \delta P_S$-continuous.

Proposition 9.6.13: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and let $\mathcal{B}$ be any basis for σ_i in Y, then f is $(i,j)\ \delta P_S$-continuous if and only if for each $B \in i\ \mathcal{B}, f^{-1}(B)$ is an $(i,j)\ \delta P_S$-open subset of X.

Proof: Necessity. Suppose that f is $(i,j)\delta P_S$- continuous. Then since each $B \in i\ \mathcal{B}$ is an i-open subset of Y and f is $(i,j)\ \delta P_S$-continuous. $f^{-1}(B)$ is an $(i,j)\ \delta P_S$-open subset of X by Theroem 9.6.5(b).

Sufficiency. Let V be any i open subset of Y. Then $V = \cup\{B_k : k \in I\}$ where every B_k is a member of $i\ \mathcal{B}$ for a suitable index set I, it follows that $f^{-1}(V) = = f^{-1}(\cup\{B_k : k \in I\}) = \cup f^{-1}(\{B_k : k \in I\})$. Since $f^{-1}(B_k)$ is an $(i,j)\delta P_S$-open subset of X for each $k \in I$, by hypothesis $f^{-1}(V)$is the union of a family of $(i,j)\delta P_S$-open sets of X and hence is $(i,j)\ \delta P_S$open set of X, by Proposition 9.2.8. Therefore, by Theorem 9.6.5(b), f is $(i,j)\ \delta P_S$- continuous.

Proposition 9.6.14: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be (i,j)-perfectly continuous and i-contra continuous, then f is $(i,j)\ \delta P_S$-continuous.

Proof: Let V be any i-open subset of Y. Since f is (i,j)-perfectly continuous, so $f^{-1}(V)$ is j-open in X. Since f is i-contra continuous, so $f^{-1}(V)$ is i-closed in X. Therefore, by

Proposition 9.2.16 $f^{-1}(V)$ is (i,j) δP_S-open set in X, so by Theorem 9.6.5(b), f is (i,j) δP_S-continuous.

Proposition 9.6.15: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i,j) δP_S-continuous, if Y is an i-open subset of a bitopological space (Z, η_1, η_2), then $f: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is (i,j) δP_S-continuous.

Proof: Let V be any i-open set in Z, then $V \cap Y$ is an i-open set in Y. Since f is (i,j) δP_S-continuous, so by Proposition 9.6.5(b), $f^{-1}(V \cap Y)$ is (i,j) δP_S-open set in X but $f(x) \in Y$ for each $x \in X$. Thus $f^{-1}(V) = f^{-1}(V \cap Y)$ is an (i,j) δP_S-open set in X.Therefore by Theorem 9.6.5(b), $f: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is (i,j) δP_S-continuous.

Proposition 9.6.16: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be almost (i,j) δP_S-continuous, if Y is an i-preopen subset of a bitopological space (Z, η_1, η_2), then $f: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is almost (i,j) δP_S-continuous.

Proof: Let V be any i-regular open set of Z . Since Y is i-preopen then by Lemma 1.3.15, $V \cap Y$ is i-regular open set in Y. Since $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is almost (i,j) δP_S-continuous, so by Theorem 9.6.7(e), $f^{-1}(V \cap Y)$ is (i,j) δP_S-open set in X but $f(x) \in Y$ for each $x \in X$, hence $f^{-1}(V) = f^{-1}(V \cap Y)$ is (i,j) δP_S-open set in X.Therefore by Theorem 9.6.5(b), $f: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is (i,j) δP_S-continuous.

Proposition 9.6.17: If $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be almost (i,j) δP_S-continuous and $g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is i-continuous and i-open, then the composition function $g \circ f: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is almost (i,j) δP_S-continuous.

Proof: Let $x \in X$ and V be an i-open set of Z containing $(g \circ f)(x)$. Since g is i-continuous, so $g^{-1}(V)$ is i-open set of Y containing $f(x)$. Since f is almost (i,j) δP_S-continuous, then there exists an (i,j) δP_S-open set U of X containing x such that $f(U) \subseteq i\ Int(i\ Cl(g^{-1}(V)))$, also g is i-continuous, then $(g \circ f)(U) \subseteq g(i\ Int(g^{-1}(i\ Cl(V))))$. Since g is i-open we obtain that $(g \circ f)(U) \subseteq i\ Int(i\ Cl(V))$. Therefore $g \circ f$ is almost (i,j) δP_S-continuous.

Proposition 9.6.18: If $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be almost (i,j) δP_S-continuous and Y is i-semi-regular, then f is (i,j) δP_S-continuous.

Proof: Let $x \in X$ amd let V be any i-open set of Y containing $f(x)$, by the semi-regularity of Y, there exists an i-regular open set G of Y such that $f(x) \in G \subseteq V$. Since f is almost (i,j) δP_S-continuous, so by Theorem 9.6.6(c), there exists an (i,j) δP_S-open set U of X containing x such that $f(U) \in G \subseteq V$. Therefore, f is (i,j) δP_S-continuous.

Definition 9.6.19: A function $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called **contra (i,j) δP_S-continuous** if $f^{-1}(V)$ is (i,j) δP_S-closed in X for every j-open V of Y.

