Chapter VII

CHAPTER VII

LINEAR OPERATORS PRESERVING PAIRS OF HERMITIAN MATRICES ON WHICH THE RANK IS ADDITIVE

Denote by $\mathbf{M}_n$ the linear space of all complex $n \times n$ matrices. We say that $A \in \mathbf{M}_n$ is **Hermitian** (resp. **skew Hermitian**) if $A^* = A$ (resp. $A^* = A$). Denote the set of all complex $n \times n$ Hermitian (resp. skew Hermitian) matrices by $\mathbf{H}_n$ (resp. $\mathbf{S}_n$). Clearly, $\mathbf{H}_n$ and $\mathbf{S}_n$ are linear spaces over $\mathbb{R}$. For an arbitrary field $\mathbb{F}$, $\mathbf{M}_{m,n}(\mathbb{F})$ denotes the linear space of all $m \times n$ matrices over $\mathbb{F}$. We say that a pair of $m \times n$ matrices (A,B) is **rank-additive** if rank (A+B) = rank A + rank B. For a subspace V of $\mathbf{M}_{m,n}(\mathbb{F})$, let Θ_+^V be the subset $V \times V$ consisting of all rank-additive pairs. A linear map $f : V \to V$ is said to preserve the set Θ_+^V if

$$f(\Theta_+^V) \subseteq \Theta_+^V.$$

This chapter characterizes the linear preservers of the set $\Theta_+^{H_n}$.

Theorem : 7.1 [1]

Let $n \geq 2$ be an integer, and f be a bijective map from $\mathbf{H}_n$ onto itself. If f preserves rank one matrix, i.e., rank (A) = 1 implies rank f(A) = 1 for all $A \in \mathbf{H}_n$, then f is either of the form

$$f(A) = cPAP^* \qquad , \forall\ A \in \mathbf{H}_n , \qquad \text{-------------------------(1)}$$

or of the form

$$f(A) = cPA^tP^* \qquad , \forall\ A \in \mathbf{H}_n , \qquad \text{-------------------------(2)}$$

where $c \in \mathbb{R}^*$, and $P \in GL_n$.

Based on the above theorem, the linear preservers of the set $\Theta_+^{H_n}$ are characterized as follows.

Theorem : 7.2

Let $n \geq 2$ be an integer, and f : $\mathbf{H}_n \rightarrow \mathbf{H}_n$ be a linear map preserving the set $\Theta_+^{H_n}$. Then either $f \equiv 0$ or of the form

$$f(A) = cPAP^* \qquad , \forall\ A \in \mathbf{H}_n ,$$

or of the form

$$f(A) = cPA^tP^* \qquad , \forall\ A \in \mathbf{H}_n ,$$

where $c \in R^*$, and $P \in GL_n$.

Proof :

We distinguish two cases :

(i) Suppose f is not injective. Then f(A) = 0 for some $A \in \mathbf{H}_n$ with rank A = $s \geq 1$. Without loss of generality, we may assume that

$$f\left(\sum_{i=1}^{s} a_i E_{ii}\right) = 0 , \qquad \text{-------------------------(3)}$$

where $a_i \in R^*$, i = 1,2,……,s. By the definition of f and $\left(a_1 E_{11}, \sum_{i=1}^{s} a_i E_{ii}\right) \in \Theta_+^{H_n}$,

we have $\left(f(a_1 E_{11}), f\left(\sum_{i=1}^{s} a_i E_{ii}\right)\right) \in \Theta_+^{H_n}$, or equivalently,

$$\text{rank } f\left(\sum_{i=1}^{s} a_i E_{ii}\right) = \text{rank } \left(f(a_1 E_{11}) + f\left(\sum_{i=2}^{s} a_i E_{ii}\right)\right)$$

$$= \text{rank } f(a_1 E_{11}) + \text{rank } f\left(\sum_{i=2}^{s} a_i E_{ii}\right).$$

This, together with (3), yield that rank $f(a_1E_{11}) = 0$.

Thus,

$$f(E_{11}) = 0. \qquad \text{-------------------------(4)}$$

For any $2 \leq j \leq n$, $x \in R^*$, it follows from $\left(x^{-1}E_{11} + E_{1j} + E_{j1} + xE_{jj}, -xE_{jj}\right) \in \Theta_+^{H_n}$ that

$$\text{rank } f\left(x^{-1}E_{11} + E_{1j} + E_{j1}\right) = \text{rank } f\left(x^{-1}E_{11} + E_{1j} + E_{j1} + xE_{jj}\right) + \text{rank } f\left(-xE_{jj}\right).$$

This, together with (4) and the linearity of f, gives that

$$\text{rank } f\left(E_{1j} + E_{j1}\right) = \text{rank } f\left(E_{1j} + E_{j1} + xE_{jj}\right) + \text{rank } f\left(E_{jj}\right).$$

Let x = 1 and 2 in the above equality, it is easy to see that

$$\text{rank } f(E_{1j}+E_{j1}+E_{jj}) = \text{rank } f(E_{1j}+E_{j1}+2E_{jj}) \text{--------------------(5)}$$

On the other hand, from $(E_{11}+E_{1j}+E_{j1}+E_{jj}, E_{jj}) \in \Theta_+^{H_n}$ and (4), we have

$$\text{rank } f(E_{1j}+E_{j1}+2E_{jj}) = \text{rank } f(E_{1j}+E_{j1}+E_{jj}) + \text{rank } f(E_{jj}).$$

This, together with (5), implies rank $f(E_{jj}) = 0$, i.e.,

$$f(E_{jj}) = 0 \qquad , \forall \ 2 \le j \le n \qquad \text{----------------------(6)}$$

For any positive integers i,j with $1 \le i \le j \le n$ and $c \in C^*$, because of $(c\bar{c}E_{ii}+cE_{ij}+\bar{c}E_{ji}+E_{jj}, c\bar{c}E_{ii}-cE_{ij}-\bar{c}E_{ji}+E_{jj}) \in \Theta_+^{H_n}$ from (4) and (6) that

$$0 = \text{rank } f(0) = \text{rank } f(cE_{ij}+\bar{c}E_{ji}) + \text{rank } f(-cE_{ij}-\bar{c}E_{ji}).$$

Consequently,

$$f(cE_{ij}+\bar{c}E_{ji}) = 0 \qquad , \forall \ 1 \le i \le j \le n \ , c \in C^* \qquad \text{----------------------(7)}$$

By the arbitrariness of i and j and the linearity of f, we obtain from (4), (6) and (7) that $f \equiv 0$.

(ii) Suppose f is injective. Now, we will show that f preserves rank one matrices, i.e., if $A \in \mathbf{H}_n$ with rank A = 1, then rank f(A) = 1. Let $A \in \mathbf{H}_n$ with rank A = 1. Then there is $P \in GL_n$ such that $A = d_{11}PE_{11}P^*$, where $d_1 \in R^*$. Let $A_i = PE_{ii}P^*$ $(2 \le i \le n)$. Thus

$$\text{rank } (A + A_2 + \ldots\ldots + A_n) = \text{rank } A + \text{rank } A_2 + \ldots\ldots + \text{rank } A_n .$$

Since f is an injective linear map preserving $\Theta_+^{H_n}$, we have

$$n \ge \text{rank } f\left(A + \sum_{i=1}^{n} A_i\right) = \text{rank } f(A) + \sum_{i=1}^{n} \text{ rank } f(A_i) \ \ge \ n$$

which implies rank $f(A) = 1$.

Thus, by Theorem 7.1, we see that f has one of the forms (1) or (2). The proof is completed.

As a corollary to Theorem 7.2, we can easily obtain the following result.

Corollary : 7.3

Let $n \geq 2$ be an integer, and f : $\mathbf{H}_n \rightarrow \mathbf{H}_n$ be a nonzero map. Then the following three conditions are equivalent :

(i) f is a linear preserver of rank, i.e., rank f(A) = rank A for all $A \in \mathbf{H}_n$.

(ii) f is a linear preserver of the set $\Theta_{+}^{H_n}$.

(iii) f has one of the forms (1) or (2).