The following result is proved easily:

Proposition 9.6.20: The function $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is contra (i,j) δP_S-continuous if and only if for each j-closed set A containing $f(x)$, there exist an (i,j) δP_S-open set G containing x such that $f(G) \subseteq A$.

Proposition 9.6.21: If a function $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is contra (i,j) δP_S-continuous at x, then for each j-closed set A containing $f(x)$, there exists an i-semi-closed set F such that $f(F) \subseteq A$.

Proof: Let A be any j-closed set containing $f(x)$. Since f is contra (i,j) δP_S-continuous, so by Proposition 9.6.12, there exists an (i,j) δP_S-open set G containing x such that $f(G) \subseteq A$. Since G is (i,j) δP_S-open set, for all $x \in G$ there exists an i-semi-closed set F such that $x \in F \subseteq G$. This implies that $f(F) \subseteq f(G) \subseteq A$ which completes the proof.

Theorem 9.6.22: The following statements are equivalent for a function $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$:

a) f is contra (i,j) δP_S-continuous,
b) The inverse image of every j-closed set of Y is (i,j) δP_S-open in X.
c) For each $x \in X$ and each j-closed set B in Y with $f(x) \in B$, there exists an (i,j) δP_S-open set A in X such that $x \in A$ and $f(A) \subseteq B$.
d) $f\big((i,j)\ \delta P_S Cl(A)\big) \subseteq j\ \ker(f(A))$, for every subset A of X.
e) $(i,j)\ \delta P_S Cl(f^{-1}(A)) \subseteq f^{-1}(j\ \ker(A))$, for every subset A of Y.

Proof: $(a) \Rightarrow (c)$: Let $x \in X$ and B be j-closed set in Y with $f(x) \in B$. By (a), it follows that $f^{-1}(Y-B) = X - f^{-1}(B)$ is (i,j) δP_S-closed in X and so $f^{-1}(B)$ is (i,j) δP_S-open. Take $A = f^{-1}(B)$, we obtain that $x \in A$ and $f(A) \subseteq B$.

$(c) \Rightarrow (b)$: Let B be j-closed set in Y with $x \in f^{-1}(B)$. Since $f(x) \in B$, so by (c) there exists an (i,j) δP_S-open set A in X containing x such that $f(A) \subseteq B$. It follows that $x \in A \subseteq f^{-1}(B)$, hence $f^{-1}(B)$ is (i,j) δP_S-open set in X.

$(b) \Rightarrow (a)$: Follows directly from Theorem 5.2.7.

$(b) \Rightarrow (d)$: Let A be any subset of X and let $y \in j\ \ker(f(A))$, then there exists a j-closed set F containing y such that $f(A) \cap F = \emptyset$. Hence, we have $A \cap f^{-1}(F) = \emptyset$ and $(i,j)\ \delta P_S Cl(A) \cap \left(f^{-1}(F)\right) = \emptyset$, so $f\left((i,j)\ \delta P_S Cl(A)\right) \cap F = \emptyset$ and $y \notin f\left((i,j)\ \delta P_S Cl(A)\right)$. Thus $f\left((i,j)\ \delta P_S Cl(A)\right) \subseteq j\ \ker(f(A))$.

$(d) \Rightarrow (e)$: Let A be any subset of Y, so by (d), $f\left((i,j)\ \delta P_S Cl(f^{-1}(B))\right) \subseteq j\ \ker(B)$. Hence $(i,j)\ \delta P_S Cl(f^{-1}(B)) \subseteq f^{-1}(j\ \ker(B))$.

$(e) \Rightarrow (a)$: Let B be any j-open set of Y, by (e), $(i,j)\ \delta P_S Cl\left(f^{-1}(B)\right) \subseteq f^{-1}(j\ \ker(B)) = f^{-1}(B)$ and hence $(i,j)\ \delta P_S Cl\left(f^{-1}(B)\right) = f^{-1}(B)$. Therefore, we obtain that $f^{-1}(B)$ is $(i,j)\ \delta P_S$-closed in X, so f is contra $(i,j)\ \delta P_S$-continuous.

Proposition 9.6.23: If a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is both j-contra continuous and $j\ i$- perfectly continuous, then f is contra $(i,j)\ \delta P_S$-continuous.

Proof: Let V be any j-open set in Y. Since f is $j\ i$-perfectly continuous so $f^{-1}(V)$ is both i-open and i-closed in X and since f is j-contra continuous, so $f^{-1}(V)$ is j-closed in X. Hence $f^{-1}(V)$ is a $(i,j)\ \delta P_S$-closed set in X, so f is contra $(i,j)\ \delta P_S$-continuous.

Proposition 9.6.24: If a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is contra $(i,j)\ \delta P_S$-continuous and $g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ is j-R-map, then $g \circ f$ is contra $(i,j)\ \delta P_S$-continuous.

Proof: Let V be any j-regular open set of Z. Since g is j-R-map, so $g^{-1}(V)$ is a j-regular open set in Y and since every regular open set is an open set, so $g^{-1}(V)$ is a j-open set in Y. Since f is contra $(i,j)\ \delta P_S$-continuous, so $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $(i,j)\ \delta P_S$-closed set in X. Therefore, $g \circ f$ is contra $(i,j)\ \delta P_S$-continuous